Analytically Modeling Social Norms Using Evolutionary Game Theory

by

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I. Introduction and Literature Review

1.1 Introduction

In this work I apply the framework of evolutionary game theory to study the emergence and stability of social norms. Specifically, I develop an analytic version of Robert Axelrod’s computer simulated Norms Game model from his 1986 article “An Evolutionary Approach to Norms”. In this section I will first introduce and discuss the idea of social norms and then I will proceed to define some basic concepts in game theory. Next I discuss how game theory can be used to study the development of social norms and my rationale for using evolutionary game theory rather than non-cooperative strategic game theory. In the next subsection I present literature on this subject and in particular the results from Axelrod’s paper in order to establish a reference case for my results. The next several chapters will be devoted to developing and discussing my model. I conclude and discuss possible topics for future research in the final chapter.

I now turn to the concept of social norms, a subject that has gotten attention in economics, as well as other fields over the last few decades (Axelrod and Hammond, 2003; Basu, 1986; Basu, Jones and Schlicht, 1987; Centola, Macy, and Willer 2005; Florini, 1996; Grimalda, Kar, and Proto, 2008; Kolstad, 2005; Young, 2007). A concise definition is given by Axelrod: “a norm exists in a given social setting to the extent that individuals usually act in a certain way and are often punished when seen not to be acting in this way”. In this light, not cheating on an exam and shaking hands with your right hand are both norms in as much as there is usually some punishment
associated with both. However, the punishment for cheating on an exam is typically concrete and severe, whereas any punishment for shaking hands with your left hand would be more mild and ambiguous, such as a funny look. That is to say, the severity of the punishment is normally proportional the severity of the offense. Perhaps a more insightful definition is provided by (Young, 2007) “Social norms are customary rules of behavior that coordinate our interactions with others. Once a particular way of doing things becomes established as a rule, it continues in force because we prefer to conform to the rule given the expectation that others are going to conform (Lewis, 1969). As a consequence, social norms provide a mechanism to direct interactions between individuals without the express supervision of an authority (Axelrod, 1986; Young, 2007). There are many examples of social norms. They include driving on the right side of the road (in most parts of the world), not cheating on exams, queuing rather than pushing, and extending your right hand in greeting. Some conventions, such which side of the road to drive on, have proven to be so necessary that laws have been written to enforce them.

The establishment and development of social norms is a complicated and very interesting social phenomenon. It is fairly straightforward to understand why a student may wish to cheat on an exam, and a teacher may wish to punish the student for doing so. However, we may be biased by the fact that this is already an established norm.

Dueling used to also be a social norm. But at the present time most people would not consider risking their life in that fashion over an argument or humiliation. Are we to assume that people valued their life less in, say, the 19th century than they
do at the present time? I don’t think so. How does an acceptable and appropriate behavior become unthinkable? A reasonable explanation would be that more preferable means of restitution have become more readily available in our society rendering dueling no longer optimal. Currently an individual would take an adversary to court or call the police to resolve a dispute. This is an example of one norm replacing another. Similarly, there are strong norms against colonialism and slavery in the international community, whereas such practices were once accepted.

As is evident from the above examples, social norms are highly prevalent in society and often are a deeply ingrained part of our lives. This makes social norms worth studying and understanding in their own right. However, they are also subject to change in the long term, and the complicated dynamics exhibited by social norms as they change over time is an even more interesting subject of study. Understanding the dynamics of how norms develop and evolve can be very useful in understanding them. This knowledge can then be used to accomplish broader goals. For example, it can be used to effectively establish new norms in favor of some desirable behavior, or against undesirable behavior.

Game theory is widely used in the literature to model the dynamics of social norms. I will therefore present a brief introduction to game theory below, after which I will make the connection between game theory and social norms more explicit.

In the context of game theory, a (strategic) game is “an action situation where there are two or more mutually aware players and the outcome for each depends on the actions of all” (Dixit and Skeath, 2004, pp. 635). “Mutually aware” can be loosely interpreted, especially in the case of evolutionary game theory since a player does not
necessarily have to aware of another player per se, to be affected by his or her actions. A game comes with a set of strategies that players may choose from. Each strategy combination comes with a payoff, which is a numerical value that the player earns when he or she takes that action, given the actions of others. More systematically, games in *normal form* are defined by the set of players and strategy sets for each player. Payoffs are defined as functions of strategy combinations. Higher numbers are associated with more successful strategies. If an action can result in more than one outcome, we define the expected payoff as the sum of the payoffs weighted by their respective probabilities.

Rationality is usually assumed in game theory. That is, we assume that each player is capable of figuring out which of the available strategies will result in the highest payoff to that player. Further, it is assumed that each player pursues the strategy that he or she calculates as best (Dixit and Skeath, 2004, pp. 30). This is referred to as rational choice game theory. We can further subdivide this sort of game theory into cooperative and non-cooperative game theory. Cooperative simply refers to the fact that agreements can be made and enforced between players prior to taking action. Non-cooperative refers to the fact that enforcement is not possible in the game and each player acts according to his or her best interests.

Evolutionary game theory, on the other hand, allows for limited or no rationality (Dixit and Skeath, 2004, pp. 35). That is, we may assume that not all players are capable of perfectly calculating their optimal strategy. In evolutionary game theory, it is usually assumed that players are programmed with a strategy, their type, and they always play that strategy. Types that earn a higher payoff, that is, types
that are relatively more successful, will be reproduced more in the future, and less successful types will decrease in number. In evolutionary game theory a replication function is used to represent how this reproduction occurs. So, while each player does not choose his or her strategy, the evolutionary pressure dictated by the replication function selects for strategies that have proven to be relatively more successful in the past. This has its roots in evolutionary biology (Maynard Smith and Price, 1973) where the more successful members of a species are more likely to reproduce and pass on their genes which are strategies in the context of evolutionary game theory. Another way to think of this which would be more applicable to the setting of social norms is that the replication function is mimicking the effect of players learning from past experience and in the future choosing to use strategies that have proven to be more successful in the past.

Finally I will define the notion of equilibrium. In strategic game theory, equilibrium is the state where each agent is playing the best response strategy given what the other agents are doing. That is to say, no player can do better, given what the other players are doing. In non-cooperative game theory this is known as Nash equilibrium. The analogous concept in evolutionary game theory is an evolutionarily stable equilibrium (Taylor and Jonker, 1978), which is the result of the dynamical process described above. A population distribution of strategies is deemed evolutionarily stable if it is robust against mutations. That is to say, if a small number of mutants invade the population, they cannot do better or even as well as the rest of the players in the population. Otherwise the invaders (mutant strategy) will be selected for by the replication function and the population will be moved away from
the equilibrium. An ESE is equivalent to a Nash Equilibrium with the added requirement of stability, which is not necessary for the former. The stability requirement is due to the possibility of mutation in EGT that does not exist in rational choice game theory.

The connection between social norms and game theory is that social norms can be represented by equilibria in appropriately chosen games (Young, 2007). A variety of games have been used to model social norms. Examples include ultimatum games (Grimalda, Kar, and Proto, 2008) and prisoner’s dilemmas (Axelrod, 1986; and Sethi, 1996). In particular, the evolutionary approach is often employed (Axelrod, 1986; Axelrod and Hammond, 2003; Kolstad, 2005; Sethi, 1996; Young, 2007).

Rational choice game theory and evolutionary game theory each have their merits and shortcomings. While non-cooperative game theory can be a good approximation of human behavior, I make the case for using evolutionary game theory to model the development of norms instead. Non-cooperative game theory requires rational behavior which implies making very complex calculations to make even relatively simple choices. It is by no means obvious that every human being should be able to accurately make such calculations given the vast number of variables that influence everyday life. In addition, people tend to base their decisions on personal preference, beliefs, and principles. Also, a person’s actions often reflect the type of person he or she is. Therefore, it seems reasonable to represent people as types.

Many individuals learn beliefs and behaviors from their parents, similar to the way they inherit their genes. These learned beliefs and behaviors dictate the actions
that the individual will choose to take in certain situations, much like a player’s type determines the strategy that he chooses to play, in EGT. This is in line with the notion that players are “types” and may play strategies that do not necessarily give the highest payoff. Analogously, some individuals will never take a particular action if they believe it is immoral, even if it gave them significantly higher payoff than the action they are taking. The evolutionary framework allows for a player to credibly commit to not taking such an immoral action. As mentioned before, in EGT the replication function is a selection rule for choosing strategies. Using an appropriate replication function allows us to incorporate some aspects of rational choice game theory. Specifically, the replication function can be made to select for strategies with higher payoffs because we expect more people to start using a strategy that has proven relatively more successful in the past.

In addition, the evolutionary game theory permits us to study the dynamics of a changing system which is affected by random shocks. The development of norms is one such dynamic system which depends on properties of the interaction and the population being modeled (Kolstad, 2005), but which can also be influenced by chance events. Rational choice game theory does not allow for random mutations and generally does not model dynamical processes.

Another argument in favor of using evolutionary game theory to model the development of social norms is indirectly provided by Florini (1996). Florini draws three strong parallels between evolutionary biology and the development of norms. The analogy she draws is that between norms and genes. Genes, like norms, direct the behavior of the individuals in the population. Both genes and norms are passed on
from one generation to the next, either through biological reproduction or through cultural transmission. Finally, both are subject to selective forces. Genes that give individuals a reproductive advantage increase in frequency from one generation to another. Similarly, actions that result in higher payoffs are more likely to be repeated in the future. For example, parents will arguably teach their children to behave in a certain way if they believe that the behavior will make their children more successful in the future. In both settings one gene/norm can prevail over all others, or a mixture of genes/norms can coexist in a population. In game-theoretic terminology these outcomes are known as *monomorphic* and *polymorphic* equilibria respectively.

**I.2 Literature Review**

I now turn to the literature where authors have studied norms, and in particular, employed evolutionary game theory to model their development.

In their seminal work, published in 1973, Maynard Smith and Price use evolutionary game theory and computer simulation to model intra-species combat behavior in animals. Males of the same species generally fight each other for mates and territory. Intuitively, one might expect that on an individual level, “total war” strategies, where the winner seriously injures or even kills his opponent, would be selected for in this type of conflict. However, this is not the case. Animals will usually use non-lethal “limited war” tactics when fighting with members of their own species. From the point of view of the species as a whole “limited war” strategies are superior to “total war” strategies. Clearly, abundant use of dangerous tactics will lead to a high mortality rate in the population, which would limit the species’ chances of survival. The authors’ results show that “limited war” tactics are superior to more
dangerous strategies on the individual level as well. Therefore, animals which use “limited war” strategies are selected for by evolutionary forces.

Basu studies the nature and consequences of triadic relations (Basu, 1986). Here triadic simply refers to the fact that a third party can influence relations between two other players. The author presents several models. The first general conclusion he makes is that under certain conditions, a society will continue to support an unpopular regime or norm even if not supporting the norm would give every member of the population a higher payoff. Suppose each individual in the population starts with an endowment. Trading with other members of society increases each agent’s utility. However, each individual must give a unit of his endowment to the ruler (or to support the norm). Furthermore, an individual is deemed disloyal if he does not support the ruler or he does business with someone who is disloyal. The individual is punished for not supporting the ruler. Under the assumption that every agent believes that no other individual would be disloyal, every person will choose to support the king because each individual would expect to lose everyone else’s business as well as to be punished. This would make the person worse off than if he supports the ruler. Therefore the unpopular regime would be sustained.

Another scenario that Basu studies is a triadic relationship between a landlord, a laborer, and a merchant. Suppose the landlord wants to hire the laborer to work a certain amount of hours at a fixed wage. The laborer will accept the offer if it gives him at least his reservation utility, which is his utility from his next best alternative. The worker will reject the offer if the arrangement gives him less utility than his reservation utility. Now suppose 1) the laborer also receives utility from trading with
the merchant and 2) the landlord can credibly prevent the merchant from trading with the laborer if the laborer does not accept the landlord’s offer. Now the laborer will accept an offer from the landlord which gives the laborer his reservation utility minus the utility he would receive from the merchant. Under such circumstances, the author argues that the laborer will accept transactions with the landlord that will give the laborer negative utility. Both of these scenarios are examples of what I will later define as metanorms.

In his 1996 paper “Evolutionary Stability and Social Norms” Rajiv Sethi employs an analytical method to characterize the emergence of certain stable patterns of behavior that can be viewed as norms. He employs the principle of evolutionarily stable states and replicator dynamics to obtain his results. There is a two stage repeated game, where the first stage is a prisoner’s dilemma and in the second stage the players can choose to punish defection. Players within the population interact in pairs. Players can be payoff maximizers (PM’s) or not. There is a class of non-optimizing players for each strategy. (The strategies of interest are as follows: Vengeful cooperator (VC) always cooperates and punishes defection, passive defector (PD) always defects never punishes, Bully (BB) always defects and punishes cooperation). Cooperation is costly (-α) and confers a benefit (β) to the other player. Inflicting punishment is also costly (-γ) and inflicts punishment (-δ). Under the assumptions that δ>α, and payoff maximizers can perfectly recognize their opponents, Sethi obtains the following results. A population composed entirely of vengeful cooperators is an Evolutionarily Stable State (ESS) for all values of the parameters. Second, if the enforcement cost is sufficiently high, then a mixture of bullies and
optimizers is also an ESS. He also concludes that a state of all payoff maximizers is not even neutrally stable, a weaker condition than ESS. The author later drops the assumption that PM’s can perfectly recognize their opponents and attains similar results. There exist stable states that payoff maximizers cannot invade.

One conclusion is that payoff maximizing behavior does not necessarily have to earn the highest payoff. The ability to credibly follow a strategy may solve the commitment problem. Another conclusion is that many norms can be stable. Also, the existence of multiple equilibria implies that initial conditions and chance events affect which norm is established in the population.

I now turn to Axelrod’s 1986 paper “An Evolutionary Approach to Norms”. I found this paper particularly interesting because his model was more concretely grounded in a real life situation. What specifically drew my attention to this article was that the author was using evolutionary game theory to model the establishment of a no-cheating norm. The model was made more realistic because the author included the probability of a defection being observed as an exogenous parameter in addition to endogenizing two characteristics for each player. I describe his model and results in some detail below, since I used it as the basis for my own work.

In this paper Axelrod uses a computer simulation to model the evolution of norms and to analyze the dynamics of the social norms game. The basic structure is similar to an n-person prisoner’s dilemma, which he extends. The setup goes like this: there exists a population of 20 people where every individual has the opportunity to cheat. If player $i$ chooses to cheat he gains the temptation payoff, $T = 3$, which is higher than 0, which is what he would get if he chose not to cheat. However, he also
inflicts damage, $H = -1$, on all other players by cheating. The extension beyond the
prisoner’s dilemma structure is that each opportunity to cheat carries with it a
probability, $S$, between zero and one, of being observed. Player $i$ will choose to cheat
if his boldness score, $B_i$, is higher than the probability of being observed ($B_i > S$). $B_i$
can take on one of eight values 0, 1/7, 2/7…, 7/7, and is randomly assigned to each
player at the start of the game. If player $j$ observes player $i$ cheating, player $j$ may
choose to punish player $i$ by inflicting a large negative payoff $P = -9$ on player $i$.
However, player $j$ also incurs an enforcement cost, $E = -2$, on himself. The
probability that player $j$ will choose to punish player $i$ is determined by player $j$’s
vengefulness score $V_i$, which, like the boldness score, is randomly assigned and can
take on one of eight values. The larger his vengefulness score, the more likely it is
that he will choose to punish. Mutations can occur at random with a fixed probability
so that new strategies can arise. In each round each player has four chances to cheat.
At the end of each round every player’s payoffs are determined by summing
$P+E+H+T$ that the player incurred during the round. These scores are used to
determine the number of “offspring” each player has. Players with scores within one
standard deviation from the average get one offspring each. Players who have scores
one standard deviation above the average or higher get two offspring. Players with
scores one standard deviation below the average or lower do not reproduce at all. The
new total number of players is normalized to have the same total population as in the
previous round. One hundred rounds (generations) are played. The above process is
repeated five times. The results consist of the five points where each of the five
populations ends up, which are considered stable states. Two end up with high
average boldness and low average vengefulness, two are in the middle, low average boldness and vengefulness, and one is a partial establishment of the no-cheating norm, with low average boldness and moderate average vengefulness. Each of the five runs start out with an average boldness and vengefulness of about one half. At first the level of vengefulness is high enough to drive down boldness. Once boldness has decreased, there is a gradual decline of vengefulness since enforcement is costly and not worth paying to punish the occasional defector. After vengefulness falls below a certain point, bold individuals are again relatively more successful and the norm completely breaks down.

In an attempt to find a remedy for the situation described above, Axelrod introduces *metanorms*. That is, each player has the chance of observing another player failing to punish a defector, and can choose to punish the non-punisher. Again, the author uses a computer simulation to arrive at equilibria in the population. This time all five of the equilibria were in the high average vengefulness low average boldness region—i.e. the anti-cheating norm was established in all cases. This shows that *metanorms* can be used to protect established norms, inasmuch as the norm against defection becomes self-policing. However, this does require the population to start with a sufficiently high level of average boldness; otherwise the norm can still break down.

Another mechanism to maintain a norm Axelrod discusses is *internalization*. This can be modeled with a different payoff matrix, where the temptation $T$ is negative rather than positive. That is to say, violating the norm is psychologically painful, and more than negates the benefit from violating the norm. The question to
ask would be how many people have to internalize the norm for it to be stable? Another way to model internalization is to make the enforcement cost incurred positive rather than negative, which means that it gives you pleasure to punish a defector. Again, given enough people for whom E>0, and a sufficiently high probability of being observed, a non-cheating norm may be sustained.

As it is, Axelrod’s model is very interesting and insightful. However, I believe more insight can be gained by modeling his norms game analytically. In my analytic model, s, the probability of a defection being observed, as well as the size of the population are population parameters rather than random computer generated numbers or fixed altogether. Thus, I can study the effect of varying these parameters on norm establishment.

Another issue I want to investigate is stability. Axelrod is not explicit about which equilibria are stable and which are not. In both the norms game and the metanorms game his results are the points where the populations end up at after 100 generations. But how stable are those outcomes? Will those populations remain at those points after 100 more generations? More questions can be asked. Why did the populations end up where they did? What characteristics are required for a population to move to any particular average boldness and vengefulness? Can a population end up with any other combination of average vengefulness and average boldness after 100 generations? Are these results specific to populations of twenty players? I hope to investigate these questions with my analytic model. I start with a simple version with vengefulness as an exogenous parameter, which I later advance by endogenizing vengefulness.
II. Model 1: A Simple Version of the Norms Game

II.1 Model

This analytic, evolutionary game theoretic model is based on Axelrod’s computer simulated “Norms Game”. While my model shares many aspects with Axelrod’s model, it has a number of differences. Taking an analytic approach allows me to generalize some of Axelrod’s results. Specifically, I can analyze the effects of varying population parameters on the equilibrium outcome, and I am not confined to studying populations of exactly 20 individuals. Further, I can avoid idiosyncrasies resulting from chance to which Axelrod’s computer generated model is susceptible. However, to make the calculations and analysis tractable, I was forced to make some simplifications. In my model, boldness and vengefulness parameters do not take on as many values as in Axelrod’s paper. However, the benefit of my model is that I can study the conditions that will support evolutionarily stable equilibria, and in particular, conditions that make the no cheating norm stable. Furthermore, this model will allow me to draw more general conclusions about all populations rather than a few specific populations that happen to be generated by a computer simulation.

Just as the aforementioned author, I have a norms game with a population where the agents have a choice between cheating and cooperating. Like in Axelrod’s paper, a player will cheat if his boldness score is higher than the probability of being observed, and will cooperate if it is not. Cheating pays a temptation payoff but incurs a punishment if it is observed by another agent who punishes the defector. Defection by an agent also inflicts damage on all other players. Punishing a defector is costly to
the punisher. An individual’s vengefulness score is effectively the probability that the agent will punish a defector.

I use Axelrod’s Payoff Structure:

\[ \begin{align*}
T &= +3 \text{ payoff for defection (temptation)} \\
H &= -1 \text{ damage inflicted on other agents by a defector} \\
P &= -9 \text{ punishment for a defection} \\
E &= -2 \text{ enforcement cost}
\end{align*} \]

I start with a simple model, in which I keep the vengefulness score constant for the entire population. This can be viewed as everyone in the population having the same vengefulness score. This is simpler than what Axelrod does, but the model will allow me to work out how average vengefulness in the population as well as other population characteristics affect the equilibrium outcome. The variables are defined as follows:

*Let \( n \) be the size of the population.*

*Let \( s \in [0,1] \) be the probability that an individual \( j \) will observe an individual \( i \) defecting.*

*Let \( b_i \in [0,1] \) be the boldness level of individual \( i \).*

*Let \( v \in [0,1] \) denote the average vengefulness in the population.*

*Let \( H = \text{High Bold Players} = \{ \text{person } i : b_i > s; \text{these players will defect} \} \)*

*Let \( L = \text{Low Bold Players} = \{ \text{person } i : b_i \leq s; \text{these players will not defect} \} \)*

*Let \( p \in [0,1] \) denote the proportion of high bold people.*

Since the probability \( s \) is an exogenously given characteristic of each population and the boldness score, \( b_i \), is fixed for each individual, whether a particular player will cheat or not is predetermined by these two numbers. Therefore,
each player is one of two types, H for high boldness or L for low boldness, which makes this truly an evolutionary model rather than a rational choice model.

It follows, from the definition of p that there are n * p high bold players in the population and n * (1 – p) low bold players in the population. With the parameters and the payoff structure defined, we can write down the expected payoffs to the two types of players in the population.

The expected payoff to the high bold type is the sum of the following four terms:

1. +3 is the temptation to defect
2. –9 * (n – 1) * v * s is the expected value of your punishment
3. –1 * (n * p – 1) is the damage inflicted by defectors cheating.
4. –2 * (n * p – 1)v * s is the expected value of the cost of enforcement

This sum simplifies to the expression below.

\[ \pi^H = 3 - 9 * (n - 1) * v * s + (1 - n * p) * (1 + 2 * v * s) \]

The number of punishments, k, is a random variable with a binomial distribution with

\[ f(k) = \sum_{k=1}^{n-1} \binom{n - 1}{k} (vs)^k (1 - vs)^{n-1-k} \]

and

\[ E[k] = (n - 1) * v * s \]

So the expected value of the punishment is

\[ -9 * (n - 1) * v * s \]
The expected value of the enforcement cost is calculated in a similar fashion.

Players are not damaged by their own defection and a player will not enforce against him- or herself. Therefore, he will not incur enforcement cost for punishing himself; neither will he incur punishment from himself. This explains why I subtract one from n in lines two, three, and four. As shown below, we do not have to worry about this with cooperators. Since L type people do not cheat, the expression for their expected punishment and expected enforcement cost is a little simpler.

The payoff to low bold players is the sum of the following four terms.

1. \( +0 \) is the payoff to not defecting
2. \( +0 \) is the expected value of punishment
3. \(-1 \cdot n \cdot p\) is the damage inflicted by all the defectors defecting
4. \(-2 \cdot n \cdot p \cdot v \cdot s\) is the expected value of enforcement cost

This sum can be rewritten as the following expression.

\[
\text{Expected Payoff to low bold people:} \\
\pi^L = -n \cdot p \cdot (1 + 2 \cdot v \cdot s)
\]

With the payoffs defined I can now write down the replication function.

\[
\Delta p = \pi^H - \pi^L \\
= 4 + 11 \cdot v \cdot s - 9 \cdot n \cdot v \cdot s \\
= 4 + (11 - 9 \cdot n) \cdot v \cdot s
\]

In words, this means that the change in the fraction, p, of H type people is proportional to the difference in payoffs to the two types of players. If the cheaters are receiving a higher payoff than the cooperators, their numbers will increase relative to that of the cooperators.
Now that we have a replication function I will give a formal definition of an equilibrium and define what makes an equilibrium evolutionarily stable.

**Definition 1**: A population composition is an *equilibrium* if all agents in the population receive the same payoff and therefore, there is no evolutionary pressure in favor of any type over another (Sethi, 1996).

**Definition 2**: An equilibrium population proportion $p^*$ is an *evolutionarily stable equilibrium*, henceforth ESE, if one of the following conditions is satisfied:

1. $p^* = 1$ and $\Delta p > 0$, for small mutations away from $p = 1$.
2. $p^* = 0$ and $\Delta p < 0$, for small mutations away from $p = 0$.
3. $p^* \in (0,1)$, $\Delta p = 0$, and $\frac{d(\Delta p)}{dp} < 0$ at equilibrium.

$$\frac{d(\Delta p)}{dp} < 0 \text{ must also hold for small mutations away from } p^*$$

The second definition requires some explanation. The general idea is that for an equilibrium to be evolutionarily stable, mutations away from the equilibrium value must be removed by evolutionary pressure, so that the population returns to its original equilibrium composition. The equilibrium $p^* \in (0,1)$ is often referred to as an interior solution and $p^* = 1$ and $p^* = 0$ are referred to as corner solutions. The first type of equilibrium listed is where the population is composed entirely of cheaters and has no cooperators. This is also referred to as a *monomorphic* equilibrium because there is only one type of player present. For the population proportion $p$ to be stable at this value, any mutation away from 1 must be counteracted. That is, the replication function has to be positive to reflect the fact that the population proportion will return to the value one and mutations will be driven
out. Similarly, \( p = 0 \) is a population that has no cheaters. For this population composition to be an equilibrium, any increase in \( p \) must be reversed. Therefore the replication function needs to be negative when \( p \) is close to zero. Finally, we can have an equilibrium where both types of agents (cheaters and cooperators) are present. This is a polymorphic equilibrium. In this case \( p \) is between zero and one. For \( p \) to have a stable interior equilibrium, the replication function should be equal to zero to reflect the fact both types are receiving the same payoff, and therefore there is no tendency for the population proportion to change. The derivative of the replication function needs to be negative in order to counteract any mutation away from the equilibrium distribution.

Now we apply the stability conditions to our replication function. If \( \Delta p > 0 \), then the evolutionarily stable equilibrium proportion of cheaters is \( p^* = 1 \). If \( \Delta p < 0 \), then the evolutionarily stable equilibrium proportion of bold people is \( p^* = 0 \).

**Lemma 1:** No interior equilibrium is an ESE since

\[
\frac{d(\Delta p)}{dp} = \frac{d}{dp} (4 + (11 - 9 * n) * v * s) = 0 \forall p
\]

violates condition (3) of definition 2, which is required for an interior ESE.

Since there is no \( p \) dependence in the expression for \( \Delta p \), the equilibrium that prevails is determined solely by the exogenous initial conditions of the population: \( n \) (the size of the population), \( v \) (the average vengefulness in the population) and the parameter \( s \) (the probability of a defection being observed).
Theorem 1

1) There is a monomorphic ESE at $p^*=1$ if and only if $a$ or $b$ holds.
   
   a. $n < \frac{5}{3}$
   
   b. $n \geq \frac{5}{3}$ and $v \cdot s < \frac{4}{9 \cdot n - 11}$

2) The no cheating norm at $p^*=0$ is an ESE if and only if

   a. $n > \frac{5}{3}$ and $v \cdot s > \frac{4}{9 \cdot n - 11}$

3) The polymorphic equilibrium $0 < p^* < 1$ is never an ESE.

4) A population cannot support an ESE if

   a. $v \cdot s = \frac{4}{9 \cdot n - 11}$ and $n \geq \frac{5}{3}$

Proof:

1) The stability requirement for the equilibrium $p^*=1$ is $\Delta p > 0$. We can derive the conditions that meet that requirement with some algebra.

$$\Delta p = 4 + (11 - 9 \cdot n) \cdot v \cdot s > 0$$

$$4 > (9 \cdot n - 11) \cdot v \cdot s$$

Clearly this inequality holds when $s = 0$, $v = 0$ or $n = \frac{11}{9}$

If $n > \frac{11}{9}$, then $(9 \cdot n - 11) > 0$;

then $\Delta p > 0$ holds if and only if $v \cdot s < \frac{4}{9 \cdot n - 11}$

If $n < \frac{11}{9}$, $(9 \cdot n - 11) < 0$

then $\Delta p > 0$ holds if and only if $v \cdot s > \frac{4}{9 \cdot n - 11}$
Note that I use the property that multiplying or dividing both sides by a negative number reverses the inequality. Note that for

\[
\frac{11}{9} < n < \frac{5}{3}, \quad \frac{4}{9n - 11} > 1
\]

Therefore the inequality

\[
v \cdot s < \frac{4}{9n - 11}
\]

Holds for all allowed values of \(v\) and \(s\) and so \(\Delta p > 0\) holds trivially for

\[
\frac{11}{9} < n < \frac{5}{3}.
\]

However, for \(n > 5/3\) and \(\Delta p > 0\) the constraint on \(v \cdot s\) is binding.

So for \(n > \frac{5}{3}\) \(\Delta p > 0\) if and only if

\[
0 < v \cdot s < \frac{4}{9n - 11}
\]

Similarly, for \(n < 11/9\)

\[
\frac{4}{9n - 11} < 0
\]

And since \(0 \leq v \cdot s \leq 1\)

\[
v \cdot s > \frac{4}{9n - 11}
\]

Holds trivially. So \(\Delta p > 0\) for \(n < 11/9\)

2)

Now I derive the conditions for \(\Delta p < 0\)

\[
\Delta p = 4 + (11 - 9n) \cdot v \cdot s < 0
\]

If and only if

\[
4 < (9n - 11) \cdot v \cdot s
\]
The second inequality holds if and only if one of the following two statements hold.

1) \( n < \frac{11}{9} \) and \( v \ast s < \frac{4}{9 \ast n - 11} \)

2) \( n > \frac{11}{9} \) and \( v \ast s > \frac{4}{9 \ast n - 11} \)

Clearly the first case violates \( 0 \leq v \ast s \leq 1 \), so we are left with the only the second condition. We know that \( v \ast s \) cannot greater than 1, so (2) can hold only if:

\[
\frac{4}{9 \ast n - 11} < 1
\]

\[
4 < 9 \ast n - 11
\]

\[
15 < 9 \ast n
\]

\[
n > \frac{5}{3}
\]

3)

As lemma 1 states, no interior equilibrium is stable.

\[
\frac{d(\Delta p)}{dp} = \frac{d}{dp} (4 + (11 - 9 \ast n) \ast v \ast s) = 0 \forall p
\]

The stability requirement is violated. Since \( \Delta p \) has no \( p \) dependence, the derivative with respect to \( p \) is always 0.

4)

Since

\[
\Delta p = 4 + (11 - 9 \ast n) \ast v \ast s
\]

It is clear that when

\[
v \ast s = \frac{4}{9 \ast n \ast -11}
\]
the replication function will be zero, which violates the two corner solution equilibria. Therefore it is impossible for a population for which the above relation holds to support a stable equilibrium.

II.2 Discussion

The results tell us that either one of the monomorphic equilibria is sustainable under certain conditions. Whether or not a given population will sustain an equilibrium, and which equilibrium it will sustain, depends on the particular characteristics of the population. However, it is apparent that, for given positive levels of vengeance and supervision, the larger the population, the more likely it is that it will sustain the no cheating norm. Furthermore, given a large enough population, the higher average vengefulness in the population and the higher the probability of being observed, the more likely it is that the no cheating norm will be established in the population. More specifically, given any pair of positive values for v and s (less than or equal to one, of course), it is possible to find a minimum number N, such that all populations of size N or greater will support the norm. Here is a simple example. Let \( v = s = 0.1 \). Plugging the given values into the condition on \( v \times s \) allows us to solve for \( n \).

\[
\frac{4}{9 \times n - 11} < 0.1 \times 0.1 \leq 1
\]

\[
\frac{4}{9 \times n - 11} < 0.01
\]

\[
4 < 0.09 \times n - 0.11
\]

\[
4.11 < 0.09 \times n
\]

\[
n > \frac{4.11}{0.09}
\]
\[ n > 45.667 \]

Since 45.667 is greater than 5/3, the conclusion is, given that \( v = s = 0.1 \), all such populations of 46 players or more will support the no cheating norm. It is evident that having larger values of \( v \) and \( s \) will allow smaller populations to support the no cheating norm.

These results are in line with Axelrod’s in that, depending on the initial population characteristics, we can attain stable equilibria in any of the three regions that Axelrod does. If the average vengefulness in the population is high, it is possible for the population to attain the low average boldness and high average vengefulness equilibrium, which is the no cheating norm. Given that the initial vengefulness in the population is sufficiently low, the population can end up in the low vengefulness and high boldness region, that is, no norm is established. Finally, even if vengefulness is relatively low, for large enough populations the equilibrium with low average boldness and vengefulness can be sustained. In this last case, players do not cheat, but no norm is established, since mutant cheaters are likely to escape unpunished. In addition, it is possible to sustain the high average boldness and vengefulness equilibrium if \( s \) is small enough. Axelrod does not demonstrate that such an equilibrium is stable.

The main distinction between my results and Axelrod’s is that in my model, some populations (with the appropriate characteristics) cannot sustain any stable equilibria. Axelrod seems to assume that all populations can support an equilibrium. However, he presents only five populations. He does not give any evidence that all
twenty player populations must converge to some stable state, much less that they must do so within the first 100 generations.

One difference between the two models which can account for the variation in results is that Axelrod has the computer model randomly assign a different probability of being observed to each opportunity to cheat. That is not the case here. The probability that a defection will be observed is a characteristic of the population, and every defection carries with it that same threat of observation. Also, in this model the opportunity to defect is continuous, whereas Axelrod treats defection as discrete (separate) events. Another source of difference is that in this model $s$ is fixed for a given population and since each individual’s boldness score will always be either greater than $s$ or not, an agent cannot change his or her type for different defections. Axelrod’s model allows individuals to change their type (cheater or not a cheater) with each opportunity to defect. However, we can model the fact that the probability of being caught cheating has changed by changing $s$. This effectively gives us a new population with all of the characteristics of the old population except $s$, which we can continue to analyze.

It is hard to say if one method is better. It depends on the particular scenario being modeled. If the object is to simulate a world where it takes time to change laws that effectively control $s$, then fixing $s$ may be more appropriate. Then, a change in $s$ can represent a change in the laws or institutions of the society. In such a setting we can study the effect of an institutional change. On the other hand, if one desired to model a world where the probability of being observed is not the same for all defections, then Axelrod’s model is more appropriate.
There are more factors that can account for the difference in our results. This particular version of the model allows for only two types of players, whereas Axelrod allows for up to 64 different strategies in the population. My results require restrictions on n, the size of the population and s, the probability of being observed. Axelrod’s model does not allow one to control these parameters. The size of the population is fixed at 20 individuals and the probability of being observed is randomly generated for each opportunity to defect. The difference in replication functions can also account for the variance in results. Since there are only two types of players in this model, one type must be the most successful in order to multiply. If both types of players are receiving the same payoff they will both maintain their numbers. So a type must be receiving at least the average payoff for its proportion not to decline. In Axelrod’s model a type can be up to one standard deviation below average and still not have its population proportion fall. I discuss this issue in more detail later.

II.3 Model 1 for a Twenty Player Population

I now consider the special case of twenty player populations. A direct application of Theorem 1 for a population of twenty people gives the following result.

1) $p^* = 1$ is an ESE if and only if $v \ast s < \frac{4}{169}$

2) The no cheating norm at $p^* = 0$, is an ESE if and only if $v \ast s > \frac{4}{169}$.

3) No stable equilibrium is possible if $v \ast s = \frac{4}{169}$
II.4 Discussion

As in the general case, a population of size twenty can support either of the two equilibria or it can have no equilibria. Since $v \cdot s$ has to be greater than $4/169$ for the no cheating norm to be an ESE, 165 out of every 169 will support the norm. To support a large amount of cheating in the population either $v$ or $s$ or both have to be quite small. That is to say, the population can end up in any of the three regions that Axelrod’s populations end up. A state with low vengefulness is compatible with supporting a lot of defectors (high average boldness). However if $s$, the probability of being observed, is large enough, and average vengefulness is not too small, (i.e. their product is greater than $4/169$) the population can still end up in the low average vengefulness and low average boldness region, that is to say, with the norm not truly established. Similarly, if average vengefulness is large, the population can sustain the no cheating norm, (the state with high vengefulness low boldness), provided that the probability of being observed is sufficiently large.

Interestingly enough, a population of 20 individuals can have an ESE in the high average boldness and high average vengefulness region provided that $s$ is small enough. It makes sense that Axelrod does not attain this result since his model does not allow one to control the value of $s$. Four opportunities to cheat in each of 100 generations give us 400 randomly generated values in the closed interval between zero and one. If these numbers are truly randomly generated, we expect the average of all of them to be close to one half. It is very unlikely that 400 such numbers can be randomly generated and average out to a sufficiently low value that would allow
cheaters to prosper in a vengeful population. It would be necessary to modify the method by which the computer generates the $s$ values to achieve this result.

The lesson learned in this chapter is that given some proportion of vengeful people in the population, the norm can always be supported in sufficiently large populations. This result depends on the evolutionary stability of vengefulness. So we turn next to a version of the model in which vengefulness is subject to evolutionary pressures.
III. Model 2: The Norms Game with Endogenous Vengefulness

III.1 Model

I now present a more advanced version of the previous model where we allow each agent to have a vengefulness score of 0 or 1, rather than each agent having the same vengefulness score $v$. That is to say, I allow vengefulness in the population to be subject to evolutionary pressures. The variables are defined as in the previous model with appropriate modification.

$n = \text{size of population}$

$s \in [0,1] \text{ is the probability of a defection being observed}$

$b_i \in [0,1] \text{ agent } i \text{'s boldness score}$

$v_i \in \{0,1\} \text{ agent } i \text{'s vengefulness score}$

$p \in [0,1] \text{ is the proportion of defectors } (b_i > s)$

$q \in [0,1] \text{ is the proportion of punishers } (v_i = 1)$

I use the same payoff structure as Axelrod:

$T = 3 \text{ payoff for defection (temptation)}$  

$H = -1 \text{ damage inflicted on other agents by a defector}$  

$P = -9 \text{ punishment for a defection}$  

$E = -2 \text{ enforcement cost}$

Notice that $s$ is still a fixed parameter of the population and both $b$ and $v$ are fixed characteristics of each individual. Therefore, each player is again a type, and
has no choice in his or her actions. So this is still an evolutionary game-theoretic model rather than a rational choice model, as it might seem at first. We now have four types of players in this game, which are defined as follows:

1. Let $LL$ players $= \{ \text{people } i \text{ such that } b_i < s \text{ and } v_i = 0 \}$
2. Let $LH$ players $= \{ \text{people } i \text{ such that } b_i < s \text{ and } v_i = 1 \}$
3. Let $HL$ players $= \{ \text{people } i \text{ such that } b_i > 0 \text{ and } v_i = 0 \}$
4. Let $HH$ players $= \{ \text{people } i \text{ such that } b_i > s \text{ and } v_i = 1 \}$

Their respective payoffs are:

$$\pi^{LL} = 0 - 1 \times n \times p - 0 = -n \times p$$

1. $0$ payoff from not defecting
2. $0$ no punishment
3. $-n \times p$ damage from all defectors
4. $0$ expected enforcement cost

$$\pi^{LH} = 0 - 1 \times n \times p - 2 \times s \times n \times p$$

$$= -n \times p \times (1 + 2 \times s)$$

1. $+0$ payoff from not defecting
2. $+0$ no punishment
3. $-np$ damage from all defectors
4. $-2nps$ expected value of enforcement cost

$$\pi^{HL} = 3 - 9 \times n \times q \times s - 1 \times (np - 1) - 0$$

$$= 4 - 9 \times n \times q \times s - n \times p$$
1. +3 payoff from cheating
2. $-9 \times q \times n \times s$ expected value of punishment from all punishers
3. $-(n \times p - 1)$ damage from other defectors
4. +0 enforcement cost

\[
\pi^{HH} = 3 - 9 \times (n \times q - 1) \times s - 1 \times (n \times p - 1) - 2 \times (n \times p - 1) \times s
\]
\[
= 4 - 9 \times n \times q \times s + 11 \times s - (1 + 2 \times s) \times n \times p
\]

1. +3 payoff from cheating
2. $-9 \times (n \times q - 1)$ expected value of punishment
3. $-(n \times p - 1)$ damage from defectors
4. $-2 \times (n \times p - 1)s$ expected value of enforcement cost

I further define the following variables:

$\alpha = \text{proportion of LL type people in the population.}$

$\beta = \text{proportion of LH type people in the population.}$

$\gamma = \text{proportion of HL type people in the population.}$

$\delta = \text{proportion of HH type people in the population.}$

Necessarily: $\alpha + \beta + \gamma + \delta = 1$

I can now define p and q in terms of the above player type proportions.

$p = \gamma + \delta$

$q = \beta + \delta$

In accordance with the evolutionary game theory approach I need to write down a replication function for each type of agent. To that end I define the average payoff for the population as
$\overline{p} = \alpha \ast \pi_{LL} + \beta \ast \pi_{LH} + \gamma \ast \pi_{HL} + \delta \ast \pi_{HH}$

Which allows me to write the replication functions as

$$\Delta \alpha = \pi^{LL} - \overline{p}$$
$$\Delta \beta = \pi^{LH} - \overline{p}$$
$$\Delta \gamma = \pi^{HL} - \overline{p}$$
$$\Delta \delta = \pi^{HH} - \overline{p}$$

With my parameters and replication functions defined I can now define what constitutes an equilibrium. Let $z$ stand for one of the population proportions, $\alpha, \beta, \gamma,$ or $\delta$. Let $z^*$ be the equilibrium level of the proportion in question. One of three things can happen, $z^* = 0, z^* = 1$ or $z^*$ can be between 0 and 1 for each of the four population proportions, provided that they sum to one. The first two are considered corner solutions, and the last is called an interior solution.

**Definition 3**: an equilibrium is an *evolutionarily stable equilibrium* if for each population proportion $\alpha, \beta, \gamma,$ and $\delta$, one of the following conditions are satisfied, given the other equilibrium proportions. Let $z$ stand for $\alpha, \beta, \gamma,$ or $\delta$, as necessary.

1. $z^* = 0$ and $\Delta z < 0$, for small mutations away from $z = 0$.
2. $z^* = 1$ and $\Delta z > 0$ for small mutations away from $z = 1$.
3. $z^* \in (0,1)$ and $\Delta z = 0$ and $\frac{d(\Delta z)}{dz} < 0$ at equilibrium.

This definition is similar to definition two in chapter two. A new variable will make this definition easier to understand.

*Let $\epsilon \in (0,1)$ represent the proportion of a mutant type that has invaded a population in equilibrium.*
Since epsilon represents the rate of mutation in the population, it is conventional to assume that it is small. That is to say, we assume that only a small proportion of all players can mutate at any one time. Half the population, say, can’t all of a sudden mutate into some other type.

In words, an equilibrium is an ESE if for epsilon sufficiently small, evolutionary forces will remove the mutant members of the population and the population proportions will return to their previous equilibrium values.

Using the definition of ESE we can classify all potential equilibria as one of the following nine types:

- **Type 1.** $0 < p < 1$ and $0 < q < 1$
- **Type 2.** $p = 0$ and $0 < q < 1$
- **Type 3.** $p = 1$ and $0 < q < 1$
- **Type 4.** $0 < p < 1$ and $q = 0$
- **Type 5.** $0 < p < 1$ and $q = 1$
- **Type 6.** $p = 0$ and $q = 0$
- **Type 7.** $p = 0$ and $q = 1$
- **Type 8.** $p = 1$ and $q = 0$
- **Type 9.** $p = 1$ and $q = 1$

Intuitively, it makes sense that for a population composition to be an equilibrium, every individual has to receive the same payoff. Otherwise, there would be an incentive to do better by switching to the type that is doing better than everyone else. More than one type of player being present in the population is the interior
solution for each of the present player types. Letting $z$ represent any of $\alpha$, $\beta$, $\gamma$, or $\delta$ that are non-zero in the population, we have

$$\Delta z = 0$$

Which can be rewritten as

$$\Delta z = \pi^z - \bar{\pi} = 0$$

In words, the above equation says that all groups present must receive the same payoff—the average.

Referring back to the definitions of $p$ and $q$ in terms of beta, gamma and delta allows me to solve for $\alpha$, $\beta$, $\gamma$, and $\delta$ by setting the payoff of each type present in a given equilibrium equal to the average payoff in that equilibrium. Below I list all the possible values of $\alpha$, $\beta$, $\gamma$, and $\delta$ for each equilibrium that satisfy the condition discussed above.

**Lemma 2:** Given that all players in the population must receive the same payoff for a population composition to be in equilibrium the following population compositions are necessary to constitute equilibria.

1) Type 1: (a) $\alpha = \frac{-11+2n}{2n}, \beta = 0, \gamma = \frac{-8+99s}{18n+s}, \delta = \frac{4}{9n+s}$

2) Type 1: (b) $\alpha = \frac{-4-11s+11n+s}{11n+s}, \beta = 0, \gamma = 0, \delta = \frac{(4+11s)}{11n+s}$

3) Type 1: (c) $\alpha = 0, \beta = \frac{2s(2n+s)}{11n+s}, \gamma = \frac{-4+9s+s}{11n+s}, \delta = 0$

4) Type 2: $\alpha = 1 - \beta$ Continuum, $\gamma = 0, \delta = 0$

5) Type 3: $\alpha = 0, \beta = 0, \gamma = 1 - \delta, s = 0$ or $n = \frac{11}{2}$

6) Type 6: $\alpha = 1, \beta = 0, \gamma = 0, \delta = 0$
7) Type 7: \( \alpha = 0, \beta = 1, \gamma = 0, \delta = 0 \)

8) Type 8: \( \alpha = 0, \beta = 0, \gamma = 1, \delta = 0 \)

9) Type 9: \( \alpha = 0, \beta = 0, \gamma = 0, \delta = 1 \)

**Sketch of Proof:**

Using the fact that payoffs must be equal for all types present in the population at equilibrium, and the restrictions on the population proportions dictated by the type of the equilibrium allows us to solve for the equilibrium proportions of the player types. The nine population configurations that give equal payoffs to all players are listed above.

Now the question is which of the nine equilibria constitute ESE’s?

**Lemma 3:** Type 6 (\( \alpha^* = 1, \beta^* = 0, \gamma^* = 0, \delta^* = 0 \)) and Type 7 (\( \alpha^* = 0, \beta^* = 1, \gamma^* = 0, \delta^* = 0 \)) Monomorphic Equilibria are not ESE’s.

**Proof:**

We let \( \epsilon > 0 \) be the proportion of the invading type.

**Type 6:** Suppose \( \alpha = 1 \). Suppose a mutation causes \( \alpha = 1 - \epsilon \) and \( \gamma = \epsilon > 0 \).

\[
\alpha = 1 - \epsilon, \beta = 0, \gamma = \epsilon, \delta = 0; \ p = \epsilon; q = 0
\]

\[
\pi^LL = -n*p
\]

\[
= -n*\epsilon
\]

\[
\pi^HL = 4 - 9*q*s - n*p
\]

\[
= 4 - 9*0*s - n*\epsilon
\]

\[
= 4 - n*\epsilon
\]

Clearly HL types dominate LL types. This simply means that HL types receive a higher payoff than LL types when HL types invade a population of LL types.

Therefore, the Type 6 equilibrium is not an ESE.
Type 7: $\beta=1$. Suppose a mutation causes $\beta = 1 - \epsilon$ and $\alpha = \epsilon > 0$

$$\alpha = \epsilon, \beta = 1 - \epsilon, \gamma = 0, \delta = 0; p = 0, q = 1 - \epsilon$$

$$\pi^{LL} = -n \times p = -n \times 0 = 0$$

$$\pi^{LH} = (1 + 2 \times s) \times (-n \times p) = (1 + 2 \times s) \times (-n \times 0) = 0$$

$$\bar{\pi} = \epsilon \times 0 + (1 - \epsilon) \times 0 = 0$$

$$\Delta \alpha = \pi^{LL} - \bar{\pi} = 0 - 0 = 0$$

$$\Delta \beta = \pi^{LH} - \bar{\pi} = 0 - 0 = 0$$

$$\Delta \beta > 0 \text{ Fails so } \beta = 1 \text{ is not stable.}$$

Lemma 4: Any equilibrium where $\beta^* > 0$, (3 and 4 in Lemma 2) is not an ESE.

Proof:

$$\pi^{LL} = -n \times p$$

$$\pi^{LH} = (1 + 2 \times s) \times (-n \times p)$$

$$\pi^{HL} = 4 - 9 \times n \times q \times s - n \times p$$

$$\pi^{HH} = 4 - 9 \times n \times q \times s + 11 \times s + (1 + 2 \times s) \times (-n \times p)$$

$$\bar{\pi} = (1 - \beta - \gamma - \delta) \times \pi^{LL} + \beta \times \pi^{LH} + \gamma \times \pi^{HL} + \delta \times \pi^{HH}$$

Case 1a: $s=0, p=0, n>0$

$$\pi^{LL} = -n \times p$$

$$= -n \times 0$$

$$= 0$$

$$\pi^{LH} = (1 + 2 \times 0) \times \pi^{LL}$$

$$= -n \times p$$

$$= -n \times 0$$
Clearly this gives

\[ \Delta \beta = \pi^{LH} - \bar{\pi} \]

\[ = 0 - 0 \]

\[ = 0 \]

And \( \Delta \beta > 0 \) fails.

Case 1b: \( s=0, p>0, n>0 \)

\[ \pi^{LL} = -n \cdot p \]

\[ \pi^{LH} = (1 + 2 \cdot 0) \cdot \pi^{LL} \]

\[ = -n \cdot p \]

\[ \pi^{HL} = 4 - 9 \cdot n \cdot q \cdot 0 + \pi^{LL} \]

\[ = 4 - n \cdot p \]

\[ \pi^{HH} = 4 - 9 \cdot n \cdot q \cdot 0 + 11 \cdot 0 + (1 + 2 \cdot 0) \cdot \pi^{LL} \]

\[ = 4 - n \cdot p \]

Since \( 4 + \pi^{LL} > \pi^{LL} \), the HL and HH types both strictly dominate the LL and LH types. Therefore, the LL and LH types cannot appear in a stable equilibrium when \( s=0 \).

Case 2a: \( s>0, p>0, n>0 \)

\[ \pi^{LL} = -n \cdot p \]

\[ \pi^{LH} = (1 + 2 \cdot s) \cdot \pi^{LL} = -(1 + 2 \cdot s) \cdot n \cdot p \]

Since \( n > 0 \) and \( p > 0 \) and \( s > 0 \)

\[ 2 \cdot s > 0 \]

\[ 1 + 2 \cdot s > 1 \]
\[-(1 + 2 \cdot s) \cdot n \cdot p < -n \cdot p\]

Therefore, \(\pi^{\text{LH}} < \pi^{\text{LL}}\) and the LL type strictly dominate the LH type. Therefore the LH type cannot appear in an ESE.

Case 2b: \(s > 0, p = 0, n > 0\)

\[0 = p = \gamma + \delta \rightarrow \gamma^* = \delta^* = 0 \rightarrow \alpha^* + \beta^* = 1\]

\[\pi^{\text{LL}} = -n \cdot p\]
\[= -n \cdot 0\]
\[= 0\]

\[\pi^{\text{LH}} = (1 + 2 \cdot s) \cdot (-n \cdot p)\]
\[= (1 + 2 \cdot s) \cdot 0\]
\[= 0\]

\[\bar{\pi} = \alpha \cdot (-n \cdot p) + \beta \cdot (1 + 2 \cdot s) \cdot (-n \cdot p)\]

\[\Delta \alpha = \pi^{\text{LL}} - \bar{\pi} = (-n \cdot p) - [\alpha \cdot (-n \cdot p) + \beta \cdot (1 + s) \cdot (-n \cdot p)]\]

\[= (1 - \alpha) \cdot (-n \cdot p) - \beta \cdot (1 + s) \cdot (-n \cdot p)\]

\[\frac{d\Delta \alpha}{d\alpha} = n \cdot p = n \cdot 0 = 0\]

\[\Delta \beta = \pi^{\text{LH}} - \bar{\pi}\]
\[= (1 + 2 \cdot s) \cdot (-n \cdot p)\]
\[\quad - [\alpha \cdot (-n \cdot p) + \beta \cdot (1 + 2 \cdot s) \cdot (-n \cdot p)] =\]
\[= (1 - \beta) \cdot (1 + 2 \cdot s) \cdot (-n \cdot p) - \alpha \cdot (-n \cdot p)\]

\[\frac{d\Delta \beta}{d\beta} = (1 + 2 \cdot s) \cdot (n \cdot p) = (1 + 2 \cdot s) \cdot (n \cdot 0) = 0\]
\[ \alpha^* + \beta^* = 1 \] implies that either \( \beta^* = 0 \), or \( 0 < \beta^* \leq 1 \), or \( \beta^* = 1 \). We know from Lemma 3 that \( \beta^* = 1 \) is not a stable equilibrium. Therefore we only need to check \( 0 < \beta^* < 1 \).

Which Requires: \( \Delta \beta = 0 \) and \( \frac{d\Delta \beta}{d\beta} < 0 \)

However, \( \frac{d\Delta \beta}{d\beta} < 0 \) fails, since \( \frac{d\Delta \beta}{d\beta} = 0 \).

Therefore \( 0 < \beta^* < 1 \) is not stable and any equilibrium with \( \beta^* > 0 \) is not an ESE. Therefore we can eliminate equilibria #3, 4, and 7 (as listed in Lemma 2) as possible ESE’s.

**Theorem 2**

1) The Type 8 equilibrium \((\alpha^* = 0, \beta^* = 0, \gamma^* = 1, \delta^* = 0)\) is an ESE if and only if

\[ n > \frac{11}{2}, s > 0 \text{ and } a \text{ or } b \text{ holds.} \]

\text{a. } 0 < \varepsilon \leq \frac{2}{11}

\text{b. } \frac{2}{11} < \varepsilon < 1 \text{ and } s < \frac{4}{(11\varepsilon - 2)s}

2) The Type 9 \((\alpha^* = 0, \beta^* = 0, \gamma^* = 0, \text{ and } \delta^* = 1)\) equilibrium is an ESE if and only if \( n < \frac{11}{2}, s > 0 \text{ and both } a \text{ and } b \text{ holds.} \)

\text{a. One of the following holds}

\text{i. } n \leq \frac{15}{11}

\text{ii. } n > \frac{15}{11} \text{ and } s \leq \frac{4}{11n - 11}

\text{iii. } n > \frac{15}{11}, s > \frac{4}{11n - 11}, \text{ and } \frac{11ns - 11s - 4}{11ns} < \varepsilon

\text{b. One of the following holds}
i. $n < \frac{5}{3}$

ii. $n \geq \frac{5}{3}$ and $s < \frac{4}{9n-11}$

3) There are no other ESE’s. It follows that there exist populations that cannot support an ESE.

Proof:

By lemma 3 and 4 we eliminate Equilibrium 3, 4, 6, and 7 (as listed in Lemma 2) as possible ESE’s. We are left with five candidates for ESE: Equilibrium 1, 2, 5, 8 and 9.

Equilibrium 1, 2, (Type 1 (a) and (b))

Type 1 Equilibrium: $0 < p < 1$ and $0 < q < 1$

Requires: $\Delta p = 0$ and $\Delta q = 0$; $\frac{d\Delta p}{dp} < 0$ and $\frac{d\Delta q}{dq} < 0$

Implies: $0 < \gamma + \delta < 1$ and $0 < \beta + \delta < 1$

We know $\beta = 0$ which implies $0 < \delta < 1$, further $0 < \gamma + \delta < 1$

requires $0 < \alpha < 1$

This gives two cases:

Case 1: $0 < \alpha < 1, \beta = 0, 0 < \gamma < 1$, and $0 < \delta < 1$

Requires $\Delta \alpha = 0 \frac{d\Delta \alpha}{d\alpha} < 0$; and $\Delta \beta < 0$; $\Delta \gamma = 0 \frac{d\Delta \gamma}{d\gamma} < 0$; and $\Delta \delta = 0$

$\frac{d\Delta \delta}{d\delta} < 0$
Type 1 (a) falls in this category

\[ \alpha = \frac{-11 + 2 \cdot n}{2n}, \beta = 0, \gamma = \frac{-8 + 99 \cdot s}{18 \cdot n \cdot s}, \delta = \frac{4}{9 \cdot n \cdot s} \]

Plugging these values into the replication function

\[ \Delta \gamma = \pi^{LH} - \bar{\pi} = \]

\[ = 4 - 4 \cdot \delta - 4 \cdot \gamma + (-11 \cdot \delta + (\beta + \delta) \cdot (-9 + 11 \cdot \delta + 11 \cdot \gamma) \cdot n) \cdot s \]

\[ \frac{d(\Delta \gamma)}{d\gamma} = -4 + 11 \cdot \beta \cdot n \cdot s + 11 \cdot \delta \cdot n \cdot s \]

Gives

\[ \Delta \gamma \left( \frac{-11 + 2 \cdot n}{2n}, 0, \frac{-8 + 99 \cdot s}{18 \cdot n \cdot s}, \frac{4}{9 \cdot n \cdot s} \right) = 0 \text{ as required} \]

But \[ \frac{d}{d\gamma} \left( \Delta \gamma \left( \frac{2n - 11}{2n}, 0, \frac{99s - 8}{18ns}, \frac{4}{9ns} \right) \right) = \frac{8}{9} \]

So \[ \frac{d \Delta \gamma}{d\gamma} < 0 \text{ Fails} \]

So the Type 1(a) equilibrium is not stable.

Case 2: \( 0 < \alpha < 1 \) \( \beta = 0, \gamma = 0, 0 < \delta < 1 \)

Requires \( \Delta \alpha = 0 \frac{d \Delta \alpha}{d\alpha} < 0; \Delta \beta < 0; \Delta \gamma < 0; \text{ and } \Delta \delta = 0 \frac{d \Delta \delta}{d\delta} < 0 \)

Type 1 (b) falls in this category

\[ \alpha = \frac{-4 - 11 \cdot s + 11 \cdot n \cdot s}{11 \cdot n \cdot s}, \beta = 0, \gamma = 0, \delta = \frac{(4 + 11 \cdot s)}{11 \cdot n \cdot s} \]

Which we acquire by solving the system of equations below for \( \alpha \) and \( \delta \)

\[ \pi^{LL} = \pi^{HH} \]
\[ \alpha + \delta = 1 \]

Where

\[ \pi^{LL} = -n * p \]
\[ \pi^{HH} = 4 - 9nqs + (1 + 2s)(-n * p) + 11s \]

These payoffs give the following replication function for LL types:

\[ \Delta \alpha = \pi^{LL} - \bar{\pi} \]
\[ = \pi^{LL} - (\alpha \star \pi^{LL} + \delta \star \pi^{HH}) \]
\[ = (1 - \alpha) \star \pi^{LL} - \delta \pi^{HH} \]

Since the payoffs have no \( \alpha \) dependence we treat them as constants

\[ \frac{d\Delta \alpha}{d\alpha} = \frac{d}{d\alpha} ((1 - \alpha) \star \pi^{LL} - \delta \pi^{HH}) \]
\[ = -\pi^{LL} \]
\[ = -1 \star (-n \star \delta) \]
\[ = n \star \delta > 0 \]

\[ \frac{d\Delta \alpha}{d\alpha} < 0 \text{ Fails} \]

Therefore we conclude that there is no stable equilibrium of type 1.

Equilibrium 5:

Type 3 Equilibrium: \( p = 1 \) and \( 0 < q < 1 \)

Gives: \( \gamma + \delta = 1 \) and \( 0 < \beta + \delta < 1 \)

Implies \( \alpha = 0, \beta = 0, 0 < \gamma < 1 \) and \( 0 < \delta < 1 \), which gives \( q = \delta \)

Requires: \( \Delta \alpha < 0; \Delta \beta < 0; \Delta \gamma = 0 \frac{d\Delta \gamma}{dy} < 0; \) and \( \Delta \delta = 0 \frac{d\Delta \delta}{d\delta} < 0; \)
Since there are only two types of people, we know that their payoffs must be equal in equilibrium. Plugging $p = \gamma + \delta = 1$ and $q = \delta + 0 = \delta$, into the respective payoffs gives

\[
\pi^{HL} = 4 - 9 \cdot n \cdot q \cdot s - np = 4 - 9 \cdot \delta \cdot s - n
\]
\[
\pi^{HH} = 4 - 9 \cdot n \cdot q \cdot s + 11 \cdot s - (1 + 2 \cdot s)n \cdot p = 4 - 9 \cdot \delta \cdot n \cdot s + 11 \cdot s - (1 + 2 \cdot s) \cdot n
\]

Now we solve

\[
\pi^{HL} = \pi^{HH}
\]

\[
\gamma + \delta = 1
\]

Gives

\[
4 - n \cdot \delta \cdot s - n = 4 - 9 \cdot \delta \cdot n \cdot s + 11 \cdot s - (1 + 2 \cdot s) \cdot n
\]

\[
-n = 11 \cdot s - (1 + 2 \cdot s) \cdot n
\]

\[
0 = 11 \cdot s - 2 \cdot n \cdot s
\]

\[
s = 0 \text{ or}
\]

\[
0 = 11 - 2 \cdot n
\]

\[
n = 11/2
\]

And

\[
\delta = 1 - \gamma
\]

Case 1: $s=0$

\[
\pi^{HL} = 4 - n
\]

\[
\pi^{HH} = 4 - n
\]
As can be seen, as long as \( p = \gamma + \delta = 1 \), the payoff for the HL and HH types are not dependent on the particular split in the population between HL and HH types. Therefore, if there is a mutation that causes a change in the division between HL and HH types, there is no incentive to return to the previous distribution since all parties still get a payoff of \( 4-n \).

Case 2: \( n = 11/2 = 5.5 \)

We define the average payoff as

\[
\bar{\pi} = \gamma \pi^{HL} + \delta \pi^{HH} = \gamma \pi^{HL} + (1 - \gamma) \pi^{HH}
\]

This gives the following replication functions

\[
\Delta \gamma = \pi^{HL} - \bar{\pi} = -(-1 + \gamma) \times (-11 + 2 \times (\delta + \gamma) \times n) \times s
\]

Plugging in \( \gamma + \delta = 1 \) and \( n = 11/2 \) gives

\[
\Delta \gamma = -(-1 + \gamma) \times \left(-11 + 2 \times 1 \times \frac{11}{2}\right) \times s = (1 - \gamma) \times 0 = 0
\]

as desired.

Taking the derivative with respect to \( \gamma \) gives

\[
\frac{d\Delta \gamma}{d\gamma} = 2(1 - \gamma) \times n \times s - (-11 + 2(\delta + \gamma) \times n) \times s
\]

Again, using \( \gamma + \delta = 1 \) and \( n = 11/2 \) gives

\[
\frac{d\Delta \gamma}{d\gamma} = 2 \delta \times n \times s - (-11 + 2 \times n) \times s = 2 \times n \times \delta - 11 + 11 \times s = 2 \times n \times \delta > 0
\]
Therefore, Equilibrium 5 is not an ESE.

Equilibrium 8

Type 8 Equilibrium: \( p = 1 \) and \( q = 0 \)

Implies: \( \alpha = 0, \beta = 0, \gamma = 1, \delta = 0 \)

Requires: \( \Delta \alpha < 0, \Delta \beta < 0, \Delta \gamma > 0 \) and \( \Delta \delta < 0 \)

There are three cases to consider, a) an LL type mutation, b) an LH type mutation and c) an HH type mutation.

a) \( \alpha = \epsilon, \beta = 0, \gamma = 1 - \epsilon, \delta = 0 \rightarrow p = 1 - \epsilon, q = 0 \)

\[
\pi^{LL} = -n * p
= -n * (1 - \epsilon)
\]

\[
\pi^{HL} = 4 - 9 * q * s - n * p
= 4 - 0 - n * (1 - \epsilon)
= 4 - n * (1 - \epsilon)
\]

As before HL types dominate LL types and LL types cannot invade this equilibrium.

b) \( \alpha = 0, \beta = \epsilon, \gamma = 1 - \epsilon, \delta = 0; p = 1 - \epsilon, q = \epsilon \)

\[
\pi^{LH} = (1 + 2 * s) * (-n * p)
= (1 + 2 * s) * (-n * (1 - \epsilon))
\]

\[
\pi^{HL} = 4 - 9 * n * q * s - n * p
= 4 - 9 * n * \epsilon * s - n * (1 - \epsilon)
\]

\[
\Delta \beta = -4 * (1 - \epsilon) - 2 * n * s * (1 - \epsilon) + 11 * n * \epsilon * s * (1 - \epsilon) < 0
\]

\[
\Delta \gamma = 4 * \epsilon + 2 * n * \epsilon * s - 11 * n * \epsilon * s * \epsilon > 0
\]

Assuming, \( 0 \leq s \leq 1, n > 0 \) and \( 0 < \epsilon < 1, \Delta \beta < 0 \) and \( \Delta \gamma > 0 \) hold if and only if
The last inequality holds if and only if one of the following two conditions holds.

1) \( \epsilon \leq \frac{2}{11} \)

2) \( \epsilon > \frac{2}{11} \) and \( s < \frac{4}{11 * \epsilon * n - 2 * n} \)

c) \( \alpha = 0, \beta = 0, \gamma = 1 - \epsilon, \delta = \epsilon; p = 1, q = \epsilon \)

\[
\pi^{HL} = 4 - 9 * n * q * s - n * p
\]
\[
= 4 - 9 * n * \epsilon * s - n
\]

\[
\pi^{HH} = 4 - 9 * n * q * s + 11 * s + (1 + 2 * s) * (-n * p)
\]
\[
= 4 - 9 * n * \epsilon * s + 11 * s - n * (1 + 2 * s)
\]

\( \Delta \gamma = -11 * \epsilon * s + 2 * n * \epsilon * s > 0 \)

\( \Delta \delta = 11 * (1 - \epsilon) * s - 2 * n * s + 2 * n * \epsilon * s < 0 \)

Assuming, \( 0 \leq s \leq 1, n > 0 \) and \( 0 < \epsilon < 1, \Delta \gamma > 0 \) and \( \Delta \delta < 0 \) hold if and only if

\(-11 * \epsilon * s + 2 * n * \epsilon * s > 0 \)

If and only if \( s > 0 \) and

\(-11 + 2 * n > 0 \)

\( n > \frac{11}{2} \)

Type 9

Type 9 Equilibrium: \( p = 1 \) and \( q = 1 \)

Implies: \( \alpha = 0, \beta = 0, \gamma = 0, \) and \( \delta = 1 \)

Requires: \( \Delta \alpha < 0, \Delta \beta < 0, \Delta \gamma < 0, \) and \( \Delta \delta > 0 \)
Again, there are three cases to consider, when there is an LL type mutation, an LH type mutation and an HH type mutation.

a) \( \alpha = \epsilon, \beta = 0, \gamma = 0, \delta = 1 - \epsilon; p = 1 - \epsilon, q = 1 - \epsilon \)

\[ \pi^{LL} = -n \cdot p \]
\[ = -n \cdot (1 - \epsilon) \]

\[ \pi^{HH} = 4 - 9 \cdot n \cdot q \cdot s + 11 \cdot s - n \cdot p \cdot (1 + 2 \cdot s) \]
\[ = 4 - 9 \cdot n \cdot (1 - \epsilon) \cdot s + 11 \cdot s - n \cdot (1 - \epsilon) \cdot (1 + 2 \cdot s) \]

\[ \Delta \alpha = 4 \cdot (-1 + \epsilon) + 11 \cdot s \cdot (-1 + \epsilon) + 11 \cdot n \cdot s \cdot (-1 + \epsilon)^2 < 0 \]

\[ \Delta \delta = 4 \cdot \epsilon + 11 \cdot \epsilon \cdot s + 11 \cdot n \cdot \epsilon \cdot s \cdot (-1 + \epsilon) > 0 \]

Assuming, \( 0 < \epsilon < 1, \Delta \alpha < 0 \) and \( \Delta \delta > 0 \) hold if and only if

\[ 4 > 11 \cdot n \cdot s \cdot (1 - \epsilon) - 11 \cdot s \]

Further assuming \( 0 \leq s \leq 1 \) and \( n > 0 \) the above inequality holds if and only if one of the following three conditions holds

1) \( n \leq \frac{15}{9} \)

2) \( n > \frac{15}{9} \) and \( s \leq \frac{4}{11 \cdot n - 11} \)

3) \( n > \frac{15}{9}, s > \frac{4}{11 \cdot n - 11}, \) and \( \frac{11 \cdot n \cdot s - 11 \cdot s - 4}{11 \cdot n \cdot s} < \epsilon \)

b) \( \alpha = 0, \beta = \epsilon, \gamma = 0, \delta = 1 - \epsilon; p = 1 - \epsilon, q = 1 \)

\[ \pi^{LH} = -n \cdot p \cdot (1 + 2 \cdot s) \]
\[ = -n \cdot (1 - \epsilon) \cdot (1 + 2 \cdot s) \]

\[ \pi^{HH} = 4 - 9 \cdot n \cdot q \cdot s + 11 \cdot s - n \cdot p \cdot (1 + 2 \cdot s) \]
\[ = 4 - 9 \cdot n \cdot s + 11 \cdot s - n \cdot (1 - \epsilon) \cdot (1 + 2 \cdot s) \]
\[ \Delta \beta = 4 * (-1 + \epsilon) + 11 * s * (-1 + \epsilon) + 9 * n * s * (1 - \epsilon) < 0 \]
\[ \Delta \delta = 4 * \epsilon + 11 * \epsilon * s - 9 * n * \epsilon * s > 0 \]

Assuming \( 0 < \epsilon < 1, 0 \leq s \leq 1 \) and \( n > 0 \), the above expression holds if and only if
\[ 4 > 9 * n * s - 11 * s \]

Assuming \( 0 < \epsilon < 1, 0 \leq s \leq 1 \) and \( n > 0 \) the above expression holds if and only if one of the following conditions is met

1) \( n < \frac{5}{3} \)

2) \( n \geq \frac{5}{3} \) and \( s < \frac{4}{9 * n - 11} \)

c) \( \alpha = 0, \beta = 0, \gamma = \epsilon, \delta = 1 - \epsilon; p = 1, q = 1 - \epsilon \)

\[ \pi^{HL} = 4 - 9 * n * q * s - n * p \]
\[ = 4 - 9 * n * (1 - \epsilon) * s - n \]
\[ \pi^{HH} = 4 - 9 * n * q * s + 11 * s - n * p * (1 + 2 * s) \]
\[ = 4 - 9 * n * (1 - \epsilon) * s + 11 * s - n * (1 + 2 * s) \]
\[ \Delta \gamma = 11 * s * (-1 + \epsilon) + 2 * n * s * (1 - \epsilon) < 0 \]
\[ \Delta \delta = 11 * \epsilon * s - 2 * n * \epsilon * s > 0 \]

Again, assuming \( 0 < \epsilon < 1, 0 \leq s \leq 1 \) and \( n > 0 \), \( \Delta \gamma < 0 \) \( \Delta \delta > 0 \) hold if and only if
\[ 11 * s - 2 * n * s > 0 \]

The above expression holds if and only if
\[ s > 0, \text{and } n < \frac{11}{2} \]

Definition 4: An equilibrium is temporarily norm supporting if it is defector free and cannot be invaded by defectors (HL and HH types) but cannot remove LL
type mutations. In terms of replication functions this means, $\Delta \gamma < 0$ and $\Delta \delta < 0$ for small numbers of HL or HH type mutations, but $\Delta \alpha \geq 0$.

**Lemma 5:** The Type 7 equilibrium ($\alpha^*=0$, $\beta^*=1$, $\gamma^*=0$, $\delta^*=0$) as well as certain Type 2 Equilibria ($\alpha^* = \epsilon$, $\beta^* = 1-\epsilon$, $\gamma^*=0$, $\delta^*=0$) are temporarily norm supporting if and only if:

$$n > \frac{26}{9}, \quad s > \frac{4}{9n-22}, \quad 0 < \epsilon < \frac{9 \cdot n \cdot s - 11 \cdot s - 4}{11 \cdot n \cdot s},$$

$$0 < \theta < \frac{9 \cdot n \cdot s - 4 - 11 \cdot \epsilon \cdot n \cdot s}{11 \cdot n \cdot s}$$

Where $\epsilon$ is the proportion of LL players and $\theta$ is the proportion of HL or HH invading cheaters.

The lemma is to be interpreted thus: The type 7 monomorphic equilibrium, as well as some of the Type 2 polymorphic equilibria can support a norm temporarily, that is, prevent cheating in the population as long as the proportion of LL types is sufficiently small. However, this will not last since LL players dominate LH players in the presence of defectors.

**Proof:**

Let $0 < \theta < 1$ be the proportion of defecting mutants of some type. We assume $\theta$ to be small.

**Case 1:** $\alpha = 0$, $\beta = 1 - \theta$, $\gamma = \theta$, $\delta = 0$

$$\pi^{LH} = -\theta \cdot n \cdot (1 + 2 \cdot s)$$

$$\pi^{HL} = 4 - \theta \cdot n - 9 \cdot (1 - \theta) \cdot n \cdot s$$

$$\Delta \beta = \pi^{LH} - \pi$$
\[ = \theta \ast (-4 + (9 - 11 \ast \theta) \ast n \ast s) > 0 \]

\[ \Delta \gamma = \pi^{HL} \ast -\pi \]

\[ = (1 - \theta) \ast (4 + (-9 + 11 \ast \theta) \ast n \ast s) < 0 \]

Assuming \( 0 < \theta < 1 \), \( 0 \leq s \leq 1 \) and \( n > 0 \), \( \Delta \beta > 0 \) and \( \Delta \gamma < 0 \) hold if and only if

\[
\begin{align*}
    n &> \frac{4}{9}, \\
    \frac{4}{9 \ast n} &< s \leq 1, \\
    0 &< \theta < \frac{9 \ast n \ast s - 4}{11 \ast n \ast s}
\end{align*}
\]

However, if \( \varepsilon > 0 \), it is at least \( 1/n \). Therefore, for this equilibrium to push out at least one HL mutation we need:

\[
\begin{align*}
    \frac{9 \ast n \ast s - 4}{11 \ast n \ast s} &> \frac{1}{n} \\
    9 \ast n \ast s - 4 &> 11 \ast s \\
    9 \ast n \ast s - 11 \ast s &> 4
\end{align*}
\]

\[ s > \frac{4}{9 \ast n - 11} \]

Therefore, we need:

\[ n > \frac{11}{9} \]

The new requirements are

\[
\begin{align*}
    n &> \frac{11}{9}, \\
    \frac{4}{9 \ast n - 11} &< s, \\
    0 &< \theta < \frac{9 \ast n \ast s - 4}{11 \ast n \ast s}
\end{align*}
\]

But since \( 0 \leq s \leq 1 \) we also need

\[
\begin{align*}
    \frac{4}{9 \ast n - 11} &\leq 1 \\
    4 &\leq 9 \ast n - 11 \\
    15 &\leq 9 \ast n \\
    \frac{5}{3} &\leq n
\end{align*}
\]
Gives:
\[
\frac{n}{3} < s \leq 1, \quad 0 < \theta < \frac{9n - 4}{11n - s}
\]

Case 2: \(\alpha = 0, \beta = 1 - \theta, \gamma = 0, \delta = \theta\)

\[
\pi^{LH} = -n \cdot \theta \cdot (1 + 2s)
\]
\[
\pi^{HH} = 4 + 11s - 9ns - n\theta(1 + 2s)
\]
\[
\Delta \beta = \theta(-4 + (-11 + 9n)s) > 0
\]
\[
\Delta \delta = (-1 + \theta)(-4 + (-11 + 9n)s) < 0
\]

Assuming \(0 < \theta < 1, \ 0 \leq s \leq 1\) and \(n > 0\), \(\Delta \beta > 0\) and \(\Delta \delta < 0\) hold if and only if

\[
n > \frac{5}{3}, \quad \frac{4}{9n - 11} < s \leq 1, \quad 0 < \theta < 1
\]

As we’ve seen before, a population of LH players cannot remove LL mutations.

Case 3: \(\alpha = \epsilon, \beta = 1 - \epsilon, \gamma = 0, \delta = 0\)

\[
\pi^{LL} = -np
\]
\[
= -n \cdot 0 = 0
\]
\[
\pi^{LH} = -np(1 + 2s)
\]
\[
= -n \cdot 0(1 + 2s) = 0
\]
\[
\bar{\pi} = \alpha \cdot \pi^{LL} + \beta \cdot \pi^{LH}
\]
\[
\Delta \beta = \pi^{LH} - \bar{\pi}
\]
\[
= \pi^{LH} - (\alpha \cdot \pi^{LL} + \beta \cdot \pi^{LH})
\]
\[
= (1 - \beta) \cdot \pi^{LH} - \alpha \cdot \pi^{LL}
\]
\[
= (1 - \beta) \cdot (-n \cdot p) \cdot (1 + 2s) - \alpha(-n \cdot p)
\]
Now suppose there are two mutations, so that in the population of LH types there is a mutation of LL types and HL types.

Case 3a: \( \alpha = \epsilon, \beta = 1 - \epsilon - \theta, \gamma = \theta, \delta = 0 \)

\[
\pi^{LL} = -\theta * n
\]
\[
\pi^{LH} = -\theta * n * (1 + 2 * s)
\]
\[
\pi^{HL} = 4 - \theta * n - 9 * n * (1 - \epsilon - \theta) * s
\]

The requirements for temporarily supporting a norm are

\[
\Delta\alpha = -\theta * (4 + 11 * (-1 + \epsilon + \theta) * n * s) > 0
\]
\[
\Delta\beta = -\theta * (4 + (-9 + 11 * (\epsilon + \theta)) * n * s) > 0
\]
\[
\Delta\gamma = 4 - 4 * \theta - 9 + 11 * \theta * (-1 + \epsilon + \theta) * n * s < 0
\]

Assuming \( \epsilon \geq 1/n, \theta \geq 1/n, \) and \( 0 \leq s \leq 1 \) \( n > 0, \Delta\beta > 0 \) holds if and only if

\[
n > \frac{26}{9}, \quad \frac{4}{9n - 22} < s \leq 1, \quad 0 < \epsilon < \frac{9 * n * s - 11 * s - 4}{11 * n * s},
\]
\[
0 < \theta < \frac{9 * n * s - 4 - 11 * \epsilon * n * s}{11 * n * s}
\]

\( \Delta\gamma < 0 \) holds under the weaker requirement

\[
0 < \theta < \frac{4}{22 * n * s} + \frac{20 - 11 * \epsilon}{22} - \frac{\sqrt{16 + ns(16 + 88\epsilon + (n - 11 + \epsilon^2)ns)}}{22 * n * s}
\]

which holds automatically when the \( \Delta\beta > 0. \) So, if \( \Delta\beta > 0 \) then \( \Delta\gamma < 0. \)
\[ \Delta \alpha > 0 \text{ holds trivially when } \Delta \beta > 0 \text{ and } \Delta \gamma < 0 \text{ since } \Delta \alpha > \Delta \beta \text{ when } s > 0. \]

**Case 3b** \( \alpha = \epsilon, \beta = 1 - \epsilon - \theta, \gamma = 0, \delta = \theta \)

\[
\begin{align*}
\pi_{LL} &= -\theta * n \\
\pi_{LH} &= -\theta * n * (1 + 2 * s) \\
\pi_{HH} &= 4 + 11 * s - 9 * (1 - \epsilon) * n * s - \theta * n * (1 + 2 * s)
\end{align*}
\]

\[ \Delta \alpha = -\theta * (4 + 11 * (1 + (-1 + \epsilon) * n) * s) > 0 \]

\[ \Delta \beta = -\theta * (4 + (11 + (-9 + 11 * \epsilon) * n) * s) > 0 \]

\[ \Delta \delta = 4 + (11 + 9 * (-1 + \epsilon) * n) * s - \theta * (4 + (11 - 9 * n + 11 * \epsilon * n) * s) < 0 \]

*Assuming \( \epsilon \geq 1/n, \theta \geq 1/n, \text{ and } 0 \leq s \leq 1 \), \( n > 0, \Delta \alpha > 0, \Delta \beta > 0, \text{ and } \Delta \delta < 0 \) hold if and only if*

\[ n > \frac{26}{9}, \quad \frac{4}{9n - 22} < s \leq 1, \quad 0 < \epsilon < \frac{9 * n * s - 11 * s - 4}{11 * n * s}, 0 < \theta < 1 \]

\[ \Delta \alpha > 0 \text{ holds trivially when } \Delta \beta > 0 \text{ and } \Delta \gamma < 0, \text{ since } \Delta \alpha > \Delta \beta \text{ when } s > 0. \]

**III.2 Discussion**

Overall, the results differ somewhat from Axelrod’s. According my model, there are two equilibria that are ESE’s. The type eight, where individuals have a high boldness score and a low vengefulness score, is an ESE for sufficiently large populations, \( n > 5.5 \). That is to say, the state where the norm has completely broken down, or has never existed is a stable state. The conditions on \( s \) and epsilon are also insightful. A population composed entirely of HL players will have low average vengefulness, zero in fact. Therefore, it is not necessary to restrict \( s \). What does need to be restricted is the rate of mutation, since that will effectively control the average vengefulness. In this model, the probability of being punished depends both on the state variable \( s \), as well as average vengefulness in the population—i.e. the proportion of LH and HH players. Therefore, as long as we keep the mutation rate sufficiently
low, the average vengefulness will also be sufficiently low, and s will not matter. The
type 9 equilibrium, where individuals have both high vengefulness and high boldness
is an ESE for sufficiently small populations, n<5.5. In terms of norm establishment
this is a strange equilibrium. Essentially, this says that for small populations, the
benefit of cheating outweighs the cost of punishment and enforcement. The norm is
not truly established. This makes sense, since there are fewer players to punish any
particular player and each player can enforce against fewer defections. The stability
conditions are more complicated here because of the small population size. Even one
mutant will give a mutation rate of at 20% or more. This makes it much easier for a
single mutant to invade a population.

The type 8 ESE does not support a non-cheating norm; in fact the conduct at
this equilibrium is exactly the opposite of norm guided behavior. Two of the
populations end up in this category in Axelrod’s paper. They both have average
boldness very close to 1 and average vengefulness very close to zero. On the other
hand, the type 9 equilibrium corresponds to high average boldness and high average
vengefulness. Axelrod does not get this result at all. This makes sense, since the
population is too large for the temptation payoff to compensate for so many potential
punishments and enforcements. This is supported by my result that a population of 20
players is too large to support the type 9 equilibrium as an ESE.

In this model I do not attain a stable equilibrium with low average boldness
and low average vengefulness, like Axelrod does, where players do not cheat even
though there is no strict norm against it. It seems probable that HL types will take
advantage of the low average vengefulness and be able to invade this equilibrium, and
cause the norm to break down completely. Therefore, it is interesting that any populations end up in this state in Axelrod’s model after 100 generations because clearly defectors should be able to move the population away from this state. In fact, they probably would, if the simulation was continued past the first 100 generations.

In the norms game Axelrod also has one population that ends up with moderately high average vengefulness and low average boldness after 100 generations. That is to say, a partial norm is established after 100 generations. However, he does not make it clear if this is a stable equilibrium. According to my results, it is not. Lemma five shows that the type 7 equilibrium (all LH players) prevents mutant defectors from invading only temporarily. Certain type 2 equilibria (mixes of LL and LH types) are also immune to invasion by defectors, provided that the proportion of LL types is sufficiently small. However, the problem is that LL types receive a higher payoff than LH types in the presence of defectors. Observing the replication functions in part 3 (a) and 3(b) of the proof of Lemma 5 leads to the immediate conclusion that if the LH replication function is positive, then so is the LL replication function. Therefore LL types will also increase in number. That is, the HL or HH mutants will be replaced by both LL and LH types; therefore the LL proportion will increase after the defectors are removed. If this process continues, it is evident that eventually the proportion of LL types will be too high, or equivalently, the average vengefulness will be too low to prevent defectors from invading the population. Therefore, the conclusion is that the type 7 and some type 2 equilibria can support a norm, maybe even for an extended period of time, but not permanently. So the norm is not stable.
The difference in our results can have several sources. First of all, the computer simulation may simply not have generated an appropriate population that would have ended up in the high average boldness and high average vengefulness region. Second, Axelrod’s replication dynamic is rather more permissive than mine. According to his replication function, if a player’s payoff is within one standard deviation from the mean, he gets one offspring in the next generation. If one’s payoff is greater than one standard deviation above the mean, one gets two offspring. Only if a player’s payoff is more than one standard deviation below the mean, does he not replicate at all. Therefore it is feasible that a player’s payoff is less than average but his population proportion does not decrease. This happens when the extra offspring of the best-performing types replace the worst performing types and the types in the middle do not change their proportion. According to my model, if a player’s payoff is below average then his type’s proportion necessarily declines.

The average is a very logical breaking point, in terms of who increases their proportion and who declines in number. However, Axelrod’s replication function is also reasonable. Which replication function is preferable depends on what particular scenario is being modeled. How competitive is the society which is being modeled? In my opinion, my replication function more accurately reflects the level of competitiveness in our society than Axelrod’s.

One more important difference in between the models is that in my model s is a population parameter and is held fixed for each population. Looking at the stability requirements for the Type 9 equilibrium, we see that both the population and the probability of being observed have to be sufficiently small. In particular, a population
of 20 individuals is too large for the type 9 equilibrium to be an ESE, which may be one reason Axelrod does not obtain this result. Another reason, previously mentioned, is that in Axelrod’s model, the probability of being observed cannot be controlled, and it is highly unlikely that the computer will continually generate sufficiently low values of s.

Another difference in our results comes from the way we present them. Axelrod plots the equilibria as average boldness in the population versus average vengefulness in the population. He does not actually tell us the underlying population composition. Many different population compositions can lead to the same population averages. Furthermore, we have to take into account the fact that in my norms game model vengefulness can take only one of two values, whereas in Axelrod’s paper it can take on one of eight values. It seems likely that the finer gradations in vengefulness and boldness would result in “adjacent types” having less variation in payoff than “adjacent types” in my model. There are only four possible types in the current model, while the computer simulation allows for up to 64 different types (all possible combinations of vengefulness and boldness scores). Therefore, if his population consists of similar types, in as much as the difference between their boldness scores and the difference between their respective vengefulness scores are not too big, slight shifts in the proportions of the present types would not lead to big differences in the average vengefulness and average boldness, which would in turn look like stability. So it may be possible to have changes in the population composition and still have stability as long as the changes do not affect the averages very much. This is actually not very clear from the way Axelrod presents his results.
In my model, there is greater variation in payoffs between the different types and therefore even small changes in population proportions can have large effects on the average payoff and therefore on the replication functions. Essentially, my model may be more sensitive to changes in population composition and therefore it produces fewer stable states. Moreover, it makes sense that my model is more sensitive because all that is required is for a particular type to get below average payoff for the population proportion to change. Axelrod’s replication dynamic allows below average types to maintain their numbers relatively more successfully than my model.

III.3 Model 2 for a Twenty Player Population

Again, we assume that $\epsilon$ is the proportion of an invading type. Applying Theorem 2 to a population of twenty players gives the following result:

1) The Type 8 ($\alpha=0, \beta=0, \gamma=1, \text{and } \delta=0$) equilibrium is the unique ESE if and only if $s>0$ and:

   a. $\epsilon \leq \frac{2}{11}$

   b. $\epsilon > \frac{2}{11}$ and $s < \frac{4}{20+11*\epsilon-2}$

2) There are no other ESE’s.

Applying lemma 5 to a population of 20 players gives the following result:

The Type 7 equilibrium ($\alpha^*=0, \beta^*=1, \gamma^*=0, \delta^*=0$) as well as certain Type 2 Equilibria ($\alpha^*=\epsilon, \beta^*=1-\epsilon, \gamma^*=0, \delta^*=0$) are temporarily norm supporting when:

$$\frac{4}{158} < s \leq 1, \quad 0 < \epsilon < \frac{169 * s - 4}{220 * s},$$

$$0 < \theta < \frac{180 * s - 4 - 220 * \epsilon * s}{220 * s}$$

Where $\epsilon$ is the proportion of LL players and $\theta$ is the proportion of invading cheaters.
III.4 Discussion

Of the two potential stable equilibria n=20 satisfies the requirement \( n > 11/2 \). Therefore, type 8 is the only possible ESE. If there are no more than three mutations at a time, i.e. epsilon satisfies condition 1 (a) in theorem 2, the population is guaranteed to have a stable equilibrium. For epsilon to satisfy \( \varepsilon > 2/11 \), at least four mutations are required, which is one fifth of the population. This is an unconventionally high mutation rate in the population, so we can reasonably ignore this possibility. Thus, we conclude that a population of twenty agents will settle at the type 8 stable equilibrium. Since this is the only evolutionarily stable state, populations of twenty players can never support the no-cheating norm as an ESE. Two of the five populations ended up in this equilibrium in Axelrod’s paper.

The main difference is that I do not reproduce his result of a stable high vengefulness and low boldness (norm supporting) equilibrium. However, even Axelrod does not exhibit a true anti-defection norm, which would correspond to my type 7 equilibrium, (average boldness=0 and average vengefulness=1). The closest he gets is an equilibrium with average boldness about halfway between zero and 1/7 and average vengefulness a little over 5/7. For me to get a population in this region, it would have to be composed largely of LH types, that is, individuals who do not cheat but do punish. Theorem 2 restricted to populations of 20 players, shows that neither norm supporting equilibrium (type 7 or type 2) is an ESE. Lemma 5 shows that a population of 20 people can temporarily support a norm assuming that the probability of being observed is sufficiently large, the population is composed largely of LH types and the proportion of LL types is sufficiently small. However, since LL types
do better than LH types in the presence of defectors, eventually, the proportion of LL types will increase beyond the level necessary to remove mutant defectors. Therefore, the population will not sustain the norm permanently.
IV. Model 3: A Simple Version of the Metanorms Game

IV.1 Model

In order to understand this model we first need a definition of *metanorms*. According to Axelrod the definition is as follows:

**Definition 5:** A *metanorm* is the existence of a norm to punish non-punishers as well as defectors.

As we have seen in the previous chapter, the no-cheating norm is not an ESE for any population size in the norms game. The goal of this chapter is to determine if metanorms will permit the establishment of a no-cheating norm as an evolutionarily stable equilibrium. Specifically I will test the type 7 equilibrium for stability. I will also test the type 8 equilibrium for stability. In Axelrod’s metanorms game, all five populations end up in the high average vengefulness and low average boldness state. However, the author admits that this result is contingent on the population starting with sufficiently high average vengefulness. Otherwise the norm breaks down. I will study the effect of average vengefulness on the establishment of the norm.

I have already established that a player’s *vengefulness score* is the probability that he or she will punish a defector. I now define the *meta-vengefulness score* as the probability that the agent will punish someone who he observes not punishing a defection. To make the model more tractable I assume, like Axelrod, that an agent’s vengefulness against a non-punisher is the same as against a cheater.

I modify Model 1 by adding the metanorm. The basic structure is still the same; with the modification that average vengefulness in the population is also the
average metavengefulness, which affects the expected payoffs of the two groups of people (cheaters and cooperators).

The payoff to high bold players is the sum of the following 6 terms

1. \(+3\) is the temptation to defect
2. \(-9 \times (n - 1) \times v \times s\) is the expected punishment
3. \(-1 \times (n \times p - 1)\) is the damage inflicted by other defectors
4. \(-2 \times (n \times p - 1) \times v \times s\) is the expected value of the cost of enforcement
5. \(-9 \times (n \times p - 1) \times s \times (1 - v) \times v \times s\) is the meta—punishment
6. \(-2 \times p \times (-1 + n \times (1 - n + p)) \times s^2 \times (-1 + v) \times v\)

which simplifies to the following expression.

\[
\begin{align*}
\text{Payoff to high bold people:} \\
\pi^H & = 3 - 9 \times (n - 1) \times v \times s + (1 - n \times p) \times (1 + 2 \times v \times s) \\
& - 9 \times (n \times p - 1) \times s \times (1 - v) \times v \times s \\
& - 2 \times p \times (-1 + n \times (1 - n + p)) \times s^2 \times (-1 + v) \times v
\end{align*}
\]

Similarly, the payoff to low bold players is the sum of the following six terms

1. \(+0\) is the payoff to not defecting
2. \(+0\) is the expected value of punishment
3. \(-1 \times n \times p\) is the damage inflicted by all the defectors
4. \(-2 \times n \times p \times v \times s\) is the expected value of enforcement cost
5. \(-9 \times n \times p \times s \times (1 - v) \times v \times s\) is the meta—punishment
6. \(2 \times n \times p \times (-2 + n + p) \times s^2 \times (-1 + v) \times v\)
is the meta – enforcement cost

which sums to the following expression.

\[ \pi^1 = -n \times p \times (1 + 2 \times v \times s) - 9 \times n \times p \times s \times (1 - v) \times v \times s \]

\[ + 2 \times n \times p \times (-2 + n + p) \times s^2 \times (-1 + v) \times v \]

The meta-punishment part of the payoff is again an expectation of a random variable with a binomial distribution. And again, cooperators can observe and punish all defections that occur in the game, whereas cheaters will have a chance to observe and punish all defections except their own. It is obvious that a cheater must necessarily observe himself or herself cheating, but we will assume that the cheater does not self-punish. This fact now also affects the meta-punishment and meta-enforcement payoffs through its effect on expected punishment and expected enforcement cost. Players do not meta-punish themselves either. Clearly, a player knows whether or not he punished a defection that he observed, and in theory could meta-punish himself if he did not punish the defection. But we assume that he does not do that. The meta-punishment payoff is fairly straightforward. It is the expected number of observed defections that the player does not punish times the probability that these non-punishments will be observed and punished by a third party, times -9, the punishment. The meta-enforcement cost is a little more complicated. Again it is different for the two types of players to account for the fact that a player will not meta-enforce against him- or herself. Fully written out, the expected meta-enforcement cost for H types is:
The meta-enforcement cost for L types is:

\[-2 \times [(1 - p) \times n \times n \times p \times s \times (1 - v) + p \times (n - 1) \times (n \times p - 1) \times s \times (1 - v)] \times v \times s\]

The meta-enforcement cost for L types is:

\[-2 \times [(1 - p) \times (n - 1) \times n \times p \times s \times (1 - v) + p \times n \times (n \times p - 1) \times s \times (1 - v)] \times v \times s\]

What I do here is take the weighted sum of the unpunished observations for each type to get the total number of unpunished observed defections that each player can meta-enforce against. This number is different for each player type because we have to subtract each player’s own non-punishments from the total number of non-punishments, since the player will not meta-enforce against him- or herself.

The new replication function, compared to the replication function from Model 1 (which I will refer to as the old replication function) is:

\[New\Delta p = Old\Delta p + \Delta(meta - punishment) + \Delta(meta - enforcement)\]

Which can be written as:

\[\Delta p = 4 + (11 - 9 \times n) \times v \times s + (-9 + 2 \times p + 2 \times n \times p - 4 \times n \times p^2) \times s^2 \times v \times (v - 1)\]

Unlike in Model 1, the replication function now depends on p. Also, it is rather more complicated. While it is possible to write down the stability conditions for each type of equilibrium, they are very complex expressions and hard to decipher just by looking at them. In order to make sense out of my results, I decided to look directly at the case of n=20. Plugging the population size into the above expression, we get:

\[\Delta p = 4 + s \times (-169 - (9 - 42 \times p + 80 \times p^2) \times s \times (v - 1)) \times v\]

From this replication function I derive the following theorem:
Theorem 3

1) For 20 player populations the no cheating norm \((p^*=0)\) is an ESE when \(a\) and \(b\) hold, assuming the mutation away from \(p^*=0\) is small.

\[
a. \quad \frac{4}{169} < s \leq 1
\]
\[
b. \quad \frac{-1690+71s+\sqrt{2867460-239980s+5041s^2}}{142}\times s < v \leq 1
\]

2) For 20 player populations the equilibrium \(p^*=1\) is an ESE when \(a\) or \(b\) hold assuming the mutation away from \(p^*=1\) is small.

\[
a. \quad 0 \leq s < \frac{4}{169} \quad \text{and} \quad 0 \leq v \leq 1
\]

or

\[
b. \quad \frac{4}{169} \leq s \leq 1 \quad \text{and}
\]
\[
0 \leq v < \frac{-1690 + 413s + \sqrt{2922180 - 1395940s + 170569s^2}}{826s}
\]

3) For 20 player populations the interior equilibrium \(0<p<1\) is stable when \(a\) or \(b\) hold.

\[
a. \quad p = \frac{1}{80} \left(21 + \frac{s^2 + \sqrt{s^2 + (1-v) + v(-320+s(13520+279s+158\cdot s^2))}}{s^2 + (v-1)\cdot s}\right)
\]

and \(\frac{4}{169} < s \leq 1\) and

\[
\frac{-13520 + 279s + \sqrt{183147520 - 7544160s + 77841s^2}}{558s} < v
\]

\[
\leq \frac{-169 + 9s + \sqrt{28705 - 3042s + 81s^2}}{18s}
\]

\[
b. \quad v = \frac{1}{2} \left(1 - \frac{169}{(9-42p+80p^2)s} + \sqrt{1 + \frac{28705-3042s-4p(-21+40p)(169s-8)}{(9-42p+80p^2)^2s^2}}\right)
\]
Proof:

1) First, remember that for \( p = 0 \) to be an ESE, a small deviation from \( p = 0 \) must be counteracted by evolutionary pressure. So we want to show that for a small deviation from 0, the replication function \( \Delta p \) is negative. It is fairly innocuous to assume that mutations do not happen very often. Therefore we can assume that multiple mutations would not happen simultaneously. Therefore all we have to do is check that \( \Delta p < 0 \) when \( p = 1/20 \). Suppose we start with a population composed entirely of cooperators (\( p = 0 \)). A mutation occurs so that one player turns into a defector. Now \( p = 1/20 \). If \( \Delta p < 0 \), then that one defector is doing worse than the rest of the cooperators, the mutant is promptly removed from the population and we return to \( p = 0 \), and therefore \( p = 0 \) is stable. So when does this happen?

Plugging in \( p = 1/20 \) into the expression for \( \Delta p \) gives

\[
\Delta p = 4 + s \left( -169 \frac{71}{10} * s * (v - 1) \right) * v
\]

We need:

\[
\Delta p = 4 + \left( -169 \frac{71}{10} * s * (v - 1) \right) * v * s < 0
\]

I will omit the algebra this time. The procedure is similar to what was done earlier to derive conditions for the other theorems, although a little more complicated. A software package such as Mathematica can be used to make the necessary calculations. The bottom line is that

\[
\Delta p < 0
\]
Holds if and only if

\[ \frac{4}{169} < s \leq 1 \]

and

\[ \frac{-1690 + 71 \cdot s + \sqrt{2867460 - 239980 \cdot s + 5041 \cdot s^2}}{142 \cdot s} < v \leq 1 \]

Note also that when the condition on s is satisfied, the lower bound on v is less than 1.

Therefore the interval that bounds v is non-empty.

2)

Next, assume that we have a population made up entirely of defectors, i.e. \( p=1 \). A mutation occurs where one defector becomes a cooperator, so \( p=19/20 \). Now the thing to check is \( \Delta p > 0 \) when \( p=19/20 \). Plugging \( p=19/20 \) into the replication function gives us:

\[ \Delta p = 4 + (-169 - 413 \cdot s \cdot (v - 1)) \cdot v \cdot s \]

Some algebra yields \( \Delta p > 0 \) if and only if

\[ 0 \leq s < \frac{4}{169} \text{ and } 0 \leq v \leq 1 \]

or

\[ \frac{4}{169} \leq s \leq 1 \text{ and } 0 \leq v < \frac{-1690 + 413 \cdot s + \sqrt{2922180 - 1395940 \cdot s + 170569 \cdot s^2}}{826 \cdot s} \]

3)

Finally we check the interior equilibrium. Remember, we need

\[ \Delta p = 0 \text{ and } \frac{d\Delta p}{dp} < 0 \]

So we solve
\[ \Delta p = 4 + s \times (-169 - (9 - 42 \times p + 80 \times p^2) \times s \times (v - 1)) \times v = 0 \]

for \( p \), and get two solutions

\[
p_1 = \frac{1}{80} \times \left( 21 - \sqrt{\frac{s^2 \times (1 - v) \times v \times (-320 + s \times (13520 + 279 \times s \times (v - 1)) \times s)}{s^2 \times (v - 1) \times v} } \right)
\]

And

\[
p_2 = \frac{1}{80} \times \left( 21 + \sqrt{\frac{s^2 \times (1 - v) \times v \times (-320 + s \times (13520 + 279 \times s \times (v - 1)) \times s)}{s^2 \times (v - 1) \times v} } \right)
\]

First, we need the expression under the square root to be positive. This happens when

\[ \frac{4}{169} < s \leq 1 \]

and

\[ \frac{-13520 + 279 \times s + \sqrt{183147520 - 7544160 \times s + 77841 \times s^2}}{558 \times s} < v < 1 \]

We also need to restrict the values of \( s \) and \( v \) so that \( p \) is a proportion. This adds the additional constraint that

\[ v \leq \frac{-169 + 9 \times s + \sqrt{28705 - 3042 \times s + 81 \times s^2}}{18 \times s} \]

Next, we take the derivative of \( \Delta p \) with respect to \( p \) and get.

\[ \frac{d\Delta p}{dp} = (160 \times p - 42) \times s^2 \times (1 - v) \times v \]

Plug in each of the solutions into the above equation and get:

\[ \frac{d\Delta p}{dp} (p_1) = 2 \times \sqrt{s^2 \times (1 - v) \times (-320 + s \times (13520 + 279 \times s \times (v - 1)) \times v)} \]

\[ \frac{d\Delta p}{dp} (p_2) = -2 \times \sqrt{s^2 \times (1 - v) \times (-320 + s \times (13520 + 279 \times s \times (v - 1)) \times v)} \]
Since we necessarily must take the positive square root, the derivative of $\Delta p$ is negative only when evaluated at $p_2$. In general,

$$\frac{d\Delta p}{dp} = (160 * p - 42) * s^2 * (1 - \nu) * \nu < 0$$

Holds if and only if

$$0 < \nu < 1, and \ 0 \leq p < \frac{21}{80}, and \ 0 < s \leq 1$$

The conditions that $p$ is a proportion already give us $p < 21/80$, therefore, the requirement for the derivative to be negative is already met.

For part $b$,

$$\nu = \frac{1}{2} * \frac{169}{(9 - 42p + 80p^2)s} + \sqrt{1 + \frac{28705 - 3042s - 4p(-21 + 40p)(169s - 8)}{(9 - 42p + 80p^2)^2s^2}}$$

Satisfies $\Delta p = 0$. We add the constraints

$$\frac{4}{169} < s \leq 1 and \ 0 \leq p < \frac{21}{80}$$

which gets us that the expression for $\nu$ automatically meets the necessary conditions

$$0 < \nu < 1$$

so that

$$\frac{d\Delta p}{dp} < 0$$

holds.

**IV.2 Discussion**

The results are what one would expect. Part one of the theorem in this chapter essentially boils down to requiring $\nu$ and $s$ to be large enough for the population to support the no-cheating norm. This makes sense since both $\nu$ and $s$ determine the
likelihood that a defection will be punished. The numbers in the first part of the theorem were calculated for \( p = 1/20 \), to make the calculations manageable. Changing the value of \( p \) will only change the lower bound of \( v \), but it does not change the substance of the theorem: it would still say that \( v \) and \( s \) have to be sufficiently large for \( p = 0 \) to be an ESE. We are only interested in knowing when the proportion will return to zero if there is a small deviation from zero.

Similarly, part two of the theorem says that \( p = 1 \) is a stable equilibrium when either \( v \) or \( s \) is sufficiently small. In this case, the numbers are for the specific case of \( p = 19/20 \) but reducing \( p \) a little will not change the substance of the theorem, and only make the algebra more complicated to look at. Besides, our primary concern is to find out if the population proportion will return to unity if there is a small deviation from one, and there is no harm in assuming that deviations from one will happen one at a time. Besides, 1 mutation out of 20 players is already 5% of the population, which satisfies our assumption that the mutation is small.

The interior equilibrium is a little more complicated. Just by looking at the numbers it is not always immediately apparent what part three of the theorem says about the values of \( p \), \( s \), and \( v \). The first case of part three tells us that \( s \) has to be greater than \( 4/169 \), which is essentially saying that \( s \) can’t be quite that close to zero. The condition on \( v \) is a little more complicated. However, rearranging the bounds on \( s \) to be in terms of \( v \), we can get purely numeric bounds on \( v \), which would be somewhere between 0.024 and 0.025. That is to say, \( v \) has to be quite small. Furthermore, we are given the specific stable equilibrium proportion, which is determined by the population parameters \( s \) and \( v \). This could be problematic since we
assume that the number of players in integer valued, and therefore the number of
defectors has to be an integer. The calculated proportion $p$ may not necessarily lead to
an integer number of defectors and cooperators. The other case when the interior
equilibrium is stable does not give us the specific interior proportion that is going to
be stable but we are given the required relationship between $v$, $p$ and $s$, and we are
given a range of $p$ values that can potentially be stable interior equilibria. The bound
on $s$ is the same as before.

The important conclusion to be drawn here is that given a population with the
right characteristics, a no-cheating norm can be established. This is part one of
theorem three. Given that $s$ is large enough, a population with high average
vengefulness can support a no cheating norm (the $p=0$ equilibrium) as an ESE.
Similarly, part 2 of the theorem tells us that a population with the right characteristics
can support an ESE with all cheaters, given that average vengefulness and or the
probability of being observed is small enough. A population can even support the
somewhat counterintuitive equilibrium of high average boldness and high average
vengefulness, as an ESE, if the probability of being observed is very small. This is
again the case where the payoff to cheating compensates for the cost of punishment
and enforcement. That is to say, the punishment does not effectively support the
norm. It is possible to achieve any of the five stable equilibria that Axelrod attains in
this model. But as before, we are able to attain equilibria in the fourth region (high
average boldness and high average vengefulness) that Axelrod does not.
V. Model 4: The Metanorms Game with Endogenous Vengefulness

V.1 Model

In this chapter I extend model three by endogenizing vengefulness. The variable n, s, b, vi, p and q as well as the payoff structure are defined as in model 2. I continue to assume that the *meta-vengefulness score*, denoted mv, which is the probability that an agent will punish someone whom he observes not punishing a defection, is equal to the *vengefulness score*, the probability that he or she will punish a defector. Therefore, punishers, vi=1, will punish non-punishers with certainty, just as they punish defectors. Non-punishers, vi=0, will not punish non-punishers, just as they do not punish defectors. Thus, as in the norms game from model 2, we still have four types of players, with the addition that the proportion of meta-punishers is q, same as punishers. Similarly, the proportion of non-meta-punishers is (1-q). The difference between this model and model 2 is that the payoffs to each agent type changes. The agent types and payoffs are defined below.

1. Let LL players = \{people i such that \( b_i < s \) and \( v_i = mv_i = 0 \} 
2. Let LH players = \{people i such that \( b_i < s \) and \( v_i = mv_i = 1 \} 
3. Let HL players = \{people i such that \( b_i > 0 \) and \( v_i = mv_i = 0 \} 
4. Let HH players = \{people i such that \( b_i > s \) and \( v_i = mv_i = 1 \} 

The respective payoffs are listed below.

\[
\pi_{LL} = 0 - 1 * n * p - 0 - 9 * n * p * s * n * q * s
\]
\[
= -n * p * (1 + 9 * n * q * s^2)
\]
1. +0 payoff from not defecting
2. +0 payoff from no punishment
3. −n * p damage from all defectors
4. +0 expected enforcement cost
5. −9 * n * p * s * n * q * s expected meta − punishment
6. +0 expected meta enforcement cost

\[ \pi^{LH} = 0 - 1 * n * p - 0 - 2 * s * n * p - 0 - 2 \\
\quad \times ((1 - p) * n * n * p * s + p * n * (n * p - 1)) * (1 - q) * n * s \\
= -n * p(1 + 2 * s) - 2 * ((1 - p) * n * n * p * s + p * n * (n * p - 1)) \\
\quad * (1 - q) * n * s \\
= n * p(-1 + 2 * s * (-1 + (-1 + n) * n * (-1 + q) * s)) \]

1. +0 payoff from not defecting
2. +0 payoff from no punishment
3. −n * p damage from all defectors
4. −2 * n * p * s expected value of enforcement cost
5. +0 meta punishment because you are a punisher
6. −2 * ((1 - p) * n * (n * p * s) + p * n * (n * p - 1) * s) * (1 - q) * n * s meta − \\
   enforcement cost

\[ \pi^{HL} = 3 - 9 * n * q * s - 1 * (np - 1) - 0 - 9 * (n * p - 1) * s * n * q * s - 0 = \\
= 4 - np + 9 * n * q * s * (-1 + s - n * p * s) \]

1. +3 payoff from cheating
2. −9 * q * n * s expected value of punishment from all punishers
3. \(-(n \times p - 1)\) damage from other defectors
4. +0 enforcement cost
5. \(-9 \times n \times q \times s \times (n \times p - 1) \times s\) meta – punishment
6. +0 expected meta – enforcement cost

\[
\pi^{HH} = 3 - 9 \times (n \times q - 1) \times s - 1 \times (n \times p - 1) - 2 \times (n \times p - 1) \times s - 0
\]

\[
-2 \times ((1 - p) \times n \times n \times p \times s + p \times n \times (np - 1) \times s) \times (1 - q) \times n \times s
\]

\[
= 4 + 11 \times s + n
\]

\[
* \left( -9 \times q \times s + p
\right.

\[
\left. \times (-1 + 2 \times s \times (-1 + (-1 + n) \times n \times (-1 + q) \times s)) \right)
\]

1. +3 payoff from cheating
2. \(-9 \times (n \times q - 1) \times s\) expected value of punishment
3. \(-(n \times p - 1)\) damage from defectors
4. \(-2 \times (n \times p - 1) \times s\) expected value of enforcement cost
5. +0 meta punishment cost
6. \(-2 \times ((1 - p) \times n \times (n \times p \times s) + p \times n \times (np - 1) \times s) \times (1 - q) \times n \times s\) meta – enforcement cost

The proportion of each player type is defined as in model 2.

Now the payoffs are even more complicated than in the simple version. I simplify matters by restricting the model to a population of twenty individuals. This gives the following payoffs.

\[
\pi^{LL} = -20 \times p \times (1 + 180 \times q \times s^2)
\]

\[
\pi^{LH} = -20 \times p \times (1 + 2 \times s + 760 \times (1 - q) \times s^2)
\]
\[ \pi^{HI} = 4 - 20 \cdot p + 180 \cdot q \cdot s \cdot (-1 + s - 20 \cdot p \cdot s) \]
\[ \pi^{HH} = 4 + (11 - 180 \cdot q) \cdot s + 20 \cdot p \cdot (-1 - 2 \cdot s - 760 \cdot (1 - q) \cdot s^2) \]

We are back to the situation of nine possible classes of equilibria.

**Type 1:** \(0 < p < 1\) and \(0 < q < 1\)

\[0 < \gamma + \delta < 1, \quad 0 < \beta + \delta < 1\]

**Type 2:** \(p = 0, 0 < q < 1 \rightarrow \gamma = 0, \delta = 0, 0 < \alpha < 1, 0 < \beta < 1\)

\[\alpha = 1 - \beta\]

**Type 3:** \(p = 10 < q < 1\)

\[\alpha = 0, \quad \beta = 0, \quad \gamma + \delta = 1, \quad 0 < \gamma < 1, \quad 0 < \delta < 1\]

**Type 4:** \(0 < p < 1, \text{ and } q = 0\)

\[\beta = 0, \quad \delta = 0, \quad 0 < \alpha < 1, \quad 0 < \gamma < 1\]

**Type 5:** \(0 < p < 1 \text{ and } q = 1\)

\[\alpha = 0, \quad \gamma = 0, \quad \beta + \delta = 1, \quad 0 < \beta < 1, \quad 0 < \delta < 1\]

**Type 6:** \(p = 0 \text{ and } q = 0\)

\[\alpha = 1, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0\]

**Type 7:** \(p = 0 \text{ and } q = 1\)

\[\alpha = 0, \quad \beta = 1, \quad \gamma = 0, \quad \delta = 0\]

**Type 8:** \(p = 1 \text{ and } q = 0\)

\[\alpha = 0, \quad \beta = 0, \quad \gamma = 1, \quad \delta = 0\]

**Type 9:** \(p = 1 \text{ and } q = 1\)

\[\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 1\]

**Lemma 6:** If a population of twenty players satisfies \(\beta \geq 0.9\) and \(0.0897 < s \leq 1\), then \(\Delta \alpha \leq 0, \Delta \beta \geq 0, \Delta \gamma < 0, \text{ and } \Delta \delta < 0\).
Proof:

Using the payoffs defined in this chapter and

\[ \Delta z = \pi^z - \bar{\pi} \]

We get the following results

Case 1: \( \epsilon = \frac{1}{20} \)

a) \( \alpha = \frac{1}{20}, \beta = \frac{19}{20}, \gamma = 0, \delta = 0 \)

\[ \Delta \alpha = 0 \quad \forall \ s > 0 \]

\[ \Delta \beta = 0 \quad \forall \ s > 0 \]

b) \( \alpha = 0, \beta = \frac{19}{20}, \gamma = \frac{1}{20}, \delta = 0 \)

\[ \Delta \beta = \frac{1}{20} \times (-4 + 169 \times s - 38 \times s^2) > 0 \]

\[ \Delta \gamma = \frac{19}{20} \times (4 - 169 \times s + 38 \times s^2) < 0 \]

for \( 0.0238 < s \leq 1 \)

c) \( \alpha = 0, \beta = \frac{19}{20}, \gamma = 0, \delta = \frac{1}{20} \)

\[ \Delta \beta = \frac{1}{20} \times (-4 + 169 \times s) > 0 \]

\[ \Delta \delta = \frac{19}{20} \times (4 - 169 \times s) < 0 \]

for \( \frac{4}{169} = 0.0237 < s \leq 1 \)

Case 2: \( \epsilon = \frac{2}{20} \)

a) \( \alpha = \frac{2}{20}, \beta = \frac{18}{20}, \gamma = 0, \delta = 0 \)

\[ \Delta \alpha = 0 \quad \forall \ s > 0 \]
$\Delta \beta = 0 \forall \ s > 0$

b) $\alpha = 0$, $\beta = \frac{18}{20}$, $\gamma = \frac{2}{20}$, $\delta = 0$

$\Delta \beta = \frac{1}{25} * (-10 + s * (395 + 386 * s)) > 0$

$\Delta \gamma = -\frac{9}{25} * (-10 + s * (395 + 386 * s)) < 0$

for $0.0247 < s \leq 1$

c) $\alpha = 0$, $\beta = \frac{18}{20}$, $\gamma = 0, \delta = \frac{2}{20}$

$\Delta \beta = \frac{1}{10} * (-4 + 169 * s) > 0$

$\Delta \delta = -\frac{9}{10} * (-4 + 169 * s) < 0$

for $\frac{4}{169} = 0.0237 < s \leq 1$

d) $\alpha = \frac{1}{20}$, $\beta = \frac{18}{20}$, $\gamma = \frac{1}{20}$, $\delta = 0$

$\Delta \alpha = \frac{1}{10} * (-2 + 9 * (11 - 95 * s) * s) < 0$

for $0 \leq s < 0.026 \ OR \ 0.0897 < s \leq 1$

$\Delta \beta = \frac{1}{10} * (-2 + s * (79 + 5 * s)) > 0$

for $0.025 < s \leq 1$

$\Delta \gamma = \frac{1}{10} * (38 + 95 * (-169 + 85 * s)) < 0$

for $0.025 < s \leq 1$

e) $\alpha = \frac{1}{20}$, $\beta = \frac{18}{20}$, $\gamma = 0, \delta = \frac{1}{20}$

$\Delta \alpha = \frac{1}{20} * (-4 + (198 - 2527 * s) * s) < 0$
for $0 \leq s \leq 1$

$$\Delta \beta = \frac{1}{20} \times (-4 + s \times (158 + 133 \times s)) > 0$$

for $0.024 < s \leq 1$

$$\Delta \delta = \frac{1}{20} \times (76 + s \times (-3042 + 133 \times s)) < 0$$

for $0.025 < s \leq 1$

\[ f) \quad \alpha = 0, \quad \beta = \frac{18}{20}, \quad \gamma = \frac{1}{20}, \quad \delta = \frac{1}{20} \]

$$\Delta \beta = \frac{1}{100} \times (-40 + s \times (1635 + 836 \times s)) > 0$$

for $0.024 < s \leq 1$

$$\Delta \gamma = \frac{18}{5} - \frac{1}{100} \times s \times (15065 + 15884 \times s) < 0$$

for $0.0233 < s \leq 1$

$$\Delta \delta = \frac{1}{100} \times (360 + s \times (-14365 + 836 \times s)) < 0$$

for $0.025 < s \leq 1$

\[ \text{Lemma 6a: A twenty player population composed of 17 LH players and 2 LL players} \]
\[ \text{can still remove a mutant and the LL players for } 0.205 < s \leq 1. \]

\[ \text{Proof:} \]

\[ a) \quad \alpha = \frac{2}{20}, \quad \beta = \frac{17}{20}, \quad \gamma = \frac{1}{20}, \quad \delta = 0 \]

$$\Delta \alpha = \frac{1}{20} \times (-4 + 17 \times s \times (11 - 48 \times s)) < 0$$

for $0 \leq s < 0.02389 \text{ OR } 0.205 < s \leq 1$
Lemma 6 and 6a imply that the as long as mutations do not introduce too many LL players into a population of LH players (the type 7 equilibriu), mutant defectors are removed along with the LL players, provided that the probability of a defection being observed is high enough. We also assume that only a small percentage of players can mutate into cheaters at a time. Since the population is so small, one cheater already makes up 5% of the population, it is sufficient to show that one or two cheater can be removed from a population composed primarily of LH players. Lemma 6 is stronger than Lemma 5 in model 2 since a few LL players no longer dominate LH players in the presence of defectors, and $\Delta \alpha < 0$ rather than $\Delta \alpha > 0$.

\[
\Delta \beta = \frac{1}{20} \cdot (-4 + 3s) \cdot (49 - 12s) > 0
\]

for $0.0274 < s \leq 1$

\[
\Delta \gamma = \frac{1}{20} \cdot (-76 + 17s) \cdot (-169 + 132s) < 0
\]

for $0.027 < s \leq 1$

\[b) \quad \alpha = \frac{2}{20}, \quad \beta = \frac{18}{20}, \quad \gamma = 0, \quad \delta = \frac{1}{20}\]

\[
\Delta \alpha = -\frac{1}{20} \cdot (-1 + 36s) \cdot (-4 + 43s) < 0
\]

for $0 \leq s < 0.027$ OR $0.093 < s \leq 1$

\[
\Delta \beta = \frac{1}{20} \cdot (-4 + s) \cdot (147 + 172s) > 0
\]

for $0.0264 < s \leq 1$

\[
\Delta \delta = \frac{1}{20} \cdot (76 + s(-2873 + 172s)) < 0
\]

for $0.0265 < s \leq 1$
Therefore, the only way that the norm can break down is if sufficiently many LL players are introduced by mutation before a mutant cheater shows up.

**Lemma 7:** The Type 8 Equilibrium is an ESE for 20 player populations provided that the population at some point gains 18 HL players and $0.0075 < s < 0.256$.

**Proof:**

*Using the payoffs defined in this chapter and*

$$\Delta z = \pi^z - \pi$$

*We get the following results*

**Case 1:** $\epsilon = \frac{1}{20}$

a) $\alpha = \frac{1}{20}, \beta = 0, \gamma = \frac{19}{20}, \delta = 0$

$$\Delta \alpha = -\frac{19}{5} < 0 \forall s > 0$$

$$\Delta \gamma = \frac{1}{5} > 0 \forall s > 0$$

b) $\alpha = 0, \beta = \frac{1}{20}, \gamma = \frac{19}{20}, \delta = 0$

$$\Delta \beta = -\frac{19}{200} \star (40 + s \star (290 + 5239 \star s)) < 0 \forall s > 0$$

$$\Delta \gamma = \frac{1}{200} \star (40 + s \star (290 + 5239 \star s)) > 0 \forall s > 0$$

c) $\alpha = 0, \beta = 0, \gamma = \frac{19}{20}, \delta = \frac{1}{20}$

$$\Delta \gamma = \frac{29}{20} \star s \star (1 + 19 \star s) > 0 \forall s > 0$$

$$\Delta \delta = -155 \star s \star (1 + 19 \star s) < 0 \forall s > 0$$

**Case 2:** $\epsilon = \frac{2}{20}$
a) $\alpha = \frac{2}{20}, \beta = 0, \gamma = \frac{18}{20}, \delta = 0$

$$\Delta \alpha = -\frac{18}{5} < 0 \forall s > 0$$

$$\Delta \gamma = \frac{2}{5} > 0 \forall s > 0$$

b) $\alpha = 0, \beta = \frac{2}{20}, \gamma = \frac{18}{20}, \delta = 0$

$$\Delta \beta = -\frac{9}{25} \times (10 + 9 \times s \times (5 + 86 \times s)) < 0 \forall s > 0$$

$$\Delta \gamma = \frac{1}{25} \times (10 + 9 \times s \times (5 + 86 \times s)) > 0 \forall s > 0$$

c) $\alpha = 0, \beta = 0, \gamma = \frac{18}{20}, \delta = \frac{2}{20}$

$$\Delta \gamma = \frac{1}{10} \times s \times (29 + 3425 \times s) > 0 \forall s > 0$$

$$\Delta \delta = -\frac{9}{10} \times s \times (29 + 3425 \times s) < 0 \forall s > 0$$

d) $\alpha = \frac{1}{20}, \beta = \frac{1}{20}, \gamma = \frac{18}{20}, \delta = 0$

$$\Delta \alpha = \frac{9}{100} \times (-40 + s \times (110 + 181 \times s)) < 0$$

For $0 \leq s < 0.255891$

$$\Delta \beta = -\frac{9}{100} \times (40 + s \times (290 + 5239 \times s)) < 0 \forall s \geq 0$$

$$\Delta \gamma = \frac{1}{100} \times (40 + 9 \times s \times (10 + 282 \times s)) > 0 \forall s \geq 0$$

e) $\alpha = \frac{1}{20}, \beta = 0, \gamma = \frac{18}{20}, \delta = \frac{1}{20}$

$$\Delta \alpha = \frac{1}{200} \times (-760 + s \times (1980 + 3529 \times s)) < 0$$

For $0 \leq s < 0.2617$
\[ \Delta \gamma = \frac{1}{200} \times (40 + s \times (180 + 3529 \times s)) > 0 \ \forall s \geq 0 \]

\[ \Delta \delta = \frac{1}{200} \times (40 - s \times (5220 + 99451 \times s)) < 0 \]

For 0.00679 < s \leq 1

\[ \alpha = 0, \ \beta = \frac{1}{20}, \ \gamma = \frac{18}{20}, \ \delta = \frac{1}{20} \]

\[ \Delta \beta = \frac{1}{100} \times (-380 - 1765 \times s - 29322 \times s^2) < 0 \ \forall s \geq 0 \]

\[ \Delta \gamma = \frac{1}{100} \times (20 + s \times (235 + 3258 \times s)) > 0 \ \forall s \geq 0 \]

\[ \Delta \delta = \frac{1}{100} \times (20 - s \times (2465 + 29322 \times s)) < 0 \]

For 0.00745 < s \leq 1

V.2 Discussion

These results here are very unexpected. Unfortunately it is possible for the no-cheating norm to break down even under metanorms provided that enough LH players mutate into LL players before an LH player mutates into a cheater. The problem is due to the fact that LL players are not removed by LH players in the absence of defectors, LL players can accumulate before some player finally mutates into a cheater. Studying the proof of Lemma 6 leads us to the following conclusion: Provided that the probability of being observed is sufficiently high, a no-cheating norm can be sustained near the Type 7 equilibrium (all LH players) as long as it is challenged, in other words, the no-cheating norm can be sustained as long as mutations do not introduce too many LL players into a population of LH players before a cheating mutant arrives. For a twenty player population, it has been shown that an LL mutant will receive a below average payoff in the presence of a cheater
when the rest of the population is composed of LH players. Therefore, a small proportion of LL players will be removed from the population along with the cheaters when the population is composed primarily of LH players. However, when the population is composed of a combination of LL and LH players, both types do equally well and LL mutants are not removed. The problem arises when LH players repeatedly mutate into LL players and no mutant cheaters appear until the average vengefulness in the population falls low enough to permit cheaters to invade the population. If we could guarantee that a mutant cheater shows up before the vengefulness in the population falls below the critical level necessary to remove cheaters, we could guarantee the persistence of the no-cheating norm. A twenty player population is relatively small, so relatively few consecutive LH to LL mutations in the absence of cheaters are necessary for the vengefulness in the population to fall below the critical level. Also, we run into the problem that LL players no longer do worse than average when there are three LL players, an HL mutant and the rest of the players are LH types, so it gets increasingly difficult to sustain the norm with more LL players.

The most that can be said for a population of 20 players is that if it reaches a composition of 18 LH players, it is likely that it will remain there for some extended period of time, but no guarantee can be made that the norm will not eventually break down.

It seems reasonable to assume that larger populations would be better able to sustain the no-cheating norm. I showed that as long as $\beta \geq 0.9$ cheaters present in the population will be removed. Suppose that $\beta \geq 0.9$ for any population of more than 20
people, just as for the 20 player population. First, more LH players would have to mutate into LL players (that is more LL players would have to accumulate) before the average vengefulness is deteriorated to the level of 90%. But the more consecutive mutations necessary to deteriorate vengefulness, the less likely it is that enough LL players will accumulate before someone mutates into a cheater. Next, suppose that there are only a few cheating mutants, (and vengefulness and the probability of being observed are sufficiently high) they would not be able to inflict very much damage on the remaining LH players (and maybe some LL players). But since the population is much bigger now, there are many more LH players that can inflict a lot of punishment on the few defecting types. So if fewer LH types can remove some defectors, more LH players should have no problem removing the same number of defectors. In fact, they should be able to remove more defectors—that is they can remove the same proportion of cheaters.

The key difference between the metanorms model and the norms game is the fact that there exist values of $s$ that will allow LH types to push out LL types in the presence of defectors. This does not happen in the norms game. LL types will strictly dominate LH types as soon as a defector appears in the population at which point there is evolutionary pressure for the LL types to multiply and increase their numbers relative to LH types. In the metanorms game LL types cannot multiply in a population primarily composed of LH types except through mutation.

Not surprisingly, the type 8 equilibrium, where the entire population is composed of defectors, is an ESE even in the metanorms game. Axelrod does not explicitly come up with a population in his metanorms game where the norm
collapses, but he acknowledges the possibility. My results verify that. But the results also show why Axelrod was not able to produce a population that would sustain such an equilibrium. The reason of course is that he uses a computer to randomly generate initial vengefulness scores for the players. Given that the computer randomly generates 20 values between zero and 1, the average should be near one half, which is too high for defectors to succeed in the metanorms game. Nevertheless it is still possible to generate 20 numbers between 0 and 1 with a sufficiently small average, but highly unlikely, which is why not one of the five populations in Axelrod’s norms game had a sufficiently low average vengefulness. Note one more thing: HH players will actually do better than HL players for very small values of s, since they avoid the metapunishment, whereas the HL types do not. But that region is practically negligible.

The argument for the type 8 equilibrium is similar. There would still be values of s for which type 8 is an ESE for larger populations. First, we know that HL types strictly dominate LL types when any number of LL types invade a population of HL players. Next, if a population on of HL players are invaded by a few LH mutants, the few mutants would not be able to inflict very much punishment or meta-punishment on the HL players, but the LH players would be seriously damaged by all the defection in the population, and on top of that they would incur a lot of enforcement and meta enforcement cost since there would be a lot of defection and non-punishment in the population. Therefore evolutionary pressure would remove the mutants. Finally HH players would not be able to invade a population of HL players for some values of s, since there would be a lot of HL players and only a few HH
players. The HL players would not be damaged much by a few punishments and meta-punishments. The enforcement cost to the HH players however would be quite large, since there would be so many defections to punish. In particular, the enforcement cost would be larger than the punishment they inflict on the HL players, and the mutants would again be removed.
VI. Model 5: A Simple Version of the Norms Game with Internalization

VI.1 Model

The simple version of internalization is identical to Model 1 except for the enforcement cost. As suggested by Axelrod, I let the enforcement cost be positive rather than negative. Therefore the new payoff structure becomes

\[ T = +3 \text{ payoff for defection (temptation)} \]
\[ H = -1 \text{ hurt inflicted on other agents by a defector} \]
\[ P = -9 \text{ punishment for a defection} \]
\[ E = +2 \text{ enforcement cost} \]

The payoffs become

\[ \text{Payoff to high bold people:} \]
\[ \pi^H = 3 - 9 \times (n - 1) \times v \times s + (1 - n \times p) \times (1 - 2 \times v \times s) \]

Where:

1. +3 is the temptation to defect
2. \(-9 \times (n - 1) \times v \times s\) is the expected value of your punishment
3. \(-1 \times (n \times p - 1)\) is the damage inflicted by all defectors
4. +2 \times (n \times p - 1) \times v \times s\) is the expected cost of enforcement

This sum simplifies to the above expression.

\[ \text{Payoff to low bold people:} \]
\[ \pi^L = -n \times p \times (1 - 2 \times v \times s) \]
Where:

1. \( +0 \) is the payoff to not defecting
2. \( -1 \times n \times p \) is the damage inflicted by all defectors
3. \( +2 \times n \times p \times v \times s \) is the expected value of enforcement cost

This sum can be rewritten as the above expression.

With the payoffs defined I can now write down the replication function.

\[
\Delta p = \pi^H - \pi^L \\
= 4 + 7 \times v \times s - 9 \times n \times v \times s \\
= 4 + (7 - 9 \times n) \times v \times s
\]

The replication function gives us the following theorem.

**Theorem 4**

1) There is a monomorphic ESE at \( p^* = 1 \) if and only if \( a \) or \( b \) holds.

   a. \( n < \frac{11}{9} \)

   b. \( n \geq \frac{11}{9} \) and \( v \times s < \frac{4}{9 \times n - 7} \)

2) The no cheating norm at \( p^* = 0 \) is an ESE if and only if

   a. \( n > \frac{11}{9} \) and \( v \times s > \frac{4}{9 \times n - 7} \)

3) Equilibrium 3: \( 0 < p^* < 1 \) is never an ESE.

4) A population cannot support an ESE if

   a. \( v \times s = \frac{4}{9 \times n - 7} \) and \( n \geq \frac{11}{9} \)

I omit the proof since it follows the proof of theorem 1 verbatim with one modification—replace the number eleven with the number seven where appropriate.
VI.2 Model 5 for a Twenty Player Population

A direct application of theorem four to a population of twenty players gives us the following result:

1) \( p^* = 1 \) if and only if \( v \cdot s < \frac{4}{173} \)

2) \( p^* = 0 \) if and only if \( v \cdot s > \frac{4}{173} \)

3) No ESE is possible if \( v \cdot s = \frac{4}{173} \)

VI.3 Discussion

Interestingly enough, the results do not differ from the results of Model 1. As before, the equilibrium attained by a population depends on its characteristics. Furthermore, we can still attain any of the equilibria attained in Model 1 (depending on the population, of course). Sufficiently low values of \( v \) will allow for the high average boldness, low average vengefulness equilibrium. A population with a sufficiently low probability of being observed and a high average vengefulness will allow for the high average boldness high average vengefulness equilibrium. A population with sufficiently high average vengefulness and \( s \) value will support a no-cheating norm. Finally, if the population is large enough it can sustain low boldness even with relatively low vengefulness and observation probability. And again, there do exist populations that will not be able to support any stable equilibrium. The only difference in the results is the values of the parameters required for the equilibria to be stable.
Something else to keep in mind is that any given population can support at most one ESE. If a population does meet the stability requirements for an equilibrium, it must attain that equilibrium and no other.

When we restrict ourselves to a population of twenty individuals, the results are identical to those of model 1 under the same conditions, again with the exception of the difference in numbers. It is still surprising that changing the payoff structure does not significantly change the resulting equilibria.
VII. Model 6: The Norms Game with Endogenous Vengefulness and Internalization

VII.1 Model

In this chapter I model 5 by endogenizing vengefulness. This model follows Model 2 exactly with the modification of making the enforcement cost a positive two rather than a negative two. All other aspects of the model remain the same.

Just as in model two, the four player types are:

5. Let LL players = \{people i such that b_i < s and v_i = 0\}
6. Let LH players = \{people i such that b_i < s and v_i = 1\}
7. Let HL players = \{people i such that b_i > 0 and v_i = 0\}
8. Let HH players = \{people i such that b_i > s and v_i = 1\}

Their respective payoffs are as follows:

\[ \pi^{LL} = 0 - 1 * n * p - 0 = -n * p \]

1. +0 payoff from not defecting
2. +0 no punishment
3. -n * p damage from all defectors
4. 0 expected enforcement cost

\[ \pi^{LH} = 0 - 1 * n * p - 2 * s * n * p \]

\[ = -n * p * (1 - 2 * s) \]
1. +0 payoff from not defecting
2. +0 no punishment
3. \(-n \cdot p\) damage from all defectors
4. +2 \(n \cdot p \cdot s\) expected value of enforcement cost

\[
\pi^{HL} = 3 - 9 \cdot n \cdot q \cdot s - 1 \cdot (np - 1) - 0
\]

\[= 4 - 9 \cdot n \cdot q \cdot s - n \cdot p\]

1. +3 payoff from cheating
2. \(-9 \cdot q \cdot n \cdot s\) expected value of punishment from all punishers
3. \(-(n \cdot p - 1)\) damage from other defectors
4. +0 enforcement cost

\[
\pi^{HH} = 3 - 9 \cdot (n \cdot q - 1) \cdot s - 1 \cdot (n \cdot p - 1) - 2 \cdot (n \cdot p - 1) \cdot s
\]

\[= 4 - 9 \cdot n \cdot q \cdot s + 11 \cdot s - (1 - 2 \cdot s) \cdot n \cdot p\]

1. +3 payoff from cheating
2. \(-9 \cdot (n \cdot q - 1)\) expected value of punishment
3. \(-(n \cdot p - 1)\) damage from other defectors
4. +2 \((n \cdot p - 1)s\) expected value of enforcement cost

There are still 9 classes of potential equilibria, as listed in Model 2. This is due to the fact metavengefulness does not affect the role of q in determining the classes of equilibria.
Lemma 8: Given that all players in the population must receive the same payoff for a population composition to be in equilibrium the following population compositions constitute equilibria.

1) Type 1 (a): $\alpha = \frac{-4-11s+7\cdot n\cdot s}{7\cdot n\cdot s}, \beta = 0, \gamma = 0, \delta = \frac{4+11s}{7\cdot n\cdot s}$

2) Type 1 (b): $\alpha = 0, \beta = \frac{4-2\cdot n\cdot s}{7\cdot n\cdot s}, \gamma = \frac{-4+9\cdot n\cdot s}{7\cdot n\cdot s}, \delta = 0$

3) Type 2: $\alpha = 1 - \beta, Continuum, \gamma = 0, \delta = 0$

4) Type 6: $\alpha = 1, \beta = 0, \gamma = 0, \delta = 0$

5) Type 7: $\alpha = 0, \beta = 1, \gamma = 0, \delta = 0$

6) Type 8: $\alpha = 0, \beta = 0, \gamma = 1, \delta = 0$

7) Type 9: $\alpha = 0, \beta = 0, \gamma = 0, \delta = 1$

Lemma 9: Type 6 ($\alpha=1, \beta=0, \gamma=0, \delta=0$) and Type 7 ($\alpha=0, \beta=1, \gamma=0, \delta=0$) Monomorphic Equilibria are not stable.

Proof:

Type 6: Suppose $\alpha=1$. Suppose a mutation causes $\alpha = 1 - \epsilon$ and $\gamma = \epsilon > 0$.

$$\alpha = 1 - \epsilon, \beta = 0, \gamma = \epsilon, \delta = 0; p = \epsilon; q = 0$$

$$\pi^{LL} = -n \cdot p = -n \cdot \epsilon$$

$$\pi^{HL} = 4 - 9 \cdot q \cdot s - n \cdot p = 4 - 9 \cdot 0 \cdot s - n \cdot \epsilon = 4 - n \cdot \epsilon$$

Clearly, LL types are dominated by HL types.

Type 6 is not stable.

Type 7: $\beta=1$. Suppose a mutation causes $\beta = 1 - \epsilon$ and $\alpha = \epsilon > 0$

$$\alpha = \epsilon, \beta = 1 - \epsilon, \gamma = 0, \delta = 0; p = 0, q = 1 - \epsilon$$

$$\pi^{LL} = -n \cdot p = -n \cdot 0 = 0$$
Lemma 10: Any equilibrium where $\alpha^* > 0$, (other) is not an ESE.

Proof:

\[ \pi^{LL} = -n \cdot p \]
\[ \pi^{LH} = (1 - 2 \cdot s) \cdot (-n \cdot p) \]
\[ \pi^{HL} = 4 - 9 \cdot n \cdot q \cdot s - n \cdot p \]
\[ \pi^{HH} = 4 - 9 \cdot n \cdot q \cdot s + 11 \cdot s + (1 - 2 \cdot s) \cdot (-n \cdot p) \]
\[ \pi = (1 - \beta - \gamma - \delta) \cdot \pi^{LL} + \beta \cdot \pi^{LH} + \gamma \cdot \pi^{HL} + \delta \cdot \pi^{HH} \]

Case 1a: $s=0, p>0, n>0$

\[ \pi^{LL} = -n \cdot p \]
\[ \pi^{LH} = (1 - 2 \cdot 0) \cdot \pi^{LL} \]
\[ = -n \cdot p \]
\[ \pi^{HL} = 4 - 9 \cdot n \cdot q \cdot 0 + \pi^{LL} \]
\[ = 4 - n \cdot p \]
\[ \pi^{HH} = 4 - 9 \cdot n \cdot q \cdot 0 + 11 \cdot 0 + (1 - 2 \cdot 0) \cdot \pi^{LL} \]
\[ = 4 - n \cdot p \]

Since $4 + \pi^{LL} > \pi^{LL}$, the HL and HH types both strictly dominate the LL and LH types.
Case 1b: $s=0$, $p=0$, $n>0$

\[ \pi^{LL} = -n \times 0 \]
\[ = 0 \]
\[ \pi^{LH} = (1 - 2 \times 0) \times \pi^{LL} \]
\[ = -n \times 0 \]
\[ = 0 \]

These two types are indistinguishable when $s=0$ and $p=0$, so there is no incentive to return to any particular distribution of LL and LH types under these conditions. Moreover, as shown above, as soon as an HL or HH type appears, and $p>0$, both LL and LH types are dominated by the invading party. Therefore the LL and LH types cannot appear in a stable equilibrium when $s=0$.

Case 2a: $s>0$ $p>0$ $n>0$

\[ \pi^{LL} = -n \times p \]
\[ \pi^{LH} = (1 - 2 \times s) \times \pi^{LL} \]
\[ = -(1 - 2 \times s) \times n \times p \]

Since $n > 0$ and $p > 0$ and $s > 0$

\[ 2 \times n \times p \times s > 0 \]
\[ -n \times p + 2 \times n \times p \times s > -n \times p \]
\[ -(1 - 2 \times s) \times n \times p > -n \times p \]

Therefore, $\pi^{LL} < \pi^{LH}$ and the LH type strictly dominates the LL type.

Therefore the LL type cannot appear in a stable equilibrium when $s > 0$ and $p > 0$.

Case 2b: $p=0$
\[ 0 = p = \gamma + \delta \rightarrow \gamma^* = \delta^* = 0 \rightarrow \alpha^* + \beta^* = 1 \]

\[ \pi_{LL} = -n \ast p \]
\[ = -n \ast 0 \]
\[ = 0 \]

\[ \pi_{LH} = (1 - 2 \ast s) \ast (-n \ast p) \]
\[ = (1 - 2 \ast s) \ast 0 \]
\[ = 0 \]

\[ \overline{\pi} = \alpha \ast (-n \ast p) + \beta \ast (1 - 2 \ast s) \ast (-n \ast p) \]

\[ \Delta \alpha = \pi_{LL} - \overline{\pi} = (-n \ast p) - [\alpha \ast (-n \ast p) + \beta \ast (1 - 2 \ast s) \ast (-n \ast p)] \]
\[ = (1 - \alpha) \ast (-n \ast p) - \beta \ast (1 - 2 \ast s) \ast (-n \ast p) \]

\[ \frac{d\Delta \alpha}{d\alpha} = n \ast p \]
\[ = n \ast 0 \]
\[ = 0 \]

\[ \Delta \beta = \pi_{LH} - \overline{\pi} \]
\[ = (1 - 2 \ast s) \ast (-n \ast p) - [\alpha \ast (-n \ast p) + \beta \ast (1 - 2 \ast s) \ast (-n \ast p)] = \]
\[ = (1 - \beta) \ast (1 - 2 \ast s) \ast (-n \ast p) - \alpha \ast (-n \ast p) \]

\[ \frac{d\Delta \beta}{d\beta} = (1 - 2 \ast s) \ast (n \ast p) \]
\[ = (1 - 2 \ast s) \ast (n \ast 0) \]
\[ = 0 \]

\[ \alpha^* + \beta^* = 1 \text{ implies that either } \alpha^* = 0, \text{ or } 0 < \alpha^* < 1 \text{ or } \alpha^* = 1 \]

We know from Lemma 9 that \( \alpha^* = 1 \) is not a stable equilibrium.

Therefore we only need to check \( 0 < \alpha^* < 1 \)
Which Requires: $\Delta \alpha = 0$ and $\frac{d\Delta \alpha}{d\alpha} < 0$

However, $\frac{d\Delta \alpha}{d\alpha} < 0$ fails, since $\frac{d\Delta \alpha}{d\alpha} = 0$.

Therefore $0 < \alpha^* < 1$ is not stable and any equilibrium with $\alpha^* > 0$ is not stable.

This allows us to eliminate Equilibrium 1 (Type 1(a)), Equilibrium 3 (Type 2), and Equilibrium 4 (Type 6). We only have to check Equilibrium 2 (Type 1(b)), Equilibrium 6 (Type 8), and Equilibrium 7 (Type 9).

Theorem 5

1) The Type 9 ($\alpha=0, \beta=0, \gamma=0, \text{ and } \delta=1$) equilibrium is the unique ESE if and only if $s>0$ and condition a. and condition b. hold.

   a. One of the following must hold:

      i. $n \leq \frac{15}{7}$

      ii. $n > \frac{15}{7}$ and $s \leq \frac{4}{-11+7n}$

      iii. $n > \frac{15}{7}$, and $s > \frac{4}{-11+7n}$, and $\epsilon > \frac{-4-11s+7n+s}{7n+s}$

   b. One of the following must hold:

      i. $n < \frac{5}{3}$,

      ii. $n \geq \frac{5}{3}$ and $s \leq \frac{4}{-11+9n}$

2) There are no other ESE’s. It follows that there exist populations that cannot support an ESE.

Proof:
We eliminate Equilibrium 1 (Type 1(a)), Equilibrium 3 (Type 2), Equilibrium 4 (Type 6), and Equilibrium 5 (Type 7) off the list in Lemma 8, by using Lemmas 8 and 9. We only have to check Equilibrium 2 (Type 1(b)), Equilibrium 6 (Type 8), and Equilibrium 7 (Type 9).

Equilibrium 2: Type 1 (b)

Type 1 Equilibrium: \(0 < p < 1\) and \(0 < q < 1\)

Implies: \(0 < \gamma + \delta < 1\) and \(0 < \beta + \delta < 1\)

Requires: \(\Delta p = 0\) and \(\Delta q = 0\); \(\frac{d\Delta p}{dp} < 0\) and \(\frac{d\Delta q}{q} < 0\)

Type 1 (b): \(\alpha = 0, \quad \beta > 0, \quad \gamma > 0, \quad \delta = 0\)

\[\alpha = 0, \quad \beta = \frac{4 - 2n * s}{7n * s}, \quad \gamma = \frac{9n * s - 4}{7n * s}, \quad \delta = 0\]

\[\pi^{LH} = -n * p * (1 - 2 * s)\]

\[= -n * \frac{9n * s - 4}{7n * s} * (1 - 2 * s)\]

\[\pi^{HL} = 4 - 9n * q * s - n * p\]

\[= 4 - 9 * \frac{4 - 2n * s}{7n * s} * n * s - n * \frac{9n * s - 4}{7n * s}\]

\[\Delta \beta = -4 * (\gamma + \delta) + (-11 * \delta + (2 + 7 * (\beta + \delta)) * (\gamma + \delta) * n) * s\]

\[\Delta \gamma = 4 - (\gamma + \delta) + (11 * \delta + (\beta + \delta) * (-9 + 7 * (\gamma + \delta)) * n) * s\]

\[\Delta \beta \left(0, \frac{4 - 2n * s}{7n * s}, \frac{9n * s - 4}{7n * s}, 0\right) = 0\]

\[\Delta \gamma \left(0, \frac{4 - 2n * s}{7n * s}, \frac{9n * s - 4}{7n * s}, 0\right) = 0\]
Stability requires:

\[
\frac{d}{d\beta} (\Delta \beta) = 7 \cdot (\gamma + \delta) \cdot n \cdot s
\]

\[
\frac{d}{dy} (\Delta \gamma) = -4 + 7 \cdot (\beta + \delta) \cdot n \cdot s
\]

To have an interior equilibrium of LH and HL types we need

\[
0 < \beta < 1 \text{ and } 0 < \gamma < 1
\]

Which implies

\[
0 < \frac{4 - 2 \cdot n \cdot s}{7 \cdot n \cdot s} < 1
\]

\[
0 < 4 - 2 \cdot n \cdot s < 7 \cdot n \cdot s
\]

\[
2 \cdot n \cdot s < 4 < 9 \cdot n \cdot s
\]

However we also need

\[
\frac{d\Delta \beta}{d\beta} \left(0, \frac{4 - 2 \cdot n \cdot s}{7 \cdot n \cdot s}, \frac{9 \cdot n \cdot s - 4}{7 \cdot n \cdot s}, 0\right) = -4 + 9 \cdot n \cdot s < 0
\]

Which holds if and only if

\[
-4 + 9 \cdot n \cdot s < 0
\]

\[
9 \cdot n \cdot s < 4
\]

Clearly, \(4 < 9 \cdot n \cdot s \) and \(4 > 9 \cdot n \cdot s \) cannot hold simultaneously, so we have a contradiction. Therefore Type 1 (b) equilibrium is not stable.

Equilibrium 6 (Type 8)
Type 8 Equilibrium: \( p = 1 \) and \( q = 0 \)

Implies: \( \alpha = 0, \beta = 0, \gamma = 1, \delta = 0 \)

Suppose there is a mutation such that some HH types appear.

\[ \alpha = 0, \quad \beta = 0, \quad \gamma = 1 - \epsilon, \quad \delta = \epsilon \quad \rightarrow p = 1 \text{ and } q = \epsilon \]

\[ \pi^{HL} = 4 - 9 * n * q * s - n * p \]

\[ = 4 - 9 * n * \epsilon * s - n \]

\[ \pi^{HH} = 4 - 9 * n * q * s - n * p(1 - 2 * s) + 11 * s \]

\[ = 4 - 9 * n * \epsilon * s - n * (1 - 2 * s) + 11 * s \]

\[ = 4 - 9 * n * \epsilon * s - n + 2 * n * s + 11 * s \]

\[ = \pi^{LH} + (11 + 2 * n) * s \]

Clearly, HH types dominate HL types when \( s > 0 \). If \( s = 0 \) then both types get a payoff of 4-\( n \) and the two groups are indistinguishable. If a mutation causes the division between HH and HL types to shift in favor of one group or the other, there would be no incentive to return to the previous division. Therefore this equilibrium is not stable when \( s > 0 \) or when \( s = 0 \).

Equilibrium 7 (Type 9)

Type 9 Equilibrium: \( p = 1 \) and \( q = 1 \)

Implies: \( \alpha = 0, \beta = 0, \gamma = 0, \text{and } \delta = 1 \)

Requires: \( \Delta \alpha < 0, \Delta \beta < 0, \Delta \gamma < 0, \text{and } \Delta \delta > 0 \)

Again, there are three cases to consider, when there is an LL type mutation, an LH type mutation and an HH type mutation.

a) \( \alpha = \epsilon, \beta = 0, \gamma = 0, \delta = 1 - \epsilon; p = 1 - \epsilon, q = 1 - \epsilon \)

\[ \pi^{LL} = -n * p = -(1 - \epsilon) * n \]
\[ \pi^{HH} = 4 - 9 \cdot n \cdot q \cdot s - n \cdot p \cdot (1 - 2 \cdot s) + 11 \cdot s \]
\[ = 4 - 9 \cdot (1 - \epsilon) \cdot n \cdot s - n \cdot (1 - \epsilon) \cdot (1 - 2 \cdot s) + 11 \cdot s \]
\[ \Delta \alpha = (-1 + \epsilon) \cdot (4 + (11 + 7 \cdot (-1 + \epsilon) \cdot n) \cdot s) < 0 \]
\[ \Delta \delta = \epsilon \cdot (4 + (11 + 7 \cdot (-1 + \epsilon) \cdot n \cdot s) > 0 \]

**Assuming** \( 0 \leq s \leq 1, 0 < \epsilon < 1, \text{and } n > 0, \Delta \alpha < 0 \text{ and } \Delta \delta > 0 \) **hold if and only if one of the following holds**

1) \( n \leq \frac{15}{7} \),

2) \( n > \frac{15}{7} \) and \( s \leq \frac{4}{-11 + 7 \cdot n} \)

3) \( n > \frac{15}{7}, s > \frac{4}{-11 + 7 \cdot n}, \text{and } \epsilon > \frac{-4 - 11 \cdot s + 7 \cdot n \cdot s}{7 \cdot n \cdot s} \)

**b)** \( \alpha = 0, \beta = \epsilon, \gamma = 0, \delta = 1 - \epsilon; p = 1 - \epsilon, q = 1 \)

\[ \pi^{LH} = -n \cdot p \cdot (1 - 2 \cdot s) \]
\[ = -(1 - \epsilon) \cdot n \cdot (1 - 2 \cdot s) \]
\[ \pi^{HH} = 4 - 9 \cdot n \cdot q \cdot s + 11 \cdot s - n \cdot p \cdot (1 - 2 \cdot s) \]
\[ = 4 - 9 \cdot n \cdot s + 11 \cdot s - n \cdot (1 - \epsilon) \cdot (1 - 2 \cdot s) \]
\[ \Delta \beta = (1 - \epsilon) \cdot (-4 + (-11 + 9 \cdot n) \cdot s) < 0 \]
\[ \Delta \delta = \epsilon \cdot (4 + (11 - 9 \cdot n) \cdot s) > 0 \]

**Assuming** \( 0 \leq s \leq 1, 0 < \epsilon < 1, \text{and } n > 0, \Delta \beta < 0 \text{ and } \Delta \delta > 0 \) **hold if and only if one of the following holds**

1) \( n < \frac{5}{3} \)

2) \( n \geq \frac{5}{3} \), and \( s \leq \frac{4}{-11 + 9 \cdot n} \)

**c)** \( \alpha = 0, \beta = 0, \gamma = \epsilon, \delta = 1 - \epsilon; p = 1, q = 1 - \epsilon \)
\[ \pi^{HL} = 4 - 9 * n * q * s - n * p \]
\[ = 4 - n - 9 * (1 - \epsilon) * n * s \]
\[ \pi^{HH} = 4 - 9 * n * q * s - n * p * (1 - 2 * s) + 11 * s \]
\[ = 4 - 9 * n * (1 - \epsilon) * s - n * (1 - 2 * s) + 11 * s \]
\[ \Delta \gamma = (-1 + \epsilon) * (1 + 2 * n) * s < 0 \]
\[ \Delta \delta = \epsilon * (11 + 2 * n) * s > 0 \]

Assuming \( 0 \leq s \leq 1, \ 0 < \epsilon < 1, \text{ and } n > 0, \Delta \beta < 0 \text{ and } \Delta \delta > 0 \) hold if and only if

\[ s > 0 \]

As before, the requirements can be derived with some algebra. A computational package such as Mathematica can be used to acquire results. I omit the algebra here.

**VII.2 Model 6 for a Twenty Player Population**

When we restrict ourselves to a population of twenty people theorem five becomes the following result. This is obtained by replacing \( n \) with 20 where appropriate

**Application of Theorem 5 to \( n=20 \)**

1) The unique Type 9 (\( \alpha=0, \beta=0, \gamma=0, \) and \( \delta=1 \)) equilibrium is an ESE when:
   
   a. \( 0 < s \leq \frac{4}{169} \)

2) There are no other stable equilibria.

3) No stable equilibrium can be sustained by a population if \( s=0 \) or \( s>4/169 \)
Lemma 11: If a population of twenty players satisfies $\beta \geq 0.9$ and $0.032 \leq s \leq 1$, then $\Delta \alpha \geq 0$, $\Delta \beta \geq 0$, $\Delta \gamma < 0$, and $\Delta \delta < 0$. If a population of twenty players satisfies $\beta \geq 0.9$ and $0.025 \leq s \leq 0.032$, then $\Delta \alpha \leq 0$, $\Delta \beta \geq 0$, $\Delta \gamma < 0$, and $\Delta \delta < 0$.

Proof:

Using the payoffs defined in this chapter and

$$\Delta z = \pi^z - \pi$$

We get the following results

Case 1: $\epsilon = \frac{1}{20}$

a) $\alpha = \frac{1}{20}$, $\beta = \frac{19}{20}$, $\gamma = 0$, $\delta = 0$

$$\Delta \alpha = 0 \forall s > 0$$

$$\Delta \beta = 0 \forall s > 0$$

b) $\alpha = 0$, $\beta = \frac{19}{20}$, $\gamma = \frac{1}{20}$, $\delta = 0$

$$\Delta \beta = \frac{1}{20} \cdot (-4 + 173 \cdot s) > 0$$

$$\Delta \gamma = \frac{19}{20} \cdot (4 - 173 \cdot s) < 0$$

For $\frac{4}{174} = 0.023 < s \leq 1$

c) $\alpha = 0$, $\beta = \frac{19}{20}$, $\gamma = 0$, $\delta = \frac{1}{20}$

$$\Delta \beta = \frac{1}{20} \cdot (-4 + 169 \cdot s) > 0$$

$$\Delta \delta = \frac{19}{20} \cdot (4 - 169 \cdot s) < 0$$

For $\frac{4}{169} = 0.0237 < s \leq 1$
Case 2: \( \epsilon = \frac{2}{20} \)

a) \( \alpha = \frac{2}{20}, \ \beta = \frac{18}{20}, \ \gamma = 0, \delta = 0 \)

\[ \Delta \alpha = 0 \ \forall \ s > 0 \]
\[ \Delta \beta = \forall \ 0 \ s > 0 \]

b) \( \alpha = 0, \ \beta = \frac{18}{20}, \ \gamma = \frac{2}{20}, \ \delta = 0 \)

\[ \Delta \beta = \frac{1}{5} \ (-2 + 83 \ s) > 0 \]
\[ \Delta \gamma = -\frac{9}{5} \ (-2 + 83 \ s) < 0 \]

for \( 0.024 = \frac{2}{83} < s \leq 1 \)

c) \( \alpha = 0, \ \beta = \frac{18}{20}, \ \gamma = 0, \ \delta = \frac{2}{20} \)

\[ \Delta \beta = \frac{1}{10} \ (-4 + 169 \ s) > 0 \]
\[ \Delta \delta = -\frac{9}{10} \ (-4 + 169 \ s) < 0 \]

for \( \frac{4}{169} = 0.0237 < s \leq 1 \)

d) \( \alpha = \frac{1}{20}, \ \beta = \frac{18}{20}, \ \gamma = \frac{1}{20}, \ \delta = 0 \)

\[ \Delta \alpha = \frac{1}{10} \ (-2 + 63 \ s) < 0 \]

for \( 0 \leq s < \frac{2}{63} = 0.032 \)

\[ \Delta \beta = \frac{1}{10} \ (-2 + 83 \ s) > 0 \]

for \( 0.024 = \frac{2}{83} < s \leq 1 \)
\[ \Delta \gamma = \frac{1}{10} \cdot (38 + 1557 \cdot s) < 0 \]

For 0.0244 = \( \frac{38}{1557} \) \( s \leq 1 \)

\[ e) \quad \alpha = \frac{1}{20}, \beta = \frac{18}{20}, \gamma = 0, \delta = \frac{1}{20} \]

\[ \Delta \alpha = \frac{1}{10} \cdot (-2 + 61 \cdot s) < 0 \]

For 0 \( \leq s < \frac{2}{61} = 0.0328 \)

\[ \Delta \beta = \frac{1}{10} \cdot (-2 + 81 \cdot s) > 0 \]

For 0.0247 = \( \frac{2}{81} \) \( s \leq 1 \)

\[ \Delta \delta = \frac{1}{10} \cdot (38 - 1519 \cdot s) < 0 \]

For 0.025 = \( \frac{38}{1519} \) \( s \leq 1 \)

\[ f) \quad \alpha = 0, \beta = \frac{18}{20}, \gamma = \frac{1}{20}, \delta = \frac{1}{20} \]

\[ \Delta \beta = -\frac{2}{5} + \frac{67}{4} \cdot s > 0 \]

For 0.0239 = \( \frac{8}{335} \) \( s \leq 1 \)

\[ \Delta \gamma = \frac{18}{5} - \frac{633}{4} \cdot s < 0 \]

For 0.0227 = \( \frac{24}{1055} \) \( s \leq 1 \)

\[ \Delta \delta = \frac{18}{5} - \frac{573}{4} \cdot s < 0 \]

For 0.025 = \( \frac{24}{955} \) \( s \leq 1 \)
The result in Lemma 11 has the same interpretation as Lemma 5: the no cheating norm can be sustained temporarily, near the type 7 equilibrium by a 20 player population since LL players do better than average in the presence of defectors for \( s \geq 0.32 \). The second result is similar to the result from Lemma 6 from model 4. When \( s \) satisfied \( 0.025 < s < 0.032 \), the only way that the norm can break down is if enough LL players are introduced by mutation only, before a mutant defector shows up.

**VII.3 Discussion**

This model gives the strangest results so far. While the simple version of Internalization paralleled its counterpart (Model 1) perfectly, the Norms Game version of internalization differs from its original counterpart (Model 2). The simple version of internalization allows a wider range of populations to attain an ESE. It also allows for a wider range of ESE’s. In the simple version, the only parameter that varies is the proportion of cheaters, and the proportion of cheaters will stabilize at 0 or 1 except in the case of \( \nu \star s = \frac{4}{9m-7} \). The set of equilibria include all possible values of \( \nu \) paired with either \( p=0 \) or \( p=1 \). That is to say, there are infinitely many ESE’s that could potentially be stable. But I remind the reader again, any one specific population can at most attain one equilibrium, which is determined by its characteristics.

On the other hand, in the norms game version of internalization, most populations will not be able to support a stable equilibrium. This is particularly obvious when we consider a population of 20 people. The stability condition essentially collapses to \( s \leq 4/169 \).
Observe that in the norms game version of the internalization model the average vengefulness of the population is not fixed but is determined by the population composition, which is susceptible to change by the evolutionary forces of differential payoff. The type 9 equilibrium dictates the population composition and therefore determines the average boldness and average vengefulness in the population at equilibrium. The conclusion, specifically for a population of 20 individuals, is very strong here. Any 20 agent population that satisfies $s \leq 4/169$ must eventually converge to the type 9 equilibrium, no matter what its initial composition. All other 20 player populations will never support the ESE no matter what their other initial compositions are.

In the norms game with internalization, populations with $s > 0.032$ will be able to support the no cheating norm at least temporarily, provided that the population attains a population distribution near $\beta = 1$ at least once. This is similar to result to Lemma 5 from Model 2. Lemma 5 tells us that a sufficiently large population ($n > 26/9$), made up primarily of LH types will be able to push out defectors provided that there are not too many mutant defectors in the population and average vengefulness is sufficiently high, i.e.—not too many LL players. That is to say, such a population can support the no cheating norm temporarily. The result also applies to the internalization model but even more strongly because in the internalization model, LH players dominate LL players in the presence of defectors not the other way around. The result in Lemma 5 was obtained with LL players dominating LH players in the presence of defectors. Therefore, we extend the result from Lemma 11 to larger populations. Since $20 > 26/9$, we can say that all populations bigger than 20 players
will be able to sustain the no cheating norm at least temporarily for some values of $s$. Furthermore, such large populations will likely be able to support the no cheating norm permanently. The one thing to test is this: Can a large population composed primarily of LH types push out both HL and HH types simultaneously in the presence of LL types. Lemma does not say anything about simultaneous HL and HH mutations, and a population of 20 players is too small to have at least one mutation of each non-LH type and have $\beta \geq 0.9$.

If there are at least 18 LH players in the population and $0.025 < s < 0.032$, a population of LH players will be able to remove LL players in the presence of a defecting mutant. This makes it likely that the no-cheating norm will be sustained longer than if $s \geq 0.032$, but we cannot guarantee that too many LH players will not mutate into LL players and average vengefulness falls too low before one mutates into a cheater.
VIII. Conclusion and Future Work

VIII.1 Conclusion

In general, the no cheating norm is difficult to sustain indefinitely. The results show that when the level of vengefulness, which is the probability that players will punish defection, is an exogenous parameter, the no cheating norm can be maintained by a population with the appropriate characteristics. However, once vengefulness is endogenized, a metanorm or internalization is required for a population to permanently support the no cheating norm under certain conditions.

The first model presented in this work is a very simple one where average vengefulness, v, and the probability of a defection being observed, s, are fixed population characteristics. Populations can support one of two evolutionarily stable states depending on their size, average vengefulness and their s characteristic. Populations with sufficiently high v and s values will be able to support a norm against cheating. Populations where either v or s is too small will not be able to sustain the norm, and will end up with a stable composition of all cheaters. Furthermore, some populations will not be able to support any evolutionarily stable state. Specifically if the relation vs=4/(11-9n) holds for the population, it will never attain an evolutionarily stable equilibrium. Finally, given any two values of v and s between zero and one, it is possible to find a population which will be sufficiently large to sustain the no cheating norm under the specified conditions.

In model 2, I endogenize vengefulness. This changes the results significantly. The norm cannot be permanently sustained by populations of sufficient size under any conditions. There are only two possible evolutionarily equilibria. The
monomorphic equilibrium composed entirely of HL type players is an ESE for sufficiently large populations for any value of the parameter s, provided that the rate of mutation is sufficiently small. Under certain conditions, the monomorphic equilibrium composed entirely of HH players is an ESE for sufficiently small populations. The equilibrium composed entirely of LH players, those who adhere to the norm, is not an evolutionarily stable state; neither is any other norm supporting equilibrium. Therefore, a no-cheating norm cannot be permanently sustained in this game. However, a norm potentially can be sustained near the type 7 equilibrium in the short term provided that the population is large enough, the probability of cheating being observed is high enough, and the population is composed primarily of LH players. A high proportion of LH players is necessary to maintain a sufficiently high average vengefulness in the population so that cheaters cannot invade the population. The reason that this configuration cannot be permanently sustained is that LL players dominate LH players in the presence of defectors. Therefore, even if a population attains a sufficiently high level of vengefulness to eradicate mutant defectors, LL players will also increase in number with every invasion of a cheating mutant. Eventually the LL players will reach a high enough proportion to deteriorate the norm. That is, there will be so many LL players that the average vengefulness will no longer be high enough to remove mutant defectors. This is in line with Axelrod’s results; once cheaters have been eliminated from the population there is little incentive to maintain vengefulness, since enforcement is costly, and average vengefulness in the population will be reduced by evolutionary pressure as described above, and the norm will break down.
Incorporating metanorms into the model allows norms to be established by twenty player populations under certain conditions. A metanorm is a mechanism for supporting norms where players can punish others not just for cheating, but for not punishing defection as well. First, I consider a simple version of the model where \( v \) is exogenously fixed. In this case the results are similar to those in model 1. The monomorphic norm supporting equilibrium can be maintained provided that average vengefulness and the probability of defection being observed are sufficiently high in the population. The all-cheater configuration is an ESE provided that vengefulness, the probability of being observed, or both are sufficiently low. Unlike in the previous model, the polymorphic equilibrium composed of cheaters and cooperators is also an ESE here under some conditions.

In the more advanced version of the metanorms game I derive two results. Twenty player populations can support the no cheating norm provided that the probability of cheating being observed is sufficiently high and not too many LH players mutate into LL payers in the absence of cheaters. The second result is that the type 8 equilibrium is an ESE, provided that the probability of a defection being observed is sufficiently small.

As previously discussed, these results could be extended to twenty player populations. Intuitively this makes sense. Given that a smaller population can support a norm as long as not too many LH players mutate into LL players before one mutates into a cheater, it should be easier for a larger population to support the norm since more LH players would have to mutate into LL players to deteriorate the average vengefulness in the population. Also, there would be more norm conforming players.
to punish invading defectors, therefore the mutant defector will be more severely punished in a larger population, and be eliminated by evolutionary pressure. Likewise, in a larger population of HL players who cheat and do not punish, cooperators will not survive since they will not get the temptation payoff. Even HH players will not survive because they will incur the enforcement and meta-enforcement cost and will not be able to inflict enough damage on the rest of the population through punishment to counteract the evolutionary force against them.

Internalization is another mechanism for supporting norms. In this model I assume that the entire population internalizes the norm. That is, players get a positive payoff from enforcing the norm. The results from the simple version of this model parallel the results from Model 1 exactly. The results from the advanced version of internalization are more interesting. Some of them are counterintuitive. Specifically, the type 8 equilibrium, composed entirely of HL players, is not an ESE under any conditions, but the type 9 equilibrium (all HH players) is an ESE provided that the appropriate conditions on $s$ and the mutation rate are satisfied. Furthermore, unlike in model two, here the type 9 equilibrium can potentially be sustained by populations of all sizes and not just the small ones.

A more optimistic result is that some populations of twenty players can support the no cheating norm, albeit for a very small range of $s$ values as long as not too many LH players mutate into LL players before a player mutes into a cheater. Populations with larger $s$ values may sustain the norm only temporarily. This is due to the fact that at larger values of $s$, LL players do better than average when small numbers of cheaters invade the population and therefore can increase their numbers.
once the defectors are eliminated. This brings us back to the problem we had with the original norms game. If LL players are able to increase their numbers because of the presence of defectors, they will eventually reach a critical proportion at which point the norm breaks down. This result can be extended to populations larger than twenty players by the same logic used to extend the results in the metanorms game.

**VIII.2 Future Work**

This work can be taken in several directions in the future. One option is to study the metanorms game further. Extending the results to larger populations explicitly would be a good first step. Next, the remaining seven classes of equilibria need to be characterized and ESE conditions, if any, established. In addition, we can generalize the metanorms game by dropping the assumption that vengefulness and metavengefulness are equal. The internalization model can also be explored further.

The next logical extension of this work is the inclusion of more player types. Specifically, the vengefulness score should be allowed to take more than two values. This can further be extended to vengefulness ranging continuously through the interval between zero and one. More player types can be created by allowing boldness to take on one of several values, which can be interpreted as the probability of a player defecting rather than players defecting or cooperating with certainty depending on the parameter s. Considering the complexity of the model with only four types of players, (especially the metanorms game), a more sophisticated computational approach would be necessary to be able to deal with large systems of equations in many variables.
Metanorms and internalization studied here are only two mechanisms for supporting norms. Axelrod suggests dominance, deterrence, social proof, membership, law, and reputation as additional mechanisms which can be modeled. Axelrod further suggests that dominance can be simulated by dividing the population into two groups where one group is relatively more resistant to punishment from members of the other group, compared to punishment by members of their group.

Another avenue which can be pursued is parametrizing the payoff structure. The results obtained here are based on specific values of temptation, punishment, enforcement cost, and damage from defection. Turning these into state variables will permit the researcher to test the robustness of my results to slight perturbations in the payoff structure. More generally, it would be very interesting to study the effect of these four factors on the establishment of norms.

An even more complex extension would involve having the payoff structure be a function of population parameters. The enforcement cost could logically depend on the size of the population as well as the desired probability of a defection being observed. The player type (cheater or cooperator) can depend on the temptation payoff in addition to the probability of being observed.

Finally, other replication functions can be used when modeling social norms. A particularly competitive society can be modeled by a replication function where a player reproduces in the future only if his or her type received the highest payoff in the population. This would look like \( \Delta x = \pi^x - \max \{ \pi^y \} \). Where the maximum is taken over all player types \( y \) present in the population.
The subject of social norms is very interesting but very complex. All but the simplest representations used to model the dynamics of this phenomenon are going to be very complicated when done analytically. Sophisticated methods are going to be required to work with such models. However, such research can lead to new insights that can help us better understand how norms are established, maintained, and transformed. Such knowledge may then be used to promote the establishment of norms in favor of desirable behaviors and practices.
References


