

Dimension Theory in Dense Regular Groups

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Abstract

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A dense regular group is a densely ordered group elementarily equivalent to a subgroup of \mathbb{R} . If a dense regular group is divisible then it is o-minimal. In this thesis we examine the structure of the definable sets in dense regular groups which may not be divisible, but have only finitely many congruence classes mod n for every positive integer n . We define an appropriate notion of “cell”, prove a cell-decomposition theorem similar to that which is known for o-minimal structures, and use cells to develop a dimension theory for the definable sets. In 2009 Zilber proposed a set of axioms which a dimension theory on a *topological* structure might satisfy, and he can show that these axioms lead to a quantifier elimination result. These axioms apply to \mathbb{R} , \mathbb{Q}_p , and every o-minimal structure with their standard topologies. We define a natural topology for the groups under investigation here — one in which the definable closed sets have positive quantifier-free definitions — and show that the dimension theory defined here satisfies all of Zilber’s axioms.

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Introduction

In his recently published book on model-theoretic geometry [11], Zilber identifies a collection of properties which a ‘good’ dimension theory on a topological structure might have [11, §3.1]. These properties essentially axiomatize the type of dimension theory that the definable sets of \mathbb{C} have with respect to the Zariski topology, and by assuming that these properties hold on a given topological structure, Zilber is able to prove certain interesting results concerning e.g. quantifier elimination and Morley rank [11, §3.2]. But Zilber’s semi-properness axiom fails in structures like \mathbb{R} and the p -adics, whose definable sets nevertheless have a well-developed dimension theory [2, 6].

In a series of lectures at the University of Oxford in 2009, Zilber proposed an alternate collection of axioms, sharing some of the properties of his earlier set, which hold in \mathbb{R} and \mathbb{Q}_p and which still lead to a quantifier elimination result similar to that of the original case [2, 6]¹. A structure M obeys these alternate axioms just in case every definable subset S of M^n , for any $n \geq 1$,

¹In the case of \mathbb{Q}_p , all the axioms except (wSI) are immediate consequences of results in [6, §3], and P. Scowcroft has verified (wSI) in unpublished work. B. Zilber has obtained the quantifier elimination result in unpublished work.

has a dimension, $\dim(S)$, obeying:

$$(Z) \dim(S) = -\infty \text{ iff } S = \emptyset$$

$$(DP) \dim(S) = 0 \text{ whenever } S \text{ is finite}$$

$$(DU) \dim(S_1 \cup S_2) = \max\{\dim(S_1), \dim(S_2)\}$$

$$(wSI) \text{ For every } S \text{ there is } \bar{S} \supseteq S \text{ closed such that } \dim(\bar{S} \setminus S) < \dim(S)$$

$$(wAF) \text{ For every } S \subseteq M^n \text{ and projection map } \pi : M^n \rightarrow M^m \text{ which is}$$

$$\text{equidimensional on fibers of } S, \dim(S) = \dim(\pi(S)) + \dim(\text{fiber})$$

$$(FC^*) \text{ If } S \subseteq M^n \text{ is closed and } d \geq 0, \text{ then } \{a \in \pi(S) \mid \dim(S_a) \geq d\} \text{ is}$$

definable (here S_a is the fiber of S over a).

Zilber actually treats the notion of “closed” axiomatically, rather than insisting that there is a topology present, but the definable sets that are closed with respect to a given topology will frequently obey his axioms, as will those sets which are definable by some positive quantifier-free formula. In fact, in the structures examined in this thesis the definable sets which are closed with respect to the relevant topology are all definable by positive quantifier-free formulas (see Theorem 3 below).

An important class of structures obeying these dimension axioms is the class of so-called o-minimal structures: ordered structures in which the definable subsets of the domain are just finite unions of points and intervals [2]. An *ordered abelian group* is an abelian group G with a translation-invariant

linear ordering. Not all ordered abelian groups are o-minimal — in fact, most are not — but the divisible ones are, and they therefore satisfy these dimension axioms.

An ordered abelian group G is said to be *dense regular* if for any positive integer n and any $a < b \in G$, there exists $c \in G$ such that c is divisible by n and $a < c < b$. For any positive integer n , the n^{th} *congruence invariant* of an ordered abelian group G is the maximum number of elements of G which are mutually incongruent modulo n . In this thesis we examine the class of dense regular ordered abelian groups, all of whose congruence invariants are finite. This class includes the divisible ordered abelian groups, and we will show that every such group has a dimension theory which satisfies the axioms above.

The thesis proceeds roughly as follows. The main goal of Chapters 2 and 3 is to prove a cell-decomposition theorem similar to the well-known one for o-minimal structures [2, Thm. 2.11]. In Chapter 2 we lay the groundwork by examining definable subsets of the domain and introducing the idea of a *change point* which is a key tool in the proof of cell-decomposition. In Chapter 3 we prove our cell-decomposition theorem.

Cell-decomposition provides an obvious definition for dimension, though we put this off until Chapter 5 in order to develop in Chapter 4 some of the machinery needed to check the more complicated axioms. Two of Zilber's axioms — (wAF) and (FC*) — concern projection maps, but cells do not behave as well as one might hope under projection. To deal with this prob-

lem Chapter 4 introduces *flats*, which are hyperplanes restricted by certain congruence conditions. After Lemma 16 shows that projections of flats are finite unions of flats, one may handle projections of arbitrary definable sets by viewing cells as subsets of appropriate flats. One consequence of cell-decomposition is that all definable functions are (finitely) piecewise linear; a large portion of Chapter 4 is devoted to understanding how flats behave under linear functions as well.

In Chapter 5 we finally define dimension and check that every axiom except (wSI) holds for this definition. This task is relatively easy following our work in Chapter 4. In Chapter 6 we define a natural topology and, as mentioned earlier, show that definable closed sets are definable by positive quantifier-free formulas. We finish by showing that under this topology (wSI) holds.

Chapter 1

Preliminaries

Let $\mathcal{L} = \langle +, \leq, 0, \equiv_1, \equiv_2, \dots \rangle$ be a language where $+$ is a binary function symbol, \leq is a binary relation symbol, 0 is a constant symbol, and $\equiv_1, \equiv_2, \dots$ are each binary relation symbols. In what follows G always denotes an \mathcal{L} -structure where $(G, +, \leq, 0)$ forms a dense regular ordered abelian group with finite congruence invariants in the obvious way (\leq is the weak ordering of G) and \equiv_n is interpreted as congruence modulo n for each $n \geq 1$. The class of such \mathcal{L} -structures is an elementary class by [4], and the complete \mathcal{L} -theory of any dense regular ordered abelian group admits quantifier elimination [8].

By *definable* we will always mean definable with parameters unless explicitly stated otherwise. We use $a \equiv b \pmod{n}$ and $a \equiv_n b$ interchangeably and we use $(a, b)_{\equiv_n c}$ as shorthand for

$$\{x \in G \mid a < x < b \text{ and } x \equiv c \pmod{n}\}.$$

We begin by exploiting quantifier elimination in G to show that any \mathcal{L} -formula is equivalent in G to one of a particularly nice form.

Definition 1. Let $\phi(x_1, \dots, x_m)$ be an \mathcal{L}_G -formula. If

$$\phi(x_1, \dots, x_m) = \bigvee_{i=1}^{\alpha} \bigwedge_{j=1}^{\beta_i} \theta_{ij}(x_1, \dots, x_m)$$

where α and every $\beta_i \in \mathbb{N}^+$ and each θ_{ij} is an identity, strict inequality, or congruence relation, then we say ϕ is in *special disjunctive normal form*, abbreviated SDNF.

Lemma 1. *Every \mathcal{L}_G -formula $\phi(x_1, \dots, x_m)$ is equivalent in G to a formula in special disjunctive normal form.*

Proof. By quantifier elimination ϕ is equivalent to a quantifier-free formula in disjunctive normal form. That is, ϕ is equivalent to a formula ϕ' of the form $\bigvee \bigwedge \psi_{ij}(x_1, \dots, x_m)$ where each ψ_{ij} is an atomic or negated atomic formula (also known as a *literal*).

In \mathcal{L} , the atomic formulae are identities, weak inequalities, and congruences. In G , negated weak inequalities are equivalent to strict inequalities: so if every ψ_i is an identity, congruence or negated weak inequality then we are done. Let us say (for this proof) that identities, negated weak inequalities, and congruences are *good literals* and negated identities, weak inequalities, and negated congruences are *bad literals*.

We will show that any disjunct λ_i of ϕ' containing at least one bad literal is equivalent to a disjunction of conjunctions of literals, where each disjunct contains fewer bad literals than λ_i . This is sufficient to prove the lemma by induction.

Let $\lambda_i = \bigwedge_{j=1}^{\beta_i} \psi_{ij}$, and without loss of generality, assume ψ_{i1} is a bad literal. If ψ_{i1} is a negated identity then it is equivalent to the disjunction of two strict inequalities. Similarly, if ψ_{i1} is a weak inequality then it is equivalent to the disjunction of an identity and a strict inequality. If ψ_{i1} is a negated congruence, say $\neg(t_1(\bar{x}) \equiv_n t_2(\bar{x}))$ for some $n \in \mathbb{N}^+$, then since we have only finitely many equivalence classes in G with respect to n , ψ_{i1} is equivalent to $\bigvee_{r \in \mathcal{R}} \theta_r$ where \mathcal{R} is a finite set consisting of exactly one member of each non-zero equivalence class mod n and θ_r is $t_1(\bar{x}) \equiv_n t_2(\bar{x}) + r$.

In each of these cases, ψ_{i1} is equivalent to $\bigvee_{k=1}^{\gamma} \theta_k$ where each θ_k is an identity, a strict inequality, or a congruence relation. Therefore λ_i is equivalent in G to $(\bigvee_{k=1}^{\gamma} \theta_k) \wedge (\bigwedge_{j=2}^{\beta_i} \psi_{ij})$ which is logically equivalent to $\bigvee_{k=1}^{\gamma} (\theta_k \wedge \bigwedge_{j=2}^{\beta_i} \psi_{ij})$. Clearly each disjunct of this formula has fewer bad literals than λ_i , and so the proof is complete. □

For formulas in one variable we can say a bit more, as shown in the following Lemma and Corollary.

Lemma 2. *(a) Any strict inequality in one variable, x , is equivalent in G to $0 = 0$, to $0 \neq 0$, or to an inequality of the form $kx < a$ or $a < kx$ for some*

$k \in \mathbb{N}^+$ and $a \in G$.

(b) Any congruence $(\text{mod } n)$ in one variable is equivalent in G to $0 = 0$, to $0 \neq 0$, or to a congruence of the form $x \equiv c \pmod{n'}$ for some $n' \in \mathbb{N}$ and $c \in G$, where n' divides n .

Proof. It is clear that any term, $t(x)$, of \mathcal{L} is equivalent in G to $kx + a$ for some $k \in \mathbb{N}$ and $a \in G$. Thus any strict inequality, $t_1(x) < t_2(x)$, is equivalent in G to $k_1x + a_1 < k_2x + a_2$ for some $k_1, k_2 \in \mathbb{N}$ and $a_1, a_2 \in G$. If $k_1 \neq k_2$ then this is equivalent to either $(k_1 - k_2)x < a_2 - a_1$ if $k_1 > k_2$, or $a_1 - a_2 < (k_2 - k_1)x$ if $k_1 < k_2$. If $k_1 = k_2$, then $k_1x + a_1 < k_2x + a_2$ is equivalent to $0 < a_2 - a_1$ which is equivalent to either $0 = 0$ or $0 \neq 0$ depending on a_1 and a_2 .

Similarly, if we have the congruence $t_1(x) \equiv t_2(x) \pmod{n}$ for some n , then it is equivalent in G to $k_1x + a_1 \equiv k_2x + a_2 \pmod{n}$. If $k_1 \neq k_2$, then without loss of generality assume $k_1 > k_2$ and let $k = k_1 - k_2$. Then the congruence is equivalent to $kx \equiv a_2 - a_1 \pmod{n}$. If $a_1 - a_2$ is divisible by (k, n) , familiar arguments¹ show that this congruence is equivalent to a congruence $x \equiv c \pmod{n'}$ with n' dividing n ; if $a_1 - a_2$ is not divisible by (k, n) , $kx \equiv a_1 - a_2 \pmod{n'}$ is equivalent to $0 \neq 0$.

If $k_1 = k_2$, then $k_1x + a_1 \equiv k_2x + a_2 \pmod{n}$ is equivalent to $0 \equiv a_2 - a_1 \pmod{n}$ which is equivalent to either $0 = 0$ or $0 \neq 0$ depending on a_1 and a_2 .

□

Corollary 1. *Let $\phi(x)$ be an \mathcal{L}_G -formula. Then ϕ is equivalent in G to a*

¹See, for example, the proofs of Lemmas 5.3.1 and 5.3.2 in [7].

formula $\phi'(x)$ of the form $\bigvee \bigwedge \theta_i(x)$ where each θ_i is an identity, a strict inequality of the form $kx < a$ or $a < kx$ with $k \in \mathbb{N}^+$, or a congruence relation of the form $x \equiv c \pmod{n}$.

Moreover, if $\phi(x)$ is in special disjunctive normal form, then $if \equiv_{n'}$ appears in ϕ' , n' divides $\text{lcm}\{n \mid \equiv_n \text{ appears in } \phi\}$.

Proof. Immediate from the previous two lemmas. □

Note: From now on, when an \mathcal{L} -formula in one variable is said to be in SDNF, this stronger form from Corollary 1 is meant.

These results imply that much as in the o-minimal case, any definable subset of G is a disjoint union of finitely many points and “intervals”. Here though, an “interval” may have endpoints which are not in G (that is, it may end at a cut), and it may consist only of elements obeying some congruence condition. One way to think of these is as intervals within a given congruence class (with endpoints possibly outside of G). These remarks are made precise below.

Lemma 3. *Let $A \subseteq G$ be a definable set. Then A is a (pairwise) disjoint union $S \sqcup I_1 \sqcup \cdots \sqcup I_l$, where S is a finite set of elements of G and I_1, \dots, I_l are sets of the form $\{x \mid a < kx < b, x \equiv c \pmod{n}\}$ where $a, b \in G \cup \{\pm\infty\}$, $c \in G$ and $k, n \in \mathbb{N}^+$.*

Moreover, if $\phi(x)$ is a formula in SDNF, defining A , then such a decomposition exists so that every k as above divides $\text{lcm}\{k_i \mid k_i x < a \text{ or } k_i x >$

a appears in ϕ }, and every n as above divides the $lcm\{n_i \mid \equiv_{n_i} \text{ appears in } \phi\}$.

Note: At times we will refer to a set of the form $\{x \mid a < kx < b, x \equiv c \pmod{n}\}$ as above, as an “interval with a congruence relation” to make the argument easier to read.

Proof. If $A \subseteq G$ is definable then by Corollary 1 we have a formula $\phi(x)$ defining A which is a disjunction of conjunctions of identities, strict inequalities, and congruence relations. Say $\phi(x)$ is $\bigvee_{i=1}^{\alpha} \lambda_i(x)$ where each $\lambda_i(x)$ is $\bigwedge_{j=1}^{\beta_i} \theta_{ij}(x)$.

We argue by induction on α . By Lemma 2, any strict inequality in which x occurs nontrivially is equivalent in G to one of the form $kx < a$ or $a < kx$ with $k \in \mathbb{N}^+$. Therefore any strict inequality can actually be thought of as having the form $a < kx < b$ where $a, b \in G \cup \{\pm\infty\}$. The conjunction of two such statements $a_1 < k_1x < b_1$ and $a_2 < k_2x < b_2$ can be written as

$$\max\left\{\frac{L}{k_1}a_1, \frac{L}{k_2}a_2\right\} < Lx < \min\left\{\frac{L}{k_1}b_1, \frac{L}{k_2}b_2\right\},$$

where $L = lcm\{k_1, k_2\}$. Thus the conjunction of any finite collection of strict inequalities of x defines a set of the form $\{x \mid a < Lx < b\}$, where L divides $lcm\{k \mid kx \text{ appears in an inequality in } \phi\}$.

Similarly, any congruence is equivalent in G to one of the form $x \equiv c \pmod{n}$ by Lemma 2. The conjunction of any finite number of congruences $x \equiv c_i \pmod{n_i}$ either defines the empty set or is equivalent in G to a single congruence $x \equiv c \pmod{n}$ with $n = lcm\{n_i\}$ by the Chinese Remainder

Theorem (the proof in [7, Thm. 5.4.2] remains correct in the present context). Thus, if $\phi(x)$ is a conjunction of identities, strict inequalities, and congruence relations, then ϕ defines either the empty set, a point in G , or a set of the form $\{x \mid a < kx < b, x \equiv c \pmod{n}\}$ where $a, b \in G \cup \{\pm\infty\}$, $c \in G$ and $k, n \in \mathbb{N}^+$ with k dividing $lcm\{k \mid kx \text{ appears in an inequality in } \phi\}$ and n dividing $lcm\{n_i \mid x \text{ appears in } \equiv_{n_i} \text{ in } \phi\}$. This concludes the base case $\alpha = 1$.

Now let $\alpha > 1$ and assume that the Corollary holds for any set which can be defined by the disjunction of α or fewer conjunctions of identities, strict inequalities, and congruences. Suppose $\phi(x)$ is $\bigvee_{i=1}^{\alpha+1} \lambda_i(x)$. Then by induction $\bigvee_{i=1}^{\alpha} \lambda_i(G)$ is the disjoint union of points and intervals with congruence relations. That is, $\bigvee_{i=1}^{\alpha} \lambda_i(G) = S \sqcup I_1 \sqcup \cdots \sqcup I_l$, where S is a finite set of points in G and I_1, \dots, I_l are intervals with congruence relations. Also, the base case of the induction implies that $\lambda_{\alpha+1}(G)$ is either empty, a point, or an interval with a congruence relation.

The only case in which we are not obviously done is when $\lambda_{\alpha+1}$ defines an interval with a congruence relation. Suppose that $\lambda_{\alpha+1}(G)$ is an interval with congruence relation and also that it has nonempty intersection with I_1 . Say $\lambda_{\alpha+1}(G) = \{x \mid a_1 < k_1x < b_1, x \equiv c_1 \pmod{n_1}\}$ and $I_1 = \{x \mid a_2 < k_2x < b_2, x \equiv c_2 \pmod{n_2}\}$. It will help momentarily to think of these sets instead (equivalently) as $\lambda_{\alpha+1}(G) = \{x \mid \frac{L}{k_1}a_1 < Lx < \frac{L}{k_1}b_1, x \equiv c_1 \pmod{n_1}\}$ and $I_1 = \{x \mid \frac{L}{k_2}a_2 < Lx < \frac{L}{k_2}b_2, x \equiv c_2 \pmod{n_2}\}$, where $L = lcm\{k_1, k_2\}$.

By the Chinese Remainder Theorem, there are c_3 and $n_3 = lcm\{n_1, n_2\}$

such that $x \equiv c_1 \pmod{n_1} \wedge x \equiv c_2 \pmod{n_2} \Leftrightarrow x \equiv c_3 \pmod{n_3}$. Notice that the equivalence class of $c_1 \pmod{n_1}$ is equal to the union of some finite number of equivalence classes $\pmod{n_3}$, with representatives g_1, \dots, g_p . Without loss of generality assume that $g_1 \equiv c_3 \pmod{n_3}$.

Suppose first that $\frac{L}{k_1}a_1 < \frac{L}{k_2}a_2 < \frac{L}{k_1}b_1 < \frac{L}{k_2}b_2$, and consider $\lambda_{\alpha+1}(G) \setminus I_1$, the part of $\lambda_{\alpha+1}(G)$ that is disjoint from I_1 . This set can be written as a disjoint union of intervals with congruence relations and a finite set as follows:

$$\begin{aligned}
& \{x \mid \frac{L}{k_1}a_1 < Lx < \frac{L}{k_2}a_2, x \equiv c_1 \pmod{n_1}\} \\
\sqcup & \{x \mid \frac{L}{k_2}a_2 < Lx < \frac{L}{k_1}b_1, x \equiv g_2 \pmod{n_3}\} \\
\sqcup & \{x \mid \frac{L}{k_2}a_2 < Lx < \frac{L}{k_1}b_1, x \equiv g_3 \pmod{n_3}\} \\
& \dots \\
\sqcup & \{x \mid \frac{L}{k_2}a_2 < Lx < \frac{L}{k_1}b_1, x \equiv g_p \pmod{n_3}\} \\
\sqcup & \{x \in \lambda_{\alpha+1}(G) \mid k_2x = a_2\}.
\end{aligned}$$

The other possible arrangements of $\frac{L}{k_1}a_1$, $\frac{L}{k_2}a_2$, $\frac{L}{k_1}b_1$, and $\frac{L}{k_2}b_2$ can all be handled in a similar fashion, so that whatever the case, $\lambda_{\alpha+1}(G) \setminus I_1$ is the disjoint union of intervals with congruence relations. Notice that induction allows us to insist that k_1 and k_2 each divide $\text{lcm}\{k \mid kx \text{ appears in an inequality in } \phi\}$ and therefore so does L . Induction also allows us to insist that n_1 and n_2 each divide $\text{lcm}\{n_i \mid \equiv_{n_i} \text{ appears in } \phi\}$ and therefore so does n_3 .

Repeated applications of this argument with each of these new disjoint intervals in place of $\lambda_{\alpha+1}(G)$ and I_2, I_3 , etc. instead of I_1 will result in a collection of disjoint intervals with congruence relations that satisfy the

required conditions.

□

As an immediate consequence to this Lemma we have uniform finiteness as follows.

Lemma 4. *Let $A \subseteq G^{m+1}$ be definable and suppose that for each $x \in G^m$ the fiber above x in A , A_x , is finite. Then there exists a uniform bound $M \in \mathbb{N}$ such that $|A_x| \leq M$ for every $x \in G^m$.*

Proof. If $A \subseteq G^{m+1}$ is definable then by Lemma 1 we have a formula $\phi(\bar{x})$ defining A which is the disjunction of conjunctions of identities, strict inequalities, and congruence relations. Say $\phi(\bar{x})$ is $\bigvee_{i=1}^{\alpha} \lambda_i(x_1, \dots, x_{m+1})$ where each $\lambda_i(\bar{x})$ is $\bigwedge_{j=1}^{\beta_\alpha} \theta_j(x_1, \dots, x_{m+1})$. Let $\bar{a} = (a_1, \dots, a_m) \in G^m$. For $A_{\bar{a}}$ to be finite is for $\phi(a_1, \dots, a_m, G)$ to be finite, which is true if and only if $\lambda_i(a_1, \dots, a_m, G)$ is finite for each i . The base case of the previous proof shows that $\lambda_i(\bar{a}, G)$ is either empty, one element, or an infinite set.

So if A_x is finite for every $x \in G^m$ then each λ_i can define no more than one point above any $x \in G^m$. This implies that ϕ can define no more than α points above any $x \in G^m$.

□

The rest of the chapter is devoted to defining a change point for a set A , and showing that if A is definable, then the set of all of its change points is also definable. This result will be critical when proving cell-decomposition in the next chapter. The idea is that if A is a definable subset of G then,

roughly speaking, locally A should look like some finite union of congruence classes. If this local picture changes at some point $g \in G$, then we say that g is a change point of A . This idea is made precise, and extended to higher dimensions in the following definition.

Definition 2. If $A \subseteq G^m$, an m -change point of A is any point $\bar{x} = (x_1, \dots, x_m) \in G^m$ such that at least one of the following holds (the “ m ” in m -change point is to indicate that we are interested in the m^{th} coordinate):

1. \bar{x} marks a division in congruence conditions:

there are $c \in G$ and $n \in \mathbb{N}^+$ such that for every positive ϵ in G there are $\gamma_1 \in (x_m - \epsilon, x_m)$ and $\gamma_2 \in (x_m, x_m + \epsilon)$ such that

$$\{(x_1, \dots, x_{m-1}, s) \mid \gamma_1 < s < x_m, s \equiv c \pmod{n}\} \subseteq A$$

and

$$\{(x_1, \dots, x_{m-1}, r) \mid x_m < r < \gamma_2, r \equiv c \pmod{n}\} \not\subseteq A$$

2. \bar{x} marks a division in congruence conditions:

there are $c \in G$ and $n \in \mathbb{N}^+$ such that for every positive ϵ in G there are $\gamma_1 \in (x_m - \epsilon, x_m)$ and $\gamma_2 \in (x_m, x_m + \epsilon)$ such that

$$\{(x_1, \dots, x_{m-1}, s) \mid \gamma_1 < s < x_m, s \equiv c \pmod{n}\} \not\subseteq A$$

and

$$\{(x_1, \dots, x_{m-1}, r) \mid x_m < r < \gamma_2, r \equiv c \pmod{n}\} \subseteq A$$

3. \bar{x} marks a missing member of a congruence class:

There are c, ϵ in G , with $\epsilon > 0$, and $n \in \mathbb{N}^+$ such that

$$\{(x_1, \dots, x_{m-1}, s) \mid x_m - \epsilon < s < x_m, s \equiv c \pmod{n}\} \subseteq A$$

and

$$\{(x_1, \dots, x_{m-1}, r) \mid x_m < r < x_m + \epsilon, r \equiv c \pmod{n}\} \subseteq A$$

and

$$x_m \equiv c \pmod{n} \text{ but } (x_1, \dots, x_m) \notin A$$

4. \bar{x} marks an isolated member of a congruence class:

There are c, ϵ in G , with $\epsilon > 0$, and $n \in \mathbb{N}^+$ such that

$$\{(x_1, \dots, x_{m-1}, s) \mid x_m - \epsilon < s < x_m + \epsilon, s \equiv c \pmod{n}\} \cap A = \{\bar{x}\}.$$

If \bar{x} satisfies condition (k) , $k \in \{1, 2, 3, 4\}$, we may say that \bar{x} is a change point of *type* (k) . Additionally, we may also say that \bar{x} is a change point of A for the congruence class of $c \pmod{n}$.

Lemma 5. *Let A be a definable subset of G with decomposition $S \sqcup I_1 \sqcup \dots \sqcup I_l$ as in Lemma 3. Any $g \in G$ which is a 1-change point of A must be either in S or an endpoint of I_i for some $i \in \{1, \dots, l\}$.*

Proof. Let $g \in G$ be a 1-change point of A of type (1). Without loss of generality $g \notin S$. Let $c \in G, n \in \mathbb{N}$ be such that for every $\epsilon > 0$ there are γ_1 and γ_2 in $(g - \epsilon, g)$ and $(g, g + \epsilon)$ respectively so that $(\gamma_1, g)_{\equiv_n c} \subseteq A$ and $(g, \gamma_2)_{\equiv_n c} \not\subseteq A$. Such c and n must exist by definition.

Partition $\mathcal{I} = \{I_1, \dots, I_l\}$ into three disjoint subsets in the following way:

$$\begin{aligned} \mathcal{I}_1 &= \{I \in \mathcal{I} \mid \exists a_1, a_2 \in I \text{ such that } a_1 < g < a_2\} \\ \mathcal{I}_2 &= \{I \in \mathcal{I} \mid \exists \epsilon > 0 \in G \text{ such that } (g - \epsilon, g + \epsilon) \cap I = \emptyset\} \\ \mathcal{I}_3 &= \{I \in \mathcal{I} \mid I \not\subseteq (\mathcal{I}_1 \cup \mathcal{I}_2)\}. \end{aligned}$$

Notice that \mathcal{I}_3 is exactly the set of those I_i of which g is an endpoint. Also notice that the definition of change point of type (1) implies that there must be elements of A arbitrarily close to g . That is, $\mathcal{I} \neq \mathcal{I}_2$. We will show that \mathcal{I}_3 is not empty.

Suppose that \mathcal{I}_3 is empty. If this is the case, then \mathcal{I}_1 must not be empty since $\mathcal{I} \neq \mathcal{I}_2$. Let $\epsilon > 0$ be small enough such that

$$\begin{aligned} S \cap (g - \epsilon, g + \epsilon) &= \emptyset \text{ and} \\ I \in \mathcal{I}_2 &\Rightarrow I \cap (g - \epsilon, g + \epsilon) = \emptyset \text{ and} \\ I \in \mathcal{I}_1 &\Rightarrow \exists a_1, a_2 \in I \text{ such that } a_1 < g - \epsilon < g + \epsilon < a_2. \end{aligned}$$

By assumption $\exists \gamma_1 \in (g - \epsilon, g)$ such that $(\gamma_1, g)_{\equiv_n c} \subseteq A$. So by hypothesis, $(\gamma_1, g)_{\equiv_n c} \subseteq I_{i_1} \cup \dots \cup I_{i_p}$ where $\mathcal{I}_1 = \{I_{i_1}, \dots, I_{i_p}\}$. So $(\gamma_1, g)_{\equiv_n c} \subseteq (\gamma_1, g)_{\equiv_{n_1} c_1} \cup \dots \cup (\gamma_1, g)_{\equiv_{n_p} c_p}$, where $\equiv_{n_j} c_j$ is the congruence condition from I_{i_j} , and because G is dense regular, $\{x \mid x \equiv c \pmod{n}\} \subseteq \bigcup_{i=1}^p \{x \mid x \equiv c_i \pmod{n_i}\}$. But if that is the case then for any $\gamma_2 \in (g, g + \epsilon)$, $(g, \gamma_2)_{\equiv_n c} \subseteq A$ which is a contradiction.

A symmetric argument will clearly work for 1-change points of type (2) and a very similar argument will also prove the lemma for 1-change points of type (3).

Finally, if $g \in G$ is a 1-change point of type (4) then by definition, g must be in A . We will show $g \in S$. Suppose $g \notin S$, then $g \in I_i = \{x \mid a_i < k_i x < b_i, x \equiv c_i \pmod{n_i}\}$ for some i . Then for any $c \in G$, $n \in \mathbb{N}^+$, and $\epsilon > 0$ we have, by dense regularity, that $(g - \epsilon, g + \epsilon)_{\equiv_n c} \cap I_i$ is either empty or infinite. This is a contradiction to g being a 1-change point of type (4), and the proof is complete.

□

Corollary 2. *If $A \subseteq G^m$ is a definable set then the set of all m -change points of A , $cp_m(A)$, is finite over G^{m-1} . That is, for each $(x_1, \dots, x_{m-1}) \in G^{m-1}$, there are only finitely many points in G^m of the form (x_1, \dots, x_{m-1}, t) which are m -change points of A .*

Proof. Lemma 5 shows that $cp_m(A)$ is finite if $m = 1$: so assume $m \geq 2$. Let $\phi(\bar{x})$ be a formula defining A , and let $\bar{a} \in G^{m-1}$. By the definition of change point it is clear that for any $t \in G$, (\bar{a}, t) is a change point of A if and only

if t is a change point of the set defined by $\phi(\bar{a}, x)$. This is a definable subset of G , and so by Lemma 5, has only finitely many change points. Since \bar{a} is arbitrary, and there are only finitely many change points of A over it, the Corollary is shown. □

Lemma 6. *Let A be a definable subset of G and let $\phi(x)$ be a definition of A in SDNF. If $g \in G$ is a 1-change point of A , then g is a change point for a congruence class $(\text{mod } n)$, such that n divides $\text{lcm}\{n \mid \equiv_n \text{ appears in } \phi\}$.*

Proof. Lemma 3 implies that $A = S \sqcup I_1 \sqcup \cdots \sqcup I_l$ where S is finite and each I_i is of the form $\{x \mid a < kx < b, x \equiv c \pmod{n}\}$.

Let g be a 1-change point of A of type (1). The previous Lemma shows that g must be the endpoint of at least one I_i . We now show that one such I_i actually “witnesses” the fact that g is a change point. That is, if $I_i = \{x \mid a_i < k_i x < b_i, x \equiv c_i \pmod{n_i}\}$ then g is a change point for a congruence class $(\text{mod } n_i)$.

Let $\mathcal{I}_1, \mathcal{I}_2$, and \mathcal{I}_3 be defined as they are in the previous proof.

First, we show that g must be the right endpoint of some I_i (by “right endpoint” we mean g is greater than every element of I_i and less than or equal to any other element of G with that property). Suppose not. Let $\epsilon > 0$ be small enough such that for each $I_i \in \mathcal{I}_1$ there are $a, b \in I_i$ such that $a < g - \epsilon < g + \epsilon < b$, and for each $I_i \in \mathcal{I}_2$, $I_i \cap (g - \epsilon, g + \epsilon) = \emptyset$. Since g is not the right endpoint of any I_i , we may assume that $(g - \epsilon, g)$

intersects no member of \mathcal{I}_3 . Then, for any $\gamma_1 \in (g - \epsilon, g)$ and any n and c , if $(\gamma_1, g)_{\equiv_n c} \subseteq A$, then

$$(\gamma_1, g)_{\equiv_n c} \subseteq (\gamma_1, g)_{\equiv_{n_1} c_1} \cup \cdots \cup (\gamma_1, g)_{\equiv_{n_p} c_p},$$

where $\equiv_{n_1} c_1, \dots, \equiv_{n_p} c_p$ are the congruence conditions from each set in \mathcal{I}_1 . By dense regularity $\{x \mid x \equiv c \pmod{n}\} \subseteq \bigcup_{i=1}^p \{x \mid x \equiv c_i \pmod{n_i}\}$, and therefore $(g, \gamma_2)_{\equiv_n c} \subseteq A$ for any $\gamma_2 \in (g, g + \epsilon)$. This conclusion contradicts g being a change point of type (1) of A : so g must be the right endpoint of I_i for some $i = 1, \dots, l$.

Let $\{J_1, \dots, J_q\} = \{I_i \mid g \text{ is the right endpoint of } I_i\}$ and say that for each $i \in \{1, \dots, q\}$, $J_i = \{x \mid a_i < k_i x < b_i, x \equiv c_i \pmod{n_i}\}$ (recycling some notation from the previous paragraph). Suppose that for each $i \in \{1, \dots, q\}$ there is some $\epsilon_i > 0 \in G$ such that $(g, g + \epsilon_i)_{\equiv_{n_i} c_i} \subseteq A$. Let $\epsilon'_1 = \min\{\epsilon_i\}$, let ϵ'_2 be small enough such that for each i , $a_i < k_i(g - \epsilon'_2)$, and let ϵ be as it was in the previous paragraph. Let $\epsilon' = \min\{\epsilon, \epsilon'_1, \epsilon'_2\}$. Now for any $\gamma_1 \in (g - \epsilon', g)$ and any n and c , if $(\gamma_1, g)_{\equiv_n c} \subseteq A$ then

$$(\gamma_1, g)_{\equiv_n c} \subseteq \bigcup_{i=1}^q J_i \cup \bigcup_{j=1}^p (\gamma_1, g)_{\equiv_{m_j} e_j},$$

where the $\equiv_{m_j} e_j$ are the congruence conditions from the sets in \mathcal{I}_1 . So much

as before,

$$\{x \mid x \equiv_n c\} \subseteq \bigcup_{i=1}^q \{x \mid x \equiv_{n_i} c_i\} \cup \bigcup_{j=1}^p \{x \mid x \equiv_{m_j} e_j\}.$$

The assumption that $(g, g + \epsilon')_{\equiv_{n_i} c_i} \subseteq A$ for each $i \in \{1, \dots, q\}$, together with the definition of \mathcal{I}_1 , implies that $(g, \gamma_2)_{\equiv_{nc}} \subseteq A$ for any $\gamma_2 \in (g, g + \epsilon')$, and we contradict the assumption that g is a change point of type (1). So for some $i \in \{1, \dots, q\}$, $(g, g + \epsilon_i)_{\equiv_{n_i} c_i} \not\subseteq A$ for all $\epsilon_i > 0$.

Without loss of generality assume there is no ϵ_1 as described above. Then for any $\epsilon > 0$ there are $\gamma_1 \in (g - \epsilon, g)$ and $\gamma_2 \in (g, g + \epsilon)$ such that $(\gamma_1, g)_{\equiv_{n_1} c_1} \subseteq A$, but $(g, \gamma_2)_{\equiv_{n_1} c_1} \not\subseteq A$. That is, J_1 witnesses that g is a change point. Lemma 3 shows that g satisfies the conclusion of the Lemma.

The argument is symmetric for change points of type (2).

Let g be a change point of type (3), for the congruence class of $c \pmod{n}$, which is not also a change point of type (1) or (2). Let \mathcal{I}_1 , \mathcal{I}_2 , and \mathcal{I}_3 be as before, and again let $\epsilon > 0$ be small enough such that for each $I_i \in \mathcal{I}_1$ there are $a, b \in I_i$ such that $a < g - \epsilon < g + \epsilon < b$, and for each $I_i \in \mathcal{I}_2$, $I_i \cap (g - \epsilon, g + \epsilon) = \emptyset$. Let $\epsilon_1 > 0$ be small enough such that $(g - \epsilon_1, g)_{\equiv_{nc}} \subseteq A$ and $(g, g + \epsilon_1)_{\equiv_{nc}} \subseteq A$, and let $\epsilon' = \min\{\epsilon, \epsilon_1\}$. Then since $(g - \epsilon', g)_{\equiv_{nc}} \subseteq A$, it must be the case that

$$(g - \epsilon', g)_{\equiv_{nc}} \subseteq \bigcup \mathcal{I}_1 \cup \bigcup \mathcal{I}_3.$$

Let $\mathcal{I}_1 = \{I_1, \dots, I_p\}$, where the congruence condition from I_j is $\equiv_{m_j} e_j$ for each $j \in \{1, \dots, p\}$, and let $\mathcal{I}_3 = \{J_1, \dots, J_q\}$, where the congruence condition from J_i is $\equiv_{n_i} c_i$ for each $i \in \{1, \dots, q\}$. Since $g \notin A$, $g \not\equiv_{m_j} e_j$ for all $j \in \{1, \dots, p\}$, but from above,

$$\{x \mid x \equiv_n c\} \subseteq \bigcup_{j=1}^p \{x \mid x \equiv_{m_j} e_j\} \cup \bigcup_{i=1}^q \{x \mid x \equiv_{n_i} c_i\};$$

so $g \equiv_{n_i} c_i$ for some $i \in \{1, \dots, q\}$. Since $(g - \epsilon', g)_{\equiv_{n_i} c_i} \subseteq A$ and g is not a change point of type (1), $(g, g + \epsilon'')_{\equiv_{n_i} c_i} \subseteq A$ for some $\epsilon'' > 0$. So g is a change point of type (3) for the congruence class of $c_i \pmod{n_i}$. As before, Lemma 3 shows that g satisfies the conclusion of the Lemma.

Assume finally that g is a change point of type (4), which is not of type (1), (2), or (3). If g is an isolated point of A , then g is a change point for its congruence class \pmod{n} for any n .

If g is not isolated, then let I_{j_1}, \dots, I_{j_r} be the I_i which have nontrivial intersection with $(g - \epsilon, g + \epsilon)$ for every $\epsilon > 0$. Since g is not a change point of type (1) or (2), if $(g, g + \epsilon)_{\equiv_{nc}} \subseteq A$ for some n, c, ϵ , there is an $\epsilon' > 0$ such that $(g - \epsilon', g)_{\equiv_{nc}} \subseteq A$ and vice versa. Moreover, if $(g, g + \epsilon)_{\equiv_{nc}} \subseteq A$ then $g \not\equiv c \pmod{n}$ since dense regularity implies that $(g - \epsilon', g + \epsilon')_{\equiv_{n'\epsilon'}} \cap [g, g + \epsilon)_{\equiv_{nc}}$ is either infinite or empty for any n', ϵ' , and ϵ' .

So if for each $i \in \{1, \dots, r\}$, $I_{j_i} = \{x \mid a_{j_i} < k_{j_i}x < b_{j_i}, x \equiv c_{j_i} \pmod{n_{j_i}}\}$, then $g \not\equiv c_{j_i} \pmod{n_{j_i}}$ for any $i \in \{1, \dots, r\}$. Each condition $x \equiv c_{j_i} \pmod{n_{j_i}}$ is equivalent in G to a finite disjunction of congruences

$(\text{mod } n^* = \text{lcm}\{n_{j_i}\})$. Let $c_1^*, \dots, c_{k^*}^*$ be a list of representatives for all such congruence conditions obtained from every I_{j_i} : then $g \not\equiv c_i^* \pmod{n^*}$ for every $i \in \{1, \dots, k^*\}$. However, g must be congruent to some $c^* \pmod{n^*}$, and for all $\epsilon > 0$, $(g - \epsilon, g + \epsilon)_{\equiv_{n^*} c^*}$ is disjoint from I_{j_1}, \dots, I_{j_r} . So there exists some $\epsilon > 0$ such that $(g - \epsilon, g + \epsilon)_{\equiv_{n^*} c^*} \cap A = \{g\}$. Clearly n^* divides $\text{lcm}\{n \mid \equiv_n \text{ appears in } \phi\}$, so the proof is complete. □

Lemma 7. *Given a definable set $A \subseteq G^m$ the set of all m -change points of A , $cp_m(A)$, is a definable set.*

Proof. Assume $m \geq 2$ and let $\bar{a} \in G^{m-1}$. As pointed out in Corollary 2, the m -change points of A above \bar{a} are exactly the same as the 1-change points of the fiber $A_{\bar{a}}$. Moreover, (\bar{a}, b) is a change point of A for a congruence class \pmod{n} if and only if b is a change point of $A_{\bar{a}}$ for the same congruence class \pmod{n} . Let $\phi(\bar{x})$ be a formula in SDNF defining A . If we think of x_1, \dots, x_{m-1} as parameters then we can apply Corollary 1 to get a new formula $\phi'(\bar{x})$, equivalent to $\phi(\bar{x})$ and also in SDNF, such that each inequality in ϕ' that contains x_m takes the form $kx_m < t(x_1, \dots, x_{m-1})$ (or $>$), and each congruence relation in ϕ' that contains x_m takes the form $x_m \equiv t(x_1, \dots, x_{m-1}) \pmod{n}$ where t is an \mathcal{L} -term.

Then $\phi'(\bar{a}, y)$ is a formula defining $A_{\bar{a}}$ which is in SDNF. The previous Lemma says that any change point of $A_{\bar{a}}$ must be a change point for a congruence class \pmod{p} where p divides $\text{lcm}\{n \mid \equiv_n \text{ appears in } \phi'(\bar{a}, y)\} = \text{lcm}\{n \mid \equiv_n \text{ appears in } \phi'(\bar{x})\}$. Since \bar{a} was arbitrary, this shows that every

change point of A is a change point for a congruence class $(\text{mod } p)$ where p divides $\text{lcm}\{n \mid \equiv_n \text{ appears in } \phi'(\bar{x})\}$. Lemma 6 says this is true when $m = 1$ as well.

In particular, the above shows that the number of congruence moduli needed to recognize all of the change points of A is finite. This is enough to see that we can obtain first-order definitions for the change points of A from the definition above. Here is a formula, $\psi_1(\bar{x}, y)$, for the change points of type (1) where we let $N = \text{lcm}\{n \mid \equiv_n \text{ appears in } \phi'(\bar{x})\}$:

$$\begin{aligned} \exists c \forall \epsilon > 0 \exists \gamma_1, \gamma_2 & \left[(y - \epsilon < \gamma_1 < y) \wedge (y < \gamma_2 < y + \epsilon) \right. \\ & \wedge \bigvee_{i=1}^N (\forall s (\gamma_1 < s < y \wedge s \equiv_i c \rightarrow (\bar{x}, s) \in A) \\ & \left. \wedge \neg \forall r (y < r < \gamma_2 \wedge r \equiv_i c \rightarrow (\bar{x}, r) \in A) \right] \end{aligned}$$

Similar formulae can be written down for types (2), (3), and (4).

□

Chapter 2

Cell Decomposition

We continue to parallel the o-minimal case in this chapter by proving a cell decomposition theorem. Lemma 3 tells us that definable subsets of G cannot be too complicated; the Theorem below tells us that this is the case for definable sets in higher dimensions as well. Roughly speaking, we will show that definable sets are made up of graphs of linear functions and sets of points between two linear functions. Of course, one must also account for the congruence relations in \mathcal{L} , but this is done in a natural way.

We will begin with a somewhat technical lemma needed to prove Theorem 1. A weakness of the definition of *change point* is that we insist they be elements of the group. Therefore, if a “change” in some set A occurs at a cut, then the set of change points of A , $cp_m(A)$ does not detect it. The lemma below shows that if you multiply the m^{th} coordinate of A by a sufficiently large integer, you can guarantee that all “changes” actually occur in the

group.

Lemma 8. *Let A be a definable subset of G and let $\phi(x)$ be a definition of A in SDNF. Let K be the product of all k such that kx appears in an inequality in $\phi(x)$, and let N be the lcm $\{K, n \mid \equiv_n \text{ appears in } \phi\}$. Let $\hat{A} = \{Kx \mid x \in A\}$. If there is no change point of \hat{A} in (g_1, g_2) and c is any element of G , then $(g_1, g_2)_{\equiv_N c}$ is either contained in \hat{A} or disjoint from \hat{A} .*

Proof. A definition $\hat{\phi}(x)$ for \hat{A} can be obtained from $\phi(x)$ by replacing each inequality $kx < a$ by $x < \frac{K}{k}a$ (likewise for $>$), replacing each identity $x = a$, by $x = Ka$, and replacing each congruence $x \equiv_n c$ by $x \equiv_{Kn} Kc$. The formula $\hat{\phi}(x)$ is in SDNF and Lemma 3 implies that \hat{A} is a disjoint union $S \sqcup I_1 \sqcup \dots \sqcup I_l$ where S is finite and I_1, \dots, I_l are sets of the form $\{x \mid a < x < b, x \equiv c \pmod{n}\}$.

We will establish the Lemma by showing that $(g_1, g_2) \cap \hat{A}$ is a union of sets $(g_1, g_2)_{\equiv_N c}$.

Let $h \in (g_1, g_2) \cap \hat{A}$. Then since $h \in \hat{A}$, h is either in S or in I_i for some $i \in \{1, \dots, l\}$. Suppose first that $h \in I_i$, and let $I_i = \{x \mid a_i < x < b_i, x \equiv c_i \pmod{n_i}\}$. By Lemma 3, n_i divides N which implies that $\{x \mid a_i < x < b_i, x \equiv h \pmod{N}\}$ is a subset of I_i . If $a_i \leq g_1$ and $g_2 \leq b_i$ then we have shown that $(g_1, g_2)_{\equiv_N h}$ is a subset of \hat{A} . If not, then at least one of a_i and b_i is in (g_1, g_2) .

Suppose $a_i \in (g_1, g_2)$. Since a_i is not a change point of \hat{A} and $(a_i, b_i)_{\equiv_N h} \subseteq \hat{A}$, there must exist $\gamma < a_i$ such that $(\gamma, b_i)_{\equiv_N h} \subseteq \hat{A}$. Since $(\gamma, a_i)_{\equiv_N h}$ is infinite, there must be $h' \in (\gamma, a_i)_{\equiv_N h}$ which is not in S , and clearly not

in I_i , and so, in I_j for some $j \neq i$. Repeat the argument for h' and I_j . If $a_j \leq g_1$ stop, otherwise repeat again. Eventually some $a_{j'}$ will be found such that $a_{j'} \leq g_1$ and $(a_{j'}, b_i)_{\equiv_N h} \subseteq \hat{A}$, since otherwise we would have found a change point between g_1 and g_2 which is a contradiction. If $b_i < g_2$ then a symmetric argument shows we can find some $b_{j''}$ such that $g_2 \leq b_{j''}$ and $(a_{j'}, b_{j''})_{\equiv_N h} \subseteq \hat{A}$. So we have shown for any $h \in (g_1, g_2) \cap \hat{A}$, if $h \notin S$ then $(g_1, g_2)_{\equiv_N h} \subseteq \hat{A}$.

Suppose now that $h \in (g_1, g_2) \cap \hat{A}$ and $h \in S$. Since h is not a change point of \hat{A} , for any $\epsilon > 0$ it must be the case that $(h - \epsilon, h + \epsilon)_{\equiv_N h} \cap \hat{A} \neq \{h\}$. In fact, $(h - \epsilon, h + \epsilon)_{\equiv_N h} \cap \hat{A}$ must be infinite, since otherwise a smaller ϵ could be found such that equality held. Assume ϵ is small enough such that $g_1 < h - \epsilon$ and $h + \epsilon < g_2$, then there must exist $h' \in (h - \epsilon, h + \epsilon)_{\equiv_N h}$ which is not in S and hence in some I_i . The previous argument shows then that $(g_1, g_2)_{\equiv_N h'} \subseteq \hat{A}$, and since $h \equiv h' \pmod{N}$, $(g_1, g_2)_{\equiv_N h} \subseteq \hat{A}$.

In any case, if $h \in (g_1, g_2) \cap \hat{A}$ then $(g_1, g_2)_{\equiv_N h} \subseteq \hat{A}$, so the proof is complete. □

Definition 3. We call a function $f : A \subseteq G^m \rightarrow G$ *linear* if there are a constant $\gamma \in G$ and, for all $i = 1, \dots, m$, elements $c_i \in G$ and integers p_i, n_i , with $n_i > 0$, such that $a_i - c_i \equiv 0 \pmod{n_i}$ and

$$f(\bar{a}) = \sum_{i=1}^m p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma$$

for all $\bar{a} = (a_1, \dots, a_m) \in A$.

When a function is written in this way, we will say that it is written in *standard linear form*.

Definition 4. A (0) -cell is a point $\{a\} \subseteq G$.

A (1) -cell is a set of the form

$$\{x \in G \mid f < kx < g, x \equiv c \pmod{n}\},$$

with $f < g \in G \cup \{\pm\infty\}$, $c \in G$, and $k, n \in \mathbb{N}^+$.

Let $i_j \in \{0, 1\}$ for $j = 1, \dots, m$ and $\bar{x} = (x_1, \dots, x_m)$.

An $(i_1, \dots, i_m, 0)$ -cell is a set of the form

$$\{(\bar{x}, t) \in G^{m+1} \mid \bar{x} \in D, f(\bar{x}) = t\},$$

with $f : D \rightarrow G$ a linear function and $D \subseteq G^m$ an (i_1, \dots, i_m) -cell.

An $(i_1, \dots, i_m, 1)$ -cell is a set C of the form

$$C = \{(\bar{x}, t) \in G^{m+1} \mid \bar{x} \in D, f(\bar{x}) < kt < g(\bar{x}), t \equiv c \pmod{n}\},$$

with $D = \pi_m(C)$ (the projection of C into the first m coordinates) an (i_1, \dots, i_m) -cell, $f, g : D \rightarrow G$ linear functions or $\pm\infty$, and $c \in G$, $k, n \in \mathbb{N}^+$.

By the *type* of an (i_1, \dots, i_m) -cell, we mean the tuple (i_1, \dots, i_m) .

Definition 5. A *decomposition of G^m into cells* (or just a *decomposition of G^m*) is a certain type of partition of G^m into finitely many cells, defined inductively as follows:

- (i) A decomposition of G is a finite partition of G into 0-cells and 1-cells.
- (ii) A decomposition of G^{m+1} is a finite partition of G^{m+1} into cells C_1, \dots, C_l such that $\{\pi_m(C_1), \dots, \pi_m(C_l)\}$ is a decomposition of G^m .

Note: Here, and for the rest of the paper, π_m will denote the projection map into the first m coordinates. We will abuse notation slightly and let π_m be the projection, into the first m coordinates, from any power of G greater than m .

Theorem 1. (I_m) *Given any definable sets $A_1, \dots, A_p \subseteq G^m$ there is a decomposition of G^m partitioning each of A_1, \dots, A_p .*

(II_m) *Moreover, for each definable function $f : A \rightarrow G$, $A \subseteq G^m$, there is a decomposition \mathcal{D} of G^m partitioning A such that the restriction $f|_B : B \rightarrow G$ to each cell $B \in \mathcal{D}$ with $B \subseteq A$ is linear.*

Proof. The proof is by induction on m .

(I_1) follows immediately from Lemma 3, and (II_1) is exactly like the proof of (II_{m+1}) below.

Assuming (I_m) and (II_m), we first prove (I_{m+1}). Let A_1, \dots, A_p be definable subsets of G^{m+1} . For each A_λ , $\lambda \in \{1, \dots, p\}$ let $\phi'_{A_\lambda}(\bar{x})$ be a formula defining A_λ in SDNF. If we think of x_1, \dots, x_m as parameters, then we can apply Corollary 1, as we did in the proof of Lemma 7, to get formulae $\phi_{A_\lambda}(\bar{x})$

also in SDNF, but with the added property that each inequality containing x_{m+1} nontrivially takes the form $kx_{m+1} < t(x_1, \dots, x_m)$ (or $>$) with $k \in \mathbb{N}^+$, and each congruence relation takes the form $x_{m+1} \equiv t(x_1, \dots, x_m) \pmod{n}$. For each λ let K_λ be the product of all k such that kx_{m+1} appears in an inequality in $\phi_{A_\lambda}(\bar{x})$, and let $K = \prod_{\lambda=1}^p K_\lambda$. For each λ let $\hat{A}_\lambda = \{(x_1, \dots, x_m, Kx_{m+1}) \mid (x_1, \dots, x_m, x_{m+1}) \in A_\lambda\}$. \hat{A}_λ is definable for each λ . Let

$$Y = cp_{m+1}(\hat{A}_1) \cup \dots \cup cp_{m+1}(\hat{A}_p).$$

Then, by Lemma 7 and Corollary 2, $Y \subseteq G^{m+1}$ is definable and the fiber $Y_{\bar{a}}$ is finite for every $\bar{a} \in G^m$. Therefore, by Lemma 4, there is $M \in \mathbb{N}$ such that $|Y_{\bar{a}}| \leq M$ for all $\bar{a} \in G^m$. For each $i \in \{0, \dots, M\}$ let $B_i = \{\bar{a} \in G^m \mid |Y_{\bar{a}}| = i\}$, and define functions $f_{i1}, f_{i2}, \dots, f_{ii}$ on B_i by

$$Y_{\bar{a}} = \{f_{i1}(\bar{a}), \dots, f_{ii}(\bar{a})\}, f_{i1}(\bar{a}) < \dots < f_{ii}(\bar{a}).$$

In addition, set $f_{i0} = -\infty$ and $f_{ii+1} = +\infty$.

We define for each $\lambda \in \{1, \dots, p\}$, $i \in \{0, \dots, M\}$ and $1 \leq j \leq i$

$$C_{ij\lambda} = \{\bar{a} \in B_i \mid f_{ij}(\bar{a}) \in (\hat{A}_\lambda)_{\bar{a}}\}$$

and

$$C'_{ij\lambda} = \{\bar{a} \in B_i \mid f_{ij}(\bar{a}) \equiv 0 \pmod{K}\}.$$

Let $N = lcm\{K, n \mid \equiv \pmod{n} \text{ appears in } \phi_{A_\lambda} \text{ for some } \lambda\}$, and let c_1, \dots, c_r

be a complete list of representatives for the congruence classes $(\text{mod } N)$. Then for each $i \in \{0, \dots, M\}$, $1 \leq j \leq i$, $\lambda \in \{1, \dots, p\}$ and $l \in \{1, \dots, r\}$, we define:

$$D_{ij\lambda l} = \{\bar{a} \in B_i \mid (f_{ij}(\bar{a}), f_{ij+1}(\bar{a}))_{\equiv_N c_l} \subseteq (\hat{A}_\lambda)_{\bar{a}}\}.$$

Now let \mathcal{D} be a decomposition of G^m which partitions each set $\pi_m(\hat{A}_\lambda)$, B_i , $C_{ij\lambda}$, $C'_{ij\lambda}$, and $D_{ij\lambda l}$, and which has the property that if a cell $E \in \mathcal{D}$ is a subset of B_i , then $f_{i1}|_E, \dots, f_{ii}|_E$ are linear functions. This decomposition must exist by the inductive hypotheses (I_m) and (II_m) .

For every cell $E \in \mathcal{D}$ with $E \subseteq B_i$ we let

$$\hat{\mathcal{D}}_E = \bigcup_{j=0}^i \bigcup_{k=1}^r \{(f_{ij}|_E, f_{ij+1}|_E)_{\equiv_N c_k}\} \cup \bigcup_{j=1}^i \{\Gamma(f_{ij}|_E)\}$$

where $\Gamma(f_{ij}|_E)$ denotes the *graph* in G^{m+1} of f_{ij} restricted to E . We claim that $\hat{\mathcal{D}} := \bigcup \{\hat{\mathcal{D}}_E \mid E \in \mathcal{D}\}$ is a decomposition of G^{m+1} which partitions each set $\hat{A}_1, \dots, \hat{A}_p$.

It is clear that $\hat{\mathcal{D}}$ is a decomposition of G^{m+1} . Let $F \in \hat{\mathcal{D}}$ and suppose that $F \cap \hat{A}_\lambda \neq \emptyset$ for some λ . We must show that $F \subseteq \hat{A}_\lambda$. Let $\bar{a} = (a_1, \dots, a_{m+1}) \in F$ and let $E_F = \pi_m(F)$. Since E_F is a cell in \mathcal{D} and \mathcal{D} partitions $\pi_m(\hat{A}_\lambda)$, $E_F \subseteq \pi_m(\hat{A}_\lambda)$. Suppose first that $f_{ij}(a_1, \dots, a_m) \neq a_{m+1}$ for all i, j . Then F must take the form $(f_{ij}|_{E_F}, f_{ij+1}|_{E_F})_{\equiv_N a_{m+1}}$ for some i, j . Since the fiber of \hat{A}_λ above (a_1, \dots, a_m) has no change points

strictly between $f_{ij}(a_1, \dots, a_m)$ and $f_{ij+1}(a_1, \dots, a_m)$, Lemma 8 shows that $(f_{ij}(a_1, \dots, a_m), f_{ij+1}(a_1, \dots, a_m))_{\equiv_N a_{m+1}}$ is a subset of this fiber. Now since \mathcal{D} partitions every $D_{ij\lambda}$, we have the same result for any other fiber over E_F , hence in this case $F \subseteq \hat{A}_\lambda$.

Suppose now that $f_{ij}(a_1, \dots, a_m) = a_{m+1}$ for some i, j . Then $F = \Gamma(f_{ij}|E_F)$. Since \mathcal{D} partitions each $C_{ij\lambda}$, $f_{ij}(\bar{b}) \in \hat{A}_\lambda$ for all $\bar{b} \in E_F$. So in any case, $F \subseteq \hat{A}_\lambda$, and therefore, $\hat{\mathcal{D}}$ partitions $\hat{A}_1, \dots, \hat{A}_p$.

We obtain a decomposition \mathcal{D}' of G^{m+1} which partitions A_1, \dots, A_p from $\hat{\mathcal{D}}$ as follows. Since N is a multiple of K , every c_l with $l \in \{1, \dots, r\}$ obeys either $(x \equiv_N c_l \rightarrow x \equiv_K 0)$ or $(x \equiv_N c_l \rightarrow x \not\equiv_K 0)$ in G (depending only on whether c_l is divisible by K). So for each $F \in \hat{\mathcal{D}}$ of the form $(f_{ij}|E, f_{ij+1}|E)_{\equiv_N c_l}$ such that c_l is divisible by K define a new cell where (essentially) the last coordinate has been divided by K :

$$F' := \left\{ \begin{array}{l} (x_1, \dots, x_m) \in E \\ (x_1, \dots, x_{m+1}) \mid f_{ij}(x_1, \dots, x_m) < Kx_{m+1} < f_{ij+1}(x_1, \dots, x_m) \\ x_{m+1} \equiv \frac{c_l}{K} \pmod{\frac{N}{K}} \end{array} \right\}.$$

For every cell $F \in \hat{\mathcal{D}}$ of the form $\Gamma(f_{ij}|E)$, either the $m+1^{\text{st}}$ coordinate of every point in F is divisible by K , or the $m+1^{\text{st}}$ coordinate of no point in F is divisible by K . This is because \mathcal{D} partitions the sets $C'_{ij\lambda}$. So for each $F \in \hat{\mathcal{D}}$ of the form $\Gamma(f_{ij}|E)$ such that $f_{ij}(x)$ is divisible by K for every $x \in E$ define a new cell $F' := \Gamma(\frac{f_{ij}}{K}|E)$.

We claim that the collection of all F' defined in these two ways forms

a decomposition \mathcal{D}' of G^{m+1} which partitions A_1, \dots, A_p . For any point $(x_1, \dots, x_{m+1}) \in G^{m+1}$, (x_1, \dots, Kx_{m+1}) is in exactly one cell $F \in \hat{\mathcal{D}}$ since $\hat{\mathcal{D}}$ is a decomposition. Moreover (x_1, \dots, Kx_{m+1}) clearly has its $m + 1^{\text{st}}$ coordinate divisible by K , so F must be a cell with an associated F' , and (x_1, \dots, x_{m+1}) is in F' . So \mathcal{D}' is a decomposition. Furthermore, since the cells of $\hat{\mathcal{D}}$ partition $\hat{A}_1, \dots, \hat{A}_p$, no cell in \mathcal{D}' can contain points both in A_λ and not in A_λ for any λ . So \mathcal{D}' partitions A_1, \dots, A_p .

This concludes the proof of (I_{m+1}) .

To prove (II_{m+1}) we let $A \subseteq G^{m+1}$ and $f : A \rightarrow G$ be definable. Let $\phi(\bar{x}, z)$ be a formula in SDNF defining the graph of f in G^{m+2} , and say $\phi(\bar{x}, z) = \bigvee_{i=1}^\alpha \psi_i(\bar{x}, z)$ where each ψ_i a conjunction of identities, strict inequalities, and congruence relations.

Now partition A by

$$A_i = \{\bar{x} \in A \mid \text{there exists } z \in G, \psi_i(\bar{x}, z)\} \setminus \bigcup_{j=1}^{i-1} A_j.$$

If, for some $i \in \{1, \dots, \alpha\}$, ψ_i does not contain an identity involving z , then by dense regularity either ψ_i must define the empty set or an infinite set, in which case f would not be a function. So each nontrivial ψ_i must contain a nontrivial identity involving z : an identity equivalent (by rearranging terms) to one of the form:

$$kz = \sum_{n=1}^{m+1} p_n x_n + \gamma$$

where $k, p_1, \dots, p_{m+1} \in \mathbb{Z}$, $k > 0$, and $\gamma \in G$. Every point (\bar{x}, z) in $\Gamma(f|_{A_i})$ must satisfy this identity, so when \bar{x} is restricted to values in A_i this identity exactly describes f .

We can now partition each A_i further into sets A_{ij} , so that if r_1, \dots, r_q is a list of representatives of the congruence classes in $G \bmod k$ then for all $\bar{x} \in A_{ij}$ and $n = 1, \dots, m+1$, $x_n \equiv_k c_{jn}$ for some $c_{jn} \in \{r_1, \dots, r_q\}$. That is, $x_n - c_{jn} \equiv_k 0$ for every $\bar{x} \in A_{ij}$. Then in A_{ij} we have:

$$\begin{aligned}
kz &= \sum_{n=1}^{m+1} p_n x_n + \gamma \\
\leftrightarrow kz &= \sum (p_n x_n - p_n c_{jn} + p_n c_{jn}) + \gamma \\
\leftrightarrow kz &= \sum (p_n x_n - p_n c_{jn}) + \sum p_n c_{jn} + \gamma \\
\leftrightarrow kz &= \sum p_n (x_n - c_{jn}) + \sum p_n c_{jn} + \gamma \\
\leftrightarrow z &= \frac{\sum p_n (x_n - c_{jn})}{k} + \frac{\sum p_n c_{jn} + \gamma}{k} \\
\leftrightarrow z &= \sum p_n \left(\frac{x_n - c_{jn}}{k} \right) + \left(\frac{\sum p_n c_{jn} + \gamma}{k} \right) \\
\leftrightarrow z &= \sum p_n \left(\frac{x_n - c_{jn}}{k} \right) + \gamma' .
\end{aligned}$$

The first three equivalences here are simply using commutativity and associativity of the group. Since $\sum p_n (x_n - c_{jn})$ is divisible by k by assumption, the fourth line implies that $\sum p_n c_{jn} + \gamma$ must be divisible by k , so we are justified when we divide both sides by k in the fourth equivalence. The fifth equivalence is group arithmetic and the fact that $x_n - c_{jn}$ is divisible by k , and the last line is just the renaming of a term.

So f is linear on each A_{ij} and we can apply (I_{m+1}) to decompose all the A_{ij} simultaneously into cells, on which f is linear.

□

Chapter 3

Flats, Linear Functions, and Projections

The goal of this section is to define a special type of cell called a flat, to show that coordinate projections of flats are finite unions of flats, and to prove that in certain cases coordinate projections of cells are cells. In order to reach these conclusions we must examine linear functions and their behavior on flats.

Definition 6. Certain cells, called *flats*, are defined as follows.

All (0)-cells are flats.

A (1)-cell is a flat if and only if it is of the form

$$\{x \in G \mid x \equiv c \pmod{n}\},$$

with $c \in G$ and $n \in \mathbb{N}^+$.

An $(i_1, \dots, i_m, 0)$ -cell is a flat if and only if it is of the form

$$\{(\bar{x}, y) \in G^{m+1} \mid \bar{x} \in D, f(\bar{x}) = y\},$$

with $f : D \rightarrow G$ a linear function and $D \subseteq G^m$ a flat.

An $(i_1, \dots, i_m, 1)$ -cell is a flat if and only if it is of the form

$$\{(\bar{x}, y) \in G^{m+1} \mid \bar{x} \in D, y \equiv c \pmod{n}\},$$

with $c \in G$, $n \in \mathbb{N}^+$, and D a flat.

Note: A flat which is an (i_1, \dots, i_m) -cell may be referred to as an (i_1, \dots, i_m) -flat.

Lemma 9. *Let D_1 and D_2 be 1-flats. Then the set $D_1 + D_2 := \{x + y \mid x \in D_1, y \in D_2\}$ is a 1-flat.*

In fact, if $D_1 = \{x \mid x \equiv c_1 \pmod{n_1}\}$ and $D_2 = \{x \mid x \equiv c_2 \pmod{n_2}\}$, then $D_1 + D_2 = \{x \mid x \equiv (c_1 + c_2) \pmod{\gcd\{n_1, n_2\}}\}$

Proof. Let D_1 and D_2 be as they are in the Lemma. If $n_1 = n_2$, then the Lemma is trivial, so assume that $n_1 \neq n_2$ and let $N = \gcd\{n_1, n_2\}$.

Let $a \in D_1 + D_2$. Then there are $b_1 \in D_1$ and $b_2 \in D_2$ such that $a = b_1 + b_2$. Since $b_1 \equiv c_1 \pmod{n_1}$, we also have $b_1 \equiv c_1 \pmod{N}$ and similarly, $b_2 \equiv c_2 \pmod{N}$: so $b_1 + b_2 \equiv c_1 + c_2 \pmod{N}$. Therefore $a \in \{x \mid x \equiv (c_1 + c_2) \pmod{N}\}$ and $D_1 + D_2 \subseteq \{x \mid x \equiv (c_1 + c_2) \pmod{N}\}$.

Now let $a \in \{x \mid x \equiv (c_1 + c_2) \pmod{N}\}$ and let $s = \frac{n_1}{N}$ and $t = \frac{n_2}{N}$. Then s and t are relatively prime, so there exist $k_1, k_2 \in \mathbb{Z}$ such that $k_1s + k_2t = 1$. Since $a \equiv (c_1 + c_2) \pmod{N}$, $a = c_1 + c_2 + Nb$ for some $b \in G$. Then using the fact that $k_1sN + k_2tN = N$ we have:

$$\begin{aligned}
a &= c_1 + c_2 + Nb \\
\leftrightarrow a &= c_1 + c_2 + (k_1sN + k_2tN)b \\
\leftrightarrow a &= c_1 + c_2 + (k_1n_1 + k_2n_2)b \\
\leftrightarrow a &= c_1 + c_2 + (k_1n_1)b + (k_2n_2)b \\
\leftrightarrow a &= (c_1 + (k_1n_1)b) + (c_2 + (k_2n_2)b).
\end{aligned}$$

Clearly $c_1 + (k_1n_1)b \in D_1$ and $c_2 + (k_2n_2)b \in D_2$, so $a \in D_1 + D_2$, and $\{x \mid x \equiv (c_1 + c_2) \pmod{N}\} \subseteq D_1 + D_2$. This completes the proof. \square

Lemma 10. *Let D be a 1-flat and $f : D \rightarrow G$ be a linear function such that for some $m \in \mathbb{N}^+$, $f(a)$ is divisible by m for all $a \in D$. Then the function given by $f'(a) = \frac{f(a)}{m}$ is a linear function on D .*

Proof. Since f is linear, there are $p, n \in \mathbb{Z}$, $n > 0$, and $c, \gamma \in G$ such that

$$f(a) = p \left(\frac{a - c}{n} \right) + \gamma$$

for all $a \in D$. For all a in D , the following equalities hold by group arithmetic:

$$\begin{aligned}
 f'(a) &= \frac{p\left(\frac{a-c}{n}\right) + \gamma}{m} = \frac{\left(\frac{p(a-c)}{n}\right) + \gamma}{m} \\
 &= \frac{\left(\frac{p(a-c) + n\gamma}{n}\right)}{m} \\
 &= \frac{p(a-c) + n\gamma}{nm} \\
 &= \frac{pa - pc + n\gamma}{nm}.
 \end{aligned}$$

So if $d = -pc + n\gamma$, then $pa \equiv -d \pmod{nm}$ for all $a \in D$. Letting $c' \in D$ and continuing from above we see that

$$\begin{aligned}
 \frac{pa - pc + n\gamma}{nm} &= \frac{pa + d}{nm} \\
 &= \frac{pa - pc' + pc' + d}{nm} \\
 &= \frac{pa - pc'}{nm} + \frac{pc' + d}{nm} \quad (*) \\
 &= \frac{p(a-c')}{nm} + \gamma' \quad \left(\text{letting } \gamma' = \frac{pc' + d}{nm}\right).
 \end{aligned}$$

Note that since pa and pc' are each congruent to $-d \pmod{nm}$, $pa - pc' \equiv 0 \pmod{nm}$. Similarly, $pc' + d$ is also divisible by nm , so $(*)$ is justified.

Let $n' = \frac{nm}{\gcd\{nm, p\}}$ and $p' = \frac{p}{\gcd\{nm, p\}}$. Since $p(a - c')$ is divisible by nm for all $a \in D$, $(a - c')$ must be divisible by n' for all $a \in D$. Therefore we can

write

$$\frac{p(a - c')}{nm} + \gamma' = p' \left(\frac{a - c'}{n'} \right) + \gamma'$$

for all $a \in D$, and f' is a linear function on D .

□

Lemma 11. *If D is a 1-flat and $f_1, f_2 : D \rightarrow G$ are linear functions on D , then $f = f_1 + f_2$ is a linear function on D .*

Proof. Since f_1 and f_2 are linear, there are $p_1, p_2, n_1, n_2 \in \mathbb{Z}$ and $c_1, c_2, \gamma_1, \gamma_2 \in G$ such that $n_1, n_2 > 0$ and

$$f_1(a) = p_1 \left(\frac{a - c_1}{n_1} \right) + \gamma_1 \quad \text{and} \quad f_2(a) = p_2 \left(\frac{a - c_2}{n_2} \right) + \gamma_2$$

for all $a \in D$.

So for all a in D ,

$$\begin{aligned} f(a) &= p_1 \left(\frac{a - c_1}{n_1} \right) + \gamma_1 + p_2 \left(\frac{a - c_2}{n_2} \right) + \gamma_2 \\ &= \frac{p_1(a - c_1)}{n_1} + \frac{p_2(a - c_2)}{n_2} + \gamma_1 + \gamma_2 \\ &= \frac{n_2 p_1(a - c_1)}{n_1 n_2} + \frac{n_1 p_2(a - c_2)}{n_1 n_2} + \gamma \quad (\text{where } \gamma = \gamma_1 + \gamma_2) \\ &= \frac{n_2 p_1(a - c_1) + n_1 p_2(a - c_2)}{n_1 n_2} + \gamma \\ &= \frac{(n_2 p_1 + n_1 p_2)a - (n_2 p_1 c_1 + n_1 p_2 c_2)}{n_1 n_2} + \gamma \end{aligned}$$

Now an argument identical to the second half of the previous proof will show that f is a linear function.

□

Lemma 12. *Let D be a 1-flat in G and $f : D \rightarrow G$ be a non-constant linear function on D : then the image of D under f in G is of the form $\{x \in G \mid x \equiv c \pmod{n}\}$, and so is a 1-flat.*

Proof. Let $D = \{x \in G \mid x \equiv c_0 \pmod{n_0}\}$ — where $c_0 \in G$ and $n_0 \in \mathbb{N}^+$ — and let $f(a) = p(\frac{a-c}{n}) + \gamma$ for all $a \in D$. We will view f as the composition of various “basic” functions and show that each one of these functions takes 1-flats to 1-flats.

Let $f_1 : D \rightarrow G$ be such that $f_1(a) = a - c$ for all $a \in D$. Since

$$\{f_1(a) \mid a \in D\} = \{a - c \mid a \in D\} = \{x \mid x \equiv c_0 - c \pmod{n_0}\},$$

the image of f_1 , $\{f_1(a) \mid a \in D\}$, is a 1-flat, say D_1 . For clarity in the remainder of the proof, let us say that $D_1 = \{x \mid x \equiv c_1 \pmod{n_1}\}$.

Let $f_2 : D_1 \rightarrow G$ be such that $f_2(a) = \frac{a}{n}$ for all $a \in D_1$. The argument divides into two cases, depending on whether n divides n_1 .

Suppose n divides n_1 . Since f is a linear function on D , by definition every element of D_1 must be divisible by n , so since $c_1 \in D_1$, c_1 must be

divisible by n . So we have:

$$\begin{aligned} \{f_2(a) \mid a \in D_1\} &= \left\{ \frac{c_1 + n_1g}{n} \mid g \in G \right\} \\ &= \left\{ \frac{c_1}{n} + \left(\frac{n_1}{n}\right)g \mid g \in G \right\} \\ &= \left\{ x \mid x \equiv \frac{c_1}{n} \pmod{\frac{n_1}{n}} \right\}. \end{aligned}$$

So in this case $\{f_2(a) \mid a \in D_1\}$ is a 1-flat.

Suppose now that n does not divide n_1 . As before, c_1 must be divisible by n . For every $g \in G$, $c_1 + n_1g \in D_1$ is divisible by n , and so by $\gcd\{n, n_1\}$.

Thus

$$\frac{c_1 + n_1g}{\gcd\{n, n_1\}} = \frac{c_1}{\gcd\{n, n_1\}} + \frac{n_1}{\gcd\{n, n_1\}}g.$$

Let $N = \frac{n}{\gcd\{n, n_1\}}$. Then the left-hand side of the equation above is divisible by N , hence so is the right-hand side. Since n divides c_1 , N divides $\frac{c_1}{\gcd\{n, n_1\}}$, and so N must divide $\frac{n_1}{\gcd\{n, n_1\}}g$ for the whole expression to be divisible by N . Because $\frac{n_1}{\gcd\{n, n_1\}}$ and N are relatively prime, N must divide g . Remember that g was arbitrary in G : so we have shown that, in this case, the whole group G must be divisible by $N = \frac{n}{\gcd\{n, n_1\}}$.

We can now see that in this case we have:

$$\begin{aligned}
\{f_2(a) \mid a \in D_1\} &= \left\{ \frac{c_1 + n_1 g}{n} \mid g \in G \right\} \\
&= \left\{ \frac{c_1}{n} + \frac{n_1 g}{n} \mid g \in G \right\} \\
&= \left\{ \frac{c_1}{n} + \left(\frac{n_1}{\gcd\{n, n_1\}} \right) \left(\frac{g}{N} \right) \mid g \in G \right\} \\
&= \left\{ x \mid x \equiv \frac{c_1}{n} \pmod{\frac{n_1}{\gcd\{n, n_1\}}} \right\}.
\end{aligned}$$

So in either case $\{f_2(a) \mid a \in D_1\}$ is a 1-flat, say D_2 . Let us say that $D_2 = \{x \mid x \equiv c_2 \pmod{n_2}\}$.

Let $f_3 : D_2 \rightarrow G$ be such that $f_3(a) = pa$ for all $a \in D_2$. Then:

$$\begin{aligned}
\{f_3(a) \mid a \in D_2\} &= \{p(c_2 + n_2 g) \mid g \in G\} \\
&= \{pc_2 + pn_2 g \mid g \in G\} \\
&= \{x \mid x \equiv pc_2 \pmod{pn_2}\}.
\end{aligned}$$

So $\{f_3(a) \mid a \in D_2\}$ is a 1-flat, say D_3 .

Let $f_4 : D_3 \rightarrow G$ be such that $f_4(a) = a + \gamma$ for all $a \in D_3$. Then by the same argument as for f_1 , $\{f_4(a) \mid a \in D_3\}$ is a 1-flat. Since $f(a) = f_4(f_3(f_2(f_1(a))))$ for all a in D , $\{f(a) \mid a \in D\} = \{f_4(a) \mid a \in D_3\}$, and the image of D under f is a 1-flat.

□

Lemma 13. *Let D be a $(1, \dots, 1)$ -flat in G^m , and $f : D \rightarrow G$ be a linear*

function such that for some $n \in \mathbb{N}^+$, $f(\bar{a})$ is divisible by n for all $\bar{a} \in D$.

Then the function given by $f'(\bar{a}) = \frac{f(\bar{a})}{n}$ is a linear function on D .

Proof. Since f is linear, there are $\gamma \in G$ and, for $i = 1, \dots, m$, elements $c_i \in G$ and integers p_i, n_i , with $n_i > 0$, such that $a_i - c_i \equiv 0 \pmod{n_i}$ and

$$f(\bar{a}) = \sum_{i=1}^m p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma$$

for all $\bar{a} = (a_1, \dots, a_m) \in D$.

We proceed by induction on m .

The base case is Lemma 10.

Let $m > 1$ and assume that the Lemma holds in G^k whenever $k < m$.

Suppose that $p_j = 0$ for some $j \in \{1, \dots, m\}$. For all $i \in \{1, \dots, m-1\}$ let $\hat{p}_i = p_i$ for $i < j$ and $\hat{p}_i = p_{i+1}$ for $i \geq j$, and define \hat{c}_i and \hat{n}_i similarly. Let \hat{D} be the projection of D into every coordinate but the j^{th} . It is clear from the definition of flat that \hat{D} is a $(1, \dots, 1)$ -flat in G^{m-1} . Let $\hat{f} : \hat{D} \rightarrow G$ be the linear function on \hat{D} such that

$$\hat{f}(\bar{b}) = \sum_{i=1}^{m-1} \hat{p}_i \left(\frac{b_i - \hat{c}_i}{\hat{n}_i} \right) + \gamma$$

for all $\bar{b} = (b_1, \dots, b_{m-1}) \in \hat{D}$. Then $\hat{f}(\bar{b})$ is divisible by n for all $\bar{b} \in \hat{D}$ since by construction $\{\hat{f}(\bar{b}) \mid \bar{b} \in \hat{D}\} = \{f(\bar{a}) \mid \bar{a} \in D\}$.

By induction, the function given by $f^*(\bar{b}) = \frac{\hat{f}(\bar{b})}{n}$ is a linear function on \hat{D} . Thus we have $\gamma^* \in G$ and $p_i^*, n_i^* \in \mathbb{Z}$, $n_i^* > 0$, $c_i^* \in G$ for $i = 1, \dots, m-1$

such that

$$f^*(\bar{b}) = \sum_{i=1}^{m-1} p_i^* \left(\frac{b_i - c_i^*}{n_i^*} \right) + \gamma^*$$

for all $\bar{b} \in \hat{D}$.

Let $p'_i = p_i^*$ if $i < j$, $p'_i = 0$ if $i = j$, and $p'_i = p_i^*$ if $i > j$ for all $i \in \{1, \dots, m\}$, and define c'_i and n'_i similarly. Then by construction it is clear that if $f'(\bar{a}) = \frac{f(\bar{a})}{n}$ for all $\bar{a} \in D$, then

$$f'(\bar{a}) = \sum_{i=1}^m p'_i \left(\frac{a_i - c'_i}{n'_i} \right) + \gamma^*$$

for all $\bar{a} \in D$. So f' is a linear function on D .

Suppose now that $p_i \neq 0$ for all $i \in \{1, \dots, m\}$. For each $i \in \{1, \dots, m\}$ let D_i be the projection of D into its i^{th} coordinate. Each D_i is a 1-flat by definition of flat. If for each $i \in \{1, \dots, m\}$ f_i is the function on D_i given by $f_i(a) = p_i \left(\frac{a - c_i}{n_i} \right)$, then f_i is linear on D_i . By Lemma 12, the image of D_i under f_i is a 1-flat. Let us say $f_i(D_i) = \{x \in G \mid x \equiv C_i \pmod{N_i}\}$. There are two cases, depending on whether n divides N_i for some i .

Suppose n divides N_j for some $j \in \{1, \dots, m\}$. Then for all $\bar{a} \in D$ we can write:

$$f(\bar{a}) = p_j \left(\frac{a_j - c_j}{n_j} \right) - C_j + \sum_{i=1, i \neq j}^m p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma + C_j.$$

For all $\bar{a} \in D$, $f(\bar{a})$ is divisible by n by assumption, and $p_j \left(\frac{a_j - c_j}{n_j} \right) - C_j$ is congruent to 0 (mod N_i), hence divisible by n . Therefore $\sum_{i \neq j} p_i \left(\frac{a_i - c_i}{n_i} \right) +$

$\gamma + C_j$ must be divisible by n as well. Let us say that $h_1(\bar{a}) = p_j \left(\frac{a_j - c_j}{n_j} \right) - C_j$ and $h_2(\bar{a}) = \sum_{i \neq j} p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma + C_j$. Then we may think of h_1 and h_2 as linear functions in fewer variables (as we did with f earlier in the proof) and apply induction to conclude that $h'_1(\bar{a}) = \frac{h_1(\bar{a})}{n}$ and $h'_2(\bar{a}) = \frac{h_2(\bar{a})}{n}$ are each linear functions on the appropriate flats. Now we have an expression for $f'(\bar{a}) = \frac{f(\bar{a})}{n}$ — apply h_1 to the j^{th} coordinate and h_2 to the rest — which shows that f' is linear on D .

Finally, suppose that n divides no N_i . Because $f(\bar{a})$ is divisible by n for all $\bar{a} \in D$, $(\sum_{i=1}^m x_i) + \gamma$ is divisible by n for all $x_1 \in f_1(D_1), \dots, x_m \in f_m(D_m)$. Repeated application of Lemma 9 implies that $\{x_1 + \dots + x_m \mid x_i \in f_i(D_i)\} = \{x \mid x \equiv C \pmod{\gcd\{N_1, \dots, N_m\}}\}$ for some $C \in G$. Thus any element of G which is congruent to $C + \gamma \pmod{\gcd\{N_1, \dots, N_m\}}$ must be divisible by n .

Since n does not divide $\gcd\{N_1, \dots, N_m\}$, the proof of Lemma 12 (paragraph five) shows that every element of G must be divisible by $N = \frac{n}{\gcd\{n, N_1, \dots, N_m\}}$. So for all $\bar{a} \in D$,

$$\begin{aligned} f'(\bar{a}) &= \frac{f(\bar{a})}{n} = \frac{\sum_{i=1}^m p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma}{N \cdot \gcd\{n, N_1, \dots, N_m\}} \\ &= \frac{\sum_{i=1}^m p_i \left(\frac{a_i - c_i}{N n_i} \right) + \frac{\gamma}{N}}{\gcd\{n, N_1, \dots, N_m\}}. \quad (*) \end{aligned}$$

Recall from earlier that $f_1(a) = p_1 \left(\frac{a - c_1}{n_1} \right)$ for all $a \in D_1$, and that $f_1(D_1) =$

$\{x \in G \mid x \equiv C_1 \pmod{N_1}\}$. If $f'_1(a) = p_1 \left(\frac{a-c_1}{Nn_1} \right)$ for all $a \in D_1$, then

$$\begin{aligned} f'_1(D_1) &= \left\{ \frac{f_1(a)}{N} \mid a \in D_1 \right\} \\ &= \left\{ \frac{x}{N} \mid x \equiv C_1 \pmod{N_1} \right\} \\ &= \left\{ \frac{x}{N} \mid x = C_1 + N_1g \text{ for some } g \in G \right\} \\ &= \left\{ y \mid y = \frac{C_1}{N} + N_1 \frac{g}{N} \text{ for some } g \in G \right\}, \end{aligned}$$

and since every element of G is divisible by N , this result implies that $f'_1(D_1) = \{y \mid y \equiv \frac{C_1}{N} \pmod{N_1}\}$. Because $\gcd\{n, N_1, \dots, N_m\}$ divides N_1 , (*) may be handled by the second case of the induction step in this proof, which shows that f' is a linear function on D .

□

Corollary 3. *Let D be an (i_1, \dots, i_m) -flat in G^m and $f : D \rightarrow G$ be a linear function, with standard linear form*

$$f(\bar{a}) = \sum_{i=1}^m p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma,$$

such that $p_j = 0$ whenever $i_j = 0$. If $n \in \mathbb{N}^+$ divides $f(\bar{a})$ for all $\bar{a} \in D$, then the function given by $f'(\bar{a}) = \frac{f(\bar{a})}{n}$ is a linear function on D , with standard linear form

$$f'(\bar{a}) = \sum_{i=1}^m p'_i \left(\frac{a_i - c'_i}{n'_i} \right) + \gamma',$$

such that $p'_j = 0$ whenever $i_j = 0$.

Proof. If D is a $(0, \dots, 0)$ -flat, then the Corollary is trivial since in this case f must be a constant function.

Assume $i_j = 1$ for some $j \in \{1, \dots, m\}$. Let j_1, \dots, j_k be those indices for which $i_{j_*} = 1$, and let $\pi : G^m \rightarrow G^k$ be the coordinate projection map into the j_1, \dots, j_k coordinates. The definition of flat implies that $\hat{D} = \pi(D)$ is a $(1, \dots, 1)$ -flat in G^k , and the function \hat{f} given by

$$\hat{f}(\bar{b}) = \sum_{i=1}^k p_{j_i} \left(\frac{b_i - c_{j_i}}{n_{j_i}} \right) + \gamma$$

for $\bar{b} \in \hat{D}$ is a linear function on \hat{D} . It is also clear from construction that $\hat{f}(\bar{b})$ is divisible by n for all $\bar{b} \in \hat{D}$: so by Lemma 13 the function given by $f^*(\bar{b}) = \frac{\hat{f}(\bar{b})}{n}$ is linear on \hat{D} . If

$$f^*(\bar{b}) = \sum_{i=1}^k p'_i \left(\frac{b_i - c'_i}{n'_i} \right) + \gamma'$$

for all $\bar{b} \in \hat{D}$, then

$$f'(\bar{a}) = \sum_{i=1}^k p'_i \left(\frac{a_{j_i} - c'_i}{n'_i} \right) + \sum_{i \neq j_*} 0 \left(\frac{a_i - 0}{1} \right) + \gamma'$$

is an expression for f' of the desired form. □

Lemma 14. *Let D be an (i_1, \dots, i_m) -flat in G^m and let f be a linear function*

on D : then f has a standard linear form

$$f(\bar{a}) = \sum_{i=1}^m p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma$$

(for all $\bar{a} \in D$) such that $p_j = 0$ whenever $i_j = 0$.

Proof. The proof is by induction on m .

If D is a 0-flat, then D is a single point in G , so f may be written as a constant function, $f(a) = \gamma$. If D is a 1-flat, there is nothing to show, so the base case is complete.

Assume the Lemma holds for all $k < m$ and let j_1, \dots, j_k be the indices for which $i_{j_*} = 1$. $D' = \pi_{m-1}(D) \subseteq G^{m-1}$ is an (i_1, \dots, i_{m-1}) -flat. If f has standard linear form

$$f(\bar{a}) = \sum_{i=1}^m p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma,$$

let $f' : D' \rightarrow G$ be given by

$$f'(\bar{b}) = \sum_{i=1}^{m-1} p_i \left(\frac{b_i - c_i}{n_i} \right) + \gamma.$$

Since f' is a linear function on a flat in G^{m-1} , the induction hypothesis provides f' with a standard linear form

$$f'(\bar{b}) = \sum_{i=1}^{m-1} p'_i \left(\frac{b_i - c'_i}{n'_i} \right) + \gamma'$$

such that $p'_j = 0$ whenever $i_j = 0$. So

$$f(\bar{a}) = \sum_{i=1}^{m-1} p'_i \left(\frac{a_i - c'_i}{n'_i} \right) + p_m \left(\frac{a_m - c_m}{n_m} \right) + \gamma'$$

for all $\bar{a} \in D$, and the argument is complete if $i_m = 1$.

Suppose $i_m = 0$: then there exists a linear function $h : D' \rightarrow G$ such that for all $\bar{a} = (a_1, \dots, a_m) \in D$, $h(a_1, \dots, a_{m-1}) = a_m$. Thus the m^{th} term of f , as displayed above, may be rewritten as $p_m \left(\frac{h(a_1, \dots, a_{m-1}) - c_m}{n_m} \right)$. Since the function from D' to G given by $h(\bar{b}) - c_m$ is linear, this function has by induction hypothesis a standard linear form permitting application of Corollary 3. Let us say that this application of Corollary 3 shows that for all $\bar{b} \in D'$,

$$\frac{h(\bar{b}) - c_m}{n_m} = h'(\bar{b}) = \sum_{i=1}^{m-1} q_i \left(\frac{b_i - d_i}{k_i} \right) + \delta,$$

where $q_j = 0$ whenever $i_j = 0$. Now we have

$$f(\bar{a}) = \sum_{i=1}^{m-1} p'_i \left(\frac{a_i - c'_i}{n'_i} \right) + \sum_{i=1}^{m-1} p_m q_i \left(\frac{a_i - d_i}{k_i} \right) + 0 \left(\frac{a_m - 0}{1} \right) + \delta + \gamma'$$

for all $\bar{a} \in D$, where $p'_i = 0$ and $p_m q_i = 0$ whenever $i_i = 0$. One can combine the j^{th} terms of each summation with the help of Lemma 11 to get an expression for f of the desired form. □

Lemma 15. *Let D be a flat in G^m and $f : D \rightarrow G$ be a non-constant linear function on D : then the image of D under f in G is of the form*

$\{x \in G \mid x \equiv c \pmod{n}\}$ and is a 1-flat.

Proof. The proof is by induction on m .

If D is a 0-flat then f must be constant. If D is a 1-flat then Lemma 12 yields the desired conclusion.

Assume the Lemma holds for all $k < m$, and suppose that D is a $(\dots, 0)$ -flat. Then by Lemma 14 we can write

$$f(\bar{a}) = \sum_{i=1}^m p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma$$

for all $\bar{a} \in D$, where $p_m = 0$. Let f' be the function on $\pi_{m-1}(D)$ given by

$$f'(\bar{b}) = \sum_{i=1}^{m-1} p_i \left(\frac{b_i - c_i}{n_i} \right) + \gamma$$

for all $\bar{b} \in \pi_{m-1}(D)$. Since $f(D) = f'(\pi_{m-1}(D))$ and f is non-constant, so is f' . By the definition of flat, $\pi_{m-1}(D)$ is a flat in G^{m-1} ; so by induction, $f'(\pi_{m-1}(D))$ is a 1-flat, as is $f(D)$.

Suppose now that D is a $(\dots, 1)$ -flat and let f be as displayed above except that p_m may not equal 0. By rearranging terms we see that

$$f(\bar{a}) = \sum_{i=1}^{m-1} p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma + p_m \left(\frac{a_m - c_m}{n_m} \right)$$

for all $\bar{a} \in D$. The left two terms of this sum define a linear function, f_1 , on $\pi_{m-1}(D)$. If f_1 is constant, then $f_1(\pi_{m-1}(D))$ is a 0-flat, and if f_1 is non-constant then, by induction, $f_1(\pi_{m-1}(D))$ is a 1-flat. Let $D_1 = f_1(\pi_{m-1}(D))$.

If one thinks of f_1 as a linear function on D in the obvious way (constant in the m^{th} coordinate), then it's clear that $f_1(D) = f_1(\pi_{m-1}(D)) = D_1$.

If $\pi : G^m \rightarrow G$ is the projection map into the m^{th} coordinate, then $f_2(a) = p_m \left(\frac{a - c_m}{n_m} \right)$ defines a linear function on $\pi(D)$. If f_2 is constant then $f_2(\pi(D))$ is a 0-flat, and if f_2 is non-constant, then by induction (or Lemma 12) $f_2(\pi(D))$ is a 1-flat. Let $D_2 = f_2(\pi(D))$. If, as before, one thinks of f_2 as a linear function on D — this time constant on all coordinates except the m^{th} — then $f_2(D) = f_2(\pi(D)) = D_2$.

Now since D is a $(\dots, 1)$ -flat, the m^{th} coordinate of any element in D is independent of the first $m - 1$ coordinates. Therefore $f(D) = f_1(D) + f_2(D) = D_1 + D_2$. If D_1 and D_2 are both 0-flats then f must be constant, a contradiction. If exactly one of D_1 and D_2 is a 1-flat, then it is clear that $D_1 + D_2$ is also a 1-flat, and if both D_1 and D_2 are 1-flats, then $D_1 + D_2$ is a 1-flat by Lemma 9. The proof is now complete. □

Lemma 16. *Let D be an (i_1, \dots, i_m) -flat in G^m and $\pi : G^m \rightarrow G^n$ be any coordinate projection: then $\pi(D)$ is a finite union of flats in G^n .*

Proof. It suffices to show that the lemma holds for any coordinate projection $\pi : G^m \rightarrow G^{m-1}$ since every coordinate projection is the composition of such functions. We proceed by induction on m .

The base case of $m = 1$ is trivial.

Assume the lemma holds for all $n < m$ and let $\pi : G^m \rightarrow G^{m-1}$ be defined by deletion of the n^{th} coordinate: $\pi(a_1, \dots, a_m) = (a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_m)$

for all $\bar{a} \in G^m$. By definition of flat, $\pi_{m-1}(D)$ is an (i_1, \dots, i_{m-1}) -flat in G^{m-1} : so if $n = m$ the Lemma holds. If $n \neq m$ and $\pi' : G^{m-1} \rightarrow G^{m-2}$ is the projection map defined by deletion of the n^{th} coordinate in G^{m-1} , then by induction, $\pi'(\pi_{m-1}(D))$ is a finite union of flats $D_1 \cup \dots \cup D_k$ in G^{m-2} . We will find flats over D_1, \dots, D_k in G^{m-1} which make up $\pi(D)$.

Suppose that $i_m = 1$. In this case, there exist some fixed c_m and n_m such that $(a_1, \dots, a_m) \in D$ if and only if

$$(a_1, \dots, a_{m-1}) \in \pi_{m-1}(D) \quad \text{and} \quad a_m \equiv_{n_m} c_m.$$

Thus, $(b_1, \dots, b_{m-1}) \in \pi(D)$ if and only if

$$(b_1, \dots, b_{m-2}) \in \pi'(\pi_{m-1}(D)) \quad \text{and} \quad b_{m-1} \equiv_{n_m} c_m$$

and $(b_1, \dots, b_{m-1}) \in \pi(D)$ if and only if $(b_1, \dots, b_{m-1}) \in D_i$ for some $i \in \{1, \dots, k\}$ and $b_{m-1} \equiv_{n_m} c_m$. So $\pi(D)$ is the union of k flats in G^{m-1} .

Suppose now that $i_m = 0$ and $i_n = 0$. Since $i_m = 0$, the m^{th} coordinate of D is a linear function of the other coordinates, say $f(a_1, \dots, a_{m-1}) = a_m$ for all $\bar{a} \in D$. Since $i_n = 0$, by Lemma 14 there is an expression in standard linear form for f which does not depend on a_n . One may think of this expression as a rule for a function f' of $m - 2$ variables, by shifting the indices of every variable after n down by one. Then $(b_1, \dots, b_{m-1}) \in \pi(D)$ if and only if

$$(b_1, \dots, b_{m-2}) \in D_i \text{ for some } i \in \{1, \dots, k\} \quad \text{and} \quad b_{m-1} = f'(b_1, \dots, b_{m-2})$$

and so $\pi(D)$ is the union of k flats in G^{m-1} .

Finally, suppose that $i_m = 0$ and $i_n = 1$. We will show that the part of $\pi(D)$ above D_1 — those points in $\pi(D)$ which project into D_1 under π_{m-2} — is a finite union of flats in G^{m-1} . The argument may then be repeated for each of the other (finitely many) D_i to get the full collection of flats making up $\pi(D)$.

Let f be the linear function such that

$$f(a_1, \dots, a_{m-1}) = \sum_{i=1}^{m-1} p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma = a_m$$

for all $\bar{a} = (a_1, \dots, a_m) \in D$, and assume, by Lemma 14, that $p_j = 0$ whenever $i_j = 0$. If $p_n = 0$ then one may proceed as in the previous case: so assume $p_n \neq 0$. Let $\pi^{(n)} : G^m \rightarrow G$ be the coordinate projection map into the n^{th} coordinate. Then the expression $p_n \left(\frac{a - c_n}{n_n} \right)$ represents a non-constant linear function, f_n , on $\pi^{(n)}(D)$ which is a 1-flat, so by Lemma 12 the image of f_n is a 1-flat, say F_n .

For all $i \in \{1, \dots, m-2\}$ let $p'_i = p_i$ for $i < n$ and $p'_i = p_{i+1}$ for $i \geq n$, and define c'_i and n'_i similarly. Let $f' : D_1 \rightarrow G$ be the linear function such that

$$f'(\bar{b}) = \sum_{i=1}^{m-2} p'_i \left(\frac{b_i - c'_i}{n'_i} \right) + \gamma$$

for all $\bar{b} \in D_1$. For each $(b_1, \dots, b_{m-2}) \in D_1$,

$$\begin{aligned}
& \{x \mid (b_1, \dots, b_{m-2}, x) \in \pi(D)\} \\
&= \{x \mid \exists a_n \in \pi^{(n)}(D) \text{ such that } f(b_1, \dots, b_{n-1}, a_n, b_n, \dots, b_{m-2}) = x\} \\
&= \{x \mid \exists a_n \in \pi^{(n)}(D) \text{ such that } f'(b_1, \dots, b_{m-2}) + f_n(a_n) = x\} \\
&= \{f'(b_1, \dots, b_{m-2})\} + F_n;
\end{aligned}$$

so $\{x \mid (b_1, \dots, b_{m-2}, x) \in \pi(D)\}$ is a 1-flat, say $F_{\bar{b}}$. Because $F_{\bar{b}}$ may vary with \bar{b} , it remains to show that D_1 can be broken up into finitely many flats, E_1, \dots, E_l , such that if \bar{b} and \bar{b}' are both in E_i then $F_{\bar{b}} = F_{\bar{b}'}$.

Suppose D_1 is an (i'_1, \dots, i'_{m-2}) -flat and let j_1, \dots, j_r be the indices (in order) for which $i'_{j_k} = 1$ and $p'_{j_k} \neq 0$. For each $k \in \{1, \dots, r\}$ let $\pi^{(j_k)} : G^{m-2} \rightarrow G$ be coordinate projection into the j_k^{th} coordinate, and let f'_{j_k} be the linear function on the 1-flat $\pi^{(j_k)}(D_1)$ such that $f'_{j_k}(b) = p'_{j_k} \left(\frac{b - c'_{j_k}}{n'_{j_k}} \right)$ for all $b \in \pi^{(j_k)}(D_1)$. Let us say that $F_n = \{x \mid x \equiv_N C\}$ and for each $k \in \{1, \dots, r\}$ that $\pi^{(j_k)}(D_1) = \{x \mid x \equiv_{m_{j_k}} e_{j_k}\}$. Then for every k , $\pi^{(j_k)}(D_1)$ is equal to a finite union of 1-flats $(\text{mod } m_{j_k} n'_{j_k} N)$, that is, flats of the form $\{x \mid x \equiv e \pmod{m_{j_k} n'_{j_k} N}\}$ for some $e \in G$. Call these 1-flats $E_{k,1}, \dots, E_{k,l_k}$.

We now build a finite collection \mathcal{E} of (i'_1, \dots, i'_{m-2}) -flats covering D_1 as follows. The flat E is in \mathcal{E} if and only if the following hold:

- if $j_1 \neq 1$ then $\pi_1(E) = \pi_1(D_1)$,
- if $j_1 = 1$ then $\pi_1(E) = E_{1,s}$ for some $s \in \{1, \dots, l_1\}$,

- for $p > 1$, if $p \neq j_k$ for all $k \in \{1, \dots, r\}$ then

$$\pi_p(E) = \pi_p(D_1) \cap \{(b_1, \dots, b_p) \in G^p \mid (b_1, \dots, b_{p-1}) \in \pi_{p-1}(E)\}, \text{ and}$$

- for $p > 1$, if $p = j_k$ for some $k \in \{1, \dots, r\}$ then

$$\pi^{(p)}(E) = E_{k,s} \text{ for some } s \in \{1, \dots, l_k\}.$$

For example, to build a flat $E \in \mathcal{E}$, start by considering $\pi_1(D_1)$. If $j_1 \neq 1$ then take $\pi_1(E)$ to be $\pi_1(D_1)$. If $j_1 = 1$, then $\pi_1(D_1)$ is the 1-flat $\{x \mid x \equiv_{m_1} e_1\}$, which is equal to $E_{1,1} \cup \dots \cup E_{1,l_1}$. In this case, take $\pi_1(E)$ to be one of $E_{1,1}, \dots, E_{1,l_1}$. Now consider $\pi_2(D_1)$. If $j_k \neq 2$ for any $k \in \{1, \dots, r\}$ then take $\pi_2(E)$ to be $\{(x, y) \mid x \in \pi_1(E), (x, y) \in \pi_2(D_1)\}$. Then $\pi_2(E)$ is either the graph of a linear function on $\pi_1(E)$ or all points above $\pi_1(E)$ for which the second coordinate satisfies some congruence condition; in either case $\pi_2(E)$ is a flat in G^2 . If $j_k = 2$ for some $k \in \{1, \dots, r\}$ then $\pi^{(2)}(D_1)$ is the 1-flat $\{x \mid x \equiv_{m_2} e_2\}$, which is equal to $E_{2,1} \cup \dots \cup E_{2,l_2}$. In this case, take $\pi_2(E)$ to be $\{(x, y) \mid x \in \pi_1(E), y \in E_{2,s}\}$ which is clearly a flat in G^2 . Continue in this way to get E , a flat in G^{m-2} . Let \mathcal{E} be the collection of all flats which may be constructed in this way (subject to different choices of $E_{k,s}$).

It is now clear from the construction that \mathcal{E} is finite, and that the union of all of the flats in \mathcal{E} is equal to D_1 .

Let $\bar{b} = (b_1, \dots, b_{m-2})$ and $\bar{b}' = (b'_1, \dots, b'_{m-2})$ be in $E \in \mathcal{E}$. Then for each $k \in \{1, \dots, r\}$, $b_{j_k} \equiv b'_{j_k} \pmod{m_{j_k} n'_{j_k} N}$ and from the definition of f'_{j_k} shown we see that $f'_{j_k}(b_{j_k}) \equiv f'_{j_k}(b'_{j_k}) \pmod{p'_{j_k} m_{j_k} N}$. In particular,

$f'_{j_k}(b_{j_k}) \equiv f'_{j_k}(b'_{j_k}) \pmod{N}$ for each $k \in \{1, \dots, r\}$. Therefore, $f'(\bar{b}) \equiv_N f'(\bar{b}')$ and $\{f'(\bar{b})\} + F_n = \{f'(\bar{b}')\} + F_n$. So for each $E \in \mathcal{E}$, let $e = f'(\bar{b})$ for some $\bar{b} \in E$ and let $E' = \{(\bar{a}, x) \mid \bar{a} \in E, x \equiv e \pmod{N}\}$. We have shown that the union of the (finite) set of all E' is the part of $\pi(D)$ above D_1 : so the proof is complete. □

Note: If F_1, \dots, F_k are the flats found in the proof of Lemma 16 such that $\pi(D) = F_1 \cup \dots \cup F_k$, then the proof shows that for any $i \in \{1, \dots, k\}$, the number of ones in the type of F_i is less than or equal to the number of ones in the type of D . This fact will be used in the proof of Theorem 2 in the next chapter.

Lemma 17. *If C is an (i_1, \dots, i_m) -cell, then there is an (i_1, \dots, i_m) -flat $F \supseteq C$ with the following property: for all $c_1, \dots, c_m \in G$ and $n_1, \dots, n_m \in \mathbb{N}^+$,*

$$a_i \equiv c_i \pmod{n_i} \text{ for all } (a_1, \dots, a_m) \in C$$

if and only if

$$a_i \equiv c_i \pmod{n_i} \text{ for all } (a_1, \dots, a_m) \in F.$$

Proof. The proof is by induction on m .

Every 0-cell is also a 0-flat. If C is a 1-cell then C is of the form $\{x \mid f < kx < g, x \equiv c \pmod{n}\}$. In this case $F = \{x \mid x \equiv c \pmod{n}\}$ is a 1-flat,

containing C , which clearly possesses the desired property. Thus, the base case is complete.

Let C be an (i_1, \dots, i_m) -cell. By definition of cell, $\pi_{m-1}(C)$ is an (i_1, \dots, i_{m-1}) -cell, so by induction $\pi_{m-1}(C)$ is contained in an (i_1, \dots, i_{m-1}) -flat F' with the property specified in the lemma.

Suppose that $i_m = 1$. Then C is of the form $\{(\bar{x}, t) \mid \bar{x} \in \pi_{m-1}(C), f(\bar{x}) < kt < g(\bar{x}), t \equiv c \pmod{n}\}$ and $F = \{(\bar{x}, t) \mid \bar{x} \in F', t \equiv c \pmod{n}\}$ is an (i_1, \dots, i_m) -flat containing C with the desired property.

Suppose that $i_m = 0$, and let f be the linear function such that

$$f(a_1, \dots, a_{m-1}) = \sum_{i=1}^{m-1} p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma = a_m$$

for all $\bar{a} = (a_1, \dots, a_m) \in C$. Since for all $(a_1, \dots, a_{m-1}) \in \pi_{m-1}(C)$, $a_i \equiv_{n_i} c_i$ for each $i \in \{1, \dots, m-1\}$, the same is true for all $\bar{b} \in F'$, so we may extend f to a linear function $f' : F' \rightarrow G$ with the same description as is displayed for f above. If $F = \{(\bar{x}, t) \mid \bar{x} \in F', f'(\bar{x}) = t\}$, then F is an (i_1, \dots, i_m) -flat containing C , and for all $c_1, \dots, c_{m-1} \in G$ and $n_1, \dots, n_{m-1} \in \mathbb{N}^+$, $a_i \equiv c_i \pmod{n_i}$ for all $(a_1, \dots, a_m) \in C$ if and only if $a_i \equiv c_i \pmod{n_i}$ for all $(a_1, \dots, a_m) \in F$. We must check that this property holds when $i = m$ as well.

Fix $c \in G$ and $n \in \mathbb{N}^+$ with $a_m \equiv_n c$ for all $(a_1, \dots, a_m) \in C$. If we can show that for all $(b_1, \dots, b_m) \in F$, $b_m \equiv_n c$ then we are done since the

converse is obvious. By Lemma 14 we may assume that

$$f'(b_1, \dots, b_{m-1}) = \sum_{i=1}^{m-1} p'_i \left(\frac{b_i - c'_i}{n'_i} \right) + \gamma'$$

for all $(b_1, \dots, b_{m-1}) \in F'$, where $p'_j = 0$ whenever $i_j = 0$.

Let $(\beta_1, \dots, \beta_m) \in F$ and let j_1, \dots, j_r be the indices (in order) for which $i_{j_k} = 1$ and $p'_{j_k} \neq 0$. For each $k \in \{1, \dots, r\}$, let $\equiv_{m_{j_k}} e_{j_k}$ be the “defining congruence” of C in the j_k^{th} coordinate: that is, $\pi_{j_k}(C) = \{(\bar{x}, t) \mid \bar{x} \in \pi_{j_k-1}(C), f(\bar{x}) < kt < g(\bar{x}), t \equiv_{m_{j_k}} e_{j_k}\}$ for some f and g . Then $\beta_{j_k} \equiv_{m_{j_k}} e_{j_k}$ for all $k \in \{1, \dots, r\}$ and by dense regularity, there must be some $\bar{a} \in C$ such that $a_{j_1} \equiv \beta_{j_1} \pmod{m_{j_1} n'_{j_1} n}$. Let $(\alpha_1, \dots, \alpha_{j_1}) = (a_1, \dots, a_{j_1})$. By dense regularity and the definition of cell again, there must be some $\bar{a}' \in C$ such that $a'_i = \alpha_i$ for $i = 1, \dots, j_1$ and $a'_{j_2} \equiv \beta_{j_2} \pmod{m_{j_2} n'_{j_2} n}$. Let $(\alpha_1, \dots, \alpha_{j_2}) = (a'_1, \dots, a'_{j_2})$.

Continuing in this fashion one may find an element $\bar{a} \in C$ such that $\alpha_{j_k} \equiv \beta_{j_k} \pmod{m_{j_k} n'_{j_k} n}$ for every $k \in \{1, \dots, r\}$. An easy computation shows that

$$p'_{j_k} \left(\frac{\beta_{j_k} - c'_{j_k}}{n'_{j_k}} \right) \equiv p'_{j_k} \left(\frac{\alpha_{j_k} - c'_{j_k}}{n'_{j_k}} \right) \pmod{p'_{j_k} m_{j_k} n}$$

for all $k \in \{1, \dots, r\}$: so $f'(\beta_1, \dots, \beta_{m-1}) \equiv_n f'(\alpha_1, \dots, \alpha_{m-1}) = \alpha_m$, $\beta_m \equiv_n c$, and the proof is complete. □

Let $\mathcal{I} = (i_1, \dots, i_m)$ be a tuple of zeros and ones, $j_1 < \dots < j_r$ be the indices for which $i_{j_k} = 1$, and $\pi_{\mathcal{I}} : G^m \rightarrow G^r$ be the projection map into the coordinates j_1, \dots, j_r : $\pi_{\mathcal{I}}(a_1, \dots, a_m) = (a_{j_1}, \dots, a_{j_r})$ for all $(a_1, \dots, a_m) \in G^m$.

Lemma 18. *Let C be an \mathcal{I} -cell in G^m : then the image of C under $\pi_{\mathcal{I}}$ is a $(1, \dots, 1)$ -cell in G^r .*

Proof. The proof is by induction on m and the base case when $m = 1$ is trivial.

Assume the Lemma holds in G^n for all $n < m$.

Suppose that $i_m = 0$, let $\mathcal{I}' = (i_1, \dots, i_{m-1})$, and define $\pi_{\mathcal{I}'} : G^{m-1} \rightarrow G^r$ similarly to $\pi_{\mathcal{I}}$. Then $\pi_{\mathcal{I}'}(\pi_{m-1}(\bar{a})) = \pi_{\mathcal{I}}(\bar{a})$ for all $\bar{a} \in G^{m-1}$ and since $\pi_{m-1}(C)$ is an \mathcal{I}' -cell, $\pi_{\mathcal{I}'}(\pi_{m-1}(C))$ is a $(1, \dots, 1)$ -cell by induction. So $\pi_{\mathcal{I}}(C)$ is a $(1, \dots, 1)$ -cell.

Suppose that $i_m = 1$. Then

$$C = \{(\bar{x}, t) \mid \bar{x} \in \pi_{m-1}(C), f(\bar{x}) < kt < g(\bar{x}), t \equiv c \pmod{n}\}$$

for some linear functions f and g . By Lemma 17, $\pi_{m-1}(C)$ is contained in an (i_1, \dots, i_{m-1}) -flat, F , to which f and g can be extended. By Lemma 14 we may assume that

$$f(\bar{a}) = \sum_{i=1}^{m-1} p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma$$

for all $\bar{a} \in F$, where $p_j = 0$ whenever $i_j = 0$. As in the previous case, $\pi_{\mathcal{I}'}(\pi_{m-1}(C))$ is a $(1, \dots, 1)$ -cell, this time in G^{r-1} , and the function $f' : \pi_{\mathcal{I}'}(\pi_{m-1}(C)) \rightarrow G$ where

$$f'(b_1, \dots, b_{r-1}) = \sum_{k=1}^{r-1} p_{j_k} \left(\frac{b_k - c_{j_k}}{n_{j_k}} \right) + \gamma$$

for all $(b_1, \dots, b_{r-1}) \in \pi_{\mathcal{I}'}(\pi_{m-1}(C))$, is well-defined and

$$f(a_1, \dots, a_{m-1}) = f'(a_{j_1}, \dots, a_{j_{r-1}})$$

for all $(a_1, \dots, a_{m-1}) \in \pi_{m-1}(C)$. If g' is obtained from g in a similar way, then

$$\pi_{\mathcal{I}}(C) = \{(\bar{x}, t) \mid \bar{x} \in \pi_{\mathcal{I}'}(\pi_{m-1}(C)), f'(\bar{x}) < kt < g'(\bar{x}), t \equiv c \pmod{n}\},$$

so $\pi_{\mathcal{I}}$ is a $(1, \dots, 1)$ -cell in G^r and the proof is complete. □

Lemma 19. *If C is an (i_1, \dots, i_{m+r}) -cell in G^{m+r} , then, for any $\bar{a} \in \pi_m(C)$, the fiber $C_{\bar{a}}$ of C over \bar{a} is an $(i_{m+1}, \dots, i_{m+r})$ -cell.*

Proof. The proof is by induction on r , and the base case when $r = 1$ is immediate from the definition of cell.

Assume the lemma holds in G^{m+k} for all $k < r$, and let $\bar{b} \in \pi_m(C)$. Then since $\pi_{m+r-1}(C)$ is an (i_1, \dots, i_{m+r-1}) -cell in G^{m+r-1} , $(\pi_{m+r-1}(C))_{\bar{b}}$ must be

an $(i_{m+1}, \dots, i_{m+r-1})$ -cell.

Suppose that $i_{m+r} = 0$ and let $f : \pi_{m+r-1}(C) \rightarrow G$ be the linear function such that $f(a_1, \dots, a_{m+r-1}) = a_{m+r}$ for all $(a_1, \dots, a_{m+r-1}) \in C$. Let $f_{\bar{b}} : (\pi_{m+r-1}(C))_{\bar{b}} \rightarrow G$ be the linear function such that

$$f_{\bar{b}}(x_1, \dots, x_{r-1}) = f(\bar{b}, x_1, \dots, x_{r-1})$$

for all $(x_1, \dots, x_{r-1}) \in (\pi_{m+r-1}(C))_{\bar{b}}$. Then $C_{\bar{b}} = \{(\bar{x}, t) \mid \bar{x} \in (\pi_{m+r-1}(C))_{\bar{b}}, f_{\bar{b}}(\bar{x}) = t\}$, so $C_{\bar{b}}$ is an $(i_{m+1}, \dots, i_{m+r})$ -cell.

Suppose that $i_{m+r} = 1$: then

$$C = \{(\bar{x}, t) \mid \bar{x} \in \pi_{m+r-1}(C), g(\bar{x}) < k_0 t < h(\bar{x}), t \equiv c \pmod{n}\}$$

for some linear functions g and h . If $g_{\bar{b}}$ is to g and $h_{\bar{b}}$ is to h as $f_{\bar{b}}$ is to f , then

$$C_{\bar{b}} = \{(\bar{x}, t) \mid \bar{x} \in (\pi_{m+r-1}(C))_{\bar{b}}, g_{\bar{b}}(\bar{x}) < k_0 t < h_{\bar{b}}(\bar{x}), t \equiv c \pmod{n}\};$$

so $C_{\bar{b}}$ is an $(i_{m+1}, \dots, i_{m+r})$ -cell and the proof is complete.

□

Chapter 4

Dimension

Definition 7. We define a map $|\ast| : (G \cup G^2 \cup G^3 \cup \dots) \rightarrow G$ by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \quad \text{and} \quad |(a_1, \dots, a_m)| = \max\{|a_1|, \dots, |a_m|\}$$

for all $m \geq 2$.

Lemma 20. *All linear functions are continuous with respect to $|\ast|$ on their domains. That is, if $f : X \subseteq G^m \rightarrow G$ is linear, then for any $\bar{x} \in X$ and positive $\epsilon \in G$, there exists a positive $\delta \in G$ such that if $\bar{y} \in X$ and $|\bar{x} - \bar{y}| < \delta$ then $|f(\bar{x}) - f(\bar{y})| < \epsilon$.*

Proof. The proof is by induction on m .

Let $m = 1$, $f(x) = p\left(\frac{x-c}{n}\right) + \gamma$ for all $x \in X$, and $\epsilon > 0$. Let $\epsilon' < \epsilon$ be divisible by p (by dense regularity), and let $\delta = n\frac{\epsilon'}{p}$. Suppose that $x, y \in X$

and $|x - y| < \delta$. Then

$$\begin{aligned}
|f(x) - f(y)| &= \left| p\left(\frac{x-c}{n}\right) + \gamma - p\left(\frac{y-c}{n}\right) - \gamma \right| \\
&= \left| p\left(\frac{x-c}{n}\right) - p\left(\frac{y-c}{n}\right) \right| \\
&= \left| p\left(\frac{x-c-y+c}{n}\right) \right| \\
&= \left| p\left(\frac{x-y}{n}\right) \right| \\
&< \left| p\left(\frac{\delta}{n}\right) \right| = \left| p\left(\frac{n\epsilon'}{n}\right) \right| = \left| p\left(\frac{\epsilon'}{p}\right) \right| = |\epsilon'| < \epsilon.
\end{aligned}$$

Now let $m > 1$ and assume the Lemma is true in G^k for all $k < m$. Let $\epsilon > 0$ and

$$f(\bar{x}) = \sum_{i=1}^m p_i \left(\frac{x_i - c_i}{n_i} \right) + \gamma$$

for all $\bar{x} \in X$, and define $f_1 : \pi_{m-1}(X) \rightarrow G$ and $f_2 : \pi^{(m)}(X) \rightarrow G$ by

$$f_1(x_1, \dots, x_{m-1}) = \sum_{i=1}^{m-1} p_i \left(\frac{x_i - c_i}{n_i} \right) + \gamma \quad \text{and} \quad f_2(x_m) = p_m \left(\frac{x_m - c_m}{n_m} \right)$$

for all $(x_1, \dots, x_{m-1}) \in \pi_{m-1}(X)$ and all $x_m \in \pi^{(m)}(X)$. By induction, f_1 and f_2 are continuous. So there exist positive δ_1 and δ_2 such that for any $\bar{x}, \bar{y} \in \pi_{m-1}(X)$,

$$|\bar{x} - \bar{y}| < \delta_1 \Rightarrow |f_1(\bar{x}) - f_1(\bar{y})| < \frac{\epsilon}{2},$$

and for any $x, y \in \pi^{(m)}(X)$,

$$|x - y| < \delta_2 \Rightarrow |f_2(x) - f_2(y)| < \frac{\epsilon}{2}$$

(without loss of generality we may assume by dense regularity that ϵ is divisible by 2). Therefore, if $\delta = \min\{\delta_1, \delta_2\}$, then for any $\bar{x}, \bar{y} \in X$, $|\bar{x} - \bar{y}| < \delta \Rightarrow |f(\bar{x}) - f(\bar{y})| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

□

Definition 8. Certain $(1, \dots, 1)$ -cells, called *open boxes*, are defined as follows.

All 1-cells are open boxes.

A $(1, \dots, 1)$ -cell is an open box in G^m if and only if it is of the form

$$\{(\bar{x}, t) \in G^m \mid \bar{x} \in B, f < kt < g, t \equiv c \pmod{n}\},$$

where B is an open box in G^{m-1} , $f, g \in G \cup \{\pm\infty\}$, $c \in G$, and $k, n \in \mathbb{N}^+$. In other words, an open box is a $(1, \dots, 1)$ -cell where all of the “boundary functions” are constant.

Note: In the next chapter it will be convenient at times to think of an open box as a product of “intervals”:

$$(f_1, g_1)_{\equiv_{n_1} c_1} \times \cdots \times (f_m, g_m)_{\equiv_{n_m} c_m}.$$

Of course, not every open box can be represented in this way — if f or g above is not divisible by k for example — but it is clear that every set of this form is an open box.

Definition 9. A set $A \subseteq G^m$ is called *nowhere dense* if for any open box $B \subseteq G^m$ there is another open box $B' \subseteq B$ such that $B' \cap A = \emptyset$.

Lemma 21. Let $A_1, \dots, A_k \subseteq G^m$ be nowhere dense. Then $A_1 \cup \dots \cup A_k$ is nowhere dense.

Proof. We will show the lemma holds for the case $k = 2$. The result then follows by induction.

Let $A_1, A_2 \subseteq G^m$ be nowhere dense. Let B be some open box in G^m . Since A_1 is nowhere dense, there is an open box $B_1 \subseteq B$ such that $B_1 \cap A_1 = \emptyset$. Since A_2 is nowhere dense, there is an open box $B_2 \subseteq B_1$ such that $B_2 \cap A_2 = \emptyset$. Clearly $B_2 \subseteq B$ and $B_2 \cap (A_1 \cup A_2) = \emptyset$.

□

Lemma 22. An (i_1, \dots, i_m) -cell is nowhere dense in G^m if and only if $i_j = 0$ for some $j \in \{1, \dots, m\}$.

Proof. (\Leftarrow): Let $j \in \{1, \dots, m\}$ and let C be an (i_1, \dots, i_m) -cell with $i_j = 0$. Let B be an open box in G^m . If $B \cap C = \emptyset$ then we are done, so assume that there is some point $\bar{b} = (b_1, \dots, b_m) \in B \cap C$.

If $j = 1$ then every point in C has first coordinate equal to b_1 . But $\pi_1(B)$ is a 1-cell, which clearly contains a smaller (1)-cell B_1 which does not contain

b_1 . Then $B' = \{(x_1, \dots, x_m) \in B \mid x_1 \in B_1\}$ is clearly a non-empty open box in B and $B' \cap C = \emptyset$.

If $j > 1$, then the projection $\pi_{j-1}(B)$ is an open box in G^{j-1} and $\pi_{j-1}(C)$ is an (i_1, \dots, i_{j-1}) -cell, both of which contain the point (b_1, \dots, b_{j-1}) . Let $f : \pi_{j-1}(C) \rightarrow G$ be the linear function such that $f(a_1, \dots, a_{j-1}) = a_j$ for all $(a_1, \dots, a_j) \in \pi_j(C)$. Let us say that $\pi_j(B) = \{(\bar{x}, t) \mid \bar{x} \in \pi_{j-1}(B), g_1 < kt < g_2, t \equiv c \pmod{n}\}$. Let g_3 be such that $kf(b_1, \dots, b_{j-1}) < kg_3 < g_2$. By continuity of f there exists a positive $\delta \in G$ such that $|f(\bar{x}) - f(\bar{b})| < |g_3 - f(\bar{b})|$ whenever $|\bar{x} - \bar{b}| < \delta$. Then

$$B_1 = \pi_j(B) \cap \{(\bar{x}, t) \mid |\bar{x} - \bar{b}| < \delta, kg_3 < kt < g_2\}$$

is an open box in G^j which is disjoint from $\pi_j(C)$. So $B' = \{\bar{x} \in B \mid (x_1, \dots, x_j) \in B_1\}$ is an open box in B which is disjoint from C .

(\Rightarrow) : Now suppose that $i_j \neq 0$ for every $j \in \{1, \dots, m\}$: that is, C is a $(1, \dots, 1)$ -cell. We will show, by induction on m , the stronger result that C contains an open box, and therefore is not nowhere dense.

The base case is trivial since every 1-cell is an open box.

Let $m > 1$ and assume that the claim is true in G^n whenever $n < m$. If

$$C = \{(\bar{x}, t) \mid \bar{x} \in \pi_{m-1}(C), f(\bar{x}) < kt < g(\bar{x}), t \equiv r \pmod{s}\},$$

then we may construct an open box $B \subseteq C$ as follows.

Since $\pi_{m-1}(C)$ is a $(1, \dots, 1)$ -cell in G^{m-1} , there is an open box $B_1 \subseteq$

$\pi_{m-1}(C)$ by induction. Let $\bar{b} \in B_1$, and $t_1, t_2 \in G$ such that $f(\bar{b}) < t_1 < t_2 < g(\bar{b})$. By the continuity of f , there exists $\delta_1 > 0$ such that $|f(\bar{x}) - f(\bar{b})| < |t_1 - f(\bar{b})|$ whenever $|\bar{x} - \bar{b}| < \delta_1$, and by the continuity of g there exists $\delta_2 > 0$ such that $|g(\bar{x}) - g(\bar{b})| < |t_2 - g(\bar{b})|$ whenever $|\bar{x} - \bar{b}| < \delta_2$. Then $B_2 = B_1 \cap \{\bar{x} \in G^{m-1} \mid |\bar{x} - \bar{b}| < \min\{\delta_1, \delta_2\}\}$ is an open box in G^{m-1} and it is clear that

$$B = \{(\bar{x}, t) \in G^m \mid \bar{x} \in B_2, t_1 < kt < t_2, t \equiv r \pmod{s}\}$$

is an open box in C .

□

Definition 10. If C is an (i_1, \dots, i_m) -cell then the *dimension* of C , $\dim C$, is $i_1 + \dots + i_m$. If A is a definable set in G^m then the *dimension* of A , $\dim A$, is defined by:

$$\dim A := \max\{i_1 + \dots + i_m \mid A \text{ contains an } (i_1, \dots, i_m)\text{-cell}\}.$$

The empty set is assigned dimension $-\infty$.

The following theorem shows that one does not actually need to consider every cell within a definable set in order to determine its dimension. In fact, any decomposition into cells will suffice.

Theorem 2. *Let A be a definable set in G^m with $\dim A = n$. If A has decomposition $A = C_1 \cup \dots \cup C_k$ into cells, then $\dim C_i = n$ for some i .*

Proof. The theorem is clear if $\dim A = 0$.

Let $\dim A = n > 0$: then there exists an (i_1, \dots, i_m) -cell $C \subseteq A$ such that $i_1 + \dots + i_m = n$. Suppose that $\dim C_i < n$ for every $i \in \{1, \dots, k\}$. For each C_i let F_i be a flat containing C_i of the same type (these F_i must exist by Lemma 17). Let $A_F = F_1 \cup \dots \cup F_k$ and $\pi : G^m \rightarrow G^n$ be the projection map into those coordinates for which $i_j = 1$.

C is a subset of A_F , so $\pi(C) \subseteq \pi(A_F)$. By Lemma 18, $\pi(C)$ is a $(1, \dots, 1)$ -cell in G^n , so by the previous lemma $\pi(C)$, and therefore $\pi(A_F)$, is not nowhere dense. On the other hand, $\pi(A_F) = \pi(F_1) \cup \dots \cup \pi(F_k)$. By Lemma 16, for each $i \in \{1, \dots, k\}$, $\pi(F_i)$ is the finite union of some flats, say $D_{i_1} \cup \dots \cup D_{i_{k_i}}$. Moreover, according to the note following Lemma 16, the number of ones appearing in the type of D_{ij} is less than or equal to the number of ones appearing in the type of F_i for each i, j . So $\dim D_{ij} < n$ for all i, j , and therefore, by Lemma 22 $\pi(A_F)$ is nowhere dense. This is a contradiction, so it must not be the case that $\dim C_i < n$ for every $i \in \{1, \dots, k\}$, and so $\dim C_i = n$ for some i .

□

Lemma 23. *If A, B are definable subsets of G^m , then*

$$\dim(A \cup B) = \max\{\dim(A), \dim(B)\}.$$

Proof. Let $d = \dim(A \cup B)$. It is clear from the definition of dimension that $\max\{\dim(A), \dim(B)\} \leq d$.

By Theorem 1, there is a decomposition of G^m into cells C_1, \dots, C_n which partitions both A and B . So $A = C_{i_1} \cup \dots \cup C_{i_j}$ and $B = C_{j_1} \cup \dots \cup C_{j_k}$ for some $i_1, \dots, i_j, j_1, \dots, j_k \in \{1, \dots, n\}$. Then $\{C_{i_1}, \dots, C_{i_j}\} \cup \{C_{j_1}, \dots, C_{j_k}\}$ is a decomposition of $A \cup B$, and by Theorem 2, $\dim C_\alpha = d$ for some $\alpha \in \{i_1, \dots, i_j, j_1, \dots, j_k\}$. Thus $d \leq \max\{\dim(A), \dim(B)\}$ and $\dim(A \cup B) = \max\{\dim(A), \dim(B)\}$.

□

Lemma 24. *Let $A \subseteq G^{m+r}$ be definable and for each $\bar{x} \in G^m$ let $A_{\bar{x}}$ denote the fiber of A above \bar{x} . Then for any $d \in \{-\infty, 0, 1, \dots, r\}$ the set*

$$A(d) = \{\bar{x} \in G^m \mid \dim A_{\bar{x}} = d\}$$

is definable.

Proof. If $d = -\infty$, then $A(d) = \{\bar{x} \in G^m \mid A_{\bar{x}} = \emptyset\} = G^m \setminus \pi_m(A)$, so $A(d)$ is definable since $\pi_m(A)$ is definable.

If $d \in \{0, \dots, r\}$, then $A(d) \subseteq \pi_m(A)$. Let A be the disjoint union of cells C_1, \dots, C_k in a decomposition of G^{m+r} . It is clear from Lemma 19 that for each $i \in \{1, \dots, k\}$ and $\bar{x} \in \pi_m(C_i)$, $\dim C_i = \dim \pi_m(C_i) + \dim(C_i)_{\bar{x}}$. Consider a cell $D \in \{\pi_m(C_1), \dots, \pi_m(C_k)\}$, and let C_{i_1}, \dots, C_{i_j} be those cells which project onto D . That is, $\pi_m(C_{i_1}) = \dots = \pi_m(C_{i_j}) = D$. Then for each

$\bar{x} \in D$, $\{(C_{i_1})_{\bar{x}}, \dots, (C_{i_j})_{\bar{x}}\}$ is a decomposition of $A_{\bar{x}}$. So

$$\begin{aligned} \dim A_{\bar{x}} &= \max\{\dim(C_{i_1})_{\bar{x}}, \dots, \dim(C_{i_j})_{\bar{x}}\} \\ &= \max\{(\dim C_{i_1} - \dim D), \dots, (\dim C_{i_j} - \dim D)\} \end{aligned}$$

which is clearly independent of choice of \bar{x} . So for any

$$D \in \{\pi_m(C_1), \dots, \pi_m(C_k)\},$$

either $D \subseteq A(d)$ or $D \cap A(d) = \emptyset$. So $A(d)$ is a union of cells in G^m , hence definable.

□

Lemma 25. *Let $A \subseteq G^{m+r}$ be definable. If there exists $d \in \{-\infty, 0, 1, \dots, n\}$ such that for every $\bar{x} \in \pi_m(A)$, $\dim A_{\bar{x}} = d$, then $\dim A = \dim \pi_m(A) + d$.*

Proof. Let A be the disjoint union of cells C_1, \dots, C_k in a decomposition of G^{m+r} . By Theorem 2, $\dim C_i = \dim A$ for some $i \in \{1, \dots, k\}$. By Lemma 19, we have $\dim C_i = \dim \pi_m(C_i) + \dim(C_i)_{\bar{x}}$ for any $\bar{x} \in \pi_m(C_i)$. So by substitution

$$\dim A = \dim \pi_m(C_i) + \dim(C_i)_{\bar{x}} \leq \dim \pi_m(C_i) + d,$$

and since $\pi_m(C_i)$ is a cell contained in $\pi_m(A)$ we have $\dim A \leq \dim \pi_m(A) + d$.

By the definition of decomposition, $\{\pi_m(C_1), \dots, \pi_m(C_k)\}$ is a decom-

position of $\pi_m(A)$, so by Theorem 2 $\dim \pi_m(C_i) = \dim \pi_m(A)$ for some $i \in \{1, \dots, k\}$. Let C_{i_1}, \dots, C_{i_j} be those cells which project (via π_m) onto $\pi_m(C_i)$. Then for all $\bar{x} \in \pi_m(C_i)$, $\{(C_{i_1})_{\bar{x}}, \dots, (C_{i_j})_{\bar{x}}\}$ is a decomposition of $A_{\bar{x}}$. So

$$\begin{aligned}
d = \dim A_{\bar{x}} &= \max\{\dim(C_{i_1})_{\bar{x}}, \dots, \dim(C_{i_j})_{\bar{x}}\} \\
&= \max\{(\dim C_{i_1} - \dim \pi_m(C_{i_1})), \dots, (\dim C_{i_j} - \dim \pi_m(C_{i_j}))\} \\
&= \max\{(\dim C_{i_1} - \dim \pi_m(C_i)), \dots, (\dim C_{i_j} - \dim \pi_m(C_i))\} \\
&= \max\{\dim C_{i_1}, \dots, \dim C_{i_j}\} - \dim \pi_m(C_i) \\
&= \max\{\dim C_{i_1}, \dots, \dim C_{i_j}\} - \dim \pi_m(A).
\end{aligned}$$

So for some $s \in \{1, \dots, j\}$, $\dim C_{i_s} = \dim \pi_m(A) + d$, and since C_{i_s} is a cell contained in A , this implies that $\dim A \geq \dim \pi_m(A) + d$. Thus, $\dim A = \dim \pi_m(A) + d$.

□

Chapter 5

Topology

Lemma 26. *The set $\mathcal{B}_G = \{(a, b)_{\equiv_n c} \mid a, b, c \in G, a < b, n \in \mathbb{N}^+\}$ is a basis for a topology on G .*

Proof. For each $g \in G$, it is clear that there is an element of \mathcal{B}_G containing g .

Let $B_1, B_2 \in \mathcal{B}_G$, and let $g \in B_1 \cap B_2$. We must show that there exists $B_3 \in \mathcal{B}_G$ such that $g \in B_3$ and $B_3 \subseteq B_1 \cap B_2$. Say $B_1 = (a_1, b_1)_{\equiv_{n_1} c_1}$ and $B_2 = (a_2, b_2)_{\equiv_{n_2} c_2}$. Let $N = n_1 n_2$, and let $B_3 = (\max\{a_1, a_2\}, \min\{b_1, b_2\})_{\equiv_N g}$. Then clearly $g \in B_3$. If $h \in B_3$ then $a_1 < h < b_1$ and $h \equiv_N g \equiv_{n_1} c_1$, so $h \in B_1$, and similarly $h \in B_2$. So $B_3 \subseteq B_1 \cap B_2$.

□

Let τ_G be the topology generated by \mathcal{B}_G . We will also abuse notation and use τ_G to refer to the product topology on G^m . Note that τ_G is not the discrete topology for any dense regular G since no singleton is open. In fact,

we will show in Theorem 3 that a definable set is closed only if it can be defined by a positive quantifier-free formula in \mathcal{L}_G .

Before we can prove this theorem we need a few lemmas and definitions. In what follows let $\bar{G} = \mathbb{Q}G \supseteq G$ be the divisible hull of G . Note that \bar{G} is a dense regular group with every congruence invariant equal to 1: so all the theory developed in the previous chapters applies to \bar{G} , and in particular we have cell decomposition in \bar{G} and we may speak of linear functions $f : X \rightarrow \bar{G}$ with $X \subseteq \bar{G}^m$. At times we may wish to consider a cell from G^m as a subset of \bar{G}^m ; when there is possible confusion we will identify cells as either G -cells or \bar{G} -cells.

Lemma 27. *Let $\bar{a} = (a_1, \dots, a_m) \in G^m$ and let C be a \bar{G} -cell in \bar{G}^m which has distance 0 from \bar{a} (that is, for any positive $\epsilon \in \bar{G}$, there is some $\bar{b} \in C$ such that $|\bar{a} - \bar{b}| < \epsilon$). Let $B = (g_1, h_1) \times \dots \times (g_m, h_m)$ be an open box in \bar{G}^m , containing \bar{a} , with $g_1, h_1, \dots, g_m, h_m \in G$, and for each $n \in \mathbb{N}^+$ define $B_n \subseteq G^m \subseteq \bar{G}^m$ by*

$$B_n = (g_1, h_1)_{\equiv_n a_1} \times \dots \times (g_m, h_m)_{\equiv_n a_m}.$$

If $B_n \cap C \neq \emptyset$ for some $n \in \mathbb{N}^+$, then $B_{kn} \cap C \neq \emptyset$ for all $k \in \mathbb{N}^+$.

Proof. The proof is by induction on m .

Suppose $m = 1$. If C is a 0-cell, then for C to be distance zero from \bar{a} , C must equal $\{\bar{a}\}$, so any box containing \bar{a} will have nonempty intersection with C . If C is a 1-cell then C must be an interval (g, h) where $g < h$. If $g < \bar{a} < h$

then $\bar{a} \in C$ and we are done. If $\bar{a} = g$ then any box $B_n = (g_1, h_1)_{\equiv_n \bar{a}}$ around \bar{a} will intersect C at some point b . In fact, $(a, b)_{\equiv_n \bar{a}} \subseteq B_n \cap C$, and so $(a, b)_{\equiv_{kn} \bar{a}} \subseteq B_n \cap C$ for any $k \in \mathbb{N}^+$ as well. Since this argument is clearly symmetric if $\bar{a} = h$, the base case is complete.

Assume the Lemma holds for all $m' < m$. If $B_n \cap C \neq \emptyset$ for some n , then $\pi_{m-1}(B_n) \cap \pi_{m-1}(C) \neq \emptyset$. By definition $\pi_{m-1}(B)$ is an open box in \bar{G}^{m-1} and clearly $\pi_{m-1}(B_n) = (\pi_{m-1}(B))_n$, so by induction $(\pi_{m-1}(B))_{kn} \cap \pi_{m-1}(C) \neq \emptyset$ for all $k \in \mathbb{N}^+$. The argument divides into two cases, depending on the type of C .

Suppose that C is a $(\dots, 1)$ -cell, and let $k \in \mathbb{N}^+$, $\bar{b}' = (b_1, \dots, b_{m-1}) \in (\pi_{m-1}(B))_{kn} \cap \pi_{m-1}(C)$, and f_1 and f_2 be the \bar{G} -linear functions such that

$$f_1(x_1, \dots, x_{m-1}) < x_m < f_2(x_1, \dots, x_{m-1})$$

for all $\bar{x} \in C$. The fiber of $B \cap C$ over \bar{b}' is

$$\{(\bar{b}', x) \mid \max\{g_m, f_1(\bar{b}')\} < x < \min\{h_m, f_2(\bar{b}')\}\},$$

which is nonempty since \bar{b}' exists. Because G is dense in \bar{G} , the interval $I = (\max\{g_m, f_1(\bar{b}')\}, \min\{h_m, f_2(\bar{b}')\})$ of \bar{G} contains a nonempty open interval of G . Since G is dense regular, we conclude that there is $b_m \in G \cap I$ with $b_m \equiv a_m \pmod{kn}$, and thus $(\bar{b}', b_m) \in B_{kn} \cap C$.

Suppose now that C is a $(\dots, 0)$ -cell and let $k \in \mathbb{N}^+$ and f be the \bar{G} -linear

function such that

$$f(x_1, \dots, x_{m-1}) = x_m$$

for all $\bar{x} \in C$. Then there is a fixed linear equation

$$c_1x_1 + \dots + c_mx_m = \gamma$$

with $c_1, \dots, c_m \in \mathbb{Z}$, $\gamma \in G$, and $c_m > 0$, obeyed by all $\bar{x} \in C$. We are able to say $\gamma \in G$, as opposed to \bar{G} , by clearing the denominator if there is one. Since C is distance 0 from \bar{a} and addition is continuous with respect to $|\ast|$ in \bar{G} , \bar{a} must obey this linear equation. Let $\bar{b}' = (b_1, \dots, b_{m-1}) \in (\pi_{m-1}(B))_{c_mkn} \cap \pi_{m-1}(C)$: that is, $\bar{b}' \in \pi_{m-1}(B) \cap \pi_{m-1}(C) \cap G^{m-1}$ and $b_i \equiv a_i \pmod{c_mkn}$ for each $i \in \{1, \dots, m-1\}$. So for each $i \in \{1, \dots, m-1\}$ there exists $t_i \in G$ such that $b_i = a_i + c_mknt_i$. Since $\bar{b}' \in \pi_{m-1}(C)$, there exists $b_m \in \bar{G}$ such that $f(\bar{b}') = b_m$: so $c_1b_1 + \dots + c_mb_m = \gamma$ and in \bar{G}

$$\begin{aligned} b_m &= \frac{-(c_1b_1 + \dots + c_{m-1}b_{m-1} - \gamma)}{c_m} \\ &= \frac{-(c_1(a_1 + c_mknt_1) + \dots + c_{m-1}(a_{m-1} + c_mknt_{m-1}) - \gamma)}{c_m} \\ &= \frac{-(c_1a_1 + c_1c_mknt_1 + \dots + c_{m-1}a_{m-1} + c_{m-1}c_mknt_{m-1} - \gamma)}{c_m} \\ &= \frac{-(c_1a_1 + \dots + c_{m-1}a_{m-1} - \gamma)}{c_m} - \frac{(c_1c_mknt_1 + \dots + c_{m-1}c_mknt_{m-1})}{c_m} \\ &= a_m - kn(c_1t_1 + \dots + c_{m-1}t_{m-1}) \in G. \end{aligned}$$

So $b_m \equiv a_m \pmod{kn}$ and $(\bar{b}', b_m) \in B_{kn} \cap C$.

□

Definition 11. For any $n \in \mathbb{N}^+$, certain open boxes, called n -boxes, are defined as follows.

An open box in G is an n -box if and only if it is of the form

$$\{x \in G \mid a < kx < b, x \equiv c \pmod{n}\},$$

where $a, b \in G \cup \{\pm\infty\}$, $c \in G$, and $k \in \mathbb{N}^+$.

An open box in G^m is an n -box if and only if it is of the form

$$\{(\bar{x}, t) \in G^m \mid \bar{x} \in B, a < kt < b, t \equiv c \pmod{n}\},$$

where B is an n -box in G^{m-1} , and a, b, c, k are as above.

Definition 12. We say a set $A \subseteq G^m$ is n -open if for every $\bar{a} \in A$, there is an n' -box, B , where $n' \leq n$, such that $\bar{a} \in B \subseteq A$. We say a set is n -closed if its complement is n -open.

Note that for all $n \in \mathbb{N}^+$, n -open implies τ_G -open, and n -closed implies τ_G -closed. Also notice that while τ_G -open and τ_G -closed are not first-order properties, n -open and n -closed are.

Lemma 28. *If $A \subseteq G^m$ is τ_G -open and definable, then A is n -open for some $n \in \mathbb{N}^+$.*

Proof. If A is empty, then A is n -open for any n . Assume A is nonempty. Then A can be defined by a formula $\phi(x_1, \dots, x_m) =$

$$\bigvee_{f \in F} \left(\bigwedge_{i \in I_f} \sum_{l=1}^m p_{fil} x_l < g_{fi} \wedge \bigwedge_{j \in J_f} \sum_{l=1}^m q_{fjl} x_l \equiv_n c_{fj} \wedge \bigwedge_{k \in K_f} \sum_{l=1}^m r_{fkl} x_l = h_{fk} \right)$$

where the p 's, q 's, and r 's are integers, n is a positive integer, the c 's come from a fixed finite set of representatives for the congruence classes mod n , and the g 's and h 's are elements of G . We may assume that every disjunct of ϕ is satisfiable and that no tuple $(p_{fi1}, \dots, p_{fim})$, $(q_{fj1}, \dots, q_{fjm})$, or $(r_{fk1}, \dots, r_{fkm})$ is $(0, \dots, 0)$.

Let \mathcal{D} be a decomposition of the divisible hull, \bar{G}^m , into cells such that each formula

$$\sum_{l=1}^m p_{fil} x_l \square g_{fi} \quad \text{and} \quad \sum_{l=1}^m r_{fkl} x_l \square h_{fk},$$

where $\square \in \{=, <, >\}$, has invariant truth value on each cell $C \in \mathcal{D}$. Then for any cell $C \in \mathcal{D}$ and $\bar{a} = (a_1, \dots, a_m), \bar{b} = (b_1, \dots, b_m) \in C \cap G^m$ such that $a_l \equiv_n b_l$ for each $l \in \{1, \dots, m\}$, $\bar{a} \in A$ if and only if $\bar{b} \in A$.

Let $\bar{a} \in A$. Since A is open, $\bar{a} \in B_N \subseteq A$ for some N -box $B_N \subseteq G^m$. Because $(0, \infty)^G$ is co-initial in $(0, \infty)^{\bar{G}}$ we may assume that B_N does not intersect any cells in \mathcal{D} which have positive distance from \bar{a} (i.e. are not at distance 0 from \bar{a}). We may write

$$B_N = (g'_1, h'_1)_{\equiv_N a_1} \times \cdots \times (g'_m, h'_m)_{\equiv_N a_m}.$$

For each $s \in \mathbb{N}^+$ let

$$B_s = (g'_1, h'_1)_{\equiv_s a_1} \times \cdots \times (g'_m, h'_m)_{\equiv_s a_m} :$$

then $\bar{a} \in B_s$.

Assume $\bar{b} \in B_n$. \bar{b} belongs to some $C \in \mathcal{D}$, and by Lemma 27 $B_{Nn} \cap C \neq \emptyset$; pick $\bar{b}' \in B_{Nn} \cap C$. Since $B_{Nn} \subseteq B_N \subseteq A$, $\bar{b}' \in A$; and since $b'_l \equiv_{Nn} a_l \equiv_n b_l$ for every $l \in \{1, \dots, m\}$, $\bar{b} \in A$. Thus $\bar{a} \in B_n \subseteq A$ and A is n -open. □

The main tool for proving Theorem 3 will be the following lemma from [1]:

Lemma 29. *Let T be a first-order theory and $\phi(\bar{x})$ an \mathcal{L}_T -formula. Suppose that for each model K of T and each homomorphism $f : A \rightarrow L$ where $A \subseteq K$ and $L \models T$ we have:*

if $\bar{a} \in A^m$ and $K \models \phi(\bar{a})$, then $L \models \phi(f(\bar{a}))$.

Then there is a positive q.f. formula $\psi(\bar{x})$ such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$.

In what follows we let S be the theory of dense regular groups with the same congruence invariants as G . Let $T = S \cup \text{diag } G$ and let $K, L \models T$. The next two lemmas examine the situation described in Lemma 29. Remember that \equiv_n is a *relation symbol* in \mathcal{L} , so that in an \mathcal{L} -substructure H of a model R of S , we may have $a \equiv_n^H b$ without having a congruent mod n to b in H (i.e. without $a \equiv_n^R b$).

Lemma 30. *If A is an \mathcal{L}_G -substructure of K and $f : A \rightarrow L$ is an \mathcal{L}_G -homomorphism, then there is an extension \hat{A} of A in K , with $|A| = |\hat{A}|$, such that:*

(i) *for any $a, b \in \hat{A}$, and $n \in \mathbb{N}^+$, if $a \equiv_n^{\hat{A}} b$ then there exists $c \in \hat{A}$ such that $a + nc = b$, and*

(ii) *there exists a homomorphism $\hat{f} : \hat{A} \rightarrow L$ such that $\hat{f}|_A = f$.*

Proof. Since $K \models T$, for any $a, b \in A \subseteq K$ and $n \in \mathbb{N}^+$ such that $a \equiv_n^A b$, there exists $c \in K$ such that $a + nc = b$ is true in K . Let $\hat{A} = \{c \in K \mid a + nc = b \text{ for some } a, b \in A, n \in \mathbb{N}^+\}$. It is clear that $A \subseteq \hat{A}$ since $0 \in A$ and $0 + 1b = b$ for any $b \in A$. It is also straightforward to check that \hat{A} forms a subgroup of K .

Let $a, b \in \hat{A}$, and $n \in \mathbb{N}^+$ with $a \equiv_n^{\hat{A}} b$. Then $a = \frac{a_1 - a_2}{n_1}$ and $b = \frac{b_1 - b_2}{n_2}$ for some $a_1, a_2, b_1, b_2 \in A$ and $n_1, n_2 \in \mathbb{N}^+$, and there exists $c \in K$ such that $a + nc = b$. So

$$\begin{aligned} a + nc &= b, \\ \frac{a_1 - a_2}{n_1} + nc &= \frac{b_1 - b_2}{n_2}, \end{aligned}$$

and by clearing denominators we see that

$$n_2(a_1 - a_2) + n_2 n_1 n c = n_1(b_1 - b_2).$$

Since $n_2(a_1 - a_2)$ and $n_1(b_1 - b_2) \in A$, $c \in \hat{A}$, and (i) is true.

For each $c \in \hat{A}$, we know we can write c as $\frac{a-b}{n}$ for some $a, b \in A$ and $n \in \mathbb{N}^+$ with $a \equiv_n^A b$. Define $\hat{f}(c) := \frac{f(a)-f(b)}{n}$ in L . Since f is a homomorphism and $a \equiv_n^A b$ we have $f(a) \equiv_n^L f(b)$, and since $L \models T$, there exists $l \in L$ such that $l = \frac{f(a)-f(b)}{n}$. Moreover, suppose that $c = \frac{a_1-b_1}{n_1} = \frac{a_2-b_2}{n_2}$. Then $n_2(a_1-b_1) = n_1(a_2-b_2)$ and since f is a homomorphism $n_2(f(a_1) - f(b_1)) = n_1(f(a_2) - f(b_2))$, and $\frac{f(a_1)-f(b_1)}{n_1} = \frac{f(a_2)-f(b_2)}{n_2}$: so $\hat{f}(c)$ is well-defined. It is straightforward to check that \hat{f} is a homomorphism.

Notice that for each pair of elements $a, b \in A$ at most countably many new elements are put into \hat{A} in the above construction. Therefore, since every \mathcal{L}_G -substructure of A is infinite, we have $|\hat{A}| = |A|$ for any A .

□

Lemma 31. *Let A be an \mathcal{L}_G -substructure of K , $f : A \rightarrow L$ be an \mathcal{L}_G -homomorphism, and let \hat{A} and \hat{f} be extensions of A and f in K as in the previous lemma. If K^* is an $|\hat{A}|^+$ -saturated elementary extension of K , then there is an extension B of A in K^* such that:*

- (i) *for any $a, b \in B$, and $n \in \mathbb{N}^+$, if $a \equiv_n^B b$ then there exists $c \in B$ such that $a + nc = b$,*
- (ii) *there exists a homomorphism $f_B : B \rightarrow L$ such that $f_B|_A = f$, and*
- (iii) *$f_B(B) \models T$.*

Proof. The following argument starts by defining a chain $\hat{A} = B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$ of substructures of K^* — with $|\hat{A}| = |B_1| = |B_2| = \dots$ — and homomorphisms $f_i : B_i \rightarrow L$ such that every $f_i|_{B_{i-1}} = f_{i-1}$.

For each $n \in \mathbb{N}^+$ let $\{c_{n,1}, c_{n,2}, \dots, c_{n,i_n}\}$ be a set of representatives of the congruence classes mod n in G , one representative per class. Consider the following collection of subsets of L :

$$\mathcal{I} = \{(\hat{f}(a), \hat{f}(b))_{\equiv_n c_{n,i}} \mid a, b \in \hat{A}, \hat{f}(a) < \hat{f}(b), n, i \in \mathbb{N}^+, i \leq i_n\}.$$

Let (I_1, I_2, \dots) be a well ordering of \mathcal{I} . If $I_1 \cap \hat{f}(\hat{A}) \neq \emptyset$ then define $B_1 = \hat{A}$.

If $I_1 \cap \hat{f}(\hat{A}) = \emptyset$ then pick $\lambda \in L$ such that $\lambda \in I_1$ (such a λ must exist since $L \models T$). For each $a \in \hat{A}$ and $k \in \mathbb{N}^+$ such that $\hat{f}(a) < k\lambda$ in L , let $\phi_{a,k}(x)$ be the $\mathcal{L}_{\hat{A}}$ -formula $a < kx$. Similarly, for each $b \in \hat{A}$ and $m \in \mathbb{N}^+$ such that $\hat{f}(b) > m\lambda$ in L , let $\psi_{b,m}(x)$ be $b > mx$. For each $n \in \mathbb{N}^+$ let g_n be an element in G such that $\lambda \equiv_n g_n$, and let $\theta_n(x)$ be $x \equiv_n g_n$. Lastly, for each $c \in \hat{A}$, let $\sigma_c(x)$ be $x \neq c$. Then the set Φ of all formulas $\phi_{a,k}(x), \psi_{b,m}(x), \theta_n(x), \sigma_c(x)$ is a 1-type over \hat{A} , since it is finitely satisfiable in K as is shown in the next paragraph.

Let F be a finite subset of Φ . Then $F = \{a_i < k_i x \mid i \in I\} \cup \{b_j > m_j x \mid j \in J\} \cup \{x \equiv_{n_r} g_{n_r} \mid r \in R\} \cup \{x \neq c_s \mid s \in S\}$ for some finite index sets I, J, R and S . Since $\lambda \equiv_{n_r} g_{n_r}$ for each $r \in R$, the sentence $\exists x (\bigwedge_{r \in R} (x \equiv_{n_r} g_{n_r}))$ is consistent with T . So by dense regularity, any nonempty open interval in any model of T will contain an element c_0 with $c_0 \equiv_{n_r} g_{n_r}$ for every $r \in R$. Let κ be the product of all the k_i and m_j . Then for each $\phi_{a_i, k_i}(x)$ in F let $\phi'_{a_i, k_i}(x)$ be $\frac{\kappa}{k_i} a_i < \kappa x$, and for each $\psi_{b_j, m_j}(x)$ in F let $\psi'_{b_j, m_j}(x)$ be $\frac{\kappa}{m_j} b_j > \kappa x$. Since $K \models T$ and $K \supseteq \hat{A}$, $K \models \forall x (\phi_{a_i, k_i}(x) \leftrightarrow \phi'_{a_i, k_i}(x))$ and

$K \models \forall x(\psi_{b_j, m_j}(x) \leftrightarrow \psi'_{b_j, m_j}(x))$ for each $i \in I$ and $j \in J$. Since $L \models T$ and for each $i \in I$, $\hat{f}(a_i) < k_i \lambda$ holds in L , so does $\frac{\kappa}{k_i} \hat{f}(a_i) < \kappa \lambda$. Likewise, for each $j \in J$, $\frac{\kappa}{m_j} \hat{f}(b_j) > \kappa \lambda$ holds in L . Thus $\frac{\kappa}{k_i} \hat{f}(a_i) < \frac{\kappa}{m_j} \hat{f}(b_j)$ in L for any $i \in I$ and $j \in J$. Therefore, since \hat{f} is an order preserving homomorphism, $\frac{\kappa}{k_i} a_i < \frac{\kappa}{m_j} b_j$ in K for any $i \in I$ and $j \in J$. So by dense regularity there must exist an element c_0 such that $\max\{\frac{\kappa}{k_i} a_i\} < \kappa c_0 < \min\{\frac{\kappa}{m_j} b_j\}$ and $c_0 \equiv_{n_r} g_{n_r}$ for each $r \in R$. Also by dense regularity, c_0 can be chosen to avoid c_s for each $s \in S$. Clearly c_0 satisfies every formula in F , and so Φ is finitely satisfiable.

Since K^* is $|\hat{A}|^+$ -saturated and $\hat{A} \subseteq K^*$ there must exist k_1 which realizes Φ in K^* . Let $B'_1 = \{a + zk_1 \mid a \in \hat{A}, z \in \mathbb{Z}\} \subseteq K^*$ and let $f'_1 : B'_1 \rightarrow L$ be defined by $f'_1(a + zk_1) := \hat{f}(a) + z\lambda$. It is straightforward to check that this is a well-defined homomorphism of \mathcal{L} -structures. Now by Lemma 30 there are an extension \hat{B}'_1 of B'_1 and a homomorphism $\hat{f}'_1 : \hat{B}'_1 \rightarrow L$, extending f'_1 , such that whenever $a \equiv_n^{\hat{B}'_1} b$ there is $c \in \hat{B}'_1$ with $a = b + nc$. Let $B_1 = \hat{B}'_1$ and $f_1 = \hat{f}'_1$. Notice that $|\hat{A}| = |B_1|$.

Suppose now that B_1, \dots, B_α have been defined. Then we may use the procedure for finding B_1 and f_1 with B_α in place of \hat{A} and $I_{\alpha+1}$ in place of I_1 to find $B_{\alpha+1}$ and $f_{\alpha+1}$. If $\mu < |\hat{A}|$ is a limit ordinal, and B_β has been defined for all $\beta < \mu$, then define $B_\mu = \bigcup_{\beta < \mu} B_\beta$, and $f_\mu = \bigcup_{\beta < \mu} f_\beta$. It is, again, a straightforward check to see that B_μ, f_μ are as desired.

Let $C'_1 = \bigcup_{\alpha < |\hat{A}|} B_\alpha$, $f_{C'_1} = \bigcup_{\alpha \leq |\hat{A}|} f_\alpha$, $C_1 = \hat{C}'_1$, and $f_{C_1} = \hat{f}_{C'_1}$, where $\hat{\cdot}$ is as before. Then $|C_1| = |\hat{A}|$ and C_1 satisfies (i) and (ii) but not necessarily (iii). We have guaranteed that between any two elements of $f(\hat{A})$ there

are elements of $f_{C_1}(C_1)$ of every possible congruence class, but perhaps not between any two elements of $f_{C_1}(C_1)$: so $f_{C_1}(C_1)$ may not be dense regular.

We may now run the entire argument again, with C_1 and f_{C_1} in place of \hat{A} and \hat{f} to get C_2 . Continuing in this way we get a countable chain of \mathcal{L} -structures: $\hat{A} \subseteq C_1 \subseteq C_2 \subseteq \dots \subseteq K^*$. Let $B = \bigcup_{m \in \mathbb{N}^+} C_m$ and $f_B = \bigcup_{m \in \mathbb{N}^+} f_{C_m}$. It is clear from the construction that B satisfies properties (i) and (ii) of the lemma. Since B contains G , and f_B is an extension of f , $f_B(B)$ is a model of $\text{diag } G$; so in order to show that B satisfies (iii) it remains to show that $f_B(B) \models S$.

According to [4], if $f_B(B)$ is an ordered abelian group, is dense regular, and has the same congruence invariants as G , then $B \models S$. C_m is an ordered abelian group and has the same congruence invariants as G for every $m \in \mathbb{N}^+$. Each of these properties can be axiomatized by $\forall\exists$ sentences, so they are preserved in B . The homomorphic image of an ordered abelian group is an ordered abelian group: so $f_B(B)$ is an ordered abelian group. For each n one can easily write down a positive sentence stating that every element is equivalent mod n to one of a finite list of elements of G . Then the formulae of this kind which are true in B are preserved by f_B in $f_B(B)$ as well. This fact, together with the fact that $f_B(B) \models \text{diag } G$ shows that $f_B(B)$ has the same congruence invariants as G . For any $f_B(a) < f_B(b) \in f_B(B)$ and $n \in \mathbb{N}^+$ there is some $m \in \mathbb{N}^+$ such that $a, b \in C_m$. Then in the construction of C_{m+1} the interval $I = (f_{C_m}(a), f_{C_m}(b))_{\equiv_n 0}$ of L will have nontrivial intersection with $f_{C_{m+1}}(C_{m+1}) \subseteq f_B(B)$: so $f_B(B)$ is dense regular and the proof is

complete. □

Theorem 3. *If $X \subseteq G^m$ is definable and closed with respect to τ_G , then X is definable by a positive quantifier-free formula in \mathcal{L}_G .*

Proof. Let $\phi(\bar{x})$ be a formula defining X , and let T be as above. We aim to use Lemma 29 in order to show the existence of a positive q.f. formula $\psi(\bar{x})$ such that $T \vdash \phi(\bar{x}) \leftrightarrow \psi(\bar{x})$; this conclusion gives the desired result because $G \models T$.

Let K, L, A, f, K^* , and B be as in Lemma 31 and assume that $\text{dom}(K^*) \cap \text{dom}(L) = \text{dom}(G)$ (if not, replace L by an isomorphic structure for which this is true). So we have $G \subseteq A \subseteq B \subseteq K^*$ and $f_B : B \rightarrow L$ a homomorphism such that $f_B|_A = f$ and $f_B(B) \models T$.

Let \mathcal{L}^+ be a new language with unary relation symbols γ, β, κ , and λ , binary relation symbols F, \leq , and \equiv_n for all $n \geq 1$, a ternary relation symbol A , and a constant symbol 0 . Let \mathcal{S} be the \mathcal{L}^+ -structure, with domain $K^* \cup L$, in which $\gamma^{\mathcal{S}} = G$, $\beta^{\mathcal{S}} = B$, $\kappa^{\mathcal{S}} = K^*$, $\lambda^{\mathcal{S}} = L$, $F^{\mathcal{S}}$ is the graph of f_B , $\leq^{\mathcal{S}} = (\leq^{K^*} \cup \leq^L)$, $\equiv_n^{\mathcal{S}} = (\equiv_n^{K^*} \cup \equiv_n^L)$ for every $n \geq 1$, $A^{\mathcal{S}}$ is the union of the graphs of addition on K^* and on L , and $0^{\mathcal{S}} = 0^{K^*} = 0^L$. Let $\mathcal{S}' \succeq \mathcal{S}$ be ω_1 -saturated. Using the symbols of \mathcal{L}^+ , we may recover ω_1 -saturated \mathcal{L} -structures $B' \succeq B$, $L' \succeq L$, and $K' \succeq K^*$, and a homomorphism $f_{B'} : B' \rightarrow K'$ that extends f_B and gives $f_{B'}(B') \succeq f_B(B) \models T$.

Notice that $\ker(f_{B'})$ is a pure, convex subgroup of B' . By Lemma 2.2 in [5], there must be a subgroup R of B' such that R is isomorphic to

$B'/\ker(f_{B'})$ via the restriction to R of the quotient homomorphism: $q(b) = b + \ker(f_{B'})$. The rule $h(b + \ker(f_{B'})) = f_{B'}(b)$ also defines an isomorphism from $B'/\ker(f_{B'})$ to $f_{B'}(B')$. Since $h(q|_R) = f_{B'}|_R$, $f_{B'}|_R$ is an isomorphism from R to $f_{B'}(B')$ and $R \models T$.

Let $\bar{c} = (c_1, \dots, c_m) \in A^m$ such that $K \models \phi(\bar{c})$. If we can show that $L \models \phi(f(\bar{c}))$, then we are done by Lemma 29. Since $\bar{c} \in A^m \subseteq (B')^m$, there is $\bar{r} = (r_1, \dots, r_m) \in R^m$ such that $c_i - r_i \in \ker(f_{B'})$ for each $i \in \{1, \dots, m\}$. We claim this implies that $R \models \phi(\bar{r})$. Since X is closed and definable, X is n -closed for some $n \in \mathbb{N}^+$ by Lemma 28. So $\phi(\bar{x})$ defines an n -closed set in G^m , and by the model-completeness of T , in R^m as well. So if $R \not\models \phi(\bar{r})$ there must be an n -box of R -positive radius ϵ centered at \bar{r} in the complement of $\phi(R^m)$. Since $R \subseteq K'$ an n -box of the same radius ϵ around \bar{r} must be contained in the set defined by $\neg\phi$ in K' . Because $K \models \phi(\bar{c})$, $K \preceq K'$, and $|c_i - r_i| < \epsilon$ for $i = 1, \dots, m$, we reach a contradiction.

Now, $R \cong f_{B'}(B')$ and $R \models \phi(\bar{r})$, so $f_{B'}(B') \models \phi(f_{B'}(\bar{r}))$. Also, $f_{B'}(\bar{r}) = f_{B'}(\bar{c}) = f(\bar{c})$ and $f_{B'}(B') \subseteq L'$, so $L' \models \phi(f(\bar{c}))$, and since L' is an elementary extension of L and $\bar{c} \in L^m$, we have $L \models \phi(f(\bar{c}))$ and the proof is complete. \square

Lemma 32. *If C is a cell in G^m then there is a τ_G -closed, definable set $\bar{C} \subseteq G^m$ such that $C \subseteq \bar{C}$ and $\dim(\bar{C} \setminus C) < \dim(C)$.*

Moreover, we may assume that there is a flat $F \supseteq C$ as in Lemma 17 with $C \subseteq \bar{C} \subseteq F$: so C and F have the same type, and for all $c_1, \dots, c_m \in G$

and $n_1, \dots, n_m \in \mathbb{N}^+$,

$$a_i \equiv c_i \pmod{n_i} \text{ for all } (a_1, \dots, a_m) \in C$$

if and only if

$$a_i \equiv c_i \pmod{n_i} \text{ for all } (a_1, \dots, a_n) \in F.$$

Proof. The proof is by induction on m .

If C is a 0-cell then C is clearly closed, and is also a flat satisfying the required conditions; so let $F = \bar{C} = C$. If C is a 1-cell then if $C = \{x \mid a < kx < b, x \equiv c \pmod{n}\}$, let

$$\bar{C} = C \cup \{x \mid a = kx, x \equiv_n c\} \cup \{x \mid b = kx, x \equiv_n c\}.$$

That is, \bar{C} is C plus its endpoints as one might expect, but only if they are elements of G (of course) and congruent to $c \pmod{n}$. It is easy to check that \bar{C} is closed — n -closed in fact — and since $\bar{C} \setminus C$ has at most 2 elements $\dim(\bar{C} \setminus C) \leq 0 < 1 = \dim(C)$. If $F = \{x \mid x \equiv c \pmod{n}\}$, then $C \subseteq \bar{C} \subseteq F$ and F is as required.

Assume now that the lemma holds in G^l for all $l < m$.

Suppose that $C \subseteq G^m$ is a $(\dots, 0)$ -cell and let $f : \pi_{m-1}(C) \rightarrow G$ be the

linear function such that

$$f(a_1, \dots, a_{m-1}) = \sum_{i=1}^{m-1} p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma = a_m$$

for all $\bar{a} = (a_1, \dots, a_m) \in C$. By induction hypothesis there is a closed definable set $\bar{D} \supseteq \pi_{m-1}(C)$ such that $\dim(\bar{D} \setminus \pi_{m-1}(C)) < \dim \pi_{m-1}(C)$ and \bar{D} is contained in a flat F' as in the lemma. Let

$$\bar{C} = \left\{ (x_1, \dots, x_{m-1}, t) \in G^m \mid \bar{x} \in \bar{D} \text{ and } \sum_{i=1}^{m-1} p_i \left(\frac{x_i - c_i}{n_i} \right) + \gamma = t \right\}.$$

Since for each $i \in \{1, \dots, m-1\}$, $a_i \equiv_{n_i} c_i$ for every $\bar{a} \in C$, the same is true in F' and hence in \bar{D} : so the definition of \bar{C} makes sense. Clearly $C \subseteq \bar{C}$ and \bar{C} is definable. We must show that \bar{C} is closed, that $\dim(\bar{C} \setminus C) < \dim(C)$, and that there is a flat $F \supseteq \bar{C}$ satisfying the required congruence conditions.

Let $(\bar{a}, b) \in G^m \setminus \bar{C}$. If $\bar{a} \in G^{m-1} \setminus \bar{D}$ then since \bar{D} is closed, there is an open box $B_{\bar{a}} \ni \bar{a}$ such that $B_{\bar{a}} \cap \bar{D} = \emptyset$. Then for any $\epsilon > 0$, $\{(\bar{x}, y) \mid \bar{x} \in B_{\bar{a}}, b - \epsilon < y < b + \epsilon\}$ is an open box containing (\bar{a}, b) which does not intersect \bar{C} . If \bar{a} is in \bar{D} , then $b \neq \sum p_i \left(\frac{a_i - c_i}{n_i} \right) + \gamma$ and so by the $|\ast|$ -continuity of linear functions, there is an open box B such that $(\bar{a}, b) \in B$ but $B \cap \bar{C} = \emptyset$. So \bar{C} is closed. If

$$F = \left\{ (x_1, \dots, x_{m-1}, t) \in G^m \mid \bar{x} \in \bar{F}' \text{ and } \sum_{i=1}^{m-1} p_i \left(\frac{x_i - c_i}{n_i} \right) + \gamma = t \right\},$$

then F is clearly a flat of the appropriate type and $C \subseteq \bar{C} \subseteq F$. The proof

of Lemma 17 shows that F satisfies the required congruence conditions.

To show that $\dim(\bar{C} \setminus C) < \dim C$, let $\{C_1, \dots, C_r\}$ be a decomposition of $\bar{C} \setminus C$ into cells. Since C is a $(\dots, 0)$ -cell, $\pi_{m-1}(\bar{C} \setminus C) = \bar{D} \setminus \pi_{m-1}(C)$; so each C_i has the form $\{(\bar{x}, t) \mid \bar{x} \in D_i, f'(\bar{x}) = t\}$ where f' is the linear function on \bar{D} with same description as that displayed for f , and $\{D_1, \dots, D_r\}$ is a decomposition of $\bar{D} \setminus \pi_{m-1}(C)$. So $\dim C_i = \dim D_i$ for each i and by induction $\dim D_i < \dim \pi_{m-1}(C) = \dim C$ for each i . Therefore, $\dim(\bar{C} \setminus C) < \dim C$.

Suppose now that C is a $(\dots, 1)$ -cell. Then C is of the form $\{(\bar{x}, t) \mid \bar{x} \in \pi_{m-1}(C), f(\bar{x}) < kt < g(\bar{x}), t \equiv c \pmod{n}\}$. Again let \bar{D} be a closed definable set containing $\pi_{m-1}(C)$ such that $\dim(\bar{D} \setminus \pi_{m-1}(C)) < \dim \pi_{m-1}(C)$, and let F' be a flat satisfying all required conditions. As above, the linear functions f and g may be extended to linear functions f' and g' on all of F' in a natural way. Let

$$\bar{C} = \{(\bar{x}, t) \mid \bar{x} \in \bar{D}, f'(\bar{x}) \leq kt \leq g'(\bar{x}), t \equiv c \pmod{n}\}.$$

Again it is clear that $C \subseteq \bar{C}$ and that \bar{C} is definable; we must check the other requirements.

Let $(\bar{a}, b) \in G^m \setminus \bar{C}$. If $\bar{a} \in G^{m-1} \setminus \bar{D}$ then as before there is an open box $B_{\bar{a}}$ containing \bar{a} which does not intersect \bar{D} and any box above $B_{\bar{a}}$ containing (\bar{a}, b) will not intersect \bar{C} . If $b \not\equiv_n c$ then for any open box $B_{\bar{a}}$ around \bar{a} and any $\epsilon > 0$, $\{(\bar{x}, y) \mid \bar{x} \in B_{\bar{a}}, b - \epsilon < y < b + \epsilon, y \equiv b \pmod{n}\}$ is an open box

containing (\bar{a}, b) which does not intersect \bar{C} . If $\bar{a} \in \bar{D}$ and $b \equiv_n c$ then either $kb < f'(\bar{a})$ or $kb > g'(\bar{a})$. In either case, $|*|$ -continuity of linear functions provides an open box around (\bar{a}, kb) which is entirely below f' or above g' , and this easily implies the existence of an open box around (\bar{a}, b) avoiding \bar{C} . So \bar{C} is closed.

If $F = \{(\bar{x}, t) \mid \bar{x} \in F', t \equiv c \pmod{n}\}$ then F is clearly as required by the lemma.

Let $\bar{C}_1 = \{(\bar{x}, t) \in \bar{C} \setminus C \mid \bar{x} \in \pi_{m-1}(C)\}$ and let $\bar{C}_2 = (\bar{C} \setminus C) \setminus \bar{C}_1$. Then $\pi_{m-1}(\bar{C}_2) = \bar{D} \setminus \pi_{m-1}(C)$, so

$$\dim \bar{C}_2 \leq \dim(\bar{D} \setminus \pi_{m-1}(C)) + 1 < \dim \pi_{m-1}(C) + 1 = \dim C.$$

So if we show $\dim \bar{C}_1 < \dim C$, then we are done. Let $\{C_1, \dots, C_s\}$ be a decomposition of \bar{C}_1 into cells. Then each C_i must be of the form $\{(\bar{x}, t) \mid \bar{x} \in D_i, f'(\bar{x}) = t\}$ or $\{(\bar{x}, t) \mid \bar{x} \in D_i, g'(\bar{x}) = t\}$, where $\{D_1, \dots, D_s\}$ form a decomposition of $\pi_{m-1}(C)$. So we have

$$\dim C_i = \dim D_i \leq \dim \pi_{m-1}(C) < \dim C$$

for every $i \in \{1, \dots, s\}$. Therefore $\dim \bar{C}_1 < \dim C$ and $\dim(\bar{C} \setminus C) < \dim C$.

□

Corollary 4. *If $A \subseteq G^m$ is a definable set, then there exists a τ_G -closed set \bar{A} such that $A \subseteq \bar{A}$ and $\dim(\bar{A} \setminus A) < \dim A$.*

Proof. Immediate from cell decomposition and Lemma 32.

□

Chapter 6

Conclusion

A natural next step would be to examine the case where infinite congruence invariants are allowed. Though it may be possible to generalize some of the arguments in this thesis to handle this new case, it would not be trivial, since even the proof of Lemma 1 exploits the finiteness of congruence invariants. Furthermore, the definition of cell would have to be altered since, for example, if there are infinitely many congruence classes mod n , then even the relatively simple set defined by $\neg(x \equiv_n 0)$ cannot be written as the finite union of cells as they are defined presently. It seems plausible that loosening the definition of cell to include sets like this may still lead to a similar cell decomposition theorem, but what complications might arise beyond that point is not clear. If one wanted to study groups where dense regularity was no longer assumed new techniques would definitely be needed since only very special ordered abelian groups admit elimination of quantifiers in languages,

like \mathcal{L} , expanding the language of ordered abelian groups by constant symbols and symbols for congruences [9].

A second project would try to characterize the sort of ‘minimality’ these groups are exhibiting and attempt to find similar structure theorems for that class as a whole. One obvious difficulty in this line of research is that definable functions could no longer be assumed to be piecewise linear — a fact used quite heavily here. A dimension theory has already been developed for *weakly* o-minimal structures [10], so one might look to borrow techniques from that area as well as from standard o-minimality. In an expansion of \mathcal{L} where one has constant symbols to represent every congruence class, the groups studied here are examples of structures which have been called *weakly quasi-o-minimal* [3], though this subject is in its infancy.

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