A Construction of Rigid Analytic Cohomology Classes for Split Reductive Algebraic Groups

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This —like everything I do—

is for Brentlynn and Sterling.

Abstract

The cohomology groups $H^1(\Gamma_0(N), V_k)$ completely describe the space of classical cusp forms of weight k and level N. We study a generalization, $H^n(\Gamma, V_\lambda)$, where some algebraic group G plays a role analogous to that of GL_2 in the classical case.

Ash and Stevens proved that certain classes in $H^n(\Gamma, V_\lambda)$ may be lifted through the natural map $\rho_\lambda : H^n(\Gamma, D_\lambda) \to H^n(\Gamma, V_\lambda)$ to overconvergent classes in $H^n(\Gamma, D_\lambda)$. Pollack and Pollack were able to prove this result constructively in the case of $G = \mathrm{GL}_3$, by providing a filtration on the distribution space D_λ.

We construct a general filtration $Fil^N D_\lambda$, for a split reductive algebraic group G. Using this filtration, we are able to lift classes in $H^n(\Gamma, V_\lambda)$ to the finite dimensional spaces $H^n(\Gamma, D_\lambda / Fil^N D_\lambda)$. These lifts approximate the lifts into $H^n(\Gamma, D_\lambda)$ and improve as $N \to \infty$.

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Contents

Chapter 1

Background and Introduction

1.1 Algebraic Groups

Let k be a field, We call G an algebraic group over k if G is an algebraic variety over k with a group structure such that the multiplication and inverse maps are morphisms of varieties. We denote by $\mathbf{G}_a$ the additive group and by $\mathbf{G}_m$ the multiplicative group defined by $\mathbf{G}_m(k) = k^*$.

The maximal closed, connected, solvable, normal algebraic subgroup of G is called its radical. We call G reductive if the unipotent subgroup of the radical is trivial.

A maximal closed, connected, solvable subgroup B of G is called a Borel subgroup. Note that the radical of G is contained in every Borel subgroup. B is split over k if there are connected algebraic subgroups B_i defined over k, such that $B = B_0 \supset B_1 \supset \ldots \supset B_n = \{0\}$ and each quotient group B_i/B_{i+1} is isomorphic to either $\mathbf{G}_m$ or $\mathbf{G}_a$.

A subgroup T of G is a split torus over k if

$$T(k) \cong \mathbf{G}_m(k) \times \mathbf{G}_m(k) \times \ldots \times \mathbf{G}_m(k).$$

A subgroup T of G is a torus if it becomes a split torus over K, some field extension of k. We call a reductive group G split if it has a split maximal torus. A morphism $\alpha : G \to \mathbf{G}_m$ is called a character of G.

A representation of G is a vector space V with a homomorphism $\varphi : G(k) \to \mathrm{GL}(V)$. Let $\lambda : G \to \mathbf{G}_m(k)$ be a character of G. If, for some $v \in V$, we have $\varphi(g) \cdot v = \lambda(g)v$ for all $g \in G$, then we call λ the weight of v.

The adjoint representation refers to the tangent space $\mathfrak{G}$ to G at the identity with conjugation. For a full definition of the adjoint representation, as well as a more complete discussion of weights and roots than is given below, see [4]. When restricted to a torus T, the adjoint representation decomposes as a sum of one dimensional representations. The weights of the nontrivial representations appearing in this decomposition are called the roots of G with respect to T.

The collection, Φ, of all roots of G with respect to some T is called the system of roots for G relative to T. We may chose a subset, Φ^+, of "positive roots" such that for each root α either $\alpha \in \Phi^+$ or $-\alpha \in \Phi^+$ and $\alpha + \beta \in \Phi^+$ whenever $\alpha + \beta$ is a root and $\alpha, \beta \in \Phi^+$. The collection of all positive roots which cannot be written as a nontrivial sum of positive roots is called the fundamental system of roots. All other positive roots can be written uniquely as a positive integral linear combination of fundamental roots.

The height of a weight (or a root) is the sum of the coefficients in its expression as a sum of fundamental roots. If for all other weights μ of V, $\lambda - \mu$ is a positive integral linear combination of fundamental roots, we say λ is the highest weight of V.

The type of a group refers to the size and structure of its fundamental system of roots. The simple group types are A_n ($n \geq 1$), B_n ($n \geq 2$), C_n ($n \geq 3$), D_n ($n \geq 4$), E_6, E_7, E_8, F_4, and G_2. The subscript refers to the number of fundamental roots. For a full description of group types, the reader is referred to [3].

1.2 Overconvergent Modular Forms

Let G be an algebraic group over $\mathbb{Q}$ which splits over $\mathbb{Q}_p$. Let V_λ be the irreducible representation of G of highest weight λ and let $\Gamma \subset G(\mathbb{Q})$ a congruence subgroup of G.

We will define a space of distributions D_λ (section 2.3) on a space X, which comes equipped with an action of Γ and a Γ-equivariant map $D_\lambda \to V_\lambda$. This gives us a natural Γ-equivariant map

$$\rho_\lambda : H^n(\Gamma, D_\lambda) \to H^n(\Gamma, V_\lambda).$$

For U, a suitable Hecke operator on $H^r(\Gamma, D_\lambda)$, we say a U-eigenvector φ in $H^r(\Gamma, D_\lambda)$ is ordinary for U if $\varphi|U = a\varphi$ and a is a p-adic unit. We have the following lifting theorem.

Theorem 1 (A. Ash, G. Stevens 1995 [1], 2008 [2])**.** *If $\varphi \in H^r(\Gamma, V_\lambda)$ is an ordinary U-eigenvector then there exists a unique ordinary U-eigenvector $\psi \in H^r(\Gamma, D_\lambda)$ such that $\rho_\lambda(\psi) = \varphi$.*

The proof by Ash and Stevens only shows the existence of such lifts. Robert Pollack and Glenn Stevens [9] provided a constructive proof when $G = \mathrm{GL}_2$, working in the equivalent case of overconvergent eigensymbols, by solving the Manin relations on p-adic Modular symbols. They lifted classical eigensymbols to approximations of overconvergent eigensymbols with well behaved error terms. These approximations converged to an overconvergent eigensymbol lifting the original classical eigensymbol.

In 2007, Matthew Greenberg [6] gave another constructive proof for the case of GL_2 which placed fewer restrictions on the nature of the approximations.

David Pollack and Robert Pollack [8] generalized Greenberg's technique to $G = \mathrm{GL}_3$ by creating a filtration on D_λ which satisfies certain key properties, most notably stability under Γ. They show such a filtration is sufficient (Theorem 7) by constructing lifts to the cohomology groups $H^n(\Gamma, D_\lambda/Fil^N D_\lambda)$. At each N, there is a natural map

$$\rho_\lambda^N : H^n(\Gamma, D_\lambda/Fil^N D_\lambda) \to H^n(\Gamma, V_\lambda),$$

and they lift of elements of $H^n(\Gamma, V_\lambda)$ to preimages in $H^n(\Gamma, D_\lambda/Fil^N D_\lambda)$. These preimages converge to a preimage within $H^n(\Gamma, D_\lambda)$.

We further generalize this technique to any split reductive group G, by constructing a suitable filtration on the space D_λ.

Proving the result constructively and giving an explicit formula for the filtration allows the computation of approximations in each $H^n(\Gamma, D_\lambda/Fil^N D_\lambda)$, of the overconvergent eigenclasses in $H^n(\Gamma, D_\lambda)$. The space D_λ is too large to compute within, but for each N, $D_\lambda/Fil^N D_\lambda$ is a finite dimensional space and is therefore more manageable.

Pollack and Pollack have computed lifts for $G = \mathrm{GL}_3$. It is now possible to perform these calculations for $G = \mathrm{GL}_4$ or Sp_4.

Chapter 2

Notation and Definitions

We work toward an understanding of the module D_λ, mentioned in the previous section. We will begin by fixing a group G and a weight λ.

2.1 The Group G

Fix a prime p. Fix $\mathbb{C}_p$, an algebraic completion of a fixed algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. Let $\mathbb{O}_p$ be the ring of integers in $\mathbb{C}_p$ and let $\mathcal{M}_p$ be the maximal ideal of $\mathbb{O}_p$. Note that $\mathbb{O}_p/\mathcal{M}_p = \overline{\mathbb{F}_p}$, the algebraic closure of the finite field $\mathbb{F}_p$.

Let G be a split reductive algebraic group over $\mathbb{Q}_p$ with a split maximal torus T in G. Let Φ be the corresponding system of roots and choose $\Phi^+ \subset \Phi$, the positive roots and Φ^- the negative roots in Φ. Let Π be the fundamental system of roots. For each $\alpha \in \Phi$, denote by $\chi_\alpha : \mathbf{G}_a \to G$ the associated one parameter subgroup.

Choose a hyperspecial point x in the apartment of T. Then G can be given the structure of a group scheme over $\mathbb{Z}_p$ so that $G(\mathbb{Z}_p)$ is the stabilizer of x, [10, Section 3]. Moreover, the χ_α are defined over $\mathbb{Z}_p$ and $\chi_\alpha(\mathbb{O}_p) = G(\mathbb{O}_p) \cap \chi_\alpha(\mathbb{C}_p)$.

For each $\alpha \in \Phi$, denote by $\chi_\alpha(\mathbb{O}_p) = U_\alpha$, the root group associated to α.

It will be helpful to define a partial order on Φ.

Definition 1. *For any $\alpha, \beta \in \Phi$, we say $\alpha \succ \beta$ if any of the following hold*

1. *$\alpha \in \Phi^+$ and $\beta \in \Phi^-$,*
2. *$\alpha, \beta \in \Phi^+$ and the height of α is greater than that of β,*
3. *$\alpha, \beta \in \Phi^-$ and the height of $-\alpha$ is less than that of $-\beta$.*

Note that the partial order $\succ$ is strictly stronger than the usual partial order on roots (where $\alpha > \beta$ whenever $\alpha - \beta$ is a positive linear combination of fundamental roots). Fix a total ordering of the roots, consistent with the partial order $\succ$. This is simply assigning an order to each finite collection of roots of a given height. We will employ this order for all products taken over roots in Φ, Φ^+ or Φ^-.

Lemma 1. *[7, 26.3] $G(\mathbb{O}_p)$ is generated by the collection of $\chi_\alpha(\mathbb{O}_p)$'s along with $T(\mathbb{O}_p)$.*

Let B be the Borel subgroup defined, for any $\mathbb{Z}_p$-algebra A, by

$$B(A) = \left\{ h \prod_{\alpha \in \Phi^+} \chi_\alpha(x_\alpha) \;\middle|\; h \in T(A),\ x_\alpha \in A \right\}.$$

Define

$$\begin{aligned} B^{opp}(A) &= \left\{ h \prod_{\alpha \in \Phi^-} \chi_\alpha(x_\alpha) \;\middle|\; h \in T(A),\ x_\alpha \in A \right\} \\ N(A) &= \left\{ \prod_{\alpha \in \Phi^+} \chi_\alpha(x_\alpha) \;\middle|\; x_\alpha \in A \right\} \\ N^{opp}(A) &= \left\{ \prod_{\alpha \in \Phi^-} \chi_\alpha(x_\alpha) \;\middle|\; x_\alpha \in A \right\}. \end{aligned}$$

So N and N^{opp} are, respectively, the groups of unipotent elements in B and B^{opp}.

We have a bijective correspondence $\ell : (\mathbb{O}_p)^{|\Phi^+|} \to N(\mathbb{O}_p)$ defined by

$$\ell\left((x_\alpha)_{\alpha\in\Phi^+}\right) = \prod_{\alpha\in\Phi^+} \chi_\alpha(x_\alpha). \tag{2.1}$$

Denote by $\mathbf{x}$ an indeterminate element in N, and by $(x_\alpha)_\alpha$ the collection of indeterminates in $\mathbb{O}_p$ such that $\ell\left((x_\alpha)_{\alpha\in\Phi^+}\right) = \mathbf{x}$.

2.2 The Semigroup S

Definition 2. *(Iwahori Subgroup) The Iwahori subgroup I of $G(\mathbb{O}_p)$ is the set of elements whose reductions modulo the maximal ideal $\mathcal{M}_p$ are in $B(\overline{\mathbb{F}_p})$.*

Theorem 2. *(Iwahori Decomposition, [5]) If $\gamma \in I$, then there exist unique $\gamma^- \in N^{opp}(\mathbb{O}_p) \cap I$, $\gamma^+ \in N(\mathbb{O}_p)$, and $h_\gamma \in T(\mathbb{O}_p)$ such that*

$$\gamma = \gamma^- h_\gamma \gamma^+.$$

The Iwahori Decomposition gives us $I = (I \cap N^{opp}(\mathbb{O}_p))B(\mathbb{O}_p)$. Therefore we can express I as

$$I = \left\{ \left(\prod_{\alpha\in\Phi^-} \chi_\alpha(y_\alpha) \right) h \left(\prod_{\alpha\in\Phi^+} \chi_\alpha(y_\alpha) \right) \middle| \; h \in T(\mathbb{O}_p),\; y_\alpha \in \mathbb{O}_p,\; p|y_\alpha \forall\alpha \in \Phi^- \right\}.$$

Let X denote the image of I in $N^{opp}(\mathbb{C}_p)\backslash G(\mathbb{C}_p)$. Since $I = (I \cap N^{opp}(\mathbb{O}_p))B(\mathbb{O}_p)$, we have

$$X \cong (I \cap N^{opp}(\mathbb{C}_p))\backslash \left[(I \cap N^{opp}(\mathbb{C}_p))B(\mathbb{O}_p)\right] \cong B(\mathbb{O}_p),$$

and so X is isomorphic as a group to $B(\mathbb{O}_p)$.

Fix a congruence subgroup Γ of $G(\mathbb{O}_p)$ contained in the Iwahori.

Let $\pi \in T(\mathbb{C}_p)$ be chosen such that $p|\alpha(\pi)$ for all $\alpha \in \Phi^+$. Let S be the smallest semigroup in $G(\mathbb{C}_p)$ containing I and π.

Note that I acts on $N^{opp}(\mathbb{C}_p)\backslash G(\mathbb{C}_p)$ by right multiplication, and that π acts on $N^{opp}(\mathbb{C}_p)\backslash G(\mathbb{C}_p)$ by

$$N^{opp}(\mathbb{C}_p)\, g \cdot \pi = N^{opp}(\mathbb{C}_p)\, \pi^{-1} g \pi.$$

We therefore have an action of S on $N^{opp}(\mathbb{C}_p)\backslash G(\mathbb{C}_p)$, which induces an action of S on X.

2.3 The Weight λ

For each fundamental root $\alpha \in \Pi$, there is a unique map

$$h_\alpha : \mathbb{C}_p \to T(\mathbb{C}_p)$$

with the following property. For any $\beta \in \Phi$ and $t, u \in \mathbb{C}_p$,

$$h_\alpha(t^{-1})\chi_\beta(u)h_\alpha(t) = \chi_\alpha(ut^{r-s}),$$

where r and s are such that $-r\beta + \alpha, \ldots, \alpha, \ldots, s\beta + \alpha$ is the longest root string through α by β.

For the remainder of the paper, we fix a character λ of T. There exist integers k_α such that

$$\lambda : \prod_{\alpha \in \Pi} h_\alpha(t_\alpha) \mapsto \prod_{\alpha \in \Pi} t_\alpha^{k_\alpha}. \tag{2.2}$$

Let

$$M_\lambda = \{f : X \to \mathbb{O}_p \mid f(hg) = \lambda(h)f(g) \;\; \forall h \in T(\mathbb{O}_p), g \in X\}.$$

We define notation for a monic monomial in M_λ. If $\mathbf{a} = (a_\alpha)_{\alpha\in\Phi^+} \in \mathbb{Z}^{|\Phi^+|}$, then we set

$$f^{\mathbf{a}}(\mathbf{x}) = \prod_{\alpha\in\Phi^+} x_\alpha^{a_\alpha}.$$

We write $d_{\mathbf{a}} = \sum a_\alpha$ for the degree of $f^{\mathbf{a}}(\mathbf{x})$.

Let

$$A_\lambda = \left\{ f \in M_\lambda \;\middle|\; f|_{N(\mathbb{O}_p)}(\mathbf{x}) = \sum_i \left(c^i f^{\mathbf{a}_i}(\mathbf{x})\right) ; \; c^i \to 0 \text{ as } d_{\mathbf{a}_i} \to \infty \right\}.$$

For any element $g \in I$, denote by h_g the torus factor of the Iwahori Decomposition of g, and by g^+ and g^- the factors from $N(\mathbb{O}_p)$ and $N^{opp}(\mathbb{O}_p)$, respectively. Since $f(h_g g_+) = \lambda(h_g)f(g_+)$, any $f \in A_\lambda$ is uniquely determined by $f|_{N(\mathbb{O}_p)}$.

Define

$$F_\lambda = \{f^{\mathbf{a}} \in A_\lambda\}.$$

Define a norm on A_λ by $|f| = \max\{|c^i|_p\}$ if $f|_{N(\mathbb{O}_p)}(\mathbf{x}) = \sum_i (c^i f^{\mathbf{a}_i})$. We assign to A_λ the usual topology on the Tate Algebra, which is induced by the norm.

Definition 3. *The distribution space D_λ is*

$$D_\lambda = \mathrm{Hom}_{\mathrm{cont}}(A_\lambda, \mathbb{Z}_p),$$

the space of $\mathbb{Z}_p$-linear continuous homomorphisms from A_λ to $\mathbb{Z}_p$.

Note that elements of D_λ are uniquely determined by their values on F_λ.

We now define a left action of S on M_λ by $(\gamma \cdot f)(\mathbf{x}) = f(\mathbf{x} \cdot \gamma)$. In Corollary 1 to Lemma 6, we will show that this restricts to a left action on A_λ, and therefore induces a right action on D_λ as follows. For $\mu \in D_\lambda$, $\gamma \in S$, and $f \in A_\lambda$,

$$(\mu|\gamma)(f) := \mu(\gamma \cdot f).$$

2.4 The Subspace K_λ

The constructions in this section generalize and are inspired by those used by Pollack and Pollack in the case $G = \mathrm{GL}_3$. The lemmas stated in this section are proved in their work, [8].

Let V_λ be the irreducible finite dimensional right algebraic representation of G defined over $\mathbb{Q}_p$ with highest weight λ (with respect to B^{opp}). Let $v_\lambda \in V_\lambda(\mathbb{Q}_p)$ be a highest weight vector. Since v_λ is a highest weight vector, N^{opp} acts trivially of v_λ. Define $f_\lambda : G(\mathbb{O}_p) \to V_\lambda(\mathbb{C}_p)$ by $g \mapsto v_\lambda \cdot g$. This function induces a function on $N^{opp}(\mathbb{O}_p)\backslash G(\mathbb{O}_p)$ which restricts to a function on X. We will henceforth refer by f_λ to this induced function with domain X.

Lemma 2. *[8, Lemma 2.5]* $f_\lambda \in A_\lambda \otimes V_\lambda(\mathbb{Q}_p)$.

Define an action $\star$ of S on V_λ by $v \star \gamma = v \cdot \gamma$ for $\gamma \in I$ and $v \star \pi = \lambda(\pi)^{-1} v \cdot \pi$. Let $\{v_i\}$ be a basis for $V_\lambda(\mathbb{Q}_p)$. There exist unique $f_i \in A_\lambda$ such that

$$f_\lambda = \sum f_i \otimes v_i. \tag{2.3}$$

Define $\rho_\lambda : D_\lambda \to V_\lambda(\mathbb{Q}_p)$ by $\mu \mapsto \sum \mu(f_i) v_i$. This definition is independent of the choice of basis. Henceforth, assume we have chosen a basis with $v_1 = v_\lambda$.

Lemma 3. *[8, Lemma 2.6] When $V_\lambda(\mathbb{Q}_p)$ is taken to have the $\star$ action, ρ_λ is an S-equivariant map.*

Let $L_\lambda = \rho_\lambda(D_\lambda)$ and $K_\lambda = \ker(\rho_\lambda)$. Since ρ_λ is S-equivariant, K_λ is closed under action by S and is an $\mathbb{Z}_p[S]$-module.

2.5 Decomposing $(\gamma \cdot f^{\mathbf{a}})(\mathbf{x})$

Permanently fix a $\gamma \in I$ and $f^{\mathbf{a}} \in F_\lambda$. We introduce notations for several useful ways of writing $\gamma \cdot f^{\mathbf{a}}(\mathbf{x})$. First define, for each vector $\mathbf{b} \in \mathbb{Z}^{|\Phi^+|}$, a $c_{\mathbf{a},\mathbf{b}} \in \mathbb{O}_p$ such that

$$(\gamma \cdot f^{\mathbf{a}})(\mathbf{x}) = \sum_{\mathbf{b} \in \mathbb{Z}^{|\Phi^+|}} c_{\mathbf{a},\mathbf{b}} f^{\mathbf{b}}(\mathbf{x}) \tag{2.4}$$

The group $\Gamma \subset I$ acts on $f^{\mathbf{a}}$ by $(\gamma \cdot f^{\mathbf{a}})(\mathbf{x}) = f^{\mathbf{a}}(\mathbf{x} \cdot \gamma)$. We may express $\mathbf{x} \cdot \gamma$ in the quotient space

$$X \cong (I \cap N^{opp}(\mathbb{O}_p)) \backslash [(I \cap N^{opp}(\mathbb{O}_p))B(\mathbb{O}_p)]$$

by its Iwahori Decomposition:

$$\mathbf{x} \cdot \gamma = h_{\mathbf{x} \cdot \gamma}(\mathbf{x} \cdot \gamma)^+. \tag{2.5}$$

In Section 3.4 we will derive a power series $\mathfrak{p}_\alpha$ in $\mathbf{x}$ for eat α, so that

$$h_{\mathbf{x} \cdot \gamma} = \prod_{\alpha \in \Pi} h_\alpha(\mathfrak{p}_\alpha(\mathbf{x})).$$

By Equation 2.2, $\lambda \left(\prod h_\alpha(\mathfrak{p}_\alpha(\mathbf{x}))\right) = \prod (\mathfrak{p}_\alpha(\mathbf{x}))^{k_\alpha}$. Define

$$\mathfrak{p}(\mathbf{x}) = \prod_{\alpha \in \Pi} \left(\mathfrak{p}_\alpha(\mathbf{x})\right)^{k_\alpha}.$$

We will also derive a collection of power series $\mathfrak{q}_\alpha$, such that $\ell\left(\left(\mathfrak{q}_\alpha(\mathbf{x})\right)_\alpha\right) = \left((\mathbf{x}\cdot\gamma)^+\right)$. Since $f^{\mathbf{a}} \in F_\lambda$, we have

$$\begin{aligned} f^{\mathbf{a}}(\mathbf{x}\cdot\gamma) &= \lambda(h_{\gamma\cdot\mathbf{x}}) f^{\mathbf{a}}\left((\mathbf{x}\cdot\gamma)^+\right) \\ &= \left(\prod_{\alpha\in\Pi} \left(\mathfrak{p}_\alpha(\mathbf{x})\right)^{k_\alpha}\right) f^{\mathbf{a}}\left(\ell\left(\left(\mathfrak{q}_\alpha(\mathbf{x})\right)_\alpha\right)\right) \\ &= \mathfrak{p}(\mathbf{x}) \prod_{\alpha\in\Phi^+} \left(\mathfrak{q}_\alpha(\mathbf{x})\right)^{a_\alpha}. \end{aligned}$$

Let $\mathfrak{q}_{\mathbf{a}}(\mathbf{x}) = \prod_\alpha \left(\mathfrak{q}_\alpha(\mathbf{x})\right)^{a_\alpha}$. By expanding this product, we arrive at power series:

$$\mathfrak{q}_{\mathbf{a}}(\mathbf{x}) = \sum_{\mathbf{b}} q_{\mathbf{a},\mathbf{b}} f^{\mathbf{b}}(\mathbf{x})$$

where $q_{\mathbf{a},\mathbf{b}} \in \mathbb{O}_p$. We also define constants $p_{\mathbf{b}} \in \mathbb{O}_p$ such that

$$\mathfrak{p}(\mathbf{x}) = \sum_{\mathbf{b}} p_{\mathbf{b}} f^{\mathbf{b}}(\mathbf{x}).$$

The function $\gamma\cdot f^{\mathbf{a}}$ is a product of these series

$$\begin{aligned} \gamma\cdot f^{\mathbf{a}} &= \mathfrak{p}\mathfrak{q}_{\mathbf{a}} \\ &= \left(\sum_{\mathbf{b}} p_{\mathbf{b}} f^{\mathbf{b}}\right)\left(\sum_{\mathbf{b}} q_{\mathbf{a},\mathbf{b}} f^{\mathbf{b}}\right) \\ &= \sum_{\mathbf{b}\in\mathbb{Z}^m} c_{\mathbf{a},\mathbf{b}} f^{\mathbf{b}}. \end{aligned}$$

Then for each $\mathbf{b}$,

$$c_{\mathbf{a},\mathbf{b}} = \sum_{\mathbf{b}'} p_{\mathbf{b}'} q_{\mathbf{a},\mathbf{b}-\mathbf{b}'}.$$

Chapter 3

An S-Stable Filtration

Define a filtration on D_λ by

$$Fil^n D_\lambda = \left\{ \mu \in K_\lambda \;\middle|\; \mathrm{ord}_p(\mu(f^{\mathbf{a}})) \geq n - \left\lfloor \frac{d_{\mathbf{a}}}{t} \right\rfloor, \forall \mathbf{a} \right\},$$

where t is twice the height of the highest root.

The goal of this section is to show that $Fil^n D_\lambda$ is an S-stable (and therefore Γ-stable), decreasing filtration which satisfies several conditions. These conditions are necessary to apply Pollack and Pollack's theorem [8]. First, we introduce the concept of "discrepancy." This will give us a way of comparing the degree of a term of $\gamma \cdot f^{\mathbf{a}}(\mathbf{x})$ to the p-adic order of its coefficient. We utilize this comparison to demonstrate the S-stability of $Fil^n D_\lambda$.

3.1 Discrepancy

Definition 4. *Let $f : X \to \mathbb{O}_p$ be a function for which*

$$f|_{N^{opp}(\mathbb{O}_p)} = \sum_{\mathbf{b}} c_{\mathbf{b}} f^{\mathbf{b}}.$$

*The **discrepancy** of f is defined to be*

$$\mathfrak{D}(f) = \min \left\{ \delta \ \in \mathbb{Z}^{+} \mid d_{\mathbf{b}} \leq \delta \mathrm{ord}_p(c_{\mathbf{b}}),\ \forall \mathbf{b} \right\},$$

or if no such δ exists, we say that $\mathfrak{D}(f)$ is undefined.

Note that when $\mathrm{ord}_p(c_{\mathbf{b}}) \neq 0$ for all $\mathbf{b}$,

$$\mathfrak{D}(f) = \max_{\mathbf{b}} \left\{ \left\lceil \frac{d_{\mathbf{b}}}{\mathrm{ord}_p(c_{\mathbf{b}})} \right\rceil \right\}.$$

It is clear that the discrepancy of any constant function is 1.

Lemma 4. *Let $f|_{N^{opp}(\mathbb{O}_p)} = \sum_{\mathbf{b}} c_{\mathbf{b}} f^{\mathbf{b}}$. If $\mathfrak{D}(f)$ exists, then*

$$\mathfrak{D}(f) = \max_{\mathbf{b}} \left\{ \mathfrak{D}\left(c_{\mathbf{b}} f^{\mathbf{b}} \right) \right\}.$$

Proof. This is clear from the definition. □

Lemma 5. *Suppose f and g are both functions so that $\mathfrak{D}(f)$ and $\mathfrak{D}(g)$ exist.*

1. $\mathfrak{D}(f+g) \leq \max\{\mathfrak{D}(f), \mathfrak{D}(g)\}$.

2. $\mathfrak{D}(fg) \leq \max\{\mathfrak{D}(f), \mathfrak{D}(g)\}$ *where fg is the pointwise product of functions.*

Proof. If either f or g is the zero map, the result is trivial. Otherwise let

$$f|_{N^{opp}(\mathbb{O}_p)}(\mathbf{x}) = \sum_{\mathbf{a}} c_{\mathbf{a}} f^{\mathbf{a}},$$
$$g|_{N^{opp}(\mathbb{O}_p)}(\mathbf{x}) = \sum_{\mathbf{a}} e_{\mathbf{a}} f^{\mathbf{a}}$$

Without loss of generality, let us assume $\mathfrak{D}(f) \geq \mathfrak{D}(g)$. Then for each $\mathbf{a}$, $d_{\mathbf{a}} \leq \mathfrak{D}(f)\mathrm{ord}_p(c_{\mathbf{a}})$ and also $d_{\mathbf{a}} \leq \mathfrak{D}(g)\mathrm{ord}_p(e_{\mathbf{a}}) \leq \mathfrak{D}(f)\mathrm{ord}_p(e_{\mathbf{a}})$.

1. We have

$$(f+g)|_{N^{opp}(\mathbb{O}_p)}(\mathbf{x}) = \sum_{\mathbf{a}} (c_{\mathbf{a}} + e_{\mathbf{a}}) f^{\mathbf{a}}(\mathbf{x})$$

   For any $\mathbf{a}$, we know, $d_{\mathbf{a}} \leq \mathfrak{D}(f)\mathrm{ord}_p(c_{\mathbf{a}})$ and $d_{\mathbf{a}} \leq \mathfrak{D}(f)\mathrm{ord}_p(e_{\mathbf{a}})$. Hence

$$\begin{aligned} d_{\mathbf{a}} &\leq \mathfrak{D}(f) \min\{\mathrm{ord}_p(c_{\mathbf{a}}), \mathrm{ord}_p(e_{\mathbf{a}})\} \\ &\leq \mathfrak{D}(f)\mathrm{ord}_p(c_{\mathbf{a}} + e_{\mathbf{a}}). \end{aligned}$$

   Thus $\mathfrak{D}(f+g) \leq \mathfrak{D}(f)$.

2. For any $\mathbf{a}, \mathbf{b}$

$$c_{\mathbf{a}} f^{\mathbf{a}}(\mathbf{x}) e_{\mathbf{b}} f^{\mathbf{b}}(\mathbf{x}) = c_{\mathbf{a}} e_{\mathbf{b}} f^{\mathbf{a}+\mathbf{b}}(\mathbf{x}).$$

   Therefore

$$\begin{aligned} d_{\mathbf{a}+\mathbf{b}} &= d_{\mathbf{a}} + d_{\mathbf{b}} \\ &\leq \mathfrak{D}(f)\mathrm{ord}_p(c_{\mathbf{a}}) + \mathfrak{D}(g)\mathrm{ord}_p(e_{\mathbf{b}}) \\ &\leq \mathfrak{D}(f)\mathrm{ord}_p(c_{\mathbf{a}}) + \mathfrak{D}(f)\mathrm{ord}_p(e_{\mathbf{b}}) \\ &= \mathfrak{D}(f)(\mathrm{ord}_p(c_{\mathbf{a}}) + \mathrm{ord}_p(e_{\mathbf{b}})) \\ &= \mathfrak{D}(f)(\mathrm{ord}_p(c_{\mathbf{a}} e_{\mathbf{b}})). \end{aligned}$$

Therefore $\mathfrak{D}(f) \geq \mathfrak{D}(c_{\mathbf{a}} f^{\mathbf{a}} e_{\mathbf{b}} f^{\mathbf{b}})$. Each term of fg is a finite sum

$$\left(\sum_{\mathbf{a}+\mathbf{b}=\mathbf{d}} c_{\mathbf{a}} e_{\mathbf{b}} \right) f^{\mathbf{d}} = \sum_{\mathbf{a}+\mathbf{b}=\mathbf{d}} c_{\mathbf{a}} f^{\mathbf{a}} e_{\mathbf{b}} f^{\mathbf{b}}$$

for some fixed vector $\mathbf{d}$. By the second part of the lemma, we have for any term of fg,

$$\mathfrak{D}\left(\sum_{\mathbf{a}+\mathbf{b}=\mathbf{d}} c_{\mathbf{a}} f^{\mathbf{a}} e_{\mathbf{b}} f^{\mathbf{b}} \right) \leq \mathfrak{D}(f).$$

By the first part of the lemma, we have $\mathfrak{D}(f) \geq \mathfrak{D}(fg)$ as desired.

□

Note that the above argument extends inductively to any finite product of functions.

3.2 Commutator Formulas

In the following sections, we aim to bound $\mathfrak{D}(\gamma \cdot f^{\mathbf{a}})$ by examining $\mathfrak{D}(\mathfrak{p})$ and $\mathfrak{D}(\mathfrak{q}_{\mathbf{a}})$ (Section 3.4).

Working toward a bound on $\mathfrak{D}(\gamma \cdot f^{\mathbf{a}})$, as described above, we introduce several formulas which allow us to compute the Iwahori Decomposition of $\mathbf{x} \cdot \gamma$ and gain insight into the nature of $\mathfrak{p}$ and the $\mathfrak{q}_\alpha$'s.

If $g \in I$, then the Iwahori Decomposition of g is written $g^- h_g g^+$, where $g^- \in N^{opp}(\mathbb{O}_p) \cap I$, $h_g \in T(\mathbb{O}_p)$ and $g^+ \in N(\mathbb{O}_p)$. We know that g^- is a product of $\chi_{-\alpha}(u_{-\alpha})$, and g^+ is a product of $\chi_{+\alpha}(u_{+\alpha})$. In order to find the Iwahori decomposition of a product $g_1 g_2 = (g_1^- h_{g_1} g_1^+)(g_2^- h_{g_2} g_2^+)$, we will employ a general formulation of Chevalley's Commutator Formula.

The indeterminant $\mathbf{x}$ is in the image of $N(\mathbb{O}_p)$ in the quotient space $N^{opp}(\mathbb{O}_p)\backslash G(\mathbb{O}_p)$. We examine the action of $\gamma \in I$ on $\mathbf{x}$ by right multiplication so that we may describe the action of γ on $f^{\mathbf{a}} \in F_\lambda$ and consequently on μ.

We are interested in the Iwahori Decomposition $\mathbf{x} \cdot \gamma = (\mathbf{x} \cdot \gamma)^- h_{\mathbf{x}\cdot\gamma}(\mathbf{x} \cdot \gamma)^+$. Since $\mathbf{x} \cdot \gamma = h_{\mathbf{x}\cdot\gamma}(\mathbf{x} \cdot \gamma)^+$ in the quotient space $N^{opp}(\mathbb{O}_p)\backslash G(\mathbb{O}_p)$ we have

$$\begin{aligned} f^{\mathbf{a}}(\mathbf{x} \cdot \gamma) &= f^{\mathbf{a}}((\mathbf{x} \cdot \gamma)^- h_{\mathbf{x}\cdot\gamma}(\mathbf{x} \cdot \gamma)^+) \\ &= f^{\mathbf{a}}((h_{\mathbf{x}\cdot\gamma}(\mathbf{x} \cdot \gamma)^+) \\ &= \lambda(h_{\mathbf{x}\cdot\gamma}) f^{\mathbf{a}}((\mathbf{x} \cdot \gamma)^+). \end{aligned}$$

In the next section, we will give a general way of computing the Iwahori Decomposition of a finite product of root group elements, $\chi_\beta(x_\beta)$.

Given the product,

$$\mathbf{x} \cdot \gamma = \left(\prod \chi_\alpha(x_\alpha)\right) \cdot \left[\left(\prod \chi_{-\alpha}(y_{-\alpha})\right) h_\gamma \left(\prod \chi_\alpha(y_\alpha)\right)\right], \tag{3.1}$$

we will then be able to use the computational techniques to show

$$\mathbf{x} \cdot \gamma = \left(\prod_{\alpha \in \Phi^+} \chi_{-\alpha}(\mathfrak{q}_{-\alpha}(\mathbf{x}))\right) h_{\mathbf{x}\cdot\gamma} \left(\prod_{\alpha \in \Phi^+} \chi_\alpha(\mathfrak{q}_\alpha(\mathbf{x}))\right). \tag{3.2}$$

Since we are acting within $X \subset N^{opp}(\mathbb{O}_p)\backslash G(\mathbb{O}_p)$, we have

$$\mathbf{x} \cdot \gamma = h_{\mathbf{x}\cdot\gamma} \prod_{\alpha \in \Phi^+} \chi_\alpha(\mathfrak{q}_\alpha(\mathbf{x})).$$

Recall

$$
\begin{aligned}
(\gamma \cdot f^{\mathbf{a}})(\mathbf{x}) &= f^{\mathbf{a}}\left(\mathbf{x} \cdot \gamma\right) \\
&= f^{\mathbf{a}}\left(h_{\mathbf{x}\cdot\gamma} \prod_{\alpha \in \Phi^+} \chi_\alpha(\mathfrak{q}_\alpha(\mathbf{x}))\right) \\
&= \lambda(h_{\mathbf{x}\cdot\gamma}) f^{\mathbf{a}}\left(\prod_{\alpha \in \Phi^+} \chi_\alpha(\mathfrak{q}_\alpha(\mathbf{x}))\right) \\
&= \mathfrak{p}(\mathbf{x})\mathfrak{q}_{\mathbf{a}}(\mathbf{x}).
\end{aligned}
$$

We will bound each $\mathfrak{D}(\mathfrak{q}_\alpha)$. This will allow us to bound $\mathfrak{D}(\mathfrak{q}_{\mathbf{a}})$. To this end, we will list the formulas needed to compute the decomposition and analyze how each application of the formulas may impact $\mathfrak{q}_\alpha(\mathbf{x})$.

Theorem 3 (Commutator Formula)**.** *[7, 32.5] For α, β nonparallel roots in Φ,*

$$
\chi_\alpha(u_\alpha)\chi_\beta(u_\beta) = \chi_\beta(u_\beta)\chi_\alpha(u_\alpha) \prod_{r\alpha+s\beta \in \Phi} \chi_{r\alpha+s\beta}(e_{r,s}u_\alpha^r u_\beta^s) \tag{3.3}
$$

where $e_{a,b} \in \{\pm 1, \pm 2, \pm 3\}$ with the far right product ordered by some fixed order.

A precise formula for commutators of nonparallel root group elements within Chevalley Groups (where the formula is referred to as "The Chevalley Commutator Formula") can be found in [3].

Recall that we fixed an order on Φ (so that negative roots appear first in order of descending height, followed by positive roots in order of ascending height). For our purposes we will forgo the computation of the coefficients $e_{r,s}$, though an algorithm is given in both [3] and [7]. It suffices to note that ± 1, ± 2, and ± 3 are p-adically integral. The following formulas will also be used in computing $\mathfrak{q}_\alpha$ for our fixed γ.

1. When adjacent elements from the same root group are multiplied, their arguments are simply added:

$$\chi_\alpha(u_\alpha)\chi_\alpha(u'_\alpha) = \chi_\alpha(u_\alpha + u'_\alpha). \tag{3.4}$$

This is due to the definition of one-parameter subgroup.

2. If the torus element $h_\alpha(u_\alpha)$ is interchanged with $\chi_\beta(u_\beta)$ we have

$$\chi_\beta(u_\beta)h_\alpha(u_\alpha) = h_\alpha(u_\alpha)\chi_\beta(u_\beta u_\alpha^{r-s}) \tag{3.5}$$

where r and s are such that $-r\beta+\alpha, \ldots, \alpha, \ldots, s\beta+\alpha$ is the longest root string through α by β.

3. When elements from opposite root groups are interchanged, we may restrict our view to the case of SL_2. Since

$$\begin{pmatrix} 1 & u_\alpha \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ u_{-\alpha} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ \frac{u_{-\alpha}}{1+u_\alpha u_{-\alpha}} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+u_\alpha u_{-\alpha} & 0 \\ 0 & \frac{1}{1+u_\alpha u_{-\alpha}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{u_{-\alpha}}{1+u_\alpha u_{-\alpha}} \\ 0 & 1 \end{pmatrix},$$

whenever $u_\alpha u_{-\alpha} \neq -1$, we have

$$\begin{aligned} &\chi_\alpha(u_\alpha)\chi_{-\alpha}(u_{-\alpha}) \\ &= \chi_{-\alpha}\left(\frac{u_{-\alpha}}{1+u_\alpha u_{-\alpha}}\right) h_\alpha\left(\frac{1}{1+u_\alpha u_{-\alpha}}\right) \chi_\alpha\left(\frac{u_\alpha}{1+u_\alpha u_{-\alpha}}\right) \end{aligned} \tag{3.6}$$

where $h_\alpha\left(\frac{1}{1+u_\alpha u_{-\alpha}}\right) \in T(\mathbb{C}_p)$. When $p|u_{-\alpha}$, this is equal to

$$\chi_{-\alpha}\left(\sum_{i=0}^{\infty}(-1)^i u_{-\alpha}^{i+1}u_\alpha^i\right) h_\alpha\left(\sum_{i=0}^{\infty}(-u_{-\alpha}u_\alpha)^i\right) \chi_\alpha\left(\sum_{i=0}^{\infty}(-1)^i u_{-\alpha}^i u_\alpha^{i+1}\right).$$

Our goal is to deduce Equation (3.2) from Equation (3.1). In particular, we would like to be able to bound the discrepancy of $\mathfrak{p}$ and all $\mathfrak{q}_\alpha$.

3.3 A Game With Roots

We introduce a simplified way of working with the above formulas. This will allow us to demonstrate how the Iwahori Decomposition may be computed.

Definition 5 (Root Sequence)**.** *Define the set* $\mathfrak{R}$ *of root sequences by*

$$\mathfrak{R} = \left\{(\alpha_1, \alpha_2, \ldots, \alpha_n) | n \in \mathbb{Z}^+,\ \alpha_i \in \Phi\right\}.$$

We define an equivalence relation $\sim$ on $\mathfrak{R}$ to be the smallest equivalence relation satisfying:

1. If $\alpha_1 \in \Phi^-$,

$$(\alpha_2, \alpha_3, \ldots, \alpha_n) \sim (\alpha_1, \alpha_2, \ldots, \alpha_n). \tag{3.7}$$

2. For $k < n$,

$$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \alpha_k, \alpha_{k+2}, \ldots, \alpha_n) \sim (\alpha_1, \ldots, \alpha_n) \tag{3.8}$$

if $\alpha_k + \alpha_{k+1} \notin \Phi$.

3. For $k < n$ and $\alpha_{k+1} \in \Phi^-$,

$$(\alpha_1, \dots \alpha_{k-1}, \alpha_{k+1}, \alpha_k, \beta_1, \beta_2, \dots \beta_m, \alpha_{k+2}, \dots, \alpha_n) \sim (\alpha_1, \dots, \alpha_n) \tag{3.9}$$

where the β_i are all the distinct roots $r\alpha_k + s\alpha_{k+1} \in \Phi$ for $r, s \in \mathbb{Z}^+$ ordered via our fixed order extending $\sim$.

4. For any roots α_i,

$$(\alpha_1, \dots, \alpha_{k-1}, \alpha_k, \alpha_k, \alpha_{k+1}, \dots, \alpha_n) \sim (\alpha_1, \dots, \alpha_n). \tag{3.10}$$

Theorem 4. *Given any root sequence $(\beta_i)_{i=1}^m$, there is a sequence $(\alpha_i)_{i=1}^n$ such that $\alpha_i \in \Phi^+$ for all i and*

$$(\alpha_i)_{i=1}^n \sim (\beta_i)_{i=1}^m.$$

Proof. Inductively, it suffices to prove the following. For $k \in \mathbb{Z}^+$, let $(\beta_1, \beta_2, \dots, \beta_n)$ be a sequence where, for each i, either $\beta_i \in \Phi^+$ or $-\beta_i$ has height less than or equal to k. Then $(\beta_1, \beta_2, \dots, \beta_n)$ is equivalent to a sequence $(\alpha_1, \alpha_2, \dots, \alpha_n)$ where the height of $-\alpha_i$ is strictly less than k for each negative root α_i.

We first claim that for any positive roots β_i, there exist roots $\eta_i \succ -\beta_m$ for all i such that

$$(\eta_1, \eta_2, \dots, \eta_n) \sim (\beta_1, \beta_2, \dots, \beta_{m-1}, -\beta_m).$$

To see this, fix β_i's in Φ^+ and consider cases on the type of the group G.

Without loss of generality, we assume G is a simple group. If $G = \bigoplus G_i$, a direct sum of simple groups, then the roots of G_i are perpendicular to the roots of G_j when $i \neq j$. It would therefore suffice to prove the theorem for each G_i.

1. G is of type A_n and D_n

   Within the simply laced groups there are no root strings of length greater than 1. We have

$$(\beta_1, \dots, \beta_{m-1}, -\beta_m) \sim \begin{cases} (\beta_1, \dots, -\beta_m, \beta_{m-1}) & : \beta_{m-1} - \beta_m \notin \Phi \\ (\beta_1, \dots, -\beta_m, \beta_{m-1}, \beta_{m-1} - \beta_m) & : \beta_{m-1} - \beta_m \in \Phi \end{cases}.$$

   Since $\beta_{m-1} \in \Phi^+$, we have $\beta_{m-1} - \beta_m \succ -\beta_m$. We may employ Relations (2) and (3) to move the term $-\beta_m$ left $m-1$ times. The resulting equivalent sequence will have the original terms: β_i for $1 \leq i \leq m$ as well as $\beta_i - \beta_m$ whenever this difference is in Φ. Since $\beta_i \in \Phi^+$, any root $\beta_i - \beta_m$ will be greater than $-\beta_m$.

   After $-\beta_m$ has been moved left $m-1$ times as described above, we have

$$(-\beta_m, \eta_1, \eta_2, \dots, \eta_n) \sim (\beta_1, \beta_2, \dots, \beta_{m-1}, -\beta_m)$$

   with all $\eta_i \succ -\beta_m$. By Relation (1),

$$(\eta_1, \eta_2, \dots, \eta_n) \sim (-\beta_m, \eta_1, \eta_2, \dots, \eta_n).$$

2. G is of type G_2

   We may name the 12 roots of G as follows. Let α, β be the fundamental roots, with α being the short fundamental root. Then the negative roots are $-\alpha, -\beta, -\alpha - \beta, -2\alpha - \beta, -3\alpha - \beta, -3\alpha - 2\beta$.

   It is tempting to employ the algorithm of moving the leftmost negative root left repeatedly (using Relations (2) and (3) as in the previous case), but there is no guarantee that such a process will terminate. For instance,

$$\begin{aligned}(2\alpha+\beta,\beta,-\alpha-\beta) &\sim (2\alpha+\beta,-\alpha-\beta,\beta,-\alpha,-2\alpha-\beta)\\ &\sim (-\alpha-\beta,2\alpha+\beta,-\beta,\alpha,\beta,-\alpha,-2\alpha-\beta)\\ &\sim (-\alpha-\beta,-\beta,2\alpha+\beta,\alpha,\beta,-\alpha,-2\alpha-\beta)\\ &\sim (2\alpha+\beta,\alpha,\beta,-\alpha,-2\alpha-\beta)\\ &\sim (2\alpha+\beta,-\alpha,\alpha,\beta,-2\alpha-\beta)\\ &\sim (-\alpha,2\alpha+\beta,\alpha+\beta,\beta,\alpha,\beta,-2\alpha-\beta)\\ &\sim (2\alpha+\beta,\alpha+\beta,\beta,\alpha,\beta,-2\alpha-\beta)\\ &\sim (2\alpha+\beta,\alpha+\beta,\beta,\alpha,-2\alpha-\beta,\beta)\\ &\sim (2\alpha+\beta,\alpha+\beta,\beta,-2\alpha-\beta,\alpha,-\alpha-\beta,-\beta,\beta)\end{aligned}$$

After several iterations of the algorithm, the original terms still appear in their original order:

$$(2\alpha+\beta,\beta,-\alpha-\beta) \sim (\underline{2\alpha+\beta},\alpha+\beta,\underline{\beta},-2\alpha-\beta,\alpha,\underline{-\alpha-\beta},-\beta,\beta)$$

If we were to continue employing the algorithm, the $-\alpha-\beta$ term would eventually move past α again, followed by $2\alpha+\beta$. A new $-2\alpha-\beta$ term would be formed, followed by a new $-\alpha-\beta$. Clearly, this process would continue indefinitely.

Consider the root sequence

$$(\alpha, \beta, 3\alpha+\beta, \alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta)$$

which contains all the positive roots of G_2. For a negative root $-\eta$, define sequences S_n by

$$\begin{aligned}
S_1 &= (\alpha, \beta, 3\alpha+\beta, \alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta, -\eta) \\
S_2 &= (\alpha, \beta, 3\alpha+\beta, \alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta, \alpha, \beta, \ldots, 2\alpha+\beta, -\eta) \\
S_3 &= (\alpha, \beta, \ldots, 2\alpha+\beta, \alpha, \beta, \ldots, 2\alpha+\beta, \alpha, \beta, \ldots, 2\alpha+\beta, -\eta) \\
&\vdots
\end{aligned}$$

Any sequence of positive roots with length n is contained as a subsequence of S_n. In order to prove that $-\eta$ may move completely left through any sequence of n positive roots, it suffices to prove that S_n is equivalent to a sequence of positive roots. This can be reduced to proving S_1 is equivalent to a sequence of positive roots for any η.

If $\eta = \alpha$ or β, then $-\eta$ moves left without generating any new negative terms. Therefore, all new terms must be greater than $-\alpha$ and $-\beta$.

If $\eta = 3\alpha+2\beta$, the greatest root in Φ, then it is vacuously true that all new roots must be strictly greater than $-\eta$.

If $\eta = \alpha+\beta$, then

$$(\alpha, \beta, 3\alpha+\beta, \alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta, -\alpha-\beta)$$

$$\sim (\alpha, \beta, \alpha+\beta, 3\alpha+2\beta, 3\alpha+\beta, 2\alpha+\beta, -\alpha-\beta)$$

$$\sim (\alpha, \beta, \alpha+\beta, 3\alpha+2\beta, 3\alpha+\beta, -\alpha-\beta, 2\alpha+\beta, -\beta, \alpha, 2\alpha+\beta, 3\alpha+\beta)$$

$$\sim (\alpha, \beta, \alpha+\beta, 3\alpha+2\beta, -\alpha-\beta, 3\alpha+\beta, 2\alpha+\beta, *)$$

$$\sim (\alpha, \beta, \alpha+\beta, -\alpha-\beta, 3\alpha+2\beta, -\beta, \alpha, 2\alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 2\alpha+\beta, *)$$

$$\sim (\alpha, \beta, -\alpha-\beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, \beta, -\alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\alpha, -2\alpha-\beta, -3\alpha-\beta, \beta, -3\alpha-2\beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, -3\alpha-\beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, \alpha, -\beta, -\alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, -\beta, -\alpha, \alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-2\alpha-\beta, \alpha, -\beta, -\alpha-\beta, -3\alpha-2\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-2\alpha-\beta, -\beta, \alpha, -\alpha-\beta, -3\alpha-2\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -3\alpha-2\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, \alpha, -\beta, -3\alpha-2\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, -\beta, -3\alpha-2\beta, \alpha, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-3\alpha-\beta, \alpha, -\beta, -\alpha-\beta, -2\alpha-\beta, -3\alpha-2\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-3\alpha-\beta, -\beta, \alpha, -\alpha-\beta, -2\alpha-\beta, -3\alpha-2\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -2\alpha-\beta, -3\alpha-2\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, \alpha, -\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, -\beta, \alpha, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-2\alpha-\beta, \alpha, -\beta, -\alpha-\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-2\alpha-\beta, -\beta, \alpha, -\alpha-\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, \alpha, -\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, -\beta, \alpha, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-2\alpha-\beta, \alpha, -\beta, -\alpha-\beta, -3\alpha-2\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-2\alpha-\beta, -\beta, \alpha, -\alpha-\beta, -3\alpha-2\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -3\alpha-2\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, \alpha, -\beta, -3\alpha-2\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-\alpha-\beta, -3\alpha-2\beta, \alpha, -\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -\beta, \beta, \alpha+\beta, 3\alpha+2\beta, *)$$

as desired. In the above calculation, $*$ represents a sequence of roots all greater than $-\alpha-\beta$.

If $\eta = 2\alpha+\beta$, then we have

$$(\alpha, \beta, 3\alpha+\beta, \alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta, -2\alpha-\beta)$$

$$\sim (\alpha, \beta, \alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta, -2\alpha-\beta)$$

$$\sim (\alpha, \beta, \alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta, -2\alpha-\beta, 2\alpha+\beta)$$

$$\sim (\alpha, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, 3\alpha+2\beta, -3\alpha-\beta, -\alpha, \beta, \alpha+\beta, 2\alpha+\beta)$$

$$\sim (\alpha, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, -3\alpha-\beta, 3\alpha+2\beta, \beta, -\alpha, \beta, \alpha+\beta, 2\alpha+\beta)$$

$$\sim (\alpha, -3\alpha-\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (-3\alpha-\beta, \alpha, -3\alpha-2\beta, -2\alpha-\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (-3\alpha-\beta, -3\alpha-2\beta, \alpha, -2\alpha-\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (\alpha, -2\alpha-\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (-2\alpha-\beta, \alpha, -3\alpha-2\beta, -\alpha-\beta, -\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (-2\alpha-\beta, -3\alpha-2\beta, \alpha, -\alpha-\beta, -\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (-\alpha-\beta, \alpha, -\beta, -\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (-\alpha-\beta, -\beta, -\beta, \alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (-\alpha-\beta, \alpha, -\beta, -\beta, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (-\alpha-\beta, -\beta, -\beta, \alpha, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (\alpha, \beta, \alpha+\beta, 3\alpha+\beta, -2\alpha-\beta, *)$$

$$\sim (\alpha, \beta, \alpha+\beta, -2\alpha-\beta, 3\alpha+\beta, -3\alpha-2\beta, -\alpha-\beta, -\beta, \alpha, *)$$

$$\sim (\alpha, \beta, -2\alpha-\beta, \alpha+\beta, -3\alpha-\beta, -\alpha, \beta, 3\alpha+\beta, -3\alpha-2\beta, -\alpha-\beta, -\beta, \alpha, *)$$

$$\sim (\alpha, -2\alpha-\beta, \beta, \alpha+\beta, -3\alpha-\beta, -\alpha, \beta, 3\alpha+\beta, -3\alpha-2\beta, -\alpha-\beta, -\beta, \alpha, *)$$

$$\sim (-2\alpha-\beta, \alpha, -3\alpha-2\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -\alpha, \beta, 3\alpha+\beta, -3\alpha-2\beta, *)$$

$$\sim (-2\alpha-\beta, -3\alpha-2\beta, \alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -\alpha, \beta, 3\alpha+\beta, -3\alpha-2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -\alpha, \beta, 3\alpha+\beta, -3\alpha-2\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -\alpha, \beta, -3\alpha-2\beta, 3\alpha+\beta, -\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -\alpha, -3\alpha-2\beta, \beta, -3\alpha-\beta, 3\alpha+\beta, -\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -\alpha, -3\alpha-2\beta, -3\alpha-\beta, \beta, 3\alpha+\beta, -\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -3\alpha-\beta, -\alpha, -3\alpha-2\beta, \beta, 3\alpha+\beta, -\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -\alpha, -3\alpha-2\beta, \beta, 3\alpha+\beta, -\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -3\alpha-\beta, -3\alpha-2\beta, -\alpha, \beta, 3\alpha+\beta, -\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -3\alpha-\beta, \beta, \alpha+\beta, -3\alpha-2\beta, -\alpha, \beta, 3\alpha+\beta, -\beta, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -3\alpha-\beta, \beta, -3\alpha-2\beta, \alpha+\beta, -3\alpha-\beta, -2\alpha-\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -3\alpha-\beta, -3\alpha-\beta, \beta, -3\alpha-2\beta, \alpha+\beta, -2\alpha-\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -3\alpha-\beta, \beta, -3\alpha-2\beta, \alpha+\beta, -2\alpha-\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -3\alpha-\beta, \beta, -3\alpha-2\beta, -2\alpha-\beta, \alpha+\beta, -3\alpha-\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -3\alpha-\beta, -3\alpha-\beta, \beta, -3\alpha-2\beta, -2\alpha-\beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -3\alpha-\beta, \beta, -3\alpha-2\beta, -2\alpha-\beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -2\alpha-\beta, -3\alpha-\beta, \beta, -3\alpha-2\beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, -3\alpha-\beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (-\alpha-\beta, \alpha, -\beta, -\beta, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (-\alpha-\beta, -\beta, -\beta, \alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -2\alpha-\beta, -3\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -3\alpha-\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (-3\alpha-\beta, \alpha, -3\alpha-\beta, -2\alpha-\beta, -\alpha-\beta, -\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (-3\alpha-\beta, -3\alpha-2\beta, \alpha, -2\alpha-\beta, -\alpha-\beta, -\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -2\alpha-\beta, -\alpha-\beta, -\beta, -2\alpha-\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -2\alpha-\beta, -\alpha-\beta, -2\alpha-\beta, -\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -2\alpha-\beta, -2\alpha-\beta, -\alpha-\beta, -3\alpha-2\beta, -\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -2\alpha-\beta, -\alpha-\beta, -3\alpha-2\beta, -\beta, -3\alpha-2\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -2\alpha-\beta, -\alpha-\beta, -3\alpha-2\beta, -3\alpha-2\beta, -\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -2\alpha-\beta, -\alpha-\beta, -3\alpha-2\beta, -\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (-2\alpha-\beta, \alpha, -3\alpha-2\beta, -\alpha-\beta, -\beta, -\alpha-\beta, -3\alpha-2\beta, -\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (-2\alpha-\beta, \alpha, -3\alpha-2\beta, -3\alpha-2\beta, -\alpha-\beta, -\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (-2\alpha-\beta, -3\alpha-2\beta, -3\alpha-2\beta, \alpha, -\alpha-\beta, -\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -\alpha, *)$$

$$\sim (\alpha, -\alpha-\beta, -\beta, -\alpha-\beta, -\beta, \beta, \alpha+\beta, -\alpha, *),$$

with $*$ representing a sequence of roots all greater than $-2\alpha-\beta$.

Finally, if $\eta = 3\alpha+\beta$, then

$$(\alpha, \beta, 3\alpha+\beta, \alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta, -3\alpha-\beta)$$

$$\sim (\alpha, \alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta, \beta, 3\alpha+\beta, -3\alpha-\beta)$$

$$\sim (\alpha, \alpha+\beta, 3\alpha+2\beta, 2\alpha+\beta, -3\alpha-\beta, \beta, 3\alpha+\beta)$$

$$\sim (\alpha, \alpha+\beta, 3\alpha+2\beta, -3\alpha-\beta, 2\alpha+\beta, -\alpha, \beta, \alpha+\beta, 3\alpha+2\beta, \beta, 3\alpha+\beta)$$

$$\sim (\alpha, \alpha+\beta, -3\alpha-\beta, 3\alpha+2\beta, \beta, 2\alpha+\beta, -\alpha, \beta, \alpha+\beta, 3\alpha+2\beta, \beta, 3\alpha+\beta)$$

$$\sim (\alpha, -3\alpha-\beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-3\alpha-\beta, \alpha, -3\alpha-2\beta, -2\alpha-\beta, -\alpha-\beta, -\beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (-3\alpha-\beta, -3\alpha-2\beta, \alpha, -2\alpha-\beta, -\alpha-\beta, -\beta, \alpha+\beta, 3\alpha+2\beta, *)$$

$$\sim (\alpha, -2\alpha-\beta, -\alpha-\beta, -\beta, \alpha+\beta, 3\alpha+2\beta, *),$$

with $*$ representing a sequence of roots all greater than $-3\alpha-\beta$.

Therefore for any positive roots β_i and negative root $-\eta$ of G_2, there exist $\alpha_i \succ -\eta$ such that

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \sim (\beta_1, \beta_2, \ldots, \beta_{m-1}, -\eta).$$

3. If G is of type B_m, C_m, E_m or F_4

Let η be a positive root. For any positive root $\beta \in \Phi$, interchanging β and $-\eta$ may result in new terms depending on whether $\beta-\eta$, $2\beta-\eta$, or $\beta-2\eta$ are roots.

$$(\ldots, \beta, -\eta, \ldots) \sim \begin{cases} (\ldots, -\eta, \beta, \ldots) & : r\beta - s\eta \notin \Phi \\ (\ldots, -\eta, \beta, \beta-\eta \ldots) & : 2\beta-\eta, \beta-2\eta \notin \Phi \\ (\ldots, -\eta, \beta, \beta-\eta, 2\beta-\eta \ldots) & : \beta-2\eta \notin \Phi \\ (\ldots, -\eta, \beta, \beta-\eta, \beta-2\eta \ldots) & : 2\beta-\eta \notin \Phi \end{cases}.$$

Only the last case may create a new term less than η. Further, $\beta-2\eta \prec \eta$ only when $\beta \prec \eta$. In this case, the plane through η and β intersects the root system in a copy of B_2. We may treat this portion of the computation as though β and η are in B_2.

Any root sequence of B_2 could be viewed as a sequence within G_2. By the arguments of the previous case, we are done.

We have shown that for any roots β_i, there are roots $\eta_i \succ -\beta_m$ for all i such that

$$(\eta_1, \eta_2, \ldots, \eta_n) \sim (\beta_1, \beta_2, \ldots, \beta_{m-1}, -\beta_m).$$

Given any root sequence, we may move the negative roots fully left in the order that they appear in the sequence. At the end of this process, we will have produced an equivalent sequence of positive roots. □

Consequently, the four formulas given in Section 3.2 suffice to compute the Iwahori Decomposition. In effect, we may show via the formulas that Equation (3.1) may be turned into Equation (3.2).

Theorem 5. *It is possible, within a finite number of invocations of the four formulas from the previous section, to compute the Iwahori Decomposition for any product of root group elements and torus elements.*

Proof. Since torus elements may be interchanged with any root group element without generating new factors, we are always able to "collect" them at the end of the process. It therefore suffices to show that the commutator formulas, can be used to move all the one-parameter subgroups associated to negative roots fully left. This is a direct result of Theorem 4. □

3.4 The Discrepancy Lemmas

We now will work toward applying Theorem 5 to describe the arguments of Equation (3.2),

$$\mathbf{x} \cdot \gamma = \left(\prod_{\alpha \in \Phi^+} \chi_{-\alpha}(\mathfrak{q}_{-\alpha}(\mathbf{x})) \right) h_{\mathbf{x}\cdot\gamma} \left(\prod_{\alpha \in \Phi^+} \chi_{\alpha}(\mathfrak{q}_{\alpha}(\mathbf{x})) \right)$$

in terms of the arguments of Equation (3.1),

$$\mathbf{x} \cdot \gamma = \left(\prod \chi_\alpha(x_\alpha)\right) \cdot \left[\left(\prod \chi_{-\alpha}(y_{-\alpha})\right) h_\gamma \left(\prod \chi_\alpha(y_\alpha)\right)\right].$$

Recall that $\mathbf{x} = \prod_{\alpha \in \Phi^+} \chi_\alpha(x_\alpha)$ and $\gamma = h_\gamma \prod_{\alpha \in \Phi} \chi_\alpha(y_\alpha)$ where $h_\gamma \in T(\mathbb{O}_p)$ and $\alpha \in \Phi^-$ implies $p | y_\alpha$.

Lemma 6. *Let $\alpha \in \Phi^+$. Then $\mathfrak{q}_\alpha(\mathbf{x})$ is a sum of constants and products of the form*

$$c \prod_{\beta \in Y} y_{-\beta} \prod_{\beta \in Z} x_\beta \text{ with } \sum_{\beta \in Z} \beta - \sum_{\beta \in Y} \beta = \alpha$$

where Y and Z are multisets of positive roots. The power series

$$\mathfrak{p}(\mathbf{x}) = \prod_{\alpha \in \Pi} (\mathfrak{p}_\alpha(\mathbf{x}))^{k_\alpha}$$

is a sum of constants and products of the form

$$c \prod_{\beta \in Y} y_{-\beta} \prod_{\beta \in Z} x_\beta \text{ with } \sum_{\beta \in Z} \beta - \sum_{\beta \in Y} \beta = 0$$

where Y and Z are multisets of positive roots.

Proof. Consider Equation (3.1) and note that the original one-parameter subgroup factors $\chi_\alpha(x_\alpha)$ have nonconstant argument x_α, trivially satisfying the claim. If we factor $h_\gamma = \prod_\alpha h_\alpha(t_\alpha)$, the argument t_α of the original torus factors $h_\alpha(t_\alpha)$ is constant.

Our goal is to show that in Equation (3.2), all arguments have the desired form. We proceed inductively, showing that any argument of h_α or $\chi_{\pm\alpha}$ after the

k^{th} step of the Iwahori Decomposition computation is a sum of terms of the form

$$c \prod_{\beta \in Y} y_{-\beta} \prod_{\beta \in Z} x_\beta$$

with $\sum_{\beta \in Z} \beta - \sum_{\beta \in Y} \beta = 0$ for the argument of h_α and $\sum_{\beta \in Z} \beta - \sum_{\beta \in Y} \beta = \pm\alpha$ for the argument of the one-parameter subgroup $\chi_{\pm\alpha}$.

We have addressed the base case, when $k = 0$, above.

Now assume the product after k steps has the desired form. Let $u(\mathbf{x})$ be a power series such that $h_\alpha(u(\mathbf{x}))$ is a factor after the k^{th} invocation of the formulas. If $h_\alpha(u(\mathbf{x}))$ was a factor after the $(k-1)^{th}$ step, then it satisfies the induction hypothesis and we are done. Assume that $h_\alpha(u(\mathbf{x}))$ was created during the k^{th} step. Note that there are no steps which *alter* the argument of an existing torus factor.

It must be that $h_\alpha(u(\mathbf{x}))$ resulted from an application of Equation (3.6), interchanging a product of the form

$$\chi_\alpha(u'_\alpha(\mathbf{x}))\chi_{-\alpha}(u'_{-\alpha}(\mathbf{x})),$$

which was present in the $(k-1)^{th}$ step. Since every term of $u'_{-\alpha}$ has a factor of the form $y_{-\beta}$, p divides every term of $u'_{-\alpha}$ and so we have

$$u(\mathbf{x}) = \sum_{l=0}^{\infty} (-u'_{-\alpha}(\mathbf{x})u'_\alpha(\mathbf{x}))^l.$$

By the induction hypothesis, $u'_{-\alpha}(\mathbf{x})$ and $u'_\alpha(\mathbf{x})$ can be written in the prescribed form. Let $c_i^\pm$ be constants and $Y_i^\pm$, $Z_i^\pm$ be multisets such that

$$u'_\alpha(\mathbf{x}) = \sum_i c_i^+ \prod_{\beta\in Y_i^+} y_{-\beta} \prod_{\beta\in Z_i^+} x_\beta \text{ with } \sum_{\beta\in Z_i^+} \beta - \sum_{\beta\in Y_i^+} \beta = \alpha$$

and

$$u'_{-\alpha}(\mathbf{x}) = \sum_i c_i^- \prod_{\beta\in Y_i^-} y_{-\beta} \prod_{\beta\in Z_i^-} x_\beta \text{ with } \sum_{\beta\in Z_i^-} \beta - \sum_{\beta\in Y_i^-} \beta = -\alpha.$$

Then the product $u'_{-\alpha}(\mathbf{x})u'_\alpha(\mathbf{x})$ can be written as

$$\begin{aligned} u'_{-\alpha}(\mathbf{x})u'_\alpha(\mathbf{x}) &= \left(\sum_i c_i^+ \prod_{\beta\in Y_i^+} y_{-\beta} \prod_{\beta\in Z_i^+} x_\beta\right)\left(\sum_j c_j^- \prod_{\beta\in Y_j^-} y_{-\beta} \prod_{\beta\in Z_j^-} x_\beta\right) \\ &= \sum_{i,j}\left(c_i^+ c_j^- \prod_{\beta\in Y_i^+\bigsqcup Y_j^-} y_{-\beta} \prod_{\beta\in Z_i^+\bigsqcup Z_j^-} x_\beta\right) \end{aligned}$$

with $\bigsqcup$ representing the disjoint union of multisets. Further, we have

$$\begin{aligned} &\left(\sum_{\beta\in Z_i^+\bigsqcup Z_j^-} \beta\right) - \left(\sum_{\beta\in Y_i^+\bigsqcup Y_j^-} \beta\right) \\ &\qquad = \left(\sum_{\beta\in Z_i^+} \beta - \sum_{\beta\in Y_i^+} \beta\right) + \left(\sum_{\beta\in Z_i^-} \beta - \sum_{\beta\in Y_i^-} \beta\right) \\ &\qquad = \alpha - \alpha \\ &\qquad = 0 \end{aligned}$$

as claimed.

Similarly, any $(-u'_{-\alpha}(\mathbf{x})u'_\alpha(\mathbf{x}))^l$ can be written in the desired form. So inductively, we have that, for any factor $h_\alpha(u_\alpha)$ in 3.2, u_α will have this form. Therefore

$\mathfrak{p}(\mathbf{x}) = \lambda\left(\prod_\alpha h_\alpha(u(\mathbf{x}))\right)$, will have the prescribed form as well.

Now let $\chi_\alpha(u(\mathbf{x}))$ be a factor after the k^{th} step. Again, we may assume this factor was created or altered during the k^{th} step.

If $\chi_\alpha(u(\mathbf{x}))$ was created via an application of Equation (3.4), then the result is trivial.

Now assume $\chi_\alpha(u(\mathbf{x}))$ was altered by applying Equation (3.6) or (3.5), then

$$u(\mathbf{x}) = u_\alpha(\mathbf{x})t(\mathbf{x})^r$$

where $u_\alpha(\mathbf{x})$ was the argument of χ_α and $t(\mathbf{x})$ was the argument of a torus factor after the $(k-1)^{th}$ step.

It suffices to show that $u_\alpha(\mathbf{x})t(\mathbf{x})$ has the desired form. By the induction assumption,

$$u_\alpha(\mathbf{x}) = \sum_i c_i^\alpha \prod_{\beta \in Y_i^\alpha} y_{-\beta} \prod_{\beta \in Z_i^\alpha} x_\beta \text{ with } \sum_{\beta \in Z_i^\alpha} \beta - \sum_{\beta \in Y_i^\alpha} \beta = \alpha$$

and

$$t(\mathbf{x}) = \sum_i c_i^t \prod_{\beta \in Y_i^t} y_{-\beta} \prod_{\beta \in Z_i^t} x_\beta \text{ with } \sum_{\beta \in Z_i^t} \beta - \sum_{\beta \in Y_i^t} \beta = 0.$$

Similar to the above case, we have

$$u_\alpha(\mathbf{x})t(\mathbf{x}) = \sum_{i,j} \left(c_i^\alpha c_j^t \prod_{\beta \in Y_i^\alpha \bigsqcup Y_j^t} y_\beta \prod_{\beta \in Z_i^\alpha \bigsqcup Z_j^t} x_\beta \right)$$

with

$$\left(\sum_{\beta\in Z_i^\alpha}\beta-\sum_{\beta\in Y_i^\alpha}\beta\right)+\left(\sum_{\beta\in Z_i^t}\beta-\sum_{\beta\in Y_i^t}\beta\right)=\alpha+0$$
$$=\alpha.$$

So $u(\mathbf{x})$ satisfies the claim.

If $\chi_\alpha(u(\mathbf{x}))$ was created via equation (3.3), then $\alpha=r\beta-s\delta$ and

$$\chi_\beta(u_\beta(\mathbf{x}))\chi_{-\delta}(u_{-\delta}(\mathbf{x}))$$

was interchanged during the k^{th} step. The arguments $u_\beta(\mathbf{x})$ and $u_{-\delta}(\mathbf{x})$ satisfy the induction hypothesis and so

$$\begin{aligned}u(\mathbf{x})&=c(u_\beta(\mathbf{x}))^r(u_{-\delta}(\mathbf{x}))^s\\&=c\left(\sum_i c_i^\beta\prod_{\eta\in Y_i^\beta}y_{-\eta}\prod_{\eta\in Z_i^\beta}x_\eta\right)^r\left(\sum_j c_j^\delta\prod_{\eta\in Y_j^\delta}y_{-\eta}\prod_{\eta\in Z_j^\delta}x_\eta\right)^s\end{aligned}$$

where for all i,

$$\sum_{\eta\in Z_i^\beta}\eta-\sum_{\eta\in Y_i^\beta}\eta=\beta\text{ and }\sum_{\eta\in Z_i^\delta}\eta-\sum_{\eta\in Y_i^\delta}\eta=\delta.$$

Therefore $u(\mathbf{x})$ is a sum of terms of the form

$$c\left(c_{i_1}^{\beta}\prod_{\eta\in Y_{i_1}^{\beta}} y_{-\eta}\prod_{\eta\in Z_{i_1}^{\beta}} x_\eta\right)\cdots\left(c_{i_r}^{\beta}\prod_{\eta\in Y_{i_r}^{\beta}} y_{-\eta}\prod_{\eta\in Z_{i_r}^{\beta}} x_\eta\right)$$
$$\cdot\left(c_{j_1}^{\delta}\prod_{\eta\in Y_{j_1}^{\delta}} y_{-\eta}\prod_{\eta\in Z_{j_1}^{\delta}} x_\eta\right)\cdots\left(c_{j_s}^{\delta}\prod_{\eta\in Y_{j_s}^{\delta}} y_{-\eta}\prod_{\eta\in Z_{j_s}^{\delta}} x_\eta\right)$$
$$= c\left(\prod_{l=1}^{r} c_{i_l}^{\beta}\prod_{l=1}^{s} c_{j_l}^{\delta}\right)\prod_{\eta\in Y} y_{-\eta}\prod_{\eta\in Z} x_\eta,$$

where

$$Y=\bigsqcup_{l=1}^{r} Y_{i_l}^{\beta}\bigsqcup_{l=1}^{s} Y_{j_l}^{\delta} \text{ and } Z=\bigsqcup_{l=1}^{r} Z_{i_l}^{\beta}\bigsqcup_{l=1}^{s} Z_{j_l}^{\delta}$$

and

$$\begin{aligned}
&\left(\sum_{\eta\in Z}\eta\right)-\left(\sum_{\eta\in Y}\eta\right)\\
&\qquad=\sum_{l=1}^{r}\left(\sum_{\eta\in Z_{i_l}^{\beta}}\eta\right)+\sum_{l=1}^{s}\left(\sum_{\eta\in Z_{j_l}^{\delta}}\eta\right)-\sum_{l=1}^{r}\left(\sum_{\eta\in Y_{i_l}^{\beta}}\eta\right)-\sum_{l=1}^{s}\left(\sum_{\eta\in Y_{j_l}^{\delta}}\eta\right)\\
&\qquad=\sum_{l=1}^{r}\left(\sum_{\eta\in Z_{i_l}^{\beta}}\eta-\sum_{\eta\in Y_{i_l}^{\beta}}\eta\right)+\sum_{l=1}^{s}\left(\sum_{\eta\in Z_{j_l}^{\delta}}\eta-\sum_{\eta\in Y_{j_l}^{\delta}}\eta\right)\\
&\qquad=r\beta-s\delta\\
&\qquad=\alpha.
\end{aligned}$$

Thus $u(\mathbf{x})$ can be written in the desired form. □

We now employ the above and make note of the following corollary to the proof of Theorem 4.

Lemma 7. *At the end of the computation of the Iwahori Decomposition, all terms of one-parameter subgroup arguments with undefined discrepancy have degree 1.*

Proof. For the sake of contradiction, suppose the computation resulted in a factor $\chi_\alpha(u_\alpha)$ where $c_{\mathbf{b}} f^{\mathbf{b}}$ is a term of u_α with $d_{\mathbf{b}} \geq 2$ and $\mathfrak{D}(c_{\mathbf{b}} f^{\mathbf{b}})$ undefined. Then $\mathrm{ord}_p(c_{\mathbf{b}}) = 0$.

We claim that $\alpha \in \Phi^+$. By Lemma 6, if $\alpha \in \Phi^-$, there is some $y_{-\beta}$ dividing $c_{\mathbf{b}}$. Since p divides each $y_{-\beta}$, $\mathrm{ord}_p(c_{\mathbf{b}}) \geq 1$. Therefore $\alpha \in \Phi^+$, and no $y_{-\beta}$ may divide $c_{\mathbf{b}}$. Since no such term exists in Equation (3.1), $\chi_\alpha(u_\alpha)$ must have been created by applying Formula (3.3) to a pair of one-parameter subgroups associated to positive roots.

Recall that the Iwahori Decomposition was computed using an algorithm derived from Theorem 4. Since the "rules of the game" do not allow for noncommuting positive root group elements to be interchanged, this is impossible. □

We now employ discrepancy for a pair of lemmas. These lemmas will be used to show the Γ-stability of the filtration.

Lemma 8 (First Discrepancy Lemma)**.** *Let $\mathfrak{p}$ be as defined in Section 2.5. Let t be the maximum root height in Φ^+. Then $\mathfrak{D}(\mathfrak{p}) \leq t$.*

Proof. We have shown that $\mathfrak{p}(\mathbf{x})$ can be written as a sum of terms of the form

$$c \prod_{\beta \in Y} y_{-\beta} \prod_{\beta \in Z} x_\beta \text{ with } \sum_{\beta \in Z} \beta - \sum_{\beta \in Y} \beta = 0.$$

Note that $Y = \emptyset$ implies $Z = \emptyset$. Therefore $\mathfrak{D}(\mathfrak{p})$ is defined.

We know that each $y_{-\beta}$ is a multiple of p and so

$$\begin{aligned}\mathfrak{D}(\mathfrak{p}) &\leq \mathfrak{D}\left(c\prod_{\beta\in Y} y_{-\beta}\prod_{\beta\in Z} x_\beta\right)\\ &= \frac{\deg\left(c\prod_{\beta\in Y} y_{-\beta}\prod_{\beta\in Z} x_\beta\right)}{\operatorname{ord}_p\left(c\prod_{\beta\in Y} y_{-\beta}\right)}\\ &\leq \frac{|Z|}{|Y|}.\end{aligned}$$

For every $\beta\in\Phi$, let $\eta_1^\beta, \eta_2^\beta, \ldots \eta_{n_\beta}^\beta$ be the (not necessarily distinct) fundamental roots such that $\beta = \sum_i \eta_i^\beta$. We form the multiset

$$Z' = \left\{\eta_i^\beta \,\middle|\, \sum_i \eta_i^\beta = \beta \in Z,\ \eta_i^\beta \in \Pi\right\}.$$

Equivalently, we form

$$Y' = \left\{\eta_i^\beta \,\middle|\, -\sum_i \eta_i^\beta = -\beta \in Y,\ \eta_i^\beta \in \Pi\right\}.$$

Note that since t is the height of the highest root, each β corresponds to a maximum of t many η_i^β's. Therefore

$$t|Z| \geq |Z'| \geq |Z| \text{ and } t|Y| \geq |Y'| \geq |Y|.$$

By the linear independence of Π, since

$$\sum_{\eta\in Z'}\eta = \sum_{\beta\in Z}\beta = \sum_{\beta\in Y}\beta = \sum_{\eta\in Y'}\eta,$$

we have $Y' = Z'$. Thus

$$\begin{aligned}\mathfrak{D}(\mathfrak{p}) &\leq \frac{|Z|}{|Y|} \\ &\leq \frac{|Z'|}{|Y|} \\ &= \frac{|Y'|}{|Y|} \\ &\leq \frac{t|Y|}{|Y|} \\ &= t.\end{aligned}$$

□

Lemma 9 (Second Discrepancy Lemma). *Let $q_{\mathbf{a},\mathbf{b}}$ be as defined in Section 2.5. Let t be the maximum root height of Φ. For all γ, $\mathbf{a}$, and $\mathbf{b}$,*

$$d_{\mathbf{b}} - d_{\mathbf{a}} \leq 2t \cdot \mathrm{ord}_p(q_{\mathbf{a},\mathbf{b}}).$$

Proof. Fix $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^m$. If $d_{\mathbf{a}} \geq d_{\mathbf{b}}$, then the result is trivial.

Assume $d_{\mathbf{b}} > d_{\mathbf{a}}$. Recall that $q_{\mathbf{a},\mathbf{b}}$ is defined to be the coefficient of $f^{\mathbf{b}}(\mathbf{x})$ in the expansion of $\prod_\alpha (\mathfrak{q}_\alpha(\mathbf{x}))^{a_\alpha}$. Thus $q_{\mathbf{a},\mathbf{b}} f^{\mathbf{b}}(\mathbf{x})$ is the product of terms of the polynomials $\mathfrak{q}_\alpha(\mathbf{x})$. As $d_{\mathbf{b}} > d_{\mathbf{a}}$, at least one factor of $q_{\mathbf{a},\mathbf{b}} f^{\mathbf{b}}(\mathbf{x})$ must be a term of degree at least 2.

Let $q_{\mathbf{a},\mathbf{b}} f^{\mathbf{b}}(\mathbf{x}) = \prod_j (w_j f^{\mathbf{e}_j}(\mathbf{x}))^{\epsilon_j}$ be a factorization with each $w_j f^{\mathbf{e}_j}(\mathbf{x})$ a term of some $\mathfrak{q}_\alpha(\mathbf{x})$. Note that $d_{\mathbf{b}} = \sum_i \mathbf{e}_i \epsilon_i$ and $d_{\mathbf{a}} = \sum_i \epsilon_i$. Then

$$d_{\mathbf{b}} - d_{\mathbf{a}} = \sum_j \epsilon_j (d_{\mathbf{e}_j} - 1) \leq \sum_{d_{\mathbf{e}_j} \geq 2} \epsilon_j d_{\mathbf{e}_j}.$$

Suppose $d_{\mathbf{e}_j} \geq 2$ for some fixed j. We examine the possible values of $\mathfrak{D}(w_j f^{\mathbf{e}_j})$.

Note that by Lemma 7, $\mathfrak{D}(w_j f^{\mathbf{e}_j})$ is defined. In each case, we show that $\mathfrak{D}(w_j f^{\mathbf{e}_j}) \leq 2t$.

By Lemma 6, the nonconstant term $w_j f^{\mathbf{e}_j}(\mathbf{x})$ of $\mathfrak{q}_\alpha(\mathbf{x})$ has the form

$$w_j f^{\mathbf{e}_j}(\mathbf{x}) = c \prod_{\beta\in Y_j} y_{-\beta} \prod_{\beta\in Z_j} x_\beta \text{ with } \sum_{\beta\in Z_j} \beta - \sum_{\beta\in Y_j} \beta = \alpha,$$

where Y_j and Z_j are multisets of positive roots.

Since $\mathfrak{D}(w_j f^{\mathbf{e}_j}) \in \mathbb{Z}$, we have

$$\begin{aligned}
\mathfrak{D}(w_j f^{\mathbf{e}_j}) &= \mathfrak{D}\left(c_j \prod_{\beta\in Y_j} y_{-\beta} \prod_{\beta\in Z_j} x_\beta\right) \\
&\leq \mathfrak{D}\left(\prod_{\beta\in Y_j} y_{-\beta} \prod_{\beta\in Z_j} x_\beta\right) \\
&\leq \frac{|Z_j|}{|Y_j|}.
\end{aligned}$$

Form Z_j' and Y_j', multisets of fundamental roots as in the previous proof, that is

$$Z_j' = \left\{\eta_i^\beta \,\middle|\, \sum_i \eta_i^\beta = \beta \in Z_j,\ \eta_i^\beta \in \Pi\right\}$$

$$\text{and } Y_j' = \left\{\eta_i^\beta \,\middle|\, -\sum_i \eta_i^\beta = -\beta \in Y_j,\ \eta_i^\beta \in \Pi\right\}.$$

We have

$$\sum_{\eta\in Z_j'} \eta - \sum_{\eta\in Y_j'} \eta = \sum_{\beta\in Z_j} \beta - \sum_{\beta\in Y_j} \beta = \alpha.$$

Thus $\sum_{\eta\in Z_j'\setminus Y_j'} \eta = \alpha$; so $|Z_j'\setminus Y_j'|$ is the height of α. We have $|Z_j'| - |Y_j'| \leq t$ and again, $|Y_j'| \leq t|Y_j|$. We have

$$\begin{aligned}\mathfrak{D}(w_j f^{\mathbf{e}_j}) &\leq \frac{|Z_j|}{|Y_j|} \\ &\leq \frac{|Z_j'|}{|Y_j|} \\ &= \frac{|Z_j'| - |Y_j'|}{|Y_j|} + \frac{|Y_j'|}{|Y_j|} \\ &\leq t + t = 2t.\end{aligned}$$

We now have, for all β, $\mathfrak{D}(w_j \mathbf{x}^{\mathbf{e}_j}) \leq 2t$ and so $d_{\mathbf{e}_j} \leq 2t \operatorname{ord}_p(w_j)$. Thus

$$d_{\mathbf{b}} - d_{\mathbf{a}} \leq \sum_{d_{\mathbf{e}_j} \geq 2} d_{\mathbf{e}_j} \leq 2t \sum_{d_{\mathbf{e}_j} \geq 2} \operatorname{ord}_p(w_j).$$

Clearly $\sum_j \operatorname{ord}_p(w_j) \leq \operatorname{ord}_p(q_{\mathbf{a},\mathbf{b}})$ and so $d_{\mathbf{b}} - d_{\mathbf{a}} \leq 2t \operatorname{ord}_p(q_{\mathbf{a},\mathbf{b}})$ as claimed.

□

Corollary 1. *The action of S on X induces an action of S on A_λ.*

Proof. It suffices to show $\gamma : F_\lambda \to A_\lambda$ for any $\gamma \in S$.

Let $f^{\mathbf{a}} : X \to \mathbb{O}_p$ be in F_λ. We have

$$f^{\mathbf{a}}(\mathbf{x}) = \prod_{\alpha \in \Phi^+} x_\alpha^{a_\alpha}.$$

First note that $\pi f^{\mathbf{a}}(\mathbf{x}) = f^{\mathbf{a}}(\mathbf{x} \cdot \pi)$ and $\mathbf{x} \cdot \pi = \ell(\alpha(\pi) x_\alpha)_{\alpha \in \Phi^+}$. So

$$\begin{aligned}\pi f^{\mathbf{a}}(\mathbf{x}) &= \prod_{\alpha \in \Phi^+} (\alpha(\pi))^{a_\alpha} \prod_{\alpha \in \Phi^+} x_\alpha^{a_\alpha} \\ &= \prod_{\alpha \in \Phi^+} (\alpha(\pi))^{a_\alpha} f^{\mathbf{a}}(\mathbf{x})\end{aligned}$$

and $\prod_{\alpha\in\Phi^+}(\alpha(\pi))^{a_\alpha} f^{\mathbf{a}}(\mathbf{x}) \in A_\lambda$.

Now let $\gamma \in I$. We want to show that $\gamma f^{\mathbf{a}} = \sum_i c^i f^{\mathbf{a}_i}$ where $c^i \to 0$ as $d_{\mathbf{a}_i} \to \infty$. Let $\epsilon > 0$ and pick $k \in \mathbb{Z}^+$ so that $\dfrac{1}{p^k} < \epsilon$.

Note that each $c^i f^{\mathbf{a}_i}(\mathbf{x})$ is a product of one term of $\mathfrak{p} = \sum p_{\mathbf{b}} f^{\mathbf{b}}(\mathbf{x})$ and one term of $\mathfrak{q}_{\mathbf{a}} = \sum q_{\mathbf{a},\mathbf{b}'} f^{\mathbf{b}'}(\mathbf{x})$. By the Discrepancy Lemmas, $d_{\mathbf{b}} \leq t\mathrm{ord}_p(p_{\mathbf{b}})$ for each $\mathbf{b}$ and $d_{\mathbf{b}'} - d_{\mathbf{a}} \leq 2t \cdot \mathrm{ord}_p(q_{\mathbf{a},\mathbf{b}'})$ for each $\mathbf{b}'$.

If $c^i f^{\mathbf{a}_i}(\mathbf{x}) = \left(p_{\mathbf{b}} f^{\mathbf{b}}(\mathbf{x})\right)\left(q_{\mathbf{a},\mathbf{b}'} f^{\mathbf{b}'}(\mathbf{x})\right)$, then $d_{\mathbf{a}_i} = d_{\mathbf{b}} + d_{\mathbf{b}'}$ and $\mathrm{ord}_p(c^i) = \mathrm{ord}_p(p_{\mathbf{b}}) + \mathrm{ord}_p(q_{\mathbf{a},\mathbf{b}'})$. By the Discrepancy Lemmas, we have

$$\begin{aligned} d_{\mathbf{a}_i} &= d_{\mathbf{b}} + d'_{\mathbf{b}} \\ &\leq t\mathrm{ord}_p(p_{\mathbf{b}}) + 2t\mathrm{ord}_p(q_{\mathbf{a},\mathbf{b}'}) + d_{\mathbf{a}} \\ &\leq 2t\mathrm{ord}_p(c^i) + d_{\mathbf{a}}. \end{aligned}$$

For any $\mathbf{a}_i$ such that $d_{\mathbf{a}_i} > d_{\mathbf{a}} + 2t\ k$, we have $\mathrm{ord}_p(c^i) > k$. □

3.5 The Filtration $Fil^n D_\lambda$

We now show that our filtration satisfies certain conditions allowing for the application of a Theorem 7 by Pollack and Pollack.

Recall the definition of our filtration,

$$Fil^n D_\lambda = \left\{ \mu \in K_\lambda \;\middle|\; \mathrm{ord}_p(\mu(f^{\mathbf{a}})) \geq n - \left\lfloor \frac{d_{\mathbf{a}}}{2t} \right\rfloor, \forall \mathbf{a} \right\},$$

where t is the height of the highest weight.

Lemma 10. *Let $f^0 \in F_\lambda$ be the constant function*

$$f^0(\mathbf{x}) := \prod_\alpha x_\alpha^0 = 1.$$

Then $\mu(f^0) = 0$ for every $\mu \in K_\lambda$.

Proof. Recall the $\{v_i\}$ are chosen to be a basis for $V_\lambda(\mathbb{Q}_p)$ with $v_1 = v_\lambda$, and the $f_i \in A_\lambda$ are chosen so that $f_\lambda = \sum f_i \otimes v_i$ (Equation (2.3)). Since v_λ is a highest weight vector, we have

$$\begin{aligned} v_\lambda \cdot \mathbf{x} &= \sum_{i=1}^{M} f_i(\mathbf{x}) v_i \\ &= v_\lambda + \sum_{i=2}^{M} f_i(\mathbf{x}) v_i. \end{aligned}$$

Thus $f_1 = 1 = f^0$.

Let $\mu \in K_\lambda$. Then

$$\begin{aligned} 0 &= \rho_\lambda(\mu) \\ &= \mu(f_\lambda) \\ &= \mu\left(f^0 v_\lambda + \sum_{i=2}^{M} f_i v_i\right) \\ &= \mu(f^0) v_\lambda + \sum_{i=2}^{M} \mu(f_i) v_i \end{aligned}$$

Thus $\mu(f^0) = 0$. $\square$

Recall that π is a torus element such that $\alpha(\pi)$ is a multiple of p for all $\alpha \in \Phi^+$ and S is the smallest sub-semigroup of $G(\mathbb{C}_p)$ containing the Iwahori and π

Theorem 6. *The filtration $Fil^n D_\lambda$ satisfies*

1. $Fil^{n+1}D_\lambda \subset Fil^n D_\lambda$ *for all* n.

2. $Fil^n D_\lambda|\pi \subset Fil^{n+1}D_\lambda$ *for all* n.

3. $Fil^n D_\lambda$ *is* S*-stable.*

4. The natural map $D_\lambda \to \varprojlim D_\lambda/Fil^n D_\lambda$ *is an isomorphism.*

Proof. 1. That the filtration is nonincreasing is clear from the definition.

2. Let $\mu \in Fil^n D_\lambda$. We want to show that $\mu|\pi \in Fil^{n+1}D_\lambda$. Since elements of D_λ are uniquely determined by their actions on F_λ, we fix $f^{\mathbf{a}} \in F_\lambda$ and consider $(\mu \mid \pi)(f^{\mathbf{a}}) = \mu(\pi \cdot f^{\mathbf{a}})$.

   We know, by our definition of π that

$$\mathbf{x} \cdot \pi = \ell((b_\alpha x_\alpha)_\alpha)$$

   for some $b_\alpha \in \mathbb{O}_p$ such that $\mathrm{ord}_p(b_\alpha) \geq 1$ for all α. We have

$$\begin{aligned} f^{\mathbf{a}}(\mathbf{x} \cdot \pi) &= f^{\mathbf{a}}(\ell((b_\alpha x_\alpha)_\alpha)) \\ &= \prod_\alpha (b_\alpha x_\alpha)^{a_\alpha} \\ &= \prod_\alpha b_\alpha^{a_\alpha} \prod_\alpha x_\alpha^{a_\alpha} \\ &= \left(\prod_\alpha b_\alpha^{a_\alpha}\right) f^{\mathbf{a}}(\mathbf{x}). \end{aligned}$$

   Note that $p^{d_{\mathbf{a}}}$ divides $\prod_\alpha b_\alpha^{a_\alpha}$.

   We have two cases to check. Either $d_{\mathbf{a}} = \sum_\alpha a_\alpha = 0$ or $d_{\mathbf{a}} \geq 1$.

(a) Suppose $d_{\mathbf{a}} \geq 1$. Then

$$\begin{aligned}
\mathrm{ord}_p((\mu \cdot \pi)(f^{\mathbf{a}})) &= \mathrm{ord}_p(\mu(\pi \cdot f^{\mathbf{a}})) \\
&= \mathrm{ord}_p\left(\mu\left(\prod_{\alpha} b_{\alpha}^{a_{\alpha}} f^{\mathbf{a}}\right)\right) \\
&= \mathrm{ord}_p\left(\left(\prod_{\alpha} b_{\alpha}^{a_{\alpha}}\right) \mu\left(f^{\mathbf{a}}\right)\right) \\
&= \mathrm{ord}_p\left(\prod_{\alpha} b_{\alpha}^{a_{\alpha}}\right) + \mathrm{ord}_p\left(\mu\left(f^{\mathbf{a}}\right)\right) \\
&\geq 1 + n - \left\lfloor \frac{d_{\mathbf{a}}}{2t} \right\rfloor .
\end{aligned}$$

Therefore $\mu \mid \pi \in Fil^{n+1} D_{\lambda}$.

(b) Now suppose $d_{\mathbf{a}} = 0$. Then $f^{\mathbf{a}} = f^0$ and $a_{\alpha} = 0$ for all α. We have shown that $\mu(f^0) = 0$ in Lemma 10 and we have

$$\begin{aligned}
\pi \cdot f^0(\mathbf{x}) &= f^0(\mathbf{x} \cdot \pi) \\
&= 1 \\
&= f^0(\mathbf{x}).
\end{aligned}$$

Thus $\mu(\pi \cdot f_{\mathbf{a}}) = 0$ and we have

$$\mathrm{ord}_p((\mu \cdot \pi)(f^{\mathbf{a}})) = \infty \geq 1 + N - \left\lfloor \frac{d_{\mathbf{a}}}{2t} \right\rfloor .$$

So $\mu | \pi \in Fil^{n+1} D_{\lambda}$.

3. It has been shown that K_{λ} is S-stable (Lemma 3). We want to show $Fil^n D_{\lambda}$

is stable under S. We have

$$Fil^n D_\lambda|\pi \subset Fil^{n+1} D_\lambda \subset Fil^n D_\lambda.$$

We will show that $Fil^n D_\lambda$ is stable under action by $\gamma \in I$.

Let $\mu \in Fil^n D_\lambda$, $\gamma \in I$. We want to show that $\mu \cdot \gamma \in Fil^n D_\lambda$. In order to prove

$$\mathrm{ord}_p(\mu \cdot \gamma(f^{\mathbf{a}})) \geq n - \left\lfloor \frac{d_{\mathbf{a}}}{2t} \right\rfloor,$$

it suffices to show

$$\mathrm{ord}_p(c_{\mathbf{a},\mathbf{b}} \mu(f^{\mathbf{b}})) \geq n - \left\lfloor \frac{d_{\mathbf{a}}}{2t} \right\rfloor$$

for all $\mathbf{b}$, with $c_{\mathbf{a},\mathbf{b}}$ defined as in Section 2.5.

Recall that

$$c_{\mathbf{a},\mathbf{b}} = \sum_{\mathbf{b}'} p_{\mathbf{b}'} q_{\mathbf{a},\mathbf{b}-\mathbf{b}'}.$$

By the ultrametric triangle inequality, we have

$$\mathrm{ord}_p(c_{\mathbf{a},\mathbf{b}}) \geq \mathrm{ord}_p(p_{\mathbf{b}'}) + \mathrm{ord}_p(q_{\mathbf{a},\mathbf{b}-\mathbf{b}'})$$

for any $\mathbf{b}'$. By the Discrepancy Lemmas, $d_{\mathbf{b}'} \leq t\ \mathrm{ord}_p(p'_{\mathbf{b}})$ and $d_{\mathbf{b}-\mathbf{b}'} - d_{\mathbf{a}} \leq 2t\ \mathrm{ord}_p(q_{\mathbf{a},\mathbf{b}-\mathbf{b}'})$.

If $d_{\mathbf{b}-\mathbf{b}'} - d_{\mathbf{a}} \geq 0$, we have

$$\begin{aligned}
\mathrm{ord}_p(c_{\mathbf{a},\mathbf{b}}) &\geq \mathrm{ord}_p(p_{\mathbf{a}}) + \mathrm{ord}_p(q_{\mathbf{a},\mathbf{b}-\mathbf{b}'}) \\
&\geq \left\lceil \frac{d_{\mathbf{b}'}}{t} \right\rceil + \left\lceil \frac{d_{\mathbf{b}-\mathbf{b}'} - d_{\mathbf{a}}}{2t} \right\rceil \\
&\geq \left\lceil \frac{d_{\mathbf{b}'}}{2t} \right\rceil + \left\lceil \frac{d_{\mathbf{b}-\mathbf{b}'} - d_{\mathbf{a}}}{2t} \right\rceil
\end{aligned}$$

and if $d_{\mathbf{b}-\mathbf{b}'} - d_{\mathbf{a}} \leq 0$,

$$\begin{aligned}
\mathrm{ord}_p(c_{\mathbf{a},\mathbf{b}}) &\geq \mathrm{ord}_p(p_{\mathbf{a}}) + \mathrm{ord}_p(q_{\mathbf{a},\mathbf{b}-\mathbf{b}'}) \\
&\geq \left\lceil \frac{d_{\mathbf{b}'}}{t} \right\rceil + 0 \\
&\geq \left\lceil \frac{d_{\mathbf{b}'}}{2t} \right\rceil + \left\lceil \frac{d_{\mathbf{b}-\mathbf{b}'} - d_{\mathbf{a}}}{2t} \right\rceil .
\end{aligned}$$

In either case,

$$\begin{aligned}
\mathrm{ord}_p(c_{\mathbf{a},\mathbf{b}}) &\geq \left\lceil \frac{d_{\mathbf{b}'}}{2t} \right\rceil + \left\lceil \frac{d_{\mathbf{b}-\mathbf{b}'} - d_{\mathbf{a}}}{2t} \right\rceil \\
&\geq \left\lceil \frac{d_{\mathbf{b}'} + d_{\mathbf{b}-\mathbf{b}'} - d_{\mathbf{a}}}{2t} \right\rceil \\
&= \left\lceil \frac{d_{\mathbf{b}} - d_{\mathbf{a}}}{2t} \right\rceil .
\end{aligned}$$

As $\mu \in Fil^n D_\lambda$, we know that

$$\mathrm{ord}_p(\mu(f^{\mathbf{b}})) \geq n - \left\lfloor \frac{d_{\mathbf{b}}}{2t} \right\rfloor ,$$

and so

$$\begin{aligned}
\mathrm{ord}_p(c_{\mathbf{a},\mathbf{b}}\mu(f^{\mathbf{b}})) &= \mathrm{ord}_p\left(c_{\mathbf{a},\mathbf{b}}\right) + \mathrm{ord}_p(\mu\left(f^{\mathbf{b}}\right)) \\
&\geq \left\lceil \frac{d_{\mathbf{b}} - d_{\mathbf{a}}}{2t} \right\rceil + n - \left\lfloor \frac{d_{\mathbf{b}}}{2t} \right\rfloor \\
&= n - \left\lfloor \frac{d_{\mathbf{a}} - d_{\mathbf{b}}}{2t} \right\rfloor - \left\lfloor \frac{d_{\mathbf{b}}}{2t} \right\rfloor \\
&\geq n - \left\lfloor \frac{d_{\mathbf{a}}}{2t} \right\rfloor .
\end{aligned}$$

Thus $\mu \cdot \gamma \in Fil^n D_\lambda$ and $Fil^n D_\lambda$ is stable under the action of S.

4. Clearly, $\cap Fil^n D_\lambda = \{0\}$, the set containing only the zero distribution. Therefore the map $D_\lambda \to \varprojlim D_\lambda / Fil^n D_\lambda$ is injective. Let

$$(\mu_n)_{n\in\mathbb{N}} \in (Fil^n D_\lambda)_{n\in\mathbb{N}}$$

be a fixed sequence, compatible with the natural maps $Fil^n D_\lambda \to Fil^{n-1} D_\lambda$. For each $\mathbf{a}$, we will show that the sequence $(\mu_n(f^{\mathbf{a}}))_{n\in\mathbb{N}}$ is Cauchy. Let $\epsilon > 0$. Let $M \in \mathbb{N}$ be such that $p^{-M} < \epsilon$, and let

$$M' = M + \left\lfloor \frac{d_{\mathbf{a}}}{2t} \right\rfloor .$$

For any $M_1, M_2 \geq M'$, we have

$$\begin{aligned}\operatorname{ord}_p\left(\mu_{M_1}(f^{\mathbf{a}}) - \mu_{M_2}(f^{\mathbf{a}})\right) &\geq \max\{\operatorname{ord}_p\left(\mu_{M_1}(f^{\mathbf{a}})\right), \operatorname{ord}_p\left(\mu_{M_2}(f^{\mathbf{a}})\right)\} \\ &\geq M' - \left\lfloor \frac{d_{\mathbf{a}}}{2t} \right\rfloor \\ &= M.\end{aligned}$$

Thus $|\mu_{M_1}(f^{\mathbf{a}}) - \mu_{M_2}(f^{\mathbf{a}})|_p < p^{-M} < \epsilon$ and so $(\mu_n(f^{\mathbf{a}}))_{n\in\mathbb{N}}$ is Cauchy.

For any $\mathbf{a}$, let

$$s_{\mathbf{a}} = \lim_{n\to\infty} \mu_n(f^{\mathbf{a}}).$$

Recall that elements in D_λ are uniquely determined by their values on elements of F_λ. Particularly $\mu \in D_\lambda$ defined by $\mu(f^{\mathbf{a}}) = s_{\mathbf{a}}$ for each $\mathbf{a}$. Then the quotient map $D_\lambda \to D_\lambda/Fil^n D_\lambda$ takes μ to μ_n at each n. Thus $\mu \mapsto (\mu_n)_n$ via the natural map $D_\lambda \to \varprojlim D_\lambda/Fil^n D_\lambda$. Therefore $D_\lambda \to \varprojlim D_\lambda/Fil^n D_\lambda$ is an isomorphism.

□

Chapter 4

Result

We recall the following result of [8]

Theorem 7. *(Pollack and Pollack [8]) Let $\Gamma \subset G$ be groups, with $\pi \in G$ and let S be the sub-semigroup of G generated by Γ and π. Let R be a commutative ring and let D be a right $R[S]$-module with a decreasing $R[S]$-stable filtration $F^N D$ such that*

1. *$F^n D \cdot \pi \subset F^{n+1} D$ for each $n \geq 0$,*
2. *the natural map $D \to \varprojlim (D/F^n D)$ is an isomorphism.*

Let ϕ in $H^r(\Gamma, D/F^0 D)$ be an ordinary eigenvector for U with eigenvalue α. Then there exists $\varphi \in H^r(\Gamma, D)$ such that

1. *The image of φ in $H^r(\Gamma, D/F^0 D)$ equals ϕ,*
2. *φ in an eigenvector for U with eigenvalue α,*
3. *$\mathrm{Ann}_R(\phi) = \mathrm{Ann}_R(\varphi)$.*

Moreover, if φ' is any ordinary U-eigenlift of ϕ, then $\varphi' = \varphi$.

This allows us to use our filtration to prove the main result. Let U_π be the Hecke-operator corresponding to $\Gamma\pi\Gamma$. Recall that $L_\lambda = \rho_\lambda(D_\lambda)$ and $Fil^0 D_\lambda = K_\lambda = \ker(\rho_\lambda)$ is chosen such that $L_\lambda = D_\lambda / Fil^0 D_\lambda$. The map

$$\rho_\lambda^r : H^r(\Gamma, D_\lambda) \to H^r(\Gamma, L_\lambda)$$

is U_π-equivariant. An eigenvector is ordinary for U_π if its eigenvalue is a p-adic unit.

Theorem 8. *If $\phi \in H^r(\Gamma, L_\lambda)$ is an ordinary U_π-eigenvector whose eigenvalue is a unit, then there is a unique ordinary U_π-eigenvector φ in $H^r(\Gamma, D_\lambda)$ such that $\rho_\lambda^r(\varphi) = \phi$.*

Proof. Note that D_λ is a right Γ-module and $Fil^n D_\lambda$ a decreasing filtration. Recall that $Fil^0 D_\lambda = K_\lambda$ was chosen so that $D_\lambda / K_\lambda = L_\lambda$. By Lemma 6, the filtration $Fil^n D_\lambda$ is Γ-stable, $Fil^n D_\lambda \cdot \pi \subset Fil^{n+1} D_\lambda$ for each $n \geq 0$, and the natural map $D_\lambda \to \varprojlim (D_\lambda / Fil^n D_\lambda)$ is an isomorphism.

Let $\phi \in H^r(\Gamma, L_\lambda)$ be an ordinary U_π-eigenvector with eigenvalue α. By Theorem 7, there exists $\varphi \in H^r(\Gamma, D_\lambda)$ such that the image of φ in $H^r(\Gamma, L_\lambda)$ equals ϕ; φ in an eigenvector for U_π with eigenvalue α; the annihilators of ϕ and φ over R are identical, and φ is the unique ordinary U_π-eigenvector lifting ϕ.

□

It is worth noting that the quotient space $D_\lambda / Fil^N D_\lambda$ is finite dimensional for any N. This allows the computation of explicit approximations to φ at each level $D_\lambda / Fil^n D_\lambda$.

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