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A Discussion of Higher-Order Alexander Modules

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Abstract

We will discuss a technique to distinguish knots via an algebraic invariant, namely the Alexander module, which is a module associated to the universal abelian cover of the complement of a knot. We will present a generalization of the Alexander module by constructing modules corresponding to different regular covering spaces of the knot complement. We will present a numerical invariant that can be extracted from these modules, and a procedure by which we can compute it in practice in the case of the first higher-order Alexander module. Finally, we will argue that this procedure could feasibly be followed algorithmically, and so a computer could be programmed to compute this numerical invariant.

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1 Motiviation, Background and Definitions

If we think of knots as 1-dimensional simple closed curves in 3-dimensional space, then it is natural to ask: when are two seemingly different knots actually the same, in the sense that one of them can be continuously deformed into the other, and when are two knots different. We formalize these notions first before beginning a brief discussion of one particular approach to this question.

Definition 1. *A **knot** is an embedding (i.e. a continuous function that is a homeomorphism from its domain to its image) $K : S^1 \to S^3$. When the meaning is clear from the context, we will use K to denote the image of this embedding in S^3.*

How should we formalize the notion of two knots being equivalent? If we are looking for an equivalence relation on the continuous maps from one space to another, we might first look is the notion of homotopy, because it gives an equivalence relation on the set of such maps.

Definition 2. *Given two topological spaces X and Y, and two maps $f_0, f_1 : X \to Y$, we say that f_0 and f_1 are **homotopic** if there exists a continuous map*

$$F : X \times [0, 1] \to Y$$

*such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. This map is called a **homotopy** from f_0 to f_1.*

Intuitively, this definition says that two maps are homotopic if one can be continuously deformed into the other. One can show that the property of being homotopic induces an equivalence relation on the set of maps from X to Y. Unfortunately, this notion of equivalence is too weak to have any distinguishing power for knots, because it turns out that all embeddings of S^1 into S^3 are homotopic. The problem is that a homotopy from one knot to another allows self-intersections during the deformation. We will need a stronger condition than this to define knot equivalence. Let us consider the notion of isotopy.

Definition 3. *Given two topological spaces X and Y, and two maps $f_0, f_1 : X \to Y$, we say that f_0 and f_1 are **isotopic** if there exists a homotopy F from f_0 to f_1 with the additional property that for any $t \in [0,1]$ the map f_t defined by $f_t(x) = F(x,t)$ is an embedding of X into Y. The map F is called an **isotopy** from f_0 to f_1.*

One can show that the property of being isotopic induces an equivalence relation on the embeddings of X into Y. This notion looks more promising, because an isotopy between two knots is a continuous deformation of one into the other while never allowing self-intersection. However, this notion is still not strong enough, because any knot can be shown to be isotopic to the unknot [BZ]. Consider the yet stronger notion of ambient isotopy.

Definition 4. *Given two topological spaces X and Y, and two embeddings $f, g : X \to Y$, we say that f and g are **ambient isotopic** if there exists a continuous map*

$$F : Y \times [0,1] \to Y$$

*such that for any $t \in [0,1]$, the map $f_t : Y \to Y$ defined by $f_t(y) = F(y,t)$ is a homeomorphism, f_0 is the identity on Y, and $f_1(f(x)) = g(x)$. Such a function is called an **ambient isotopy** from f to g.*

Again, the property of being ambient isotopic induces an equivalence relation on embeddings of X into Y. But in this case, the equivalence relation induced on the embeddings of S^1 into S^3 coincides very nicely with our intuition of what it means for two knots to be equivalent. So we make the following definition.

Definition 5. *Two knots $K_1 : S^1 \to S^3$ and $K_2 : S^1 \to S^3$ are **equivalent** if there is an ambient isotopy from K_1 to K_2.*

It is clear from this definition that, if two knots K_1 and K_2 are equivalent by an ambient isotopy F, then the map $f_1(x) = F(x, 1)$ restricted to $S^3 \setminus K_1$ is a homeomorphism between $S^3 \setminus K_1$ and $S^3 \setminus K_2$. This leads us to the following statement.

Fact 1. *If K is a knot, then the homeomorphism type of its complement in S^3 is an invariant of its knot type.*

This is a very natural invariant of knot type, but not a very practical one, because there is no easy way to effectively distinguish the homeomorphism type of 3-manifolds. If we were to attempt to distinguish them, we might start by looking at the fundamental group.

Definition 6. *The **fundamental group** of a topological space X is the set of all path homotopy classes of loops in X based at a point x_0, where the group operation is concatenation. It is denoted by $\pi_1(X, x_0)$. In the specific case when X is the complement of a knot K in S^3, we will call $\pi_1(X, x_0)$ the **knot group** of K.*

A thorough treatment of the fundamental group can be found in [Mun] and [Hat]. When X is a path-connected space, the isomorphism type of its fundamental group is the same regardless of the choice of base point. So when X is path-connected, we will simply denote its fundamental group by $\pi_1(X)$.

The fundamental group is an invariant of the homeomorphism type of a space, and so the fundamental group of the complement of the knot is an invariant of the knot type. Given only a knot diagram for K, a straightforward application of the Seifert-van Kampen Theorem to $S^3 \setminus \bar{K}$, where $\bar{K}$ is a regular neighborhood of

K, allows one to write down a finite presentation for the fundamental group of a knot complement, called the **Wirtinger Presentation** for the knot group. This is thoroughly covered in [CF, Chapter VI]. Unfortunately, there is no general method by which to tell whether two groups are isomorphic given only a finite presentation for each of them. Thus, while the knot group is a natural invariant of knot type, it is also an impractical one.

Although we do not have a method to distinguish finitely generated groups in general, we do have a good structure theorem for finitely generated abelian groups. It is with this in mind that we might consider the first homology of the knot complement. Since the complement of a knot is always path-connected, the first homology group of this space will just be the abelianization of its fundamental group. This is a finitely generated abelian group which is an invariant of knot type. However, all knot groups have the property that their abelianization is the free abelian group on one generator, i.e. if $G = \pi_1(S^3 \setminus K)$ for any knot K, then $G/G^{(1)} \cong \mathbb{Z}$. The first homology of the complement of the knot, therefore, has no power to distinguish knots.

To sidestep this problem, we can instead look at a class of spaces that are closely related to the knot complement, namely its covering spaces. We will first focus our attention on the universal abelian cover of X, which will lead to the definitions of the well-studied classical integral Alexander module and rational Alexander module, which have proven quite useful as tools for distinguishing knots.

Definition 7. *Given a topological space X, a **covering space** of X is a space $\tilde{X}$ together with a map $\rho : \tilde{X} \to X$ such that for every $x_0 \in X$ there is a neighborhood U of x_0 such that the preimage $\rho^{-1}(U)$ is a disjoint union of open sets each of which is mapped homeomorphically onto U by ρ.*

Two covering spaces $\rho_1 : \tilde{X}_1 \to X$ and $\rho_2 : \tilde{X}_2 \to X$ of X are said to be ***equivalent*** *if there is a homeomorphism $\varphi : \tilde{X}_1 \to \tilde{X}_2$ so that $\varphi \circ \rho_1 = \rho_2$.*

We reproduce here a few useful facts about covering spaces. For a much more complete discussion of covering spaces, see [Hat].

Fact 2. *If X is a path-connected, locally path-connected and semi-locally simply-connected space with fundamental group $G = \pi_1(X)$, then for any subgroup $H \leq G$ there is exactly one path-connected covering space (up to equivalence) $\rho : X_H \to X$ so that $\pi_1(X_H) \cong H$.*

Fact 3. *If X is a space as above, and $H \lhd G = \pi_1(X)$, then G/H acts on X_H on the right as the group of deck transformations, so it acts on the singular chain complex $S_*(X_H)$ on the right, which makes the homology groups of X_H into right $\mathbb{Z}[G/H]$ modules. More on this in* [DK].

With these facts in hand, we are ready to define the integral Alexander module:

Definition 8. *Let K be a knot, let $X = S^3 \setminus K$, and let $G = \pi_1(X)$. Then we define the* ***integral Alexander module of K*** *to be the first homology of the universal abelian cover of X (that is, the cover corresponding to the first derived subgroup of G) viewed as a right $\mathbb{Z}[G/G^{(1)}]$-module (i.e. a right $\mathbb{Z}[t, t^{-1}]$-module).*

The isomorphism type of this module is an invariant of knot type. The integral Alexander module turns out to be a strong knot invariant, but it is not particularly easy to work with. One difficulty is that the ring $\mathbb{Z}[t, t^{-1}]$ is not a PID, so we don't have a good structure theorem to deal with the integral Alexander module. We attempt to remedy this by localizing the module as follows.

Definition 9. *Let K be a knot, let $X = S^3 \setminus K$, and let $G = \pi_1(X)$. Then we define the* ***rational Alexander module of K*** *(or simply the* ***Alexander module of K****)* *to be the first rational homology of the $G^{(1)}$-cover of X, viewed as a $\mathbb{Q}[G/G^{(1)}]$-module (that is, a $\mathbb{Q}[t, t^{-1}]$-module).*

Now we have a module over a PID whose isomorphism type is an invariant of knot type. This area of study gives rise to several important and useful knot invariants, such as the Alexander polynomial, which can be computed directly from the diagram of the knot.

2 Construction of Higher-Order Alexander Modules

This construction provokes a question: could we find a similar module for other normal subgroups of the fundamental group? If H is any normal subgroup of $G = \pi_1(S^3 \setminus K)$, then the deck group of the H-cover X_H of $S^3 \setminus K$ is G/H and it acts on the homology groups of the H-cover, making $H_1(X_H)$ a right $\mathbb{Z}[G/H]$-module.

Naturally, the normal subgroups we consider should be subgroups that are defined for all knot groups, so we will look at classes of characteristic subgroups. But even with such a choice of H, we won't know very much a priori about the integral group ring $\mathbb{Z}[G/H]$, so in general these modules will not be very accessible to us. To remedy this, we will restrict our attention to specific normal subgroups H of G, which have properties that enforce useful facts about their integral group rings.

Definition 10. *Let G be a group. We say that G is* ***poly-(torsion-free abelian)*** *(hereafter abbreviated PTFA) if there is a normal series*

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_{n-1} \triangleright G_n =< 1 >$$

so that each quotient G_i/G_{i+1} is torsion free abelian for $0 \leq i \leq n-1$.

One can show that if G is a PTFA group, then its integral group ring $\mathbb{Z}[G]$

has no zero-divisors [Str]. Further, it can be shown that $\mathbb{Z}[G]$ satisfies the right Ore condition (that the right ideals generated by any two elements intersect nontrivially), and so it embeds into its classical skew field of fractions [Lew].

If G is a knot group and we can find a normal subgroup H of G so that the quotient G/H is PTFA, then the first homology of the H-cover of the knot complement can be viewed as a module over $\mathbb{Z}[G/H]$, which has the above properties. It turns out not to be so hard to find such normal subgroups.

Definition 11. *Let G be a group. The* ***rational derived series*** *of G is defined as follows:*

$$G_r^{(0)} = G \ and$$

$$G_r^{(i)} = \{x \in G_r^{(i)} : x^k \in [G_r^{(i-1)}, G_r^{(i-1)}] for \ some \ k \in \mathbb{Z}^+\}.$$

It is clear from the definition that $G_r^{(i)}/G_r^{(i+1)}$ is torsion free abelian for all i. Additionally, for any term $G_r^{(k)}$ of the rational derived series, the quotient group $G/G_r^{(k)}$ admits the normal series

$$G/G_r^{(k)} \ \triangleright \ G_r^{(1)}/G_r^{(k)} \ \triangleright \ \ldots \ \triangleright \ G_r^{(k-1)}/G_r^{(k)} \ \triangleright \ G_r^{(k)}/G_r^{(k)} \ = <1>,$$

which has torsion-free abelian successive quotients $G_r^{(i)}/G_r^{(k)} \Big/ G_r^{(i+1)}/G_r^{(k)} \ \cong \ G_r^{(i)}/G_r^{(i+1)}$, and so $G/G_r^{(n)}$ is PTFA.

Additionally, the following useful fact is a result from [Str].

Fact 4. *In the special case where G is a knot group, the n^{th} term of the rational derived series of G is the same as the n^{th} term of its ordinary derived series, $G^{(n)}$* [Str].

So if G is a knot group, then for any term $G_r^{(n)}$ in the rational derived series we can view the first homology of its corresponding covering space $X_{G_r^{(n)}}$ as a right $\mathbb{Z}[G/G_r^{(n)}]$-module [DK], which will have the convenient properties listed above. It will also be the same as the first homology of the covering space corresponding to $X_{G^{(n)}}$ viewed as a right $\mathbb{Z}[G/G^{(n)}]$-module.

This naturally leads us to extend our concept of the integral Alexander module to apply to covering spaces corresponding with higher terms of the derived series of a knot group.

Definition 12. *Let K be a knot, let $X = S^3 \setminus K$ and let $G = \pi_1(X)$. Then if $X_{G^{(n+1)}}$ is the covering space of X corresponding to the $n+1^{st}$ term of the derived series of G, then the **n^{th} integral Alexander module** is the the first homology of $X_{G^{(n+1)}}$ as a right $\mathbb{Z}[G/G^{(n+1)}]$-module. We will denote it by $\mathcal{A}_n^{\mathbb{Z}}(K)$.*

First, observe that this definition agrees with the classical integral Alexander module in the case when $n = 0$. Additionally, note that the isomorphism type of these modules is invariant under knot equivalence.

These modules are still not easy to work with (note that they are modules over non-PIDs). We would like to use the fact that $G/G^{(n)}$ is PTFA in order to localize these modules, giving a higher-order analogue of the rational Alexander module.

Let K be a knot, X its complement and $G = \pi_1(X)$. Let H_n be the kernel of the projection map $\pi : G/G^{(n+1)} \to G/G^{(1)}$ (i.e. H_n is $G^{(1)}/G^{(n+1)}$). Since $G/G^{(n+1)}$ is PTFA, its subgroup H_n will be PTFA as well, and so $\mathbb{Z}[H_n]$ is a right Ore domain (satisfies the right Ore condition). Let $S_n = \mathbb{Z}[H_n] \setminus \{0\}$, which is a subset of $\mathbb{Z}[G/G^{(n+1)}]$. Since $\mathbb{Z}[H_n]$ is an Ore domain, S_n is a right divisor set of $\mathbb{Z}[G/G^{(n+1)}]$, and so we can localize it, giving $(\mathbb{Z}[G/G^{(n+1)}])(S_n)^{-1}$. Call this ring R_n.

We can show that this ring R_n is a noncommutative PID, and furthermore, we can establish that it is isomorphic to the ring of skew Laurent polynomials $\mathbb{K}_n[t, t^{-1}]$, where $\mathbb{K}_n$ is the skew field given by the localization $(\mathbb{Z}H_n)(\mathbb{Z}H_n \setminus \{0\})^{-1}$ [Coc].

With this convenient ring in hand, we are ready to define the higher-order analogue of the rational Alexander module.

Definition 13. *We define the* $\boldsymbol{n^{th}}$ ***rational Alexander module*** *(or simply the* $\boldsymbol{n^{th}}$ ***Alexander module****) to be* $\mathcal{A}_n^{\mathbb{Z}}(K) \otimes_{\mathbb{Z}[G/G^{(n+1)}]} R_n$*. We will denote it by* $\mathcal{A}_n(K)$*.*

The isomorphism type of these modules is an invariant of the knot type. In addition, each of these higher-order rational Alexander modules is over a PID, which will greatly simplify our calculations. In particular, because $\mathcal{A}_n(K)$ is weakly isomorphic to a $\mathbb{K}_n[t, t^{-1}]$-module, we can also extract a natural numerical invariant as follows.

Definition 14. *We define* $\boldsymbol{\delta_n(K)}$ *to be the rank of the* n^{th} *rational Alexander module of* K *as a* $\mathbb{K}_n$*-module.*

Fact 5. *If* K *is a knot, then* $\delta_n(K)$ *is an invariant of the knot type.*

One useful tool we have to aid us in practical computations of these invariants is *homology with local coefficients*. We will briefly build this machinery here before moving on to computation in practice.

If X is a space which admits a universal cover $\tilde{X}$, then the fundamental group $G = \pi_1(X)$ acts on the singular chain groups of $\tilde{X}$ on the right, and so it acts on the homology groups of $\tilde{X}$ on the right, so the singular chain complex of $\tilde{X}$ can be regarded as a chain complex of right $\mathbb{Z}G$-modules.

Definition 15. *Let* X *be a space which admits a universal cover* $\tilde{X}$*, let* $G = \pi_1(X)$*, and let* M *be any left* $\mathbb{Z}G$*-module. Consider the chain complex of tensor products*

$$\ldots \to S_{n+1}(\tilde{X}) \otimes_{\mathbb{Z}G} M \to S_n(\tilde{X}) \otimes_{\mathbb{Z}G} M \to S_{n-1}(\tilde{X}) \otimes_{\mathbb{Z}G} M \to \ldots \to S_0(\tilde{X}) \otimes_{\mathbb{Z}G} M \to 0.$$

We will call the homology of this chain complex the ***homology of X with local coefficients in M****, and we will denote it by* $H_*(X; M)$.

One useful application of homology with local coefficients is as follows: If X is a space which admits a universal cover $\tilde{X}$ and $G = \pi_1(X)$, then for any normal subgroup H of G with corresponding covering space X_H, then the $\mathbb{Z}[G/H]$-modules $H_n(X; \mathbb{Z}[G/H])$ and $H_n(X_H)$ are isomorphic for all n. A thorough discussion of homology with local coefficients can be found in [DK].

Two nice results follow from this. If K is a knot and X is its complement in S^3, then

$$\mathcal{A}_n^{\mathbb{Z}}(K) \cong H_1(X; \mathbb{Z}[G/G^{(n+1)}]) \quad \textit{and} \quad \mathcal{A}_n(K) \cong H_1(X; R_n).$$

3 Computing δ_1 in Practice

We outline here a procedure by which one can compute δ_1 of a knot from its knot diagram.

One can write down a finite presentation for the fundamental group of the complement of a knot from only its diagram. If the diagram for K has n-crossings, then a finite presentation for the knot group G is given by n generators, one for each arc, and n relations, one for each crossing. More information on how to find the Wirtinger presentation for a knot group can be found in [CF]. Note that the application of the Seifert-van Kampen theorem to find a finite presentation depends on the existence of a regular neighborhood of the knot. So this procedure for finding $\delta_1(K)$ will only work if K is a tame knot.

It is a fact about the Wirtinger presentations of knot groups that any one of the relations can be written in terms of the other $n-1$. So without loss of generality, we can discard the n^{th} relation and still have a finite presentation for the knot group.

It can be shown that in the Wirtinger presentation, under the abelianization map (that is, the projection map $\pi : G \to G/G^{(1)}$), all of the generators of G will be sent to a generator of the infinite cyclic group $G/G^{(1)} \cong \mathbb{Z}$. Later in our practical computation, we will want to find a splitting homomorphism from $G/G^{(1)}$ to $G/G^{(2)}$ in the short exact sequence:

$$1 \to G^{(1)}/G^{(2)} \to G/G^{(2)} \to G/G^{(1)} \to 1.$$

There will not be a natural choice of such a splitting if *all* of the generators of G are sent to a generator of $G/G^{(1)}$ under the abelianization map. For convenience, if $\{a_1, a_2, \dots, a_n\}$ are the generators from the Wirtinger presentation, we will rewrite our group presentation by defining a new generating set

$$\{a_1, x_2, x_3, \dots, x_n\}$$

$$where$$

$$x_i = a_i \cdot a_1^{-1} \ for\ all\ 2 \leq i \leq n,$$

and rewriting the relations as necessary.

We now have a group presentation for the knot group with the property (which we will make use of later) that exactly one of its generators is sent to the generator of $G/G^{(1)}$ under the projection map, while the rest are sent to the identity.

Fox calculus gives a procedure by which we can find an explicit presentation for the module $H_1(X, x_0; \mathbb{Z}G)$ from a finite presentation of G. We outline it here. A more detailed explanation can be found in [CF].

Definition 16. *A* ***derivative*** *in the group ring* $\mathbb{Z}G$ *is a ring homomorphism*

$d : \mathbb{Z}G \to \mathbb{Z}G$ so that for any two elements c_1 and c_2 of $\mathbb{Z}G$,

$$d(c_1 + c_2) = d(c_1) + d(c_2)$$

and

$$d(c_1c_2) = d(c_1)t(c_2) + c_1d(c_2),$$

where t is the trivializing homomorphism of the group ring.

One can show that if G is finitely generated by $\{a_1, a_2, \ldots, a_n\}$, then for each generator a_i there is exactly one derivative d_i in $\mathbb{Z}G$ with the property that

$$d_i(a_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We call it the derivative with respect to a_i. We can use these derivatives in the group ring to write down a presentation for the module $H_1(X, x_0; \mathbb{Z}G)$. Recall the following definition:

Definition 17. *Let P be an $m \times n$ matrix with entries from a (not necessarily commutative) ring R.*

$$P = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix}$$

*We say that a module M has the above matrix as its presentation, or that P is a **presentation for a right module M**, if M is isomorphic to the quotient of the free module on n generators $\{a_1, a_2, \ldots, a_n\}$ by the submodule generated by the set* *of relations:*

$$\{a_1 \cdot r_{11} + a_2 \cdot r_{12} + \ldots + a_n \cdot r_{1n},$$

$$a_1 \cdot r_{21} + a_2 \cdot r_{22} + \ldots + a_n \cdot r_{2n},$$

$$\vdots$$

$$a_1 \cdot r_{m1} + a_2 \cdot r_{m2} + \ldots + a_n \cdot r_{mn}\}.$$

If our knot group has finite presentation

$$G = \left\langle a_1, a_2, \ldots, a_n \middle| r_1, r_2, \ldots, r_{n-1} \right\rangle,$$

we can form the $(n-1) \times n$ matrix

$$\Big(d_i(a_j) \Big)$$

where $1 \leq i \leq n-1$ and $1 \leq j \leq n$. This will be a presentation matrix for $H_1(X, x_0; \mathbb{Z}G)$[CF].

Note that this is also a presentation matrix for $H_1(X, x_0; \mathbb{Z}[G/G^{(2)}])$, because from our matrix presentation we have the exact sequence

$$(\mathbb{Z}G)^{n-1} \to (\mathbb{Z}G)^n \to H_1(X, x_0; \mathbb{Z}G) \to 0,$$

where the first map is given by sending the generators of the free module $(ZG)^{n-1}$ to the relations given by the matrix presentation. Under the tensor product, the sequence

$$(ZG)^{n-1} \otimes_{\mathbb{Z}G} \mathbb{Z}[G/G^{(2)}] \to (\mathbb{Z}G)^n \otimes_{\mathbb{Z}G} \mathbb{Z}[G/G^{(2)}] \to H_1(X, x_0; \mathbb{Z}G) \otimes_{\mathbb{Z}G} \mathbb{Z}[G/G^{(2)}] \to 0,$$

or more simply

$$(\mathbb{Z}[G/G^{(2)}])^{n-1} \to (\mathbb{Z}[G/G^{(2)}])^n \to H_1(X, x_0; \mathbb{Z}[G/G^{(2)}]) \to 0$$

will still be exact. (The rewriting of the homology group is a result of the fact that $\mathbb{Z}[G/G^{(2)}]$ is a flat $\mathbb{Z}G$ module [Coc]). So the relations given in the original matrix will still be a generating set for the kernel of the map out of $(\mathbb{Z}[G/G^{(2)}])^n$, and so the matrix given by Fox calculus will be a presentation for $H_1(X, x_0; \mathbb{Z}[G/G^{(2)}])$. By the same argument, the matrix given by Fox calculus is a matrix presentation for the module $H_1(X, x_0; \mathbb{Z}[G/G^{(1)}]) \cong H_1(X, x_0; \mathbb{Z}[\mu, \mu^{-1}])$

Now recall that we rewrote the finite presentation of G so that only the generator a_1 was sent to the generator of $G/G^{(1)}$ under the projection map, while the rest were sent to the identity. This gives us a natural choice of function of $\varphi : G/G^{(1)} \to G/G^{(2)}$ which splits the exact sequence

$$0 \to G^{(1)}/G^{(2)} \to G/G^{(2)} \to G/G^{(1)} \to 0.$$

As a consequence of this splitting, we get that $G/G^{(2)}$ is isomorphic to the semi-direct product

$$G/G^{(2)} \cong G/G^{(1)} \ltimes G^{(1)}/G^{(2)}.$$

As a consequence of the semi-direct product structure of $G/G^{(2)}$, each of its elements can be written uniquely as a product of an element in $G/G^{(1)}$ and an element in $G^{(1)}/G^{(2)}$, i.e. can be written uniquely in the form $t^k \cdot r$ where t is the generator of $G/G^{(1)}$, $k \in \mathbb{Z}^+$ and $r \in G^{(1)}/G^{(2)}$ [DF].

So we can rewrite our presentation matrix to get a matrix presentation for the module $H_1(X, x_0; \mathbb{Z}[G/G^{(1)} \ltimes G^{(1)}/G^{(2)}])$.

Now, note that for a knot group we know that $G/G^{(2)}$ is PTFA, so $\mathbb{Z}[G/G^{(2)}]$ satisfies the right Ore condition. Thus we can localize, adding formal inverses for all nonzero elements of the subring $\mathbb{Z}[G^{(1)}/G^{(2)}] \subset \mathbb{Z}[G/G^{(2)}]$, yielding a presentation matrix for

$$H_1(X, x_0; (\mathbb{Z}[G/G^{(1)} \ltimes G^{(1)}/G^{(2)}])(\mathbb{Z}[G^{(1)}/G^{(2)}] \setminus \{0\})^{-1}).$$

But this is weakly isomorphic to $H_1(X, x_0; \mathbb{K}_1[t, t^{-1}])$ where $\mathbb{K}_1$ is the skew field

$$\mathbb{Z}[G^{(1)}/G^{(2)}](\mathbb{Z}[G^{(1)}/G^{(2)}] \setminus \{0\})^{-1}.$$

Since $\mathbb{K}_1[t, t^{-1}]$ is a PID, we can use row and column operations to diagonalize this matrix, and the sum of the degrees of the Laurent polynomials on the diagonal will give us the rank of $H_1(X, x_0; \mathbb{K}[t, t^{-1}])$ as a $\mathbb{K}_1$-module, which is the same as the rank of the 1^{st} Alexander module of K. Note that, from the long exact sequence of a pair

$$H_1(X, x_0; \mathbb{Z}[G/G^{(2)}]) \cong H_1(X; \mathbb{Z}[G/G^{(2)}]) \oplus H_0(x_0; \mathbb{Z}[G/G^{(2)}])$$

$$\cong H_1(X; \mathbb{Z}[G/G^{(2)}]) \oplus \mathbb{Z}[G/G^{(2)}],$$

and so one of the columns will eventually be all zeros, corresponding to a free generator. We will drop this column and continue to diagonalize the remaining square matrix, which will give $\delta_1(K)$.

However, while diagonalizing we will need a way to tell whether elements in $\mathbb{Z}[G^{(1)}/G^{(2)}]$ have inverses, i.e. whether elements in $\mathbb{Z}[G^{(1)}/G^{(2)}]$ are zero. How do we do this?

Note that $G^{(1)}/G^{(2)}$ is the same as the underlying abelian group structure of the first homology of the universal abelian cover of X, i.e. the group structure of $H_1(X;\mathbb{Z}[G/G^{(1)}])$. As we showed above, Fox calculus can find a matrix presentation for $H_1(X,x_0;\mathbb{Z}G)$, which is also a matrix presentation for $H_1(X,x_0;\mathbb{Z}[G/G^{(1)}])$, which we can write as $H_1(X,x_0;\mathbb{Z}[\mu,\mu^{-1}])$.

As above, through the long exact sequence of a pair, we get

$$H_1(X,x_0;\mathbb{Z}[\mu,\mu^{-1}]) \cong H_1(X;\mathbb{Z}[\mu,\mu^{-1}]) \oplus H_0(x_0;\mathbb{Z}[\mu,\mu^{-1}])$$

$$\cong H_1(X;\mathbb{Z}[\mu,\mu^{-1}]) \oplus \mathbb{Z}[\mu,\mu^{-1}].$$

So while diagonalizing this matrix presentation, at one point there will be a column with a unit in one entry and zeros elsewhere (the generator of $\mathbb{Z}[\mu,\mu^{-1}]$) which we can remove, leaving a matrix presentation for $H_1(X;\mathbb{Z}[\mu,\mu^{-1}])$, and the structure of the underlying abelian group of this module is $G^{(1)}/G^{(2)}$. This matrix, therefore, can be used to obtain a canonical presentation of elements in $G^{(1)}/G^{(2)}$, which gives us our desired method to check whether elements in $\mathbb{Z}[G^{(1)}/G^{(2)}]$ are invertible after we localize.

4 Developing A Computer-Based Approach

The method outlined in the previous section for computing δ_1 is feasible, but in practice can be very time consuming. It should be possible to write a computer program to assist with some if not all of the computational procedure discussed in the previous section.

Start with the Wirtinger presentation for the knot, given as a list of n generators $\{a_1, a_2, \ldots, a_n\}$ and n relations $\{w_1, w_2, \ldots, w_n\}$ where the generators are characters, and the relations are strings in those characters together with a character $\{^{-1}\}$ to denote inverses. Then we can discard the n^{th} relation and, using the procedure outlined above, rewrite the group presentation as desired, with new generating set $\{x_1 = a_1, x_2 = a_2 \cdot a_1^{-1}, \ldots, x_n = a_n \cdot a_1^{-1}\}$ and rewritten relations $\{r_1, r_2, \ldots, r_n\}$, so that only $x_1 = a_1$ is sent to a generator of $G/G^{(1)}$ under the projection map while all the other x_i are all sent to the identity.

We can algorithically use Fox calculus to find a matrix presentation for $H_1(X, x_0; \mathbb{Z}G)$. The procedure for finding the derivative with respect to some generator or x_i of the relation r_j as a string in the generators, exponents, and the symbols $\{+, -, 1\}$ is as follows:

(1) instantiate an empty string and begin reading the relation string r_j from left to right.

(2) if x_i^{-1} is read, then append "$-$" and all of the relation that has been read so far, including the x_i^{-1}, to the output string.

(3) if x_i is read without an inverse, then if the output string is nonempty, append "$+$". Then append all of the relation that has been read so far, excluding the x_i, to the output string, or the symbol 1 if the string that would be appended is empty.

This procedure will compute $d_i(r_j)$ for all i and j. Put the results in a matrix with $d_i(r_j)$ in the (i, j)-entry, which yields a matrix presentation for $H_1(X, x_0; \mathbb{Z}G)$.

As we have seen, through no explicit computation this matrix is also a presentation matrix for $H_1(X, x_0; \mathbb{Z}[\mu, \mu^{-1}])$ and $H_1(X, x_0; \mathbb{Z}[G/G^{(2)}])$.

If we then take the matrix presentation for $H_1(X, x_0; \mathbb{Z}[\mu, \mu^{-1}])$, we know we will be able to remove a column corresponding to a free generator, then bring the resulting square matrix to rational canonical form, and we will get canonical presentations for elements in $G^{(1)}/G^{(2)}$, which gives us a way to tell whether elements in $\mathbb{Z}[G^{(1)}/G^{(2)}]$ are zero (i.e. whether they have inverses after we localize below).

If we consider this matrix presentation for $H_1(X, x_0; \mathbb{Z}[G/G^{(2)}])$, recall that every element can be written uniquely as the product of an element in $G/G^{(1)}$ and an element in $G^{(1)}/G^{(2)}$. To rewrite the entries $e_{i,j}$ of this matrix in this form, find the sum of the powers of x_1 in the entry and call it k, and multiply the entry on the left by $x_1^k \cdot x_1^{-k}$. The first x_1^k is a coset representative of an element in $G/G^{(1)}$. We can see that the rest of the entry, $x^{-k} \cdot e_{i,j}$, is in $G^{(1)}/G^{(2)}$ because it becomes trivial under the abelianization map from G to $G/G^{(1)}$. We can localize, adding formal inverses for the nonzero elements of $\mathbb{Z}[G^{(1)}/G^{(2)}]$, yielding a presentation matrix for $H_1(X, x_0; \mathbb{K}_1[t, t^{-1}])$, which after removing a column corresponding to a free generator will be weakly isomorphic to $\mathcal{A}_1(K)$ and so has rank as a $\mathbb{K}_1$-module will be $\delta_1(K)$. We can compute this rank by diagonalizing the matrix. During this diagonalization we will have to drop a column of zeros corresponding to the free generator. The sum of the degrees of the terms on the diagonal will be $\delta_1(K)$.

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