On almost universal ternary inhomogeneous quadratic polynomials

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Abstract

A fundamental question in the study of integral quadratic forms is the representation problem which asks for an effective determination of the set of integers represented by a given quadratic form. A related and equally interesting problem is the representation of integers by inhomogeneous quadratic polynomials. An inhomogeneous quadratic polynomial is a sum of a quadratic form and a linear form; it is called almost universal if it represents all but finitely many positive integers. This thesis gives a characterization of almost universal ternary inhomogeneous quadratic polynomials, $H(x)$ whose quadratic parts are positive definite and anisotropic at exactly one prime. Imposing some other mild arithmetic conditions, we utilize the theory of quadratic lattices and primitive spinor exceptions to give a list of explicit conditions, under which $H(x)$ is almost universal. In the final chapter, we will give some examples of almost universal quadratic polynomials given by mixed sums of polygonal numbers.
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Contents

Dedication i

Abstract ii

Acknowledgements iii

Introduction 1

1 Preliminaries 4
  1.1 Quadratic Spaces and Lattices . . . . . . . . . . . . . . . . . . . . . . . . . . 4
  1.2 Primitive Spinor Exceptions . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
  1.3 Inhomogeneous Quadratic Polynomials . . . . . . . . . . . . . . . . . . . . . 11

2 \( p \) is odd 17
  2.1 Preliminaries . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
  2.2 Universal Ternary \( \mathbb{Z}_2 \)-lattice . . . . . . . . . . . . . . . . . . . . . 18
  2.3 Main Result . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 23

3 \( p \) is even 31
  3.1 \( Q(\nu) \in \mathbb{Z}_2^\times \) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 33
  3.2 \( Q(\nu) \in 2\mathbb{Z}_2^\times \) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 64
3.3 $Q(\nu) \in 4\mathbb{Z}_2^\times$ ......................................................... 71

4 Mixed Sums of Squares and Triangular Numbers ..................................... 75

Appendix A Computing Spinor Norms ....................................................... 83
  A.1 Non-dyadic Case ................................................................. 83
  A.2 Dyadic Case ................................................................. 84
    A.2.1 Binary Case .......................................................... 84
    A.2.2 Higher Dimensional Case ............................................. 84
  A.3 Computing Relative Spinor Norms ................................................. 86

Bibliography ................................................................. 89
Introduction

A fundamental question in the study of quadratic forms asks, given a quadratic polynomial \( f \) and an integer \( a \), how can we decide when \( f \) represents \( a \) over the integers? This question has been the driving force behind many developments in number theory over the past centuries. Very notably, in 1900, David Hilbert addressed the International Congress of Mathematicians in Paris posing his famous list of 23 problems; among them was the following:

To solve a given quadratic equation with algebraic numerical coefficients in any number of variables by integral or fractional numbers belonging to the algebraic realm of rationality determined by the coefficients [9].

Along this line of inquiry, this thesis deals with the representation problem of inhomogeneous quadratic polynomials. That is, given an integer \( a \) and an inhomogeneous quadratic polynomial

\[
f(x) = Q(x) + \ell(x) + c,
\]

in more than one variable, when is \( f(x) = a \) solvable over the integers? Here \( Q \) and \( \ell \) are homogeneous quadratic and linear polynomials, respectively, and \( c \) is a constant. Triangular numbers, defined by \( T_x = \frac{x(x-1)}{2} \) where \( x \) is an integer, give a
particularly interesting family of inhomogeneous quadratic polynomials. In 1796, Gauss observed that the sum of three triangular numbers, \( T_x + T_y + T_z \), represents all natural numbers, a property which we call \textit{universal}. This fact can be easily confirmed by completing the square, and noting that \( 8n + 3 \) is represented by the polynomial \((2x + 1)^2 + (2y + 1)^2 + (2z + 1)^2\) as a consequence of the Three Square Theorem.

In 1862 Liouville examined this problem further, by asking for which positive integer triples \((\alpha, \beta, \gamma)\) the weighted ternary sum \(\alpha T_x + \beta T_y + \gamma T_z\) is universal. He determined that only seven such triples exist, namely \((1, 1, 1), (1, 1, 2), (1, 1, 4), (1, 1, 5), (1, 2, 2), (1, 2, 3)\) and \((1, 2, 4)\). Recently, in [8], [19], and [23], Z.-W. Sun et al. effectively determine all \(\alpha x^2 + \beta T_y + \gamma T_z\) and \(\alpha x^2 + \beta y^2 + \gamma T_z\) that are universal.

An extension to the above problem is a characterization of quadratic polynomials representing all but finitely many natural numbers; that is, quadratic polynomials which are \textit{almost universal}. Kane and Sun approach this question in [14], conjecturing a list of necessary and sufficient conditions on the triple \((\alpha, \beta, \gamma)\) of positive integers for which \(\alpha T_x + \beta T_y + \gamma T_z\), \(\alpha x^2 + \beta T_y + \gamma T_z\) or \(\alpha x^2 + \beta y^2 + \gamma T_z\) are almost universal. In [14], they resolved the conjecture for \(\alpha x^2 + \beta y^2 + \gamma T_z\) and prove sufficiency for the remaining two cases. In [2], Chan and Oh resolve the conjecture for weighted sums of three triangular numbers, and in a joint paper with Chan [1], we resolve the remaining case.

From here, the problem can be further generalized to arbitrary inhomogeneous quadratic polynomials, \(f(x) = Q(x) + \ell(x) + c\). The results presented in this thesis, which will be reported in Chapter 4, will not only subsume all previous results but also allow for an examination of a wider class of quadratic polynomials. Imposing some mild arithmetic conditions on \(f\), explained in full detail in Section
1.3, this thesis is devoted to establishing a characterization of almost universal inhomogeneous quadratic polynomials.
Chapter 1

Preliminaries

1.1 Quadratic Spaces and Lattices

In this chapter we introduce the basic concepts and definitions of the arithmetic theory of quadratic forms that will be used throughout the thesis. For any other unexplained notation and terminology, the reader is referred to [20].

Let $F$ be a number field, or the localization of a number field at one of its prime spots. Let $\mathcal{O}$ be the ring of integers in $F$, and $\mathcal{O}^*$ its group of units. A quadratic space $V$ over $F$ is an $n$ dimensional $F$-vector space endowed with a symmetric bilinear map $B : V \times V \rightarrow F$ such that the associated quadratic map $Q : V \rightarrow F$ satisfies the relation

$$Q(x + y) = Q(x) + Q(y) + 2B(x, y)$$

for any $x, y \in V$. For the purposes of this thesis, we assume that $V$ is nondegenerate; that is, $B(x, V) = 0$ if and only if $x = 0$.

An $\mathcal{O}$-submodule $L \subseteq V$ is called an $\mathcal{O}$-lattice (or simply a lattice) in $V$ if
there exists some basis \( \{v_1, \ldots, v_n\} \) for \( V \) so that

\[
L \subseteq \mathcal{O}v_1 \oplus \ldots \oplus \mathcal{O}v_n.
\]

We say that \( L \) is a lattice on \( V \) if \( FL = V \). The scale of \( L \), denoted \( s(L) \), is the \( \mathcal{O} \)-module generated by the subset \( B(L, L) \) in \( F \). The norm of \( L \), denoted \( n(L) \), is the \( \mathcal{O} \)-module generated by the subset \( Q(L) \) of \( F \). Both \( n(L) \) and \( s(L) \) are fractional \( F \)-ideals, and it is not difficult to see that

\[
2s(L) \subseteq n(L) \subseteq s(L).
\]

In the case where \( n(L) = s(L) \), \( L \) is called proper; otherwise \( L \) is improper.

Fixing a basis \( \{v_1, \ldots, v_n\} \) for \( V \), the symmetric matrix \( A = [a_{ij}] \) where \( a_{ij} := B(v_i, v_j) \) for \( 1 \leq i, j \leq n \), is called the Gram matrix for \( V \) with respect to that basis. If \( A \) is a Gram matrix for \( V \) with respect to some basis, then we write \( V \cong A \). The discriminant of \( V \), denoted \( dV \), is the element \( \det(A)F^{\times 2} \in F^{\times}/F^{\times 2} \).

For simplicity, if \( \det(A) = \alpha \), then we say that \( dV = \alpha \). A quadratic space which has a Gram matrix \( A = [a_{ij}] \) with \( a_{ij} = 0 \) whenever \( i \neq j \) is denoted \( [a_{11}, \ldots, a_{nn}] \), and the lattice corresponding to this Gram matrix is denoted \( \langle a_{11}, \ldots, a_{nn} \rangle \).

A space \( V \) is isotropic if there exists a non-zero vector \( \nu \in V \) such that \( Q(\nu) = 0 \). If no such vector exists, then \( V \) is called anisotropic. We immediately observe that if \( V \) is isotropic, then any lattice \( L \) on \( V \) is also isotropic, since an isotropic vector \( \nu \in V \) can be scaled by a suitable \( a \in \mathcal{O} \) to yield an isotropic vector \( a\nu \in L \). Similarly if \( L \) is isotropic, then \( V \) is isotropic, since any vector \( \nu \in L \) is also a vector in \( V \).

An isomorphism \( \sigma \) from \( V \) to itself is called an isometry of \( V \) if \( Q(\nu) = \)
Q(σ(v)) for any v ∈ V. Two lattices L and K on V are isometric, denoted L ≃ K, if there exists an isometry of V sending one to the other. The set of all isometries on V, denoted O(V), is called the orthogonal group of V. Elements in O(V) have determinant ±1, and the special orthogonal group, denoted O+(V), is the set of all isometries in O(V) with determinant 1. It is well known that any isometry σ ∈ O(V) is a product of symmetries [20, 43:3] whose determinant are −1. Therefore, any σ ∈ O+(V) is a product of an even number of symmetries; that is, σ = τx₁ ⋅ τx₂m where τxᵢ denotes the symmetry with respect to xᵢ, which is given by the mapping τxᵢ : V → V sending x to x − 2B(x, xᵢ)₂xᵢ for all x ∈ V. The spinor norm map on the special orthogonal group is given by

\[ \theta : O^+(V) \rightarrow F^×/F^×₂ \]

\[ \sigma \mapsto Q(x₁) \cdots Q(x₂m)F^×₂. \]

The kernel of this map is denoted O′(V).

Suppose that F is a number field, O its ring of integers, and let p be a nonarchimedean prime spot in F. Then \( F_p \) denotes the completion of F with respect to the p-adic valuation, and \( O_p \) the ring of integers of \( F_p \). For a complete description of local fields and the p-adic valuation, the reader is referred to [20, §32]. Fixing a prime p in F, the quadratic map on V is extended to the quadratic \( F_p \)-space \( V \otimes_F F_p \). We will denote this \( F_p \)-space by \( V_p \), and call this the localization of V at p. In a similar manner, we have the localization \( L_p := L \otimes_O O_p \) of the lattice \( L \), which is an \( O_p \)-lattice on \( V_p \). For a more complete discussion of the classification of quadratic spaces over local fields, see [20, Chapter VI].

A nonzero vector ν ∈ \( L_p \) is primitive if it can be extended to a basis of \( L_p \). Let a be a nonzero element in \( F_p \). For the \( O_p \)-lattice \( L_p \) on \( V_p \), let \( P(L_p, a) \) denote
the set of all primitive vectors \( v \in L_p \) such that \( Q(v) = a \). Fixing a \( v \in P(L_p, a) \), let \( \chi \) denote the set of all isometries \( \sigma \in O^+(V) \) such that \( \sigma(v) \in P(L_p, a) \). Now we define the primitive relative spinor norm, \( \theta^*(L_p, a) \), to be the image of \( \chi \) under the spinor norm map. Note that for a fixed \( \nu \in P(L_p, a) \), the quadratic space \( V_p \) splits as \( Q_p[\nu] \perp W_p \) for some quadratic space \( W_p \) orthogonal to \( \nu \). Since any rotation contained in \( O^+(W_p) \) can be extended to an element of \( O^+(V_p) \) fixing \( \nu \), we are guaranteed that \( \theta(O^+(W_p)) \subseteq \theta^*(L_p, a) \).

In what follows, we assume that \( V \) is a nondegenerate quadratic \( \mathbb{Q} \)-space, and \( L \) is a \( \mathbb{Z} \)-lattice on \( V \). For each prime \( p \) in \( \mathbb{Z} \), we can consider the localizations \( V_p \) and \( L_p \) as \( \mathbb{Q}_p \)-spaces and \( \mathbb{Z}_p \)-lattices, respectively.

**Lemma 1.1.** If a ternary \( \mathbb{Z}_q \)-lattice is universal, then it is isotropic.

**Proof.** Suppose that \( L \) is a \( \mathbb{Z}_q \)-lattice which is universal. Then \( \mathbb{Q}_q L \) is universal; in particular \( \mathbb{Q}_q L \) represents \(-dL\), and therefore the quaternary quadratic space \([dL] \perp \mathbb{Q}_q L\) is isotropic (cf. [20, 42:10]). According to [20, 63:17], there is only one anisotropic quaternary quadratic space over a local field. Such a space has determinant 1, and is of the form \([1] \perp W\), where \( W \) is ternary and anisotropic. Computing the Hasse symbol for this space, we see that

\[
S_q([1] \perp W) = S_q([1]) \cdot S_q(W) \cdot (1, dW) = S_qW,
\]

and \( S_q(W) = -(-1, -1)_q \) since \( W \) is anisotropic (cf. [20, 58:6]). Consequently, an isotropic quaternary quadratic space either has determinant not equal to 1, or it has Hasse invariant equal to \((-1, -1)_q\). Clearly \([dL] \perp \mathbb{Q}_q L\) has determinant 1, and hence it must be true that

\[
(-1, -1)_q = S_q([dL] \perp \mathbb{Q}_q L) = S_q([dL]) \cdot S_q(\mathbb{Q}_q L) \cdot (dL, dL) = S_q(\mathbb{Q}_q L),
\]
and thus $Q_qL$ is isotropic (cf. [20, 58:6]). Therefore, $L$ is isotropic.

A $\mathbb{Z}_p$-lattice $L_p$ is called **unimodular** if $\mathfrak{s}(L_p) = \mathbb{Z}_p$ and $dL \in \mathbb{Z}_p^\times$, and **(a)-modular** if $L_p^{a^{-1}}$ is unimodular. Here $L_p^{a^{-1}}$ is a scaling by $a^{-1}$ of the quadratic map associated to $L_p$, that is, $Q(\nu)$ is replaced by $a^{-1}Q(\nu)$ for any $\nu \in L_p$.

As described in [20, §91 C], $L_p$ has an orthogonal basis when $p$ is odd, and an orthogonal splitting into one or two dimensional modular lattices when $p = 2$. In general, by grouping the modular components with the same scale, $L_p$ has a splitting

$$L_p = L_{(1)} \perp \cdots \perp L_{(t)}$$

where each $L_{(i)}$ is modular, and

$$\mathfrak{s}(L_{(1)}) \supset \cdots \supset \mathfrak{s}(L_{(t)}).$$

This splitting is called a **Jordan splitting** of $L_p$, and the $L_{(i)}$ are called the **Jordan components**. In general, Jordan splittings are not unique. But given two Jordan splittings for $L_p$,

$$L_{(1)} \perp \cdots \perp L_{(t)} = K_{(1)} \perp \cdots \perp K_{(T)},$$

it is known [20, §93 G] that $t = T$, $n(L_{(i)}) = n(K_{(i)})$, and $\text{rank}(L_{(i)}) = \text{rank}(K_{(i)})$ for all $i$. So, $t$, $n(L_i)$ and $\text{rank}(L_{(i)})$, $1 \leq i \leq t$ are invariants of $L_p$. They are called the **Jordan invariants** of $L_p$. When $p = 2$, any binary unimodular component of $L_2$ is either proper or improper, and in the latter case it is isometric to one of
the following two binary lattices. The first is the hyperbolic plane

\[ \mathbb{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \]

which is isotropic and represents precisely every element in \(2\mathbb{Z}_2\). The second,

\[ \mathbb{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \]

is anisotropic and represents precisely every 2-adic integer of odd order. Following

the notation set in [20, §93 B], we let \(A(\alpha, \beta)\) denote the binary \(\mathbb{Z}_2\)-lattice

\[ \begin{bmatrix} \alpha & 1 \\ 1 & \beta \end{bmatrix}, \]

where \(\alpha, \beta \in \mathbb{Z}_2\), and \(-1 + \alpha\beta \in \mathbb{Z}_2^\times\).

\[1.2\] Primitive Spinor Exceptions

For the remainder of this thesis we require that every quadratic space is positive

definite over \(\mathbb{Q}\); i.e. \(Q(\nu) > 0\) for all non-zero vectors \(\nu\). For two lattices \(L\) and

\(K\) on a quadratic space \(V\), we say that \(K\) is in the class of \(L\), denoted \(\text{cls}(L)\), if

there exists an isometry \(\sigma \in O(V)\) so that \(\sigma(L) = K\). If for every prime \(p\) there

exists a \(\sigma_p \in O(V_p)\) so that \(\sigma_p(L_p) = K_p\), then \(K\) is in the genus of \(L\), denoted

\(\text{gen}(L)\). A lattice \(K\) belongs to the (proper) spinor genus of \(L\), denoted \(\text{spn}(L)\), if

there is an isometry \(\varphi \in O^+(V)\) and \(\sigma_p \in O'(V_p)\) for every prime \(p\), so that

\(\varphi(K_p) = \sigma_p(L_p)\). The set of spinor genera in \(\text{gen}(L)\) form a partition of \(\text{gen}(L)\).

An integer \(t\) is said to be represented by \(\text{gen}(L)\) if there is some lattice in

\(\text{gen}(L)\) which represents \(t\). It is clear that in order for \(t\) to be represented by

\(L\), it is necessary for \(t\) to be represented by \(\text{gen}(L)\). However, the latter is not
sufficient to guarantee a representation of $t$ by $L$; in other words, the local-global principle of representation by lattices does not hold in general. But when $n \geq 5$, Tartakowsky [24] (see [12] for an arithmetic proof) proved an *asymptotic* local-global principle. More precisely, when $L$ is a lattice of rank $n \geq 5$, there exists a constant $C$ depending only on $L$, such that if $t$ is represented by $\text{gen}(L)$ and $t > C$, then $t$ is represented by $L$. Tartakowsky [24] proves a similar result for quaternary lattices but only when the local representations are primitive.

For ternary lattices, the existence of local representations everywhere is not sufficient to guarantee a global representation, not even if the representations are primitive. The spinor genus containing $L$ plays a vital role in determining the representation behavior for ternary quadratic lattices. It is well known that an integer which is primitively represented by the genus of $L$ will be represented primitively by either every spinor genus in the genus of $L$, or precisely half of the spinor genera. Integers satisfying the latter condition are called *primitive spinor exceptions* of $\text{gen}(L)$. Once a non-zero integer $t$ is represented primitively by $\text{gen}(L)$, work of Kneser [17] and Schulze-Pillot [21] give us the following necessary and sufficient conditions for $t$ to be a primitive spinor exception of $\text{gen}(L)$:

$(a)$ $-tdL \not\in \mathbb{Q}^{\times 2}$,

$(b)$ $\theta(O^+(L_p)) \subseteq \mathfrak{N}_p(\mathbb{Q}(\sqrt{-tdL}))$,

$(c)$ $\theta^*(L_p, t) = \mathfrak{N}_p(\mathbb{Q}(\sqrt{-tdL}))$,

for every prime $p$, where $\mathfrak{N}_p(\mathbb{Q}(\sqrt{-tdL}))$ denotes the group of local norms from $\mathbb{Q}_p(\sqrt{-tdL})$ to $\mathbb{Q}_p$. The groups $\theta(O^+(L_p))$ and $\theta^*(L_p, t)$ are determined by Earnest and Hsia [5], Earnest, Hsia and Hung [7], and Kneser [16]. Therefore, the primitive spinor exceptions of a given genus can be effectively determined. For the
convenience of the readers, the formulae for $\theta(O^+(L_p))$ and $\theta^*(L_p, t)$ obtained in these papers are given in Theorems A.1-A.6 in Appendix A. The following theorem of Duke and Schulze-Pillot [4, Corollary] incorporates the theory of spinor genera to give an asymptotic local-global principle for ternary lattices.

**Theorem 1.2.** Let $L$ be a positive definite ternary quadratic lattice. Then every sufficiently large integer represented primitively by a lattice in the spinor genus of $L$ is represented by $L$ itself.

Since $\text{gen}(L)$ typically contains several spinor genera, we rely on the spinor norm calculations above to determine which integers are represented by $\text{spn}(L)$. When $\mathbb{Z}_p^\times \subseteq \theta(O^+(L_p))$ for every prime $p$, then $\text{spn}(L) = \text{gen}(L)$ (cf. [20, 102:8a]). This case is much easier to handle, since here $\text{gen}(L)$ has only one spinor genus.

The following corollary, which is a reformulation of Theorem 1.2, addresses this special case.

**Corollary 1.3.** Let $L$ be a positive definite ternary quadratic lattice. Suppose that the genus of $L$ has only one spinor genus. Then every sufficiently large integer represented primitively by the genus of $L$ is represented by $L$ itself.

### 1.3 Inhomogeneous Quadratic Polynomials

Suppose that we have an inhomogeneous quadratic polynomial $f(x) = Q(x) + \ell(x) + c$, where $Q$ is a quadratic form, $\ell$ is a linear form, and $c$ is a constant. It is not a surprise that we can study the arithmetic of these polynomials from the geometric perspective of quadratic spaces and lattices. Indeed, $Q$ can be viewed as a quadratic map associated to some lattice $N$, with symmetric bilinear map $B$.

Under the assumption that $N$ is positive definite, $\ell(x) = 2B(\nu, x)$ for a unique
choice of vector $\nu$ in $V := \mathbb{Q}N$ which is not necessarily in $N$. The choices for $\nu$ and $N$ are completely determined by the coefficients of $Q$ and $\ell$. Since the constant $c$ does not contribute anything essential to the arithmetic of $f$, there is no harm in assuming that it is equal to zero. An easy calculation reveals that an integer $t$ is represented by $f(x)$ if and only if $Q(\nu) + t$ is represented by the coset $\nu + N$. When $n \geq 4$, Chan and Oh [3, Theorem 4.9] show how the asymptotic local-global principles for representations of lattices with approximation property by Jöchner-Kitaoka [13] and by Hsia-Jöchner [11] lead to an asymptotic local-global principle for representations of integers by cosets.

Suppose we have the integer valued inhomogeneous polynomial $Q(x)+2B(\nu, x)$ described above, where $Q$ now is ternary. For simplicity we will require that the $\mathbb{Z}$-ideal generated by $Q(\nu + x)$, for all $x \in N$, is contained in $\mathbb{Z}$. Under this assumption, it is immediate that $Q(\nu) \in \mathbb{Z}$, and the $\mathbb{Z}$-ideal generated by $Q(x)+2B(\nu, x)$ for all $x \in N$, which we denote $n(\nu, N)$, is contained in $\mathbb{Z}$. Suppose that $n(\nu, N) = \mathbb{Z}$. If $Q(x)+2B(\nu, x)$ were almost universal, this would imply that $\nu + N$ represents $Q(\nu) + n$ for all but finitely many positive integers $n$. Then the ternary $\mathbb{Z}$-lattice given by $M := \mathbb{Z}\nu + N$ is almost universal. However, it is a well known consequence of Hilbert Reciprocity that a positive definite ternary $\mathbb{Z}$-lattice is anisotropic at an odd number of finite primes, and therefore it is not universal at these primes by Lemma 1.1. If $M_p$ is not universal at some prime $p$, then by the Local Square Theorem [20, 63:1], $M_p$ fails to represent an entire square class in $\mathbb{Q}_p^\times/\mathbb{Q}_p^\times \mathbb{Z}$, and consequently $M$ itself fails to represents infinitely many positive integers. Therefore, a reasonable requirement is to fix a prime $p$, and assume that $n(\nu, N) = p^\alpha \mathbb{Z}$, with $\alpha > 0$. In this case, we define an integer
valued polynomial $H(x)$ by

$$H(x) := \frac{Q(x) + 2B(\nu, x)}{p^\alpha}.$$ 

Our goal is to obtain a list of necessary and sufficient conditions on the pair $(\nu, N)$ under which this $H(x)$ is almost universal.

**Lemma 1.4.** (1) If $H(x)$ is almost universal, then $M_q$ represents all $\mathbb{Z}_q$ whenever $q \neq p$, and $N_q$ represents all $\mathbb{Z}_q$ whenever $q \nmid [M : N]$.

(2) If $N_q$ represents all of $\mathbb{Z}_q$ then $\theta(O^+(N_q)) \supseteq \mathbb{Z}_q^\times$.

(3) If $M_q$ represents all $\mathbb{Z}_q$ for all $q \neq p$, then $Q_p M = Q_p N$ is anisotropic.

**Proof.** (1) Suppose $H(x)$ is almost universal. Then $Q(\nu) + p^\alpha n$ is represented by the coset $\nu + N$ for all but finitely many positive integers $n$. Consequently, all but finitely many positive integers are represented by $\nu + N_q$ for $q \neq p$, and therefore by $M_q$ for $q \neq p$. By the denseness of $\mathbb{Z}$ in $\mathbb{Z}_q$, each $q$-adic integer $Q(\nu) + p^\alpha n_0$ is sufficiently close to infinitely many of those $Q(\nu) + p^\alpha n$, so every $Q(\nu) + p^\alpha n_0$ is represented by $M_q$. But in fact, $\{Q(\nu) + p^\alpha n_0 : n_0 \in \mathbb{Z}_q\} = \mathbb{Z}_q$ because $p^\alpha \in \mathbb{Z}_q^\times$, so $M_q$ represents all of $\mathbb{Z}_q$. When $q \nmid [M : N]$, $\nu + N_q = N_q$, and it follows that $N_q$ represents all of $\mathbb{Z}_q$.

(2) Suppose that $N_q$ represents all of $\mathbb{Z}_q$. Then, for any unit $\delta \in \mathbb{Z}_q^\times$, there exists a vector $\nu_\delta \in N_q$ such that $Q(\nu_\delta) = \delta$; in particular, $Q(\nu_1) = 1$. Note that $\tau_{\nu_\delta} \in O(N_q)$ for all $\delta \in \mathbb{Z}_q^\times$. Now it is clear from the definition of the spinor norm map that $\theta(\tau_{\nu_\delta} \cdot \tau_{\nu_1}) = \delta$, and therefore $\theta(O^+(N_q)) \supseteq \mathbb{Z}_q^\times$.

(3) Suppose that $M_q$ represents all $\mathbb{Z}_q$ for all $q \neq p$. By Lemma 1.1, we see that $Q_q M$ is isotropic for all $q \neq p$. Since $M$ is positive definite, it follows from Hilbert Reciprocity that $Q_p M = Q_p N$ is anisotropic. 

13
We get the following as an immediate corollary to Lemma 1.4.

**Corollary 1.5.** If \( H(x) \) is almost universal, then \( \text{ord}_p(Q(\nu)) < \alpha \).

**Proof.** Suppose that \( H(x) \) is almost universal, and that \( Q(\nu) \in p^\alpha \mathbb{Z}_p \). Then, all but finitely many positive integers of the form \( Q(\nu) + p^\alpha n \) are represented by the coset \( \nu + N \) and hence by the space \( \mathbb{Q}N \). But then \( \mathbb{Q}_p N^{\frac{1}{\nu}} \) is universal, and so \( \mathbb{Q}_p N \) must be isotropic. This contradicts Lemma 1.4 part (3). So \( \text{ord}_p(Q(\nu)) < \alpha \). \( \square \)

In view of Lemma 1.4, it would be helpful to impose some restriction on the prime divisors of \([M : N]\). A reasonable assumption is that \([M : N] = p^\gamma \) for some positive integer \( \gamma \). Therefore, to recap, we have imposed the following conditions,

1. \( n(\nu, N) = p^\alpha \mathbb{Z}, \) with \( \alpha > 0; \) and

2. \( [M : N] = p^\gamma \) for \( \gamma > 0. \)

They will be assumed throughout the remainder of this thesis.

Given a primitive spinor exception \( t \) of the genus of \( M \), we define \( E := \mathbb{Q}(\sqrt{-tdM}) \). Note that if \( t \) is a primitive spinor exception of \( \text{gen}(M) \), then \( -tdM \not\in \mathbb{Q}^2 \), as shown in [16].

**Lemma 1.6.** Suppose that \( M_q \) represents all \( q \)-adic integers for every prime \( q \neq p \).

If \( t \) is a primitive spinor exception of \( \text{gen}(M) \), then

1. \( E = \mathbb{Q}(\sqrt{-p}) \) and \( p \equiv 7 \mod 8, \) when \( p > 2. \)

2. \( E = \mathbb{Q}(\sqrt{-1}) \) or \( \mathbb{Q}(\sqrt{-2}) \) when \( p = 2. \)

**Proof.** Suppose that \( M_q \) represents all \( q \)-adic integers for every prime \( q \neq p \). For any \( q \neq p, \) \( M_q = N_q \), and so \( \mathbb{Z}_q^* \subseteq \theta(O^+(M_q)) \subseteq \mathfrak{M}_p(E), \) by Lemma 1.4 part (2).
From [18, XI.4.4] it is immediate that $E$ is unramified at all $q \neq p$, and therefore no prime other than $p$ will divide the discriminant of $E$.

For odd primes $p$, there is only one choice of an imaginary quadratic extension ramified only at the prime $p$, namely, $\mathbb{Q}(\sqrt{-p})$, where $p \equiv 3 \mod 4$. By our initial assumption $M_2$ represents all integers in $\mathbb{Z}_2$, so in particular, for every $\delta \in \mathbb{Z}_2^\times \cup 2\mathbb{Z}_2^\times$, $M_2$ contains a vector $\nu_0$ such that $Q(\nu_0) = \delta$. Moreover, for each of such $\delta$, $\tau_{\nu_0} \in O(M_2)$. Now, from the definition of the spinor norm, it is clear that $\mathfrak{N}_2(E) \supseteq \theta(O^+(M_2)) = \mathbb{Q}_2^\times$. Consequently, $[\mathbb{Q}_2^\times : \mathfrak{N}_2(E)] = 1$, and hence $\mathbb{Q}_2(\sqrt{-p}) = \mathbb{Q}_2$. Therefore, $-p$ is a square in $\mathbb{Q}_2$, and hence $p \equiv 7 \mod 8$.

At the prime 2, there are two choices for an imaginary quadratic extension ramified only at 2, namely $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-2})$.

In the following lemma, we show that if $\nu$ is replaced by $\nu + x_0$, where $x_0 \in N$, then all results on almost universality will still hold.

**Lemma 1.7.** Let $\omega = \nu + x_0$, where $x_0 \in N$, and define

$$H'(x) = \frac{1}{\rho^\alpha}[Q(x) + 2B(\omega, x)].$$

Then,

(1) if $n(\nu, N) = p^\alpha \mathbb{Z}$, then $n(\omega, N) = p^\alpha \mathbb{Z}$; and,

(2) if $H(x)$ is almost universal, then $H'(x)$ is almost universal.

**Proof.** Suppose that $n(\nu, N) = p^\alpha \mathbb{Z}$. Let $\omega = \nu + x_0$ with $x_0 \in N$. First, observe that for any $x \in N$,

$$Q(x + x_0) + 2B(\nu, x + x_0) = Q(x) + Q(x_0) + 2B(x, x_0) + 2B(\nu, x) + 2B(\nu, x_0),$$
which implies that $2B(x, x_0) \equiv 0 \mod p^\alpha$ for any $x \in N$. Now, for any $x \in N$

$$Q(x) + 2B(\omega, x) = Q(x) + 2B(\nu, x) + 2B(x_0, x) \equiv 0 \mod p^\alpha,$$

and hence $n(\omega, N) \subseteq p^\alpha Z$. Switching the roles of $\nu$ and $\omega$ in the argument above, we get containment in the other direction. Therefore, $n(\omega, N) = p^\alpha Z$

Define $p^\alpha m := Q(x_0) + 2B(\nu, x_0)$. Then, for any integer $n$,

$$Q(\omega) + p^\alpha(n - m) = Q(\nu + x_0) + p^\alpha n - Q(x_0) - 2B(\nu, x_0)$$

$$= Q(\nu) + Q(x_0) + 2B(\nu, x_0) + p^\alpha n - Q(x_0) - 2B(\nu, x_0)$$

$$= Q(\nu) + p^\alpha n.$$

If $H(x)$ is almost universal, then $\nu + N$ represents $Q(\nu) + p^\alpha n$ for $n$ sufficiently large, and hence $\nu + N = \omega + N$ represents $Q(\omega) + p^\alpha(n - m)$ for all $n$ sufficiently large. Therefore, if $H(x)$ is almost universal, then $H'(x)$ is almost universal. \qed

The following remark is an immediate consequence of Lemma 1.7.

**Remark 1.8.** Much of the rest of this thesis is to demonstrate the almost universality of $H(x)$ under some arithmetic conditions imposed on $\nu$ and $N$. In view of Lemma 1.7, we can always change $\nu$ to $\nu + x_0$ with $x_0 \in N$, as long as the arithmetic conditions are unchanged.
Chapter 2

$p$ is odd

2.1 Preliminaries

In this chapter, we turn our attention to the case when $p$ is odd. Our condition from Chapter 1 becomes $n(\nu, N) = p^\alpha \mathbb{Z}$ where $p$ is odd and $\alpha > 0$. Additionally, we know from Corollary 1.5, that $\text{ord}_p(Q(\nu)) < \alpha$.

Lemma 2.1. If $n(\nu, N) = p^\alpha \mathbb{Z}$ with $\alpha > 0$, then $B(\nu, N)$ and $Q(N)$ are subsets of $p^\alpha \mathbb{Z}$.

Proof. Suppose that $n(\nu, N) = p^\alpha \mathbb{Z}$ with $\alpha > 0$. Then, $Q(x) + 2B(\nu, x) \subseteq p^\alpha \mathbb{Z}$, for all $x \in N$. Suppose, for the sake of contradiction, that $B(\nu, x_0) \not\subseteq p^\alpha \mathbb{Z}$ for some $x_0 \in N$. Then $Q(x_0) \not\subseteq p^\alpha \mathbb{Z}$, or else the initial assumption would fail. But then,

$$Q(2x_0) + 2B(\nu, 2x_0) = 4Q(x_0) + 4B(\nu, x_0) \equiv 2Q(x_0) \mod p^\alpha,$$

and clearly $2Q(x_0) \not\equiv 0 \mod p^\alpha$ since $p$ is odd. Therefore, we conclude that both $B(\nu, x)$ and $Q(x)$ are contained in $p^\alpha \mathbb{Z}$, for any $x \in N$. \qed
Since \( \text{ord}_p(Q(\nu)) < \alpha \), it is now possible to scale \( Q \) by \( p^{-\text{ord}_p(Q(\nu))} \), and then assume that \( \text{ord}_p(Q(\nu)) = 0 \). For simplicity, we define \( \epsilon := Q(\nu) \in \mathbb{Z}_p^\times \). As an immediate consequence to Lemma 2.1, we have that \( B(\nu, p^\gamma \nu) = p^\gamma \epsilon \equiv 0 \mod p^\alpha \), and therefore we conclude that \( \gamma \geq \alpha \).

An arbitrary coset of \( N \) in \( M \) is of the form \( a\nu + N \), where \( 0 \leq a \leq p^\gamma - 1 \). For any \( x \in N \), we have

\[
Q(a\nu + x) = a^2\epsilon + Q(x) + 2aB(\nu, x) \equiv a^2\epsilon \mod p^\alpha,
\]

by Lemma 2.1. Hence, if \( \epsilon + p^\alpha n \) is represented by this \( a\nu + N \), then \( \epsilon \equiv a^2\epsilon \mod p^\alpha \), and therefore \( a \equiv \pm 1 \mod p^\alpha \), because \( p \) is odd. From this we conclude that every representation of \( \epsilon + p^\alpha n \) by \( M \) must be from one of the cosets \( (bp^\alpha \pm 1)\nu + N \), \( 0 \leq b \leq p^\gamma - \alpha \). In order to be sure that a representation of \( \epsilon + p^\alpha n \) is from \( \nu + N \), we need to make an additional assumption; namely, \( \gamma = \alpha \). Under this assumption, any representation of \( \epsilon + p^\alpha n \) by \( M \) must be from \( \nu + N \) or \( -\nu + N \).

However, it is not difficult to see that these cosets represent precisely the same integers. To recap, the following assumptions are held throughout this section:

1. \( n(\nu, N) = p^\alpha \mathbb{Z}, \) with \( \alpha > 0 \); and
2. \( [M : N] = p^\alpha \).

### 2.2 Universal Ternary \( \mathbb{Z}_2 \)-lattice

A \( \mathbb{Z}_2 \)-lattice \( L \) is called \( \mathbb{Z}_2 \)-maximal if \( n(L) \subseteq \mathbb{Z}_2 \), and for any lattice \( M \) properly containing \( L \), \( n(M) \nsubseteq \mathbb{Z}_2 \) holds. Let \( V \) be an anisotropic quadratic \( \mathbb{Q}_2 \)-space. It is a well known fact (cf. [20, 91:1]) that a lattice \( L \) on \( V \) is \( \mathbb{Z}_2 \)-maximal if and only if \( L = \{ x \in V : Q(x) \in \mathbb{Z}_2 \} \). In particular, there is a unique \( \mathbb{Z}_2 \)-maximal lattice.
on $V$, and any integer represented by $V$ is represented by this unique $\mathbb{Z}_2$-maximal lattice. We have the following lemma regarding maximal lattices.

**Lemma 2.2.** Every binary $\mathbb{Z}_2$ lattice of the form $\langle \beta, 2\gamma \rangle$ with $\beta, \gamma \in \mathbb{Z}_2^\times$, is $\mathbb{Z}_2$-maximal.

**Proof.** Any binary lattice of the form $\langle \beta, 2\gamma \rangle$, $\beta, \gamma \in \mathbb{Z}_2 \times$, is anisotropic. Consequently the binary $\mathbb{Q}_2$-space $[\beta, 2\gamma]$, underlying such a lattice, is also anisotropic. Suppose that $[\beta, 2\gamma]$ represents some 2-adic integer $\delta$. Then, $\delta = \beta x^2 + 2\gamma y^2$ for $x, y \in \mathbb{Q}_2$. Since $\delta$ is a 2-adic integer, $|\delta|_2 \leq 1$, and since $\beta$ and $\gamma$ are 2-adic units, $|\beta x^2|_2 \neq |2\gamma y^2|_2$. By the principle of domination,

$$|\delta|_2 = \max\{|\beta x^2|_2, |2\gamma y^2|_2\}.$$ 

Since the maximum between $|\beta x^2|_2$ and $|2\gamma y^2|_2$ is less than or equal to 1, it is clear that $x, y \in \mathbb{Z}_2$. Therefore any integer represented by the quadratic space $[\beta, 2\gamma]$ is actually represented by the lattice $\langle \beta, 2\gamma \rangle$.

Given an integer $\beta$ with $-\beta \not\in \mathbb{Q}_2^\times$, the binary $\mathbb{Q}_2$-space $[1, \beta]$, corresponds to the norm form for the quadratic extension $\mathbb{Q}_2(\sqrt{-\beta})$. When $\beta \equiv 3 \mod 8$, then this quadratic extension is unramified, and consequently its norm group contains all of the units in $\mathbb{Z}_2^\times$; that is, $\mathcal{N}_2(\mathbb{Q}_2(\sqrt{-\beta})) = \{1, 3, 5, 7\}\mathbb{Q}_2^\times$. Although it is not immediate, it can be shown with little difficulty that the integral lattice $\langle 1, \beta \rangle$ represents 1, 3, 5, and 7. On the other hand, when $\beta \equiv 1 \mod 4$, then the quadratic extension ramifies, and in this case the norm group contains only half of the units in $\mathbb{Z}_2^\times$. More precisely, when $\beta \equiv 1 \mod 8$, then $\mathcal{N}_2(\mathbb{Q}_2(\sqrt{-\beta})) = \{1, 2, 5, 10\}\mathbb{Q}_2^\times$, and when $\beta \equiv 5 \mod 8$, then $\mathcal{N}_2(\mathbb{Q}_2(\sqrt{-\beta})) = \{1, 5, 6, 14\}\mathbb{Q}_2^\times$. 

19
Lemma 2.3. If \( K \) is a universal ternary \( \mathbb{Z}_2 \)-lattice, then \( K \) is isotropic and \( \text{ord}_2(dK) < 2 \). When this is the case, \( K \cong \langle 1, -1, -dK \rangle \) and

(1) if \( \text{ord}_2(dK) = 1 \), then \( K \) primitively represents every element of \( \mathbb{Z}_2 \) except those in \( 4\mathbb{Z}_2^\times \);

(2) if \( \text{ord}_2(dK) = 0 \), then \( K \) primitively represents every element of \( \mathbb{Z}_2 \).

Proof. Suppose that \( K \) is a universal ternary \( \mathbb{Z}_2 \)-lattice. It is immediate from Lemma 1.1 that \( K \) is isotropic.

Since \( K \) is universal, \( K \) represents 1, and therefore \( \langle 1 \rangle \) is an orthogonal summand of \( K \). Suppose that \( K \) has an orthogonal decomposition of the form \( \langle 1, \beta, 2^j\gamma \rangle \) in a basis \( \{e_1, e_2, e_3\} \) with \( \beta, \gamma \in \mathbb{Z}_2^\times \) and \( j > 0 \). If \( \langle 1, \beta \rangle \) is not isometric to \( \langle 1, -1 \rangle \) or \( \langle 1, 3 \rangle \), then \( \langle 1, \beta \rangle \) will miss a square class of units; so \( j \leq 2 \). If \( \langle 1, \beta \rangle \) is isometric to either \( \langle 1, 3 \rangle \) or \( \langle 1, -1 \rangle \), then \( \langle 1, \beta \rangle \) misses all of \( 2\mathbb{Z}_2^\times \); so \( j \leq 1 \). Therefore, regardless of our choice of \( \beta \), \( j \leq 2 \) when \( K \) is universal. Furthermore, when \( j = 2 \), then \( \langle 1, \beta, 4\gamma \rangle \cong \langle 1, \beta + 4\gamma, (\beta + 4\gamma)4\beta\gamma \rangle \). Since \( \langle 1, 1, 4\gamma \rangle \) only represents elements in \( \mathbb{Z}_2^\times \) which are congruent to 1 mod 4, and \( \langle 1, 3, 4\gamma \rangle \) fails to represent any elements in \( 2\mathbb{Z}_2^\times \), we conclude that \( j < 2 \) when \( K \) is universal.

Suppose that \( j = 0 \) and consequently \( K \cong \langle 1, \beta, \gamma \rangle \). Then \( K \) is an isotropic unimodular \( \mathbb{Z}_2 \)-lattice, and is therefore of the form \( \mathbb{H} \perp \langle -\beta\gamma \rangle \cong \langle 1, -1, -\beta\gamma \rangle \) (cf. [15, Proposition 5.2.3]). The hyperbolic plane primitively represents every element in \( 2\mathbb{Z}_2 \). Since any 2-adic unit can be written as \( 2\eta - \beta\gamma \) for some \( \eta \in \mathbb{Z}_2 \), \( K \) represents every element in \( \mathbb{Z}_2^\times \). Furthermore, we note that any representation of a unit by \( K \) will always be primitive. Therefore, \( K \) is universal and primitively represents every element of \( \mathbb{Z}_2 \).

Suppose that \( j = 1 \) and therefore \( K \cong \langle 1, \beta, 2\gamma \rangle \). From Lemma 2.2, we know that \( \langle \beta, 2\gamma \rangle \) is \( \mathbb{Z}_2 \)-maximal. Therefore, since \( K \) is isotropic, \( [\beta, 2\gamma] \) represents \(-1\).
and hence $\langle \beta, 2\gamma \rangle \cong \langle -1, -2\beta \gamma \rangle$, thus $K \cong \langle 1, -1, -2\beta \gamma \rangle$ in a basis $\{f_1, f_2, f_3\}$. Since $\langle 1, -1 \rangle \cong \langle 3, 5 \rangle$, it is easily seen that $K$ primitively represents every unit in $\mathbb{Z}_2$. The leading binary component of $K$ contains

$$L \cong \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

as a sublattice, with the basis $\{f_1 + f_2, f_1 - f_2\}$. Since every element of $2\mathbb{Z}_2^\times$ can be written in the form $-2\beta \gamma + 4\eta$ for some $\eta \in \mathbb{Z}_2$, it is clear that $L \perp \langle -2\beta \gamma \rangle$, and hence $K$, primitively represents every element in $2\mathbb{Z}_2^\times$. Let $\delta$ be an element in $\mathbb{Z}_2$. The sublattice $L$ represents $4\delta$, by the vector $(f_1 + f_2) + (f_1 - f_2)\delta = (1 + \delta)f_1 + (1 - \delta)f_2$. In particular, when $\delta$ is even, this representation gives a primitive representation of $4\delta$ by $K$. Therefore, $K$ represents every element in $4\mathbb{Z}_2$, and primitively represents every element in $8\mathbb{Z}_2$.

Suppose that $\delta \in \mathbb{Z}_2^\times$. Then there exist $x, y, z \in \mathbb{Z}_2$ such that $4\delta = x^2 - y^2 - 2\beta \gamma z^2$. It is clear that $x$ and $y$ have the same parity, or else the right hand side of the equality is odd. But then, $x^2 - y^2 \equiv (x + y)(x - y) \equiv 0 \pmod{4}$, meaning $z$ must be even, and hence $4\delta \equiv x^2 - y^2 \pmod{8}$. If $x$ and $y$ were both even, then the representation would not be primitive, as such, we assume that $x$ and $y$ are units in $\mathbb{Z}_2$, which means that $x^2 \equiv y^2 \equiv 1 \pmod{8}$. Consequently, $4\delta \equiv 0 \pmod{8}$, which is not true, since $\delta \in \mathbb{Z}_2^\times$. Therefore, any representation of elements from $4\mathbb{Z}_2^\times$ by $K$ must be imprimitive.

Suppose the unimodular component of $K$ is unary; that is, $K \cong \langle 1 \rangle \perp 2^i L$, where $i > 0$ and $s(L) = \mathbb{Z}_2$. If $i > 1$, then $K$ only represents units congruent to $1 \pmod{4}$, so we may assume that $i = 1$. Similarly, if $L$ is improper, then $K$ only represents units congruent to $1 \pmod{4}$, so we may assume that $L \cong \langle \beta, 2^j \gamma \rangle$, with
\[ j \geq 0. \] Now, \( K \cong \langle 1, 2\beta, 2^{j+1}\gamma \rangle \) and therefore we may assume that \( j = 0 \) or \( 1 \), or else \( K \) is not universal since \( \langle 1, 2\beta \rangle \) fails to represent some unit in \( \mathbb{Z}_2 \).

Suppose that \( j = 0 \), and \( K \cong \langle 1, 2\beta, 2\gamma \rangle \). Since \( K \) only represents integers of the form \( x^2 + 2\beta y^2 + 2\gamma z^2 \), the only units represented by \( K \) are those congruent to \( 1, 1 + 2\beta, 1 + 2\gamma \) or \( 1 + 2(\beta + \gamma) \mod 8 \). If \( \beta \equiv \gamma \mod 4 \), then \( 2\beta \equiv 2\gamma \mod 8 \), and consequently \( 1 + 2\beta \equiv 1 + 2\gamma \mod 8 \), which means that \( K \) will represent at most three of the four possible square classes of units. Suppose that \( \beta \not\equiv \gamma \mod 4 \), then \( \beta + \gamma \equiv 0 \mod 4 \), and so \( 1 \equiv 1 + 2(\beta + \gamma) \mod 8 \), and again \( K \) fails to represent all of the square classes of units. In any case, \( K \) fails to be universal.

Suppose that \( j = 1 \), and \( K \cong \langle 1, 2\beta, 4\gamma \rangle \). From Lemma 2.2 we know that \( \langle 1, 2\beta \rangle \) is \( \mathbb{Z}_2 \)-maximal, which implies that \( \langle 1, 2\beta \rangle \cong \langle -\gamma, -2\beta\gamma \rangle \), since \( K \) is isotropic. Consequently, \( K \cong \langle -\gamma, -2\beta\gamma, 4\gamma \rangle \), and hence \( K^{-\gamma} \cong \langle 1, 2\beta, -4 \rangle \).

Suppose that \( \eta \in \mathbb{Z}_q^\times \), with \( \eta \equiv \beta \mod 4 \) and \( \eta \not\equiv \beta \mod 8 \). If \( 2\eta \) is represented by \( K^{-\gamma} \), then \( \eta \equiv 2(x + y)(x - y) + \beta \mod 8 \) for some \( x, y \in \mathbb{Z}_2 \). If \( x + y \) is even, then this implies that \( \eta - \beta \equiv 0 \mod 8 \), a contradiction. On the other hand, if \( x + y \) is odd, then this implies that \( \frac{2-\beta}{2} \equiv 1, 3 \mod 4 \), also a contradiction. Therefore, \( K^{-\gamma} \) does not represent \( 2\eta \), and hence \( K \) is not universal.

\[ \Box \]

**Lemma 2.4.** If \( H(x) \) is almost universal, then \( N_q \) represents all of \( \mathbb{Z}_q \) whenever \( q \neq p \), and consequently, \( N_q \cong \langle 1, -1, -dN \rangle \) for \( q \neq p, 2 \), and \( N_2 \cong \langle 1, -1, -dN \rangle \) with \( \text{ord}_2(dN) \leq 1 \).

**Proof.** If \( H(x) \) is almost universal, then \( N_q \) represents all of \( \mathbb{Z}_q \) whenever \( q \neq p \) by Lemma 1.4 part (1). Combining this with Lemma 1.1, it is immediate that \( N_q \cong \langle 1, -1, -dN \rangle \) for \( q \neq p, 2 \), and it follows from Lemma 2.3 that \( N_2 \cong \langle 1, -1, -dN \rangle \) with \( \text{ord}_2(dN) \leq 1 \).

\[ \Box \]
2.3 Main Result

Lemma 2.5. Suppose that $N_q$ represents every integer in $\mathbb{Z}_q$ for all primes $q \neq p$. Then, every positive integer $\epsilon + p^n \alpha$ which is not of the form $4\delta$, where $\delta \in \mathbb{Z}_2^\times$, is represented primitively by $\text{gen}(M)$. If $\epsilon + p^n \alpha = 4\delta$, then $\frac{\epsilon + p^n \alpha}{4}$ is represented primitively by $\text{gen}(M)$.

Proof. Suppose that $\epsilon + p^n \alpha$ is not of the form $4\delta$, with $\delta \in \mathbb{Z}_2^\times$. From Lemma 2.4, it is clear that $\epsilon + p^n \alpha$ is represented primitively by $M_q = N_q$ whenever $q \neq p, 2$. When $q = 2$, then it follows from Lemma 2.3 that every positive integer $\epsilon + p^n \alpha$, which is not of the form $4\delta$, where $\delta \in \mathbb{Z}_2^\times$, is represented primitively by $M_2 = N_2$. Since $\epsilon$ is a unit in $\mathbb{Z}_p$, and clearly $\epsilon$ is represented by $M_p$, it follows that $\epsilon + p^n \alpha$ is represented primitively by $M_p$. Therefore, $\epsilon + p^n \alpha$ is represented primitively by $\text{gen}(M)$.

Suppose that $\epsilon + p^n \alpha = 4\delta$ for some $\delta \in \mathbb{Z}_2^\times$. Again, Lemma 2.4, ensures that $\frac{\epsilon + p^n \alpha}{4}$ is represented primitively by $M_q$ whenever $q \neq p, 2$. Now it follows from Lemma 2.3 that $\frac{\epsilon + p^n \alpha}{4}$ is represented primitively by $M_2$. Since $\epsilon \in \mathbb{Z}_p^\times$ and $\epsilon + p^n \alpha$ is represented primitively by $M_p$, and since $4 \in \mathbb{Z}_p^\times$, it also follows that $\frac{\epsilon + p^n \alpha}{4}$ is represented primitively by $M_p$. Therefore, in this case, $\frac{\epsilon + p^n \alpha}{4}$ is represented primitively by $\text{gen}(M)$.

Now we are ready to state the first main result.

Theorem 2.6. Suppose $p \not\equiv 7 \mod 8$. Then $H(x)$ is almost universal if and only if $N_q$ represents all $q$-adic integers whenever $q \neq p$.

Proof. Suppose that $N_q$ represents all of $\mathbb{Z}_q$ whenever $q \neq p$. First, we suppose $\epsilon + p^n \alpha \neq 4\delta$ where $\delta \in \mathbb{Z}_2^\times$. In this case $\epsilon + p^n \alpha$ is represented primitively by $\text{gen}(M)$, by Lemma 2.5. Since $p \not\equiv 7 \mod 8$, $M$ can have no primitive spinor
exceptions by Lemma 1.6. Thus, $\epsilon + p^\alpha n$ is represented primitively by the spinor genus of $M$. Now, it follows from Theorem 1.2 that $\epsilon + p^\alpha n$ is represented by the lattice $M$ for $n$ sufficiently large, and hence by the coset $\nu + N$. Next, suppose that $\epsilon + p^\alpha n = 4\delta$, with $\delta \in \mathbb{Z}_p^\times$. In this case, we know that $\delta$ is represented primitively by $\text{gen}(M)$, so by the argument above $\delta$ and hence $4\delta$, are represented by $M$ for $n$ sufficiently large. Therefore, all but finitely many $\epsilon + p^\alpha n$ are represented by $\nu + N$, and hence $H(x)$ is almost universal.

Let $\{e_1, e_2, e_3\}$ be a basis for $N$. In view of Remark 1.8, we may assume that $\nu = \frac{1}{p^\alpha}(ae_1 + be_2 + ce_3)$, with $0 \leq a, b, c \leq p^\alpha - 1$, and at least one of $a, b, c$, not divisible by $p$. There is no harm in assuming that $a \in \mathbb{Z}_p^\times$. It is clear that all $\mathbb{Z}_p$-linear combinations of $\nu, e_2,$ and $e_3$ are contained in $M_p$. Furthermore, $e_1$ can be written as $e_1 = \frac{1}{a}(p^\alpha \nu - be_2 - ce_3)$, which is a $\mathbb{Z}_p$-linear combination of $\nu, e_2,$ and $e_3$. Therefore, $\{\nu, e_2, e_3\}$ is a basis for $M_p$, and relative to this basis,

$$M_p \equiv \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mod p^\alpha$$

since $Q(x)$ and $B(\nu, x)$ are congruent to $0 \mod p^\alpha$ for every $x \in N$, by Lemma 2.1.

Now, as a consequence of the uniqueness of Jordan invariants (cf. [20, Theorem 91:9]), $M_p$ must be of the form

$$M_p \cong (\epsilon, p^i \beta, p^j \gamma)$$

in a basis $\{\nu, f_1, f_2\}$, with $\beta, \gamma \in \mathbb{Z}_p^\times$ and $0 \leq i \leq j$.

Let $\text{sf}(dN)$ denote the square-free part of the discriminant of $N$, and let $\text{sf}(dN)'$
denote the non-$p$ part of $\text{sf}(dN)$.

**Theorem 2.7.** Suppose $p \equiv 7 \mod 8$. Then, $H(x)$ is almost universal if and only if $N_q$ represents all $q$-adic integers over $\mathbb{Z}_q$ whenever $q \neq p$, and one of the following holds:

1. $\text{ord}_p(dN)$ is even;
2. $\text{sf}(dN)$ is divisible by a prime $q$ satisfying $\left(\frac{-p}{q}\right) = -1$;
3. $\left(\frac{Q(\nu)}{p}\right) = -1$;
4. $\text{sf}(dN)'$ is represented by $\nu + N$.

**Proof.** We suppose throughout that $N_q$ represents all $q$-adic integers over $\mathbb{Z}_q$ whenever $q \neq p$. If $\epsilon + p^\alpha n$ is a primitive spinor exception of the genus of $M$, then $E := \mathbb{Q}(\sqrt{-\epsilon + p^\alpha n}dN)$ is $\mathbb{Q}(\sqrt{-p})$, by Lemma 1.6. But $\text{ord}_p(dN)$ and $\text{ord}_p(dM)$ have the same parity, and in this case, they must both be odd. Therefore, if $\text{ord}_p(dN)$ is even, then $\epsilon + p^\alpha n$ cannot be a primitive spinor of exception of the genus of $M$.

Suppose that (1) fails, so $\text{ord}_p(dN)$ is odd. Suppose that $q$ is a prime with $\left(\frac{-p}{q}\right) = -1$. At such a prime, $-p$ is a non-square, and hence $E_q = \mathbb{Q}_q(\sqrt{-p})$ is a quadratic extension over $\mathbb{Q}_q$. By Theorem A.5 part (a), $\theta(O^+(M_q)) \subseteq \mathfrak{N}_q(E)$ if and only if $\text{ord}_q(dN)$ is even. If (2) holds, then $q \mid \text{sf}(dN)$ meaning that $\theta(O^+(M_q)) \not\subseteq \mathfrak{N}_q(E)$, and therefore $\epsilon + p^\alpha n$ is not a primitive spinor exception of the genus of $M$.

Suppose (1) and (2) fail, but (3) holds. Then,

$$M_p \cong \langle \epsilon, p^i \beta, p^j \gamma \rangle$$
and since \( \text{ord}_p(dN) \) is odd, \( i \) and \( j \) have different parity. Now we can compute the spinor norm of \( O^+(M_p) \) using Theorem A.1. Suppose that \( i \) is even, meaning that \( \mathbb{Q}_p M \) contains \([\epsilon, \beta]\) as a subspace. Since \( M_p \) is anisotropic by Lemma 1.4 part 2(b), we conclude that \(-\epsilon \beta\) must be a nonsquare in \( \mathbb{Q}_p \). Furthermore, since \( p \equiv 7 \mod 8 \), and therefore \(-1\) is not a square in \( \mathbb{Q}_p \), we conclude that \( \epsilon \beta \) is a square in \( \mathbb{Q}_p \). Since \( \text{sf}(dN)' \) is only divisible by primes which are squares in \( \mathbb{Q}_p \) by the failure of (2), it follows that \( \gamma \) is also a square in \( \mathbb{Q}_p \). Computing the spinor norm in this case gives

\[
\theta(O^+(M_p)) = \{1, \epsilon \beta, \epsilon \gamma p, \beta \gamma p\} \mathbb{Q}_p^{\times 2} = \{1, \epsilon p\} \mathbb{Q}_p^{\times 2},
\]

where by (3), \( \epsilon \) is non-square in \( \mathbb{Q}_p \). For \( E \) defined as above, \( \mathfrak{N}_p(E) = \{1, p\} \mathbb{Q}_p^{\times 2} \). Since, \( \theta(O^+(M_q)) \not\in \mathfrak{N}_q(E) \), therefore \( \epsilon + p^a n \) cannot be a primitive spinor exception of \( \text{gen}(M) \). The case when \( j \) is even is the same, substituting \( i \) for \( j \) and \( \beta \) for \( \gamma \) where necessary.

Suppose (1), (2), and (3) all fail. We will show that \( \text{sf}(dN)' \) is a primitive spinor exception of \( \text{gen}(M) \). Without any confusion, in what follows, let \( E \) denote the field \( \mathbb{Q}(\sqrt{-\text{sf}(dN)'dN}) = \mathbb{Q}(\sqrt{-p}) \).

First, we claim that \( \text{sf}(dN)' \) is represented primitively by the genus of \( M \). Since the prime factors of \( \text{sf}(dN)' \) can appear with order at most 1, it is clear from combining Lemma 2.3 and Lemma 2.4 that \( \text{sf}(dN)' \) is represented primitively by \( M_q \) when \( q \neq p \). By the failure of (2), \( \text{sf}(dN)' \) is only divisible by primes which are squares in \( \mathbb{Q}_p \), and therefore \( \text{sf}(dN)' \) is a square unit in \( \mathbb{Q}_p \). By the failure of (3), \( \epsilon \) is a square unit in \( \mathbb{Q}_p \). This means that \( \text{sf}(dN)' \) and \( \epsilon \) are in the same square class modulo \( p \). Since \( \epsilon \) is represented primitively by \( M_p \), it follows that \( \text{sf}(dN)' \) is also represented primitively by \( M_p \). This proves that \( \text{sf}(dN)' \) is represented
primitively by the genus of \( M \).

Since \( p \equiv 7 \mod 8 \), \( E \) splits at the prime 2. Moreover, \( E \) also splits at all primes \( q \) such that \( \left( \frac{-p}{q} \right) = 1 \), and hence \( \theta(O^+(M_q)) \subseteq \mathfrak{N}_q(E) = \mathbb{Q}_q^\times \) for these \( q \). Also, since it is always true that \( \mathfrak{N}_q(E) \subseteq \theta^*(M_q, \text{sf}(dN)'_q) \subseteq \mathbb{Q}_q^\times \), we have \( \theta^*(M_q, \text{sf}(dN)'_q) = \mathfrak{N}_q(E) \). At those primes \( q \) for which \( \left( \frac{-p}{q} \right) = -1 \), it follows from the failure of (2) that \( q \nmid \text{sf}(dN)'_q \), and \( M_q \cong \langle 1, -1, dN \rangle \) from Lemma 2.4. Therefore, from Theorem A.5 part (a), we have \( \theta(O^+(M_q)) \subseteq \mathfrak{N}_q(E) \), and since \( \text{sf}(dN)'_q \) is not in any ideal generated by \( q \) we get \( \theta^*(M_q, \text{sf}(dN)'_q) = \mathfrak{N}_q(E) \).

At the prime \( p \) we will compute the spinor norm using Theorem A.1. Since

\[
M_p \cong \langle \epsilon, \beta p^i, \gamma p^j \rangle,
\]

we get

\[
\theta(O^+(M_p)) = \{1, \epsilon \beta p^i, \epsilon \gamma p^j, \beta \gamma p^{i+j}\} \mathbb{Q}_p^\times.
\]

By the failure of (3), \( \epsilon \) is a square in \( \mathbb{Q}_p \). If \( i \) is even, then \( \epsilon \beta \) must be a square in \( \mathbb{Q}_p \) since \( \mathbb{Q}_p M \) contains the anisotropic subspace \([\epsilon, \beta]\). In this case, \( \gamma \) must also be a square in \( \mathbb{Q}_p \), by the failure of (2). Similarly, if \( i \) is odd, and \( j \) is even, then \( \gamma \) is a square in \( \mathbb{Q}_p \), since \( \mathbb{Q}_p M \) contains the anisotropic subspace \([\epsilon, \gamma]\), and \( \epsilon \) is a square. Therefore, in any case

\[
\theta(O^+(M_p)) = \{1, p\} \mathbb{Q}_p^\times,
\]

and hence \( \theta(O^+(M_p)) \subseteq \mathfrak{N}_p(E) \). Finally, by Theorem A.5 part b, \( \mathfrak{N}_p(E) = \theta^*(M_p, \text{sf}(dN)'_p) \), since \( \text{sf}(dN)'_p \) will not be in any ideal generated by \( p \).

We have shown above that when one of (1), (2), or (3) holds, then \( \text{gen}(M) \) has no primitive spinor exceptions of the form \( \epsilon + p^\alpha n \) in \( \mathbb{Q}_p \). In fact, since these
arguments rely only on the square class of $\epsilon + p^n$, they can be easily extended to show that $\text{gen}(M)$ has no primitive spinor exceptions which are of the form $t^2(\epsilon + p^n)$, where $t \in \mathbb{Z}_p^\times$. Suppose we are in the special case where $\epsilon + p^n = 4\delta$ with $\delta \in \mathbb{Z}_p^\times$. Then $\frac{\epsilon + p^n}{4}$ is represented primitively by the genus of $M$ by Lemma 2.5, and if one of (1), (2), or (3) holds, then $\frac{\epsilon + p^n}{4}$ is not a primitive spinor exception of the genus of $M$. From Theorem 1.2, we conclude that $\frac{\epsilon + p^n}{4}$, and hence $\epsilon + p^n$, are represented by $M$, for all sufficiently large $n$. Therefore, $\epsilon + p^n$ are represented by $\nu + N$ for all sufficiently large $n$, and consequently $H(x)$ is almost universal. In a similar manner, when we are not in this special case, then $\epsilon + p^n$ is represented primitively by the genus of $M$ by Lemma 2.5, and if one of (1), (2), or (3) hold, then $\epsilon + p^n$ is not a primitive spinor exception of the genus of $M$ for every positive integer $n$. Then, $\epsilon + p^n$ are represented by $M$ for all $n$ sufficiently large, and therefore $H(x)$ is almost universal.

Suppose that (1), (2), and (3) all fail. Now, we will show that $H(x)$ is almost universal if and only if (4) holds. Suppose that (4) holds, then $\text{sf}(dN)'$ is represented by $M$. Suppose we are in the special case $\epsilon + p^n = 4\delta$ with $\delta \in \mathbb{Z}_p^\times$. From Lemma 2.5, we know that $\delta$ is represented primitively by the genus of $M$. Suppose that $\delta$ is not a primitive spinor exception of the genus of $M$. Then, all sufficiently large $\delta$ and hence $4\delta$ are represented by $M$. Suppose that $\delta$ is a primitive spinor exception, then $\delta$ is a square multiple of $\text{sf}(dN)'$ and hence it is represented by $M$. Thus $4\delta$ must be represented by $M$. Now suppose we are not in the special case, and suppose $\epsilon + p^n$ is not a primitive spinor exception of $\text{gen}(M)$. Then $\epsilon + p^n$ is represented by $M$ when $n$ is sufficiently large. Otherwise, $\epsilon + p^n$ must be a square multiple of $\text{sf}(dN)'$ and hence $\epsilon + p^n$ are represented by $M$. Therefore, $\epsilon + p^n$ is represented by $M$ for all $n$ sufficiently large and hence $H(x)$ is almost universal.
Conversely, suppose that (4) fails. This implies that \( sf(dN)' \) is not represented by \( M \). Then there exist, as shown in [22], infinitely many primes \( q \) such that \( q^2 sf(dN)' \) are not represented by \( M \). Further, these \( q \) are precisely the primes that split in \( \mathbb{Q}(\sqrt{-p}) \). Let \( \mu \) be the inverse of \( sf(dN)' \) modulo \( p \). Then \( \epsilon \mu \) is a square modulo \( p^\alpha \), and thus \( \rho^2 \equiv \epsilon \mu \mod p^\alpha \) for some integer \( \rho \).

The cyclotomic extension \( \mathbb{Q}(\zeta_{p^\alpha}) \) contains a unique quadratic extension, and since \( p \) is the only prime which ramifies in this extension and since \( p \equiv 7 \mod 8 \), this quadratic extension must be \( \mathbb{Q}(\sqrt{-p}) \). The Galois group, \( G \), of \( \mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q} \) is cyclic and contains automorphisms of the form \( \varphi_a(\zeta_{p^\alpha}) = \zeta_{p^\alpha}^a \), where \( a \) and \( p \) are relatively prime. For an odd prime \( q \) not equal to \( p \), the automorphism \( \varphi_q \in G \) generates \( G_q \), the decomposition group of \( q \). The fixed field \( J \) of \( G_q \) is the maximal subfield of \( \mathbb{Q}(\zeta_{p^\alpha}) \) in which \( q \) splits completely (cf. [18, I, §5, Corollary 3]). It is well known that \( q \) splits completely in \( J \) if and only if \( q \) splits completely in any subextension of \( J \) properly containing \( \mathbb{Q} \) (cf. [18, II, §1, SC1]). Thus, by Galois theory, this implies that \( q \) splits completely in \( \mathbb{Q}(\sqrt{-p}) \) if and only if \( \varphi_q \) is in \( H := Gal(\mathbb{Q}(\zeta_{p^\alpha})/\mathbb{Q}(\sqrt{-p})) \).

The 2-element subgroup generated by complex conjugation, \( \varphi_{-1} \), has as its fixed field the maximal real subfield \( \mathbb{Q}(\zeta_{p^\alpha})^+ \), which cannot sit between \( \mathbb{Q}(\sqrt{-p}) \) and \( \mathbb{Q}(\zeta_{p^\alpha}) \). By Galois theory, \( \varphi_{-1} \) is not in \( H \), and since \( H \) is an index 2 subgroup in \( G \), this means that for any integer \( \rho \) relatively prime to \( p \), either \( \varphi_\rho \in H \), or \( \varphi_\rho \varphi_{-1} = \varphi_{-\rho} \in H \). If \( \varphi_\rho \in H \), then by the Čebotarev Density Theorem there exist infinitely many primes \( q \) such that \( \varphi_q = \varphi_\rho \in H \), and so \( q \) splits in \( \mathbb{Q}(\sqrt{-p}) \). On the other hand, if \( \varphi_\rho \not\in H \), then \( \varphi_{-\rho} \in H \), and again there exist infinitely many primes \( q \) such that \( \varphi_q = \varphi_{-\rho} \in H \). In both cases, \( q \) splits in \( \mathbb{Q}(\sqrt{-p}) \) and either \( \rho \equiv q \mod p^\alpha \), or \( -\rho \equiv q \mod p^\alpha \).

Therefore, for a fixed integer \( \rho \), we know that there are infinitely many primes.
that split in $\mathbb{Q}(\sqrt{-p})$ and either $q \equiv \rho \mod p^\alpha$ or $q \equiv -\rho \mod p^\alpha$. So we have an infinite family of positive integers, $n := \frac{q^2 sf(dN') - \epsilon}{p^\alpha}$, for which $\epsilon + p^\alpha n$ are not represented by $M$, and hence $H(x)$ is not almost universal. □
Chapter 3

$p$ is even

Now we turn our attention to the prime 2. Following the assumptions outlined in Chapter 1, in this chapter require that $n(\nu, N) = 2^\alpha$ with $\alpha > 0$, and $[M : N] = 2$. Consequently, $2\nu$ is a primitive vector in $N$; that is, $2\nu \in N$ but $\nu \not\in N$. Unlike the previous cases, where $p$ is odd, we are not guaranteed an orthogonal decomposition for $N_2$. From Lemma 1.4, we know that when $N_q$ represents all $q$-adic integers over $\mathbb{Z}_q$ for every odd prime $q$, then $N_2$ is anisotropic. So, either $N_2$ is not diagonalizable, in which case it has a binary modular Jordan component of the form

$$2^j A \simeq \begin{bmatrix} 2^{j+1} & 2^j \\ 2^j & 2^{j+1} \end{bmatrix}$$

with $j \geq 0$, or it is diagonalizable.

In the following discussion, we consider the possible cases for $\text{ord}_2(Q(\nu))$ separately. If $\text{ord}_2(Q(\nu)) = 0$, then

$$Q(2\nu) + 2B(\nu, 2\nu) = 4Q(\nu) + 4Q(\nu) = 8Q(\nu) \not\equiv 0 \mod 16,$$
so in particular, $\alpha < 4$. Because of this, we will limit all subsequent discussions only to the cases where $1 \leq \alpha \leq 3$. Under this assumption, we have $0 \leq \text{ord}_2(Q(\nu)) \leq 2$, since $\text{ord}_2(Q(\nu)) \geq 3$ would imply that $N_2$ is isotropic by Corollary 1.5.

It is also worthwhile to consider the possible cases for values of the $\mathbb{Z}_2$-ideal $B(\nu, N_2)$. To simplify our discussion, we make one additional assumption that $B(\nu, N_2) \subseteq \mathbb{Z}_2$. Furthermore, since $B(\nu, 2\nu) = 2Q(\nu)$, we may always assume that $2^{\text{ord}_2(Q(\nu))+1}\mathbb{Z}_2 \subseteq B(\nu, N_2) \subseteq \mathbb{Z}_2$. We know from Lemma 1.4 that $n(N_q) = \mathbb{Z}_q$ for every odd prime $q$. This means we can assume that $n(N)$ is a $\mathbb{Z}$-ideal generated by some power of 2. We recall also, that

$$H(x) = \frac{Q(x) + 2B(\nu x)}{2^\alpha}$$

is an inhomogeneous quadratic polynomial.

In principle, these cases could all be addressed in one comprehensive theorem, but since this would complicate things and yield a result which is less transparent to the readers, the cases are treated separately.

**Lemma 3.1.** If $H(x)$ is almost universal, then $N_q$ represents all of $\mathbb{Z}_q$ whenever for every odd prime $q$, and consequently, $N_q \cong \langle 1, -1, -dN \rangle$ for $q \neq 2$.

**Proof.** If $H(x)$ is almost universal, then $N_q$ represents all of $\mathbb{Z}_q$ for every odd prime $q$, by Lemma 1.4 part (1). Combining this with Lemma 1.1, it is immediate that $N_q \cong \langle 1, -1, -dN \rangle$ for $q \neq 2$. \qed
3.1 $Q(\nu) \in \mathbb{Z}_2^\times$

In this section we assume that $Q(\nu) \in \mathbb{Z}_2^\times$. For simplicity, we define $\epsilon := Q(\nu)$. We immediately observe that under this assumption, $B(\nu, N_2) = \mathbb{Z}_2$ or $2\mathbb{Z}_2$. So, in the theorems that follow, we consider all possible cases for $\alpha = 1, 2, 3$, $B(\nu, N_2) = \mathbb{Z}_2, 2\mathbb{Z}_2$, and $N_2$ diagonalizable or non-diagonalizable.

From Lemma 1.7, if we replace $\nu$ with $\omega = \nu + x_0$ for any vector $x_0 \in N$, then we are guaranteed that $n(\nu, N) = n(\omega, N)$. In later discussion, it will be helpful to demonstrate now that $B(\omega, N_2) = B(\nu, N_2)$ when $s(N_2) \subseteq 2\mathbb{Z}_2$. We note that regardless of our choice of $B(\nu, N_2)$, it must be true that $n(N) \subseteq 2\mathbb{Z}$, since $\alpha > 0$.

Lemma 3.2. If $s(N_2) \subseteq 2\mathbb{Z}_2$, then $B(\omega, N_2) = B(\nu, N_2)$.

Proof. Suppose that $s(N_2) \subseteq 2\mathbb{Z}_2$, and let $\omega = \nu + x_0$ for $x_0 \in N$. Then, since $2\nu$ is a primitive vector in $N$, we get that $2B(\nu, x_0) = B(2\nu, x_0) \in 2\mathbb{Z}_2$. Furthermore, since $Q(\nu) \in \mathbb{Z}_2^\times$ and $n(N) \subseteq 2\mathbb{Z}$, we conclude that $Q(\omega) = Q(\nu) + Q(x_0) + 2B(\nu, x_0) \in \mathbb{Z}_2^\times$.

Suppose that $B(\nu, N_2) = \mathbb{Z}_2$. Then, $B(\nu, x) \in \mathbb{Z}_2^\times$ for some $x \in N_2$, and by the initial assumption $B(x_0, x) \in 2\mathbb{Z}$. From this, it follows immediately that $B(\omega, x) = B(\nu, x) + B(x_0, x) \in \mathbb{Z}_2^\times$; hence $B(\omega, N_2) = \mathbb{Z}_2$.

If $B(\nu, N_2) = 2\mathbb{Z}_2$, then clearly $B(\omega, x) = B(\nu, x) + B(x_0, x) \in 2\mathbb{Z}_2$, for any $x \in N_2$. From the argument above, $Q(\omega) \in \mathbb{Z}_2^\times$, and therefore $B(\omega, 2\omega) = 2Q(\omega) \in 2\mathbb{Z}_2^\times$. Therefore, $B(\omega, N_2) = 2\mathbb{Z}_2$. \qed

Since $n(N) \subseteq 2\mathbb{Z}$, $N$ only represents even integers. Since we have assumed that $2\nu$ is primitive in $N$, we know that $M/N$ consists of precisely two cosets, namely, $N$ and $\nu + N$. Under the assumption that $Q(\nu)$ is odd, we can clearly see that any representation of $Q(\nu) + 2^\alpha n$ by $M$ must be from the coset $\nu + N$,
as desired.

**Theorem 3.3.** Suppose that $\alpha = 1$, and $Q(\nu) \in \mathbb{Z}_2^\times$. Then, $H(x)$ is almost universal if and only if $N_q$ represents all $q$-adic integers for every odd prime $q$ and one of the following holds:

1. $2s(N) = n(N) = 4\mathbb{Z}$; or,

2. $N_2$ is diagonalizable, and $\text{ord}_2(dN) = 3$; or,

3. $N_2$ is diagonalizable, $\text{ord}_2(dN) = 5$ and $B(\nu, N_2) = 2\mathbb{Z}_2$.

**Proof.** Suppose that $N_q$ represents all $q$-adic integers for every odd prime $q$, and suppose that $2s(N) = n(N) = 4\mathbb{Z}$. Then,

$$N_2 \cong 2A \perp 2^i\langle \eta \rangle,$$

in a basis $\{e_1, e_2, e_3\}$, where $i > 1$ is odd, or else $N_2$ is isotropic. Let $\nu = \frac{ae_1 + be_2 + ce_3}{2}$, where $0 \leq a, b, c \leq 1$ by Remark 1.8 and Lemma 3.2, then

$$\epsilon = \frac{4a^2 + 4b^2 + 4ab + 2^i\eta c^2}{4} = a^2 + b^2 + ab + 2^{i-2}\eta c^2.$$

Since $\epsilon \in \mathbb{Z}_2^\times$, therefore $a = 1$. Now, in the basis $\{\nu, e_2, e_3\}$, we have

$$M_2 \cong \begin{bmatrix} \epsilon & u & w \\ u & 4 & 0 \\ w & 0 & 2^i\eta \end{bmatrix},$$

where

$$u = B(\nu, e_2) = \begin{cases} \frac{2}{2} = 1 & \text{if } b = 0 \\ \frac{4+2}{2} = 3 & \text{if } b = 1, \end{cases}$$

34
and

\[ w = B(\nu, e_3) = \begin{cases} 
0 & \text{if } c = 0 \\
2^{i-1} \eta & \text{if } c = 1.
\end{cases} \]

In any case, \( M_2 \) is split by the binary unimodular sublattice,

\[
\begin{bmatrix}
\epsilon & u \\
u & 4
\end{bmatrix} \cong \langle \epsilon, \epsilon(-u^2 + 4\epsilon) \rangle \cong \epsilon\langle 1, -1 + 4\epsilon \rangle,
\]

which represents all units in \( \mathbb{Z}_2^\times \). So, \( \mathbb{Z}_2^\times \subseteq \theta(O^+(M_2)) \). Combining this with Lemma 1.4, we have \( \mathbb{Z}_2^\times \subseteq \theta(O^+(M_q)) \) for every prime \( q \), and so \( \text{gen}(M) \) has only one spinor genus. Therefore \( \text{gen}(M) \) does not have any primitive spinor exceptions. Since every \( \epsilon + 2n \) is represented primitively by \( \text{gen}(M) \), we conclude that \( \epsilon + 2n \) is represented by the lattice \( M \) itself, for every sufficiently large \( n \). Therefore, \( H(x) \) is almost universal.

Suppose that (2) holds, so \( N_2 \) is diagonalizable, and \( \text{ord}_2(dN) = 3 \). Then,

\[ N_2 \cong \langle 2\eta, 2\beta, 2\gamma \rangle, \]

in a basis \( \{e_1, e_2, e_3\} \). Letting \( \nu = \frac{ae_1 + be_2 + ce_3}{2} \), we have that

\[ \epsilon = \frac{2\eta a^2 + 2\beta b^2 + 2\gamma c^2}{4}, \]

and therefore by virtue of Remark 1.8 and Lemma 3.2 we may assume without loss of generality that \( a \) and \( b \) are equal to 1, and \( c = 0 \). Now, in the basis \( \{\nu, e_1, e_3\} \),
we have
\[ M_2 \cong \begin{bmatrix} \epsilon & \eta & 0 \\ \eta & 2\eta & 0 \\ 0 & 0 & 2\gamma \end{bmatrix} \cong \langle \epsilon, \epsilon(2\eta \epsilon - \eta^2), 2\gamma \rangle. \]

But \(2\eta \epsilon - \eta^2 \equiv -1 + 2\eta \epsilon \equiv 1 \mod 4\); hence,
\[ M_2 \cong \epsilon\langle 1, -1 + 2\eta \epsilon, 2\gamma \epsilon \rangle \]

represents all units. So, we have \(\mathbb{Z}_2^\times \subseteq \theta(O^+(M_2))\). Combining this with Lemma 1.4, we now have \(\mathbb{Z}_q^\times \subseteq \theta(O^+(M_q))\) for every prime \(q\), and so \(\text{gen}(M)\) has only one spinor genus. So, since every possible \(\epsilon + 2n\) is represented primitively by \(\text{gen}(M)\), \(\epsilon + 2n\) is represented by \(M\) itself for \(n\) sufficiently large. Therefore, \(H(x)\) is almost universal.

Now suppose that (3) holds. So \(N_2\) is diagonalizable with \(\text{ord}_2(dN) = 5\), and \(B(\nu, N_2) = 2\mathbb{Z}_2\). Then,
\[ N_2 \cong \langle 2\eta, 2^i \beta, 2^j \gamma \rangle, \]

where either \(i = 1\) and \(j = 3\) or \(i = j = 2\). Suppose that we are in the first case. Letting \(\nu = \frac{ae_1 + be_2 + ce_3}{2}\), with \(0 \leq a, b, c \leq 1\) by Remark 1.8 and Lemma 3.2, we have
\[ \epsilon = \frac{2\eta a^2 + 2\beta b^2 + 8\gamma c^2}{4} = \frac{\eta a^2 + \beta b^2}{2} + 2\gamma c^2 \in \mathbb{Z}_2^\times. \]

Thus \(a\) and \(b\) are both equal to 1. But now, \(B(\nu, e_1) = \eta\); so \(B(\nu, N_2) = \mathbb{Z}_2\), which is a contradiction. So, we must have \(i = j = 2\). In this case,
\[ N_2 \cong \langle 2\eta, 4\beta, 4\gamma \rangle, \]

36
and
\[ \epsilon = \frac{2\eta a^2 + 4\beta b^2 + 4\gamma c^2}{4}. \]

Thus \( a = 0 \), and
\[ \epsilon = \beta b^2 + \gamma c^2 \in \mathbb{Z}_2^\times. \]

So, without loss of generality, we may assume \( b = 1 \) and \( c = 0 \). So, \( \nu = \epsilon\frac{a}{2} \), and in the basis \( \{\nu, e_1, e_3\} \), we have

\[ M_2 \cong (\epsilon, 2\eta, 4\gamma). \]

Clearly \( M_2 \) primitively represents units in the square classes of \( \epsilon, \epsilon + 2\eta, \epsilon + 4\gamma, \) and \( \epsilon + 2(\eta + 2\gamma) \), which are all of the square classes of units in \( \mathbb{Z}_2^\times \). Therefore, \( \mathbb{Z}_2^\times \subseteq \theta(O^+(M_2)) \). So, combining this with Lemma 1.4, we have \( \mathbb{Z}_q^\times \subseteq \theta(O^+(M_q)) \) for every prime \( q \). So, \( \text{gen}(M) \) has only one spinor genus, and \( \text{gen}(M) \) primitively represents every \( \epsilon + 2n \). Therefore, \( H(x) \) is almost universal.

Now we will show that when (1), (2) and (3) all fail, then \( H(x) \) is not almost universal. First, suppose that \( s(N) = n(N) = 4\mathbb{Z} \). Then, \( 2B(\nu, x) = B(2\nu, x) \subseteq 4\mathbb{Z} \) and \( Q(x) \subseteq 4\mathbb{Z} \) for any \( x \in N \). Therefore, \( Q(x) + 2B(\nu, x) \equiv 0 \mod 4 \) for any \( x \), and so \( H(x) \) only represent even integers.

Next, suppose that \( n(N) = 2\mathbb{Z} \), and \( N_2 \) is not diagonalizable. Then,
\[ N_2 \cong 2^iA \perp 2^j(\eta) \]

with respect to a basis \( \{e_1, e_2, e_3\} \). If \( i > j \), then \( j = 1 \). But then \( N_2 \) cannot represent \( 4\epsilon \), since \( i > 1 \). If \( i < j \), then \( i = 0 \), and \( j > 1 \) must be even, since \( N_2 \) is anisotropic by Lemma 1.4. But \( N_2 \) must represent \( 4\epsilon \), so \( j = 2 \). Let \( \nu = \frac{ae_1 + be_2 + ce_3}{2} \).
wit $0 \leq a, b, c \leq 1$. Then,

$$\epsilon = \frac{2a^2 + 2b^2 + 2ab + 4\eta c^2}{4},$$

so $a = b = 0$ and $c = 1$, and hence $\nu = \frac{\epsilon}{2}$. Therefore,

$$M_2 \cong A \perp \langle \epsilon \rangle$$

in the basis $\{e_1, e_2, \nu\}$. But now, $M_2$ fails to represent $\epsilon + 4\delta$ with $\delta$ odd, so $H(x)$ cannot represent $2\delta$, and is therefore not almost universal.

Now suppose that $s(N) = n(N) = 2\mathbb{Z}$ and

$$N_2 \cong \langle 2\eta, 2\beta, 2^j\gamma \rangle,$$

where $j > 1$. If $\nu = \frac{ae_1 + be_2 + ce_3}{2}$, $0 \leq a, b, c \leq 1$, we have

$$\epsilon = \frac{2\eta a^2 + 2\eta b^2 + 2^j\gamma c^2}{4},$$

so $a$ and $b$ are equal to $1$. Thus, $\nu = \frac{e_1 + e_2 + ce_3}{2}$, and given an arbitrary $xe_1 + ye_2 + ze_3 \in N$, we have

$$2B(\nu, xe_1 + ye_2 + ze_3) + Q(xe_1 + ye_2 + ze_3)$$
$$= B(e_1 + e_2 + ce_3, xe_1 + ye_2 + ze_3) + Q(xe_1 + ye_2 + ze_3)$$
$$\equiv 2\eta x + 2\beta y + 2\eta x^2 + 2\beta y^2 \mod 4$$
$$\equiv 2\eta x(1 + x) + 2\beta y(1 + y) \mod 4$$
$$\equiv 0 \mod 4.$$

Therefore, $H(x)$ only represents even integers.
Now, we suppose that \( s(N) = n(N) = 2\mathbb{Z} \), and

\[
N_2 \cong \langle 2\eta, 2^i\beta, 2^j\gamma \rangle
\]

in a basis \( \{e_1, e_2, e_3\} \), where \( 1 < i \leq j \). Then, with \( \nu = \frac{ae_1 + be_2 + ce_3}{2} \), \( 0 \leq a, b, c \leq 1 \),

\[
\epsilon = \frac{2\eta a^2 + 2^i\beta b^2 + 2^j\gamma c^2}{4},
\]

so immediately \( a = 0, i = 2, j > 2 \), and \( b = 1 \). Note that we can immediately rule out \( i = j = 2 \), by the failure of (3). Now

\[
N_2 \cong \langle 2\eta, 4\beta, 2^j\gamma \rangle,
\]

and

\[
M_2 \cong \begin{bmatrix}
\epsilon & \xi & 0 \\
\xi & 2^j\gamma & 0 \\
0 & 0 & 2\eta
\end{bmatrix},
\]

in the basis \( \{\nu, e_3, e_1\} \), where

\[
\xi = \begin{cases} 
0 & \text{if } c = 0 \\
2^{j-1}\gamma & \text{if } c = 1.
\end{cases}
\]

In either case, \( 2^j\gamma \epsilon - \xi^2 \equiv 0 \mod 2^j \), so

\[
M_2 \cong \langle \epsilon, 2\eta, \epsilon(2^j\gamma \epsilon - \xi^2) \rangle,
\]

which only represents units in the square classes of \( \epsilon \) and \( \epsilon + 2\eta \). Therefore \( H(x) \)}
Lemma 3.4. Suppose that $\alpha = 2, 3$, and $Q(\nu) \in \mathbb{Z}_2^\times$. Assume that $N_2$ is anisotropic and not diagonalizable. Then, $H(x)$ is not almost universal if one of the following holds:

1. $B(\nu, N_2) = \mathbb{Z}_2$;
2. $B(\nu, N_2) = 2\mathbb{Z}_2$ and $n(N) = 2s(N)$;
3. $B(\nu, N_2) = 2\mathbb{Z}_2$, $n(N) = s(N)$, and $\alpha = 2$.

Proof. Since $N_2$ is not diagonalizable, therefore

$$N_2 \cong 2^i\langle \gamma \rangle \perp 2^jA$$

in a basis $\{e_1, e_2, e_3\}$, where $\gamma \in \mathbb{Z}_2^\times$, and $i \neq j$. First, let us suppose that $B(\nu, N_2) = \mathbb{Z}_2$. Then, $2B(\nu, N_2) = 2\mathbb{Z}_2$, and therefore $n(N) = 2\mathbb{Z}$, since we require that $n(\nu, N) = 2^\alpha\mathbb{Z}$. If $n(N) = s(N)$, then $i = 1$. Since $N_2$ represents $4\epsilon$, therefore, $j = 1$, a contradiction.

Suppose that $n(N) = 2s(N)$. Then, $j = 0$. If $i = 1$, then $N_2$ is isotropic, which contradicts the hypothesis. So $i = 2$ because $N$ represents $4\epsilon$, and

$$N_2 \cong \langle 4\gamma \rangle \perp A$$

in a basis $\{e_1, e_2, e_3\}$. Now, $\nu = \frac{ae_1+z}{2}$ with $a \in \mathbb{Z}_2$ and $z \in A$. Then $z$ is imprimitive, or else $Q(\nu) \notin \mathbb{Z}$. Thus, $\frac{z}{2} = y \in A$, and we write $y = be_2 + ce_3$. Since $B(\nu, N_2) = \mathbb{Z}_2$, either $b$ or $c$ must be a unit in $\mathbb{Z}_2$. Without loss of generality, let’s assume that $b \in \mathbb{Z}_2^\times$. Suppose first that $c \in 2\mathbb{Z}_2$. Then, $Q(e_2) + 2B(\nu, e_2) =$
$2 + 2(2b + c) \equiv 2 \mod 4$, which is a contradiction since $\alpha \geq 2$. Next, we suppose that $c \in \mathbb{Z}_2^\times$. Then, $Q(e_2 + e_3) + 2B(\nu, e_2 + e_3) = 6 + 2(2b + c + b + 2c) \equiv 2 \mod 4$; again, a contradiction.

Now, suppose that $B(\nu, N_2) = 2\mathbb{Z}_2$ and $n(N) = 2\mathfrak{s}(N)$. Then, $2B(\nu, N_2) = 4\mathbb{Z}_2$, and therefore $n(N) = 4\mathbb{Z}$ because $n(\nu, N) = 2^\alpha \mathbb{Z}$ and $N_2$ represents $4\epsilon$. So,

$$N_2 \cong 2\mathfrak{A} \perp 2^i\langle \gamma \rangle$$

in a basis $\{e_1, e_2, e_3\}$, where $i \geq 2$ and $\gamma \in \mathbb{Z}_2^\times$. Since $N_2$ is anisotropic, therefore $i \geq 3$. Now, under our assumptions, $n(\nu, N) = 2^\alpha \mathbb{Z}$. Let $\nu = \frac{ae_1 + be_2 + ce_3}{2}$ with $0 \leq a, b, c \leq 1$, and consider

$$2B(\nu, e_1) + Q(e_1) = B(ae_1 + be_2 + ce_3, e_1) + Q(e_1) = 4a + 2b + 4 = 2(b + 2a + 2).$$

For $\alpha = 2$ or $3$, we need $2(b + 2a + 2) \equiv 0 \mod 4$, so $b = 0$. Similarly,

$$2B(\nu, e_2) + Q(e_2) = B(ae_1 + be_2 + ce_3, e_2) + Q(e_2) = 2a + 4b + 4 = 2(a + 2b + 2),$$

so $a = 0$. But then $\nu = \frac{ce_3}{2}$, and so $\epsilon = \frac{2^i \gamma c^2}{4}$, which is even, a contradiction.

Finally, suppose that $B(\nu, N_2) = 2\mathbb{Z}_2$, $n(N) = \mathfrak{s}(N)$, and $\alpha = 2$. Then,
\[ n(N) = 4\mathbb{Z} \] and \( \mathbb{Z}[2\nu] \cong \langle 4\epsilon \rangle \) splits \( N_2 \) as an orthogonal summand. So,

\[ N_2 \cong \langle 4\epsilon \rangle \perp 2^i A, \]

in a basis \( \{2\nu, f_1, f_2\} \), where \( i \geq 1 \) since \( n(N) = 4\mathbb{Z} \), and \( i \) is even since \( N_2 \) is anisotropic by Lemma 1.4 part 3. With this,

\[ M_2 \cong \langle \epsilon \rangle \perp 2^i A \]

in the basis \( \{\nu, f_1, f_2\} \). But now, \( M_2 \) does not represent any unit of the form \( \epsilon + 4n \) with \( n \) odd. So, \( \epsilon + 4n \) is not represented by \( \text{gen}(M) \) for any odd \( n \). Therefore, \( M \) itself fails to represent infinitely many \( \epsilon + 4n \), and therefore \( H(x) \) is not almost universal.

\[ \square \]

**Theorem 3.5.** Suppose that \( \alpha = 2, 3 \), \( Q(\nu) \in \mathbb{Z}_2^\times \), and \( N_2 \) is not diagonalizable. Then, \( H(x) \) is almost universal if and only if \( N_q \) represents all \( q \)-adic integers for every odd prime \( q \), \( \alpha = 3 \), \( B(\nu, N_2) = 2\mathbb{Z}_2 \), and \( n(N) = s(N) \).

**Proof.** Suppose that \( \alpha = 2, 3 \), and \( N_2 \) is not diagonalizable. By Lemma 3.4, if \( H(x) \) is almost universal, then \( B(\nu, N_2) = 2\mathbb{Z}_2 \), \( n(N) = s(N) \), and \( \alpha = 3 \).

Suppose that \( N_q \) represents all \( q \)-adic integers for every odd prime \( q \), \( \alpha = 3 \), \( B(\nu, N_2) = 2\mathbb{Z}_2 \), and \( n(N) = s(N) \). Then, \( s(N) = 4\mathbb{Z} \), and \( \mathbb{Z}[2\nu] \cong \langle 4\epsilon \rangle \) splits \( N_2 \) as an orthogonal summand. So,

\[ N_2 \cong \langle 4\epsilon \rangle \perp 2^i A \]
in a basis \( \{2\nu, f_1, f_2\} \), where \( i \) is even and \( i \geq 2 \), and therefore

\[
M_2 \cong \langle \epsilon \rangle \perp 2^i \mathbb{A}
\]

in the basis \( \{\nu, f_1, f_2\} \). We observe that \( \mathbb{A} \) represents all \( 2\delta \) where \( \delta \) is a unit, so in particular, \( \mathbb{A} \) contains vectors \( u \) and \( w \) of length 2 and \( 2\delta \), respectively. Hence, \( O^+(M_2) \) contains the symmetries \( \tau_u \) and \( \tau_w \), and from here it is not difficult to see that \( 2 \cdot 2\delta \), and therefore \( \delta \), is an element in \( \theta(O^+(M_2)) \). But now, \( \mathbb{Z}_2^\times \subseteq \theta(O^+(M_2)) \). Combining this with Lemma 1.4, we have \( \mathbb{Z}_q^\times \subseteq \theta(O^+(M_q)) \) for every prime \( q \). Therefore, \( \text{gen}(M) \) has only one spinor genus. Since \( \epsilon + 8n \) is clearly primitively represented by \( \text{gen}(M) \), we conclude that \( \epsilon + 8n \) is represented by \( M \) itself if \( n \) is sufficiently large, from Theorem 1.2. So, \( H(x) \) is almost universal. \( \square \)

**Lemma 3.6.** Let \( \alpha = 2, 3 \). Assume that \( N_2 \) is diagonalizable. Then \( H(x) \) is not almost universal if one of the following holds:

1. \( \alpha = 3 \) and \( B(\nu, N_2) = \mathbb{Z}_2 \); or,

2. \( \alpha = 2 \), \( B(\nu, N_2) = 2\mathbb{Z}_2 \), and \( n(G_2) \subseteq 8\mathbb{Z}_2 \), where \( G \) is the orthogonal complement of \( \nu \) in \( N \).

**Proof.** Suppose that \( N_2 \) is diagonalizable, \( \alpha = 3 \), and \( B(\nu, N_2) = \mathbb{Z}_2 \). Then, \( 2B(\nu, N_2) = 2\mathbb{Z}_2 \), and \( n(N) = 2\mathbb{Z} \). Therefore,

\[
N_2 \cong \langle 2\eta, 2^i \beta, 2^j \gamma \rangle
\]

in a basis \( \{e_1, e_2, e_3\} \), where \( 1 \leq i \leq j \), \( \beta, \gamma \in \mathbb{Z}_2^\times \). By Remark 1.8 and Lemma 3.2, we may assume that \( \nu = \frac{ae_1 + be_2 + ce_3}{2} \), where \( 0 \leq a, b, c \leq 1 \). Then,

\[
Q(e_1) + 2B(\nu, e_1) = 2\eta + B(ae_1, e_1) = 2\eta \text{ or } 4\eta.
\]
In either case, \( Q(e_1) + 2B(\nu, e_1) \not\equiv 0 \mod 8 \), which contradicts our initial assumptions.

Now, suppose that \( \alpha = 2 \), and \( B(\nu, N_2) = 2\mathbb{Z}_2 \). As usual, \( \mathbb{Z}[2\nu] \cong \langle 4\epsilon \rangle \) splits \( N_2 \) as an orthogonal summand, so

\[
N_2 \cong \langle 4\epsilon \rangle \perp G_2.
\]

For an arbitrary \( 2x\nu + u \in N \), with \( u \in G_2 \) we have that

\[
Q(2x\nu + u) + 2B(\nu, 2x\nu + u) = 4\epsilon(x^2 + x) + Q(u) + 2B(2x\nu, u) + B(2\nu, u) \equiv Q(u) \mod 8
\]

Therefore, if \( n(G) \subseteq 8\mathbb{Z} \), then \( H(x) \) only represents even integers, thus it is not almost universal.

**Lemma 3.7.** Suppose that \( Q(\nu) \in \mathbb{Z}_2^\times \) and \( N_2 \) is diagonalizable. If \( N_q \) represents every \( q \)-adic integer for every odd prime \( q \) and

1. \( \alpha = 3 \) and \( B(\nu, N_2) = 2\mathbb{Z}_2 \); or,

2. \( \alpha = 2 \), \( B(\nu, N_2) = 2\mathbb{Z}_2 \) and \( n(G_2) = 4\mathbb{Z}_2 \), where \( G \) is the orthogonal complement of \( \nu \) in \( N \); or,

3. \( \alpha = 2 \), and \( B(\nu, N_2) = \mathbb{Z}_2 \),

then \( \epsilon + 2^\alpha n \) is represented primitively by \( \text{gen}(M) \) for every \( n \geq 1 \).

**Proof.** Since \( N_q \) represents every \( q \)-adic integer for every odd prime \( q \), \( N_q \cong \langle 1, -1, -dN \rangle \) by Lemma 1.4, and so \( \epsilon + 2^\alpha n \) is represented primitively by \( N_q \) at every odd prime \( q \).
First, suppose that $\alpha = 3$, and $B(\nu, N_2) = 2\mathbb{Z}_2$, meaning that $n(N) = 4\mathbb{Z}$. It is immediate that $\mathbb{Z}[\nu] \cong \langle \epsilon \rangle$ splits $M_2$ as an orthogonal summand. Therefore, $\epsilon + 8n$ is represented by $M_2$ for all possible choices of $n$, and so $\text{gen}(M)$ represents all $\epsilon + 8n$ primitively.

Next, suppose that $\alpha = 2$, $B(\nu, N_2) = 2\mathbb{Z}_2$ and $n(G_2) = 4\mathbb{Z}_2$. Again, $\mathbb{Z}[2\nu] \cong \langle 4\epsilon \rangle$ splits $N_2$ as an orthogonal summand, and now

$$M_2 \cong \langle \epsilon \rangle \perp G_2.$$  

Clearly, $M_2$ represents all units that are congruent to $\epsilon \mod 4$, and since they are units, those representations must be primitive. So, $\text{gen}(M)$ primitively represents all $\epsilon + 4n$ in this case.

Finally, suppose that $\alpha = 2$ and $B(\nu, N_2) = \mathbb{Z}_2$. Then, $n(N) = 2\mathbb{Z}$, so we may assume that

$$N_2 \cong \langle 2\eta, 2^i \beta, 2^j \gamma \rangle$$

in a basis $\{e_1, e_2, e_3\}$, where $\eta, \beta, \gamma \in \mathbb{Z}_2^\times$, and $1 \leq i \leq j$. Let $\nu = \frac{ae_1 + be_2 + ce_3}{2}$, with $0 \leq a, b, c \leq 1$ by Remark 1.8 and Lemma 3.2. Then

$$Q(e_1) + 2B(\nu, e_1) = Q(e_1) + B(ae_1 + be_2 + ce_3, e_1) = 2\eta + 2an\eta,$$

which means that $a = 1$. Now,

$$Q(\nu) = \frac{2\eta + 2^i \beta b + 2^j \gamma c}{4},$$

which makes $i = 1$ and $b = 1$. Therefore, $\{\nu, e_2, e_3\}$ is a basis for $M_2$, and in this
basis

\[ M_2 \cong \begin{bmatrix} \epsilon & \beta & \xi \\ \beta & 2\beta & 0 \\ \xi & 0 & 2^j \gamma, \end{bmatrix} \]

where \( \xi := B(\nu, e_3) = 0 \) or \( 2^j \gamma \) depending on the parity of \( c \). In any case, \( M_2 \) contains the binary sublattice

\[ \begin{bmatrix} \epsilon & \beta \\ \beta & 2\beta \end{bmatrix} \cong \langle \epsilon, \epsilon(2\epsilon \beta - \beta^2) \rangle \]

which represents all units congruent to \( \epsilon \mod 4 \). Therefore, \( \epsilon + 4n \) is represented primitively by \( \text{gen}(M) \) for any possible choice of \( n \).

For the following theorem, we define an integer \( \delta \) by

\[ \delta := \begin{cases} 1 & \text{if } \text{ord}_2(dN) \text{ is even,} \\ 2 & \text{if } \text{ord}_2(dN) \text{ is odd.} \end{cases} \]

**Theorem 3.8.** Suppose that \( \alpha = 3 \), \( Q(\nu) \in \mathbb{Z}_2^\times \), and \( N_2 \) is diagonalizable. Let \( G \) be the orthogonal complement of \( \nu \) in \( N \). Then \( H(x) \) is almost universal if and only if \( N_q \) represents every \( q \)-adic integer in \( \mathbb{Z}_q \) for all odd primes \( q \), \( B(\nu, N_2) = 2\mathbb{Z}_2 \), and one of the following holds:

1. \( n(G_2) = 8\mathbb{Z}_2 \), and \( \text{ord}_2(dN) \) is even or \( \text{ord}_2(dN) = 9 \); or,

2. \( n(G_2) = 16\mathbb{Z}_2 \), and \( \text{ord}_2(dN) \) is odd; or,

3. \( \text{sf}(dN)' \) is divisible by a prime \( q \) satisfying \( \left( \frac{-\delta}{q} \right) = -1 \); or,

4. \( \text{sf}(dN)' \not\equiv Q(\nu) \mod 8 \); or,
(5) \( n(G_2) = 8\mathbb{Z}_2 \), and \( 8Q(\nu) \) is not represented by \( G_2 \); or,

(6) \( \frac{sf(dN') - Q(\nu)}{8} \) is represented by \( H(x) \).

Proof. Suppose that \( N_q \) represents every \( q \)-adic integer in \( \mathbb{Z}_q \) for all odd primes \( q \).
Assuming that \( \alpha = 3 \) and \( B(\nu, N_2) = 2\mathbb{Z}_2 \), we have \( \epsilon + 8n \) represented primitively
by \( \text{gen}(M) \) for all \( n \), by Lemma 3.7. Since \( B(\nu, N_2) = 2\mathbb{Z}_2 \), therefore \( n(N) = 4\mathbb{Z} \).
Hence, \( \mathbb{Z}_2[2\nu] \cong \langle 4\epsilon \rangle \) splits \( N_2 \) as an orthogonal summand. Thus,

\[
N_2 \cong \langle 4\epsilon \rangle \perp G_2
\]
in a basis \( \{2\nu, f_1, f_2\} \), and hence,

\[
M_2 \cong \langle \epsilon \rangle \perp G_2
\]
in the basis \( \{\nu, f_1, f_2\} \). When \( G_2 \) is diagonalizable, we write \( G_2 \cong \langle 2^i\beta, 2^j\gamma \rangle \)
with \( \beta, \gamma \in \mathbb{Z}_2^\times \) and \( 2 \leq i \leq j \). Note that if \( n(G_2) \subseteq 16\mathbb{Z}_2 \), then \( G_2 \) must
be diagonalizable. Given an odd primitive spinor exception \( t \) of \( \text{gen}(M) \), let \( E := \mathbb{Q}(\sqrt{-tdN}) \). Since \( \mathbb{Z}_q^\times \subseteq \theta(O^+(N_q)) \) for every odd prime, we may conclude
that \( E \) is \( \mathbb{Q}(\sqrt{-\delta}) \).

Suppose that (1) holds, so \( n(G_2) = 8\mathbb{Z}_2 \). When \( G_2 \) is improper, then \( N_2 \cong \langle 4\epsilon \rangle \perp 4\mathbb{A} \), so \( M_2 \cong \langle \epsilon \rangle \perp 4\mathbb{A} \). It follow that for any \( \delta \in \mathbb{Z}_2^\times \) there is a vector \( \nu_\delta \in 4\mathbb{A} \) such that \( Q(\nu_\delta) = 8\delta \). In particular, \( \mathbb{Z}_2^\times \subseteq \theta(O^+(M_2)) \). Now, \( \mathbb{Z}_q^\times \subseteq \theta(O^+(M_q)) \)
for every prime \( q \), and therefore \( M \) has no primitive spinor exceptions.

Now assume that \( G_2 \) is proper. Then, in the basis \( \{\nu, f_1, f_2\} \), we have

\[
M_2 \cong \langle \epsilon, 8\beta, 2^j\gamma \rangle,
\]
with \( j \geq 3 \). If \( \text{ord}_2(dN) \) is even, then we are in the situation of Theorem A.6 part (a). The “\( r \)” there is 3 in our case, which is odd, and the \( \mathbb{Z}_2 \)-ideal \((2^3-3)\) contains all odd integers; so \( \theta^*(M_2,t) \neq \mathfrak{N}_2(E) \). In this case, \( \text{gen}(M) \) has no odd primitive spinor exceptions. If \( \text{ord}_2(dN) = 9 \), then \( j = 4 \) and hence \( M_2 \) is of Type \( E \) defined in Appendix A. In this case, \( \theta(O^+(M_2)) = \mathbb{Q}_2^\times \), so \( \text{gen}(M) \) has no odd primitive spinor exceptions.

Suppose that (2) holds, so \( n(G_2) = 16\mathbb{Z}_2 \), and

\[
M_2 \cong \langle \epsilon, 16\beta, 2^j\gamma \rangle
\]

where \( j \geq 4 \) is odd. Then we are in the situation of Theorem A.6 part (b). The “\( r \)” there is 4 in our case, and \( t \in (2^r - 4) \) for any odd integer \( t \). Therefore, \( \theta^*(M_2,t) \neq \mathfrak{N}_2(E) \), and so \( \text{gen}(M) \) has no odd primitive spinor exceptions in this case.

Suppose that (1) and (2) fail, and (3) holds, so \( \text{sf}(dN)' \) is divisible by a prime \( q \) for which \(-\delta\) is a non-square in \( \mathbb{Q}_q \). In this case, \( E_q/\mathbb{Q}_q \) is a quadratic extension. From Lemma 1.4, we know that \( N_q \cong \langle 1, -1, -dN \rangle \), where \( \text{ord}_q(dN) \) is odd. Now, from Theorem A.5, we know that \( \theta(O^+(M_q)) \not\subseteq \mathfrak{N}_q(E) \). So, \( \text{gen}(M) \) has no odd primitive spinor exceptions in this case.

Suppose that (1)-(3) all fail, and (4) holds. So we have

\[
M_2 \cong \langle \epsilon, 2^i\beta, 2^j\gamma \rangle
\]

with \( \beta\gamma \not\equiv 1 \mod 8 \), and consequently for any positive integer \( n \),

\[
(\epsilon + 8n)\epsilon\beta\gamma \equiv \epsilon\epsilon\beta\gamma \equiv \beta\gamma \not\equiv 1 \mod 8.
\]
If $\epsilon + 8n$ is a primitive spinor exception of $\text{gen}(M)$, $Q(\sqrt{-(\epsilon + 8n)dN}) = Q(\sqrt{-\delta})$, by Lemma 1.6. But when (4) holds, this cannot happen. So, in this case, $\epsilon + 8n$ is not a primitive spinor exception of $\text{gen}(M)$ for any positive integer $n$.

Suppose that (1)-(4) all fail, and (5) holds. So,

$$G_2 \cong \langle 8\beta, 2^{j}\gamma \rangle$$

where $j \geq 6$ is even by the failure of (1), and $8Q(\nu)$ is not represented by $G_2$. Equivalently, $\epsilon$ is not represented by $\langle \beta, 2^{j-3}\gamma \rangle$; so in particular, $\epsilon \not\equiv \beta \mod 8$.

From the failure of (4), we have that $\beta \equiv \gamma \mod 8$, which implies that $\epsilon \gamma \equiv \epsilon \beta \not\equiv 1 \mod 8$. Since $M_2$ is anisotropic, it is clear that $\epsilon \gamma \not\equiv -1 \mod 8$. Furthermore, suppose that $\epsilon \beta \equiv 5 \mod 8$. Then the quadratic space underlying $M_2$ is

$$[1, 10, 5] \cong [1, -6, 5],$$

which is clearly isotropic, contradicting Lemma 1.4. So, we conclude that $\epsilon \beta \equiv 3 \mod 8$. From [5, Theorem 2.7], we know that $\theta(O^+(M_2)) = Q(P(U))Q(P(W))Q_2^{x^2}$, where

$$U \cong \langle 1, 24 \rangle \text{ and } W \cong 24\langle 1, 2^{j-3} \rangle,$$

and $P(U)$ is the set of all primitive anisotropic vectors whose associated symmetries are in $O(U)$ (define $O(W)$ similarly). Since $U$ represents 1, it is immediate that $\theta(O^+(U)) = Q(P(U))Q_2^{x^2}$. Moreover, $6 \cdot \theta(O^+(\langle 1, 2^{j-3} \rangle)) = Q(P(W))Q_2^{x^2}$. Using the formulas given in Theorem A.2, we get

$$Q(P(U)) = \theta(O^+(U)) = \{\gamma \in Q_2^{x^2} : (\gamma, -6) = 1\}$$

$$= \{1, 6, -1, -6\}Q_2^{x^2}$$
\[ Q(P(W)) = 6 \cdot \theta(O^+(\langle 1, 2^{j-3} \rangle)) = 6 \cdot \begin{cases} \{ \gamma \in \mathbb{Q}_2^\times : (\gamma, -2) = 1 \} & \text{if } j = 6 \\ \{1, 2\} \mathbb{Q}_2^\times & \text{if } j \geq 8 \end{cases} = 6 \cdot \begin{cases} \{1, 2, 3, 6\} \mathbb{Q}_2^\times & \text{if } j = 6 \\ \{1, 2\} \mathbb{Q}_2^\times & \text{if } j \geq 8 \end{cases} = \begin{cases} \{1, 2, 3, 6\} \mathbb{Q}_2^\times & \text{if } j = 6 \\ \{3, 6\} \mathbb{Q}_2^\times & \text{if } j \geq 8. \end{cases} \]

In any case, \(-6 \in \theta(O^+(M_2))\). Since \(\text{ord}_2(dN)\) here is odd, \(E = \mathbb{Q}(\sqrt{-2})\). Clearly \(-6 \not\in \mathcal{N}_2(E)\), so \(\text{gen}(M)\) has no odd primitive spinor exceptions in this case.

Now we show that when (1)-(5) all fail, then \(\text{sf}(dN)'\) is a primitive spinor exception of \(\text{gen}(M)\). From Lemma 1.4 it is immediate that \(\text{sf}(dN)'\) is represented primitively by \(M_q\) at every odd prime \(q\). Furthermore, from the failure of (4) it is clear that \(\text{sf}(dN)'\) is represented by \(M_2\), and the representation is primitive since \(\text{sf}(dN)'\) is odd. So, \(\text{sf}(dN)'\) is represented primitively by \(\text{gen}(M)\).

Without confusion, let \(E\) denote \(\mathbb{Q}(\sqrt{-\text{sf}(dN)'dN}) = \mathbb{Q}(\sqrt{-\delta})\). At the primes \(q\) where \((\frac{-\delta}{q}) = 1\), we have \(E_q = \mathbb{Q}_q\), and therefore \(\mathcal{N}_q(E) = \mathbb{Q}_q^\times\). Since it is always true that

\[ \mathcal{N}_q(E) \subseteq \theta^*(M_q, \text{sf}(dN)') \subseteq \mathbb{Q}_q^\times, \]

we have

\[ \theta(O^+(M_q)) \subseteq \mathcal{N}_q(E) = \theta^*(M_q, \text{sf}(dN)') = \mathbb{Q}_q^\times. \]

At the primes \(q\) where \((\frac{-\delta}{q}) = -1\), we know from the failure of (3) that such a prime does not divide \(\text{sf}(dN)'\). Combining Lemma 1.4 and Theorem A.5 part
(a), we conclude that $\theta(O^+(M_q)) \subseteq \mathfrak{N}_q(E)$ at these primes. Also, since $q$ does not divide $sf(dN)'$, we know that $sf(dN)'$ will not be contained in any $\mathbb{Z}_q$-ideal generated by $q^{2k+1}$ where $k \geq 1$, and therefore, $\mathfrak{N}_q(E) = \theta^*(M_q, sf(dN)')$.

Now we compute the spinor norm and the relative spinor norm at the prime 2. First we will deal with the case where $i = j$. In this case, $\text{ord}_2(dN)$ is even, so we may assume that $i \geq 4$, from the failure of (1), and $E = \mathbb{Q}(\sqrt{-1})$. To compute the spinor norm, we turn to Theorem A.4. Here, $M_2$ has a binary Jordan component $G_2$ with $dG_2 \equiv 1 \mod 8$, from the failure of (4). So,

$$\theta(O^+(M_2)) = \{ \gamma \in \mathbb{Q}_2^\times : (\gamma, -1) = 1 \} = \{ 1, 2, 5, 10 \} \mathbb{Q}_2^\times,$$

which is equal to $\mathfrak{N}_2(E)$ in this case. Now we may compute the relative spinor norm using Theorem A.6 part (a). Since "r" $\geq 4$ here, (i)-(iv) there immediately fail, and therefore $\theta^*(M_2, sf(dN)') = \mathfrak{N}_2(E)$.

Now suppose that $i \neq j$. From the failure of (1)-(5), we have

$$M^i_2 \cong \langle 1, 2^i \beta \epsilon, 2^j \beta \epsilon \rangle,$$

where $3 \leq i < j$. Since $M^i_2$ is anisotropic, it follows that $\beta \epsilon \equiv 1, 3 \mod 8$ when $\text{ord}_2(dN)$ is odd, and $\beta \epsilon \equiv 1 \mod 4$ when $\text{ord}_2(dN)$ is even. Recall that when $i = 3$, we have $\beta \epsilon \equiv 1 \mod 8$, from the failure of (5). Let

$$U = \langle 1, 2^i \beta \epsilon \rangle \text{ and } W = 2^i \beta \epsilon \langle 1, 2^{j-i} \rangle,$$
with $P(U)$ and $P(W)$ as defined previously. Now, using Theorem A.2, we obtain

$$Q(P(U)) = \theta(O^+(U)) = \begin{cases} 
\{\gamma \in \mathbb{Q}^2_x : (\gamma, -2) = 1\} & \text{if } i = 3 \\
\{1, \beta \epsilon, 5, 5 \beta \epsilon\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x & \text{if } i = 4 \\
\{1, 2^i \beta \epsilon\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x & \text{if } i \geq 5
\end{cases}$$

and

$$Q(P(W)) = 2^i \beta \epsilon \theta(O^+((1, 2^{j-i}))) = 2^i \beta \epsilon \begin{cases} 
\{\gamma \in \mathbb{Z}_2^x : (\gamma, -2) = 1\} & \text{if } j - i = 1, 3 \\
\{\gamma \in \mathbb{Z}_2^x \mathbb{Q}_2^x \times \mathbb{Q}_2^x : (\gamma, -1) = 1\} & \text{if } j - i = 2 \\
\{1, 5\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x & \text{if } j - i = 4 \\
\{1, 2^{j-i}\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x & \text{if } j - i \geq 5
\end{cases}$$

$$= 2^i \beta \epsilon \begin{cases} 
\{1, 2, 3, 6\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x & \text{if } j - i = 1, 3 \\
\{1, 5\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x & \text{if } j - i = 2, 4 \\
\{1, 2^{j-i}\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x & \text{if } j - i \geq 5
\end{cases}$$

If $\text{ord}_2(dN)$ is odd, then $Q(P(W)) \subseteq \{1, 2, 3, 6\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x$, since $i$ and $j$ must have opposite parity. In this case, $E = \mathbb{Q}(\sqrt{-2})$ and $\mathfrak{H}_2(E) = \{1, 2, 3, 6\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x$, and therefore we have $\theta(O^+(M_2)) \subseteq \mathfrak{H}_2(E)$. Note that when $\text{ord}_2(dN)$ is odd, $i \neq 4$, from the failure of (2). If $\text{ord}_2(dN)$ is even, then $i$ and $j$ have the same parity, meaning that $Q(P(W)) \subseteq \{1, 2, 5, 10\} \mathbb{Q}_2^x \times \mathbb{Q}_2^x$. Therefore, in this case $E = \mathbb{Q}(\sqrt{-1})$.
and $\mathfrak{N}_2(E) = \{1, 2, 5, 10\} Q_2^\times$, and hence $\theta(O^+(M_2)) \subseteq \mathfrak{N}_2(E)$. In any case $\theta(O^+(M_2)) \subseteq \mathfrak{N}_2(E)$.

To compute the relative spinor norm of $M_2$, we turn to Theorem A.6. First, suppose that $\text{ord}_2(dN)$ is odd, meaning that we are in the situation of part (c) of the theorem. If “$r$” there is even, then $i > 4$ by the failure of (2), and hence (i) fails; if “$r$” is odd, then either $K = M_2$ or “$r$” > 3, and in either case (ii) fails; and “$s$” > 2, so part (iii) fails. Suppose that $\text{ord}_2(dN)$ is even, meaning that now we are in the situation of part (b) of the theorem. If “$r$” is even, then (i)-(iii) all fail, since $\text{sf}(dN)'$ is odd, and if “$r$” is odd, then (iv) fails, since $i > 3$ by the failure of (1). In either case, $\theta^*(M_2, \text{sf}(dN)') = \mathfrak{N}_2(E)$. Therefore, $\text{sf}(dN)'$ is a primitive spinor exception of $\text{gen}(M)$.

Suppose that (1)-(5) all fail, and (6) holds. Then, $\frac{\text{sf}(dN)' - \epsilon}{8}$ is represented by $H(x)$; hence $\text{sf}(dN)'$ is represented by the coset $\nu + N$, and therefore by the lattice $M$. If $\epsilon + 8n$ is not a primitive spinor exception of $\text{gen}(M)$, then it is represented by $M$ when $n$ is sufficiently large by Theorem 1.2, and therefore all sufficiently large $n$ are represented by $H(x)$. If $\epsilon + 8n$ is a primitive spinor exception of $\text{gen}(M)$, then $\epsilon + 8n = m^2 \text{sf}(dN)'$ for some integer $m$, and therefore $\epsilon + 8n$ is represented by the lattice $M$. If (6) holds, then $H(x)$ is almost universal.

Now, suppose that (6) fails. Then, $\frac{\text{sf}(dN)' - \epsilon}{8}$ is not represented by $H(x)$, and therefore $\text{sf}(dN)'$ is not represented by $M$. Since we are assuming that (1)-(5) all fail, we know that $\text{sf}(dN)'$ is a primitive spinor exception of $\text{gen}(M)$. Furthermore, from the failure of (4), we have $\text{sf}(dN)' \equiv \epsilon \mod 8$. Now, as shown in [22], there exist infinitely many primes $q$ for which

$$n := \frac{\text{sf}(dN)'q^2 - \epsilon}{8}$$
is an integer not represented by $H(x)$. Therefore, in this case $H(x)$ is not almost universal.

\[ \square \]

**Theorem 3.9.** Suppose that $\alpha = 2$, $N_2$ is diagonalizable and $Q(\nu) \in \mathbb{Z}_2^\times$. Then $H(x)$ is almost universal if and only if $N_q$ represents every $q$-adic integer in $\mathbb{Z}_q$ for all $q$ odd, and one of the following holds:

1. $B(\nu, N_2) = \mathbb{Z}_2$, and
   
   (a) $\text{ord}_2(dN)$ is odd; or,
   
   (b) $\text{ord}_2(dN) = 4$; or,
   
   (c) $\text{sf}(dN)$ is divisible by a prime congruent to 3 mod 4; or,
   
   (d) $N_2$ has a binary Jordan component with the square-free part of its discriminant congruent to 5 mod 8.

2. $B(\nu, N_2) = 2\mathbb{Z}_2$, $n(G_2) = 4\mathbb{Z}_2$, where $G_2$ is the orthogonal complement of $\nu$ in $N_2$, and
   
   (a) $\text{ord}_2(dN)$ is odd; or,
   
   (b) $\text{ord}_2(dN) = 6$; or,
   
   (c) $\text{sf}(dN)$ is divisible by a prime congruent to 3 mod 4.

3. $\frac{\text{sf}(dN)^{\nu} - Q(\nu)}{4}$ is represented by $H(x)$.

**Proof.** First we will deal with the case when $B(\nu, N_2) = \mathbb{Z}_2$, so $n(N) = 2\mathbb{Z}$. As shown in the proof of Lemma 3.7, we may assume that $N_2$ has an orthogonal decomposition

$$N_2 \cong \langle 2\eta, 2\beta, 2^j\gamma \rangle$$
in a basis \(\{e_1, e_2, e_3\}\), where \(\eta, \beta, \gamma \in \mathbb{Z}_2^\times\), and \(j \geq 1\). So \(N_2\) has a binary sublattice

\[2\langle \eta, \beta \rangle,\]

with discriminant \(4\eta\beta\). Also, as discussed in the proof of Lemma 3.7, we may assume that \(\nu = \frac{e_1 + e_2 + ce_3}{2}\), for some \(c = 0\) or \(1\). In any case, \(\{\nu, e_2, e_3\}\) is a basis for \(M_2\), and in this basis

\[
M_2 \cong \begin{bmatrix}
\epsilon & \beta & \xi \\
\beta & 2\beta & 0 \\
\xi & 0 & 2^j\gamma,
\end{bmatrix}
\]

where \(\xi = 0\) if \(c = 0\), and \(\xi = 2^{j-1}\gamma\) is \(c = 1\). If \(c = 0\), then \(\epsilon = \frac{\eta + \beta}{2}\), giving

\[2\epsilon\beta - \beta^2 = (\eta + \beta)\beta - \beta^2 = \eta\beta\]

and

\[M_2 \cong \langle \epsilon, \epsilon\eta\beta, 2^j\gamma \rangle,\]

with \(j \geq 1\). If \(c = 1\), then \(\epsilon = \frac{\eta + \beta}{2} + 2^{j-2}\gamma\). So \(j > 2\), and

\[2\epsilon\beta - \beta^2 = (\eta + \beta + 2^{j-1}\gamma)\beta - \beta^2 = \beta(\eta + 2^{j-1}\gamma).\]

Therefore, in the basis \(\{\nu, e_2, 2^{j-1}\gamma e_1 - \eta e_3\}\),

\[M_2 \cong \langle \epsilon, \epsilon\beta(\eta + 2^{j-1}\gamma), 2^j\eta\gamma(\eta + 2^{j-1}\gamma) \rangle.\]

In any case, \(dM = 2^j\eta\beta\gamma\).

Suppose that \(\text{ord}_2(dN)\) is odd, and \(t\) is an odd primitive spinor exception of
Then, \( E = \mathbb{Q}(\sqrt{-2}) \), and we are in the situation outlined in Theorem A.6 part (b). But \( M_2 \) has a binary unimodular component, so “r” there is 0, and so \( \theta(O^+(M_2)) \not\subset \mathfrak{H}_2(E) \). Therefore, \( \text{gen}(M) \) has no odd primitive spinor exceptions. In particular, if \( \text{ord}_2(dN) \) is odd, then \( \epsilon + 4n \) is not a primitive spinor exception of \( \text{gen}(M) \).

Now, suppose that 1(a) fails, meaning that \( \text{ord}_2(dN) \) is even, and hence \( E = \mathbb{Q}(\sqrt{-1}) \). Therefore \( \mathfrak{H}_2(E) = \{1, 2, 5, 10\} \mathbb{Q}_2^\times \). Now \( j \) is even, and for any choice of \( \nu \), we have

\[
M_2 \cong \langle \epsilon, \epsilon \beta \eta, 2^j \gamma \rangle.
\]

Suppose that 1(b) holds, and hence \( \text{ord}_2(dN) = 4 \), implying that \( \text{ord}_2(dM) = 2 \), and therefore \( j = 2 \). For convenience, we let \( L_2 := \langle 1, \beta \eta, 4 \epsilon \gamma \rangle \), and now we may use Theorem A.4 part (2) to compute the spinor norm of \( L_2 \). Since \( M_2 \) must be anisotropic, it is necessary that \( \beta \eta \equiv 1 \mod 4 \), and hence

\[
\langle 1, \eta \beta \rangle \cong \begin{bmatrix} 1 & 1 \\ 1 & 1 + \eta \beta \end{bmatrix}
\]

with \( 1 + \eta \beta \equiv 2 \mod 4 \). So we are in part (3) of Theorem A.4 part (2), but since \( j = 2 \), part (iii) fails; thus \( \mathbb{Q}_2^\times = \theta(O^+(M_2)) \). Therefore, \( \theta(O^+(M_2)) \not\subset \mathfrak{H}_2(E) \), and \( \epsilon + 4n \) is not a primitive spinor exception of \( \text{gen}(M) \).

Suppose that 1(a) and 1(b) both fail, so \( \text{ord}_2(dN) \geq 6 \) is even, and hence \( E = \mathbb{Q}(\sqrt{-1}) \). Suppose that \( \text{sf}(dN) \) is divisible by a prime \( q \equiv 3 \mod 4 \). At any one of these primes \( q \), \(-1\) is not a square in \( \mathbb{Q}_q \) and \( E_q/\mathbb{Q}_q \) is a genuine quadratic extension. Now we are in the situation described in Theorem A.5 part (a). Since \( q \) divides the square free part of the discriminant of \( N \), \( \text{ord}_q(dM) \) is odd. But according to Theorem A.5 part (a), this means \( \theta(O^+(M_q)) \not\subset \mathfrak{H}_q(E) \). Therefore,
if \( q \equiv 3 \mod 4 \) divides \( sf(dN) \), then \( \text{gen}(M) \) cannot have any odd primitive spinor exceptions. As a result, if 1(c) holds, then \( \epsilon + 4n \) is not primitive spinor exception of \( \text{gen}(M) \).

Suppose now that 1(a)-(c) fail, and 1(d) holds, that is, \( \eta \beta \equiv 5 \mod 8 \). Then, since \( j \) is even, the quadratic space underlying \( L_2 \) is

\[
[1, 5, \epsilon\gamma].
\]

From the failure of 1(c), \( \gamma \equiv 1 \mod 4 \), but then, \( \epsilon \equiv 1 \mod 4 \), or else \( M_2 \) is isotropic. Therefore, we are in the situation of Theorem A.4 part (2), with the binary component \( (1, 5) \cong A(1, 6) \), and a unary component \( 2^j\langle \epsilon\gamma \rangle \), with \( \epsilon\gamma \equiv 1 \mod 4 \), and the Hilbert symbol

\[
(2^j\epsilon\gamma, -5) = 1.
\]

Furthermore, \( "r_2 - r_1" \geq 4 \) here, since \( j \geq 4 \). So,

\[
\theta(O^+(L_2)) = \theta(O^+(M_2)) = \{ \gamma \in \mathbb{Q}_2^\times : (\gamma, -5) = 1 \} = \{1, 5, 6, 14\}\mathbb{Q}_2^\times.
\]

Therefore, in this case, \( \theta(O^+(M_2)) \not\subseteq \mathfrak{N}_2(E) \), which means that \( \epsilon + 4n \) is not a primitive spinor exception in this case.

Now we consider the case where \( B(\nu, N_2) = 2\mathbb{Z}_2 \). Since we require that \( n(\nu, N) = 4\mathbb{Z} \), we have that \( n(N) = 4\mathbb{Z} \). Therefore, \( \mathbb{Z}[2\nu] \cong \langle 4\epsilon \rangle \) splits \( N_2 \) as an orthogonal summand, so

\[
N_2 \cong \langle 4\epsilon \rangle \perp G_2
\]
and
\[ M_2 \cong \langle \epsilon \rangle \perp G_2. \]

Suppose that 2(a) holds, that is, suppose that ord_2(dN) is odd. Then, \( G_2 \cong \langle 4\beta, 2^j \gamma \rangle \) with \( j \) odd, and
\[ M_2 \cong \langle \epsilon, 4\beta, 2^j \gamma \rangle. \]

Now we are in the situation described in Theorem A.6 part (b), where we have “\( r \)” = 2 here, which is even. The \( \mathbb{Z}_2 \)-ideal \( (2^{-2}) \) contains \( \mathbb{Z}_2 \); in particular, \( \epsilon + 4n \) is contained in this ideal. Therefore, (i) holds, and \( \theta^*(M_2, \epsilon + 4n) \neq \mathcal{N}_2(E) \), thus \( \epsilon + 4n \) cannot be a primitive spinor exception of \( \text{gen}(M) \).

Suppose that 2(b) holds; that is, ord_2(dN) = 6. So \( G_2 \) is 4-modular. Suppose \( G_2 \) is proper, i.e. \( G_2 \cong \langle 4\beta, 4\gamma \rangle \), then
\[ M_2 \cong \langle \epsilon, 4\beta, 4\gamma \rangle. \]

Now we compute the spinor norm of the orthogonal group of
\[ M_2^\beta \cong \langle \epsilon \beta \rangle \perp 4\langle 1, \beta \gamma \rangle \]
using Theorem A.4 part (2). Clearly \( \beta \gamma \equiv 1 \mod 4 \), or else \( M_2 \) is isotropic. Then
\[ \langle 1, \beta \gamma \rangle \cong A(1, 1 + \beta \gamma), \]
where \( 1 + \beta \gamma \equiv 2 \mod 4 \). Since “\( r_2 - r_1 \)” = 2 here, part (iii) fails, and we can conclude that \( \theta(O^+(M_2)) = \mathbb{Q}_2^\times \). Therefore, in this case \( \theta(O^+(M_2)) \not\subseteq \mathcal{N}_2(E) \), and hence \( \epsilon + 4n \) is not a primitive spinor exception of \( \text{gen}(M) \). Now suppose that \( G_2 \) is improper. Then, \( G_2 \cong 2A \) and \( M_2 \cong 4\langle \epsilon \rangle \perp 2A \). Then, \( \theta(O^+(M_2)) \supseteq \mathbb{Z}_2^\times \).

58
Since \( \theta(O^+(M_q)) \supseteq \mathbb{Z}_q^\times \) for all \( q > 2 \), therefore \( \text{gen}(M) \) does not have any primitive spinor exceptions.

Suppose that 2(a) and 2(b) both fail. In particular, \( \text{ord}_2(dN) \) is even, and hence \( E = \mathbb{Q}(\sqrt{-1}) \). Suppose that \( \text{sf}(dN) \) is divisible by a prime \( q \equiv 3 \mod 4 \). At such a \( q \), \( \mathbb{Q}_q(\sqrt{-1}) \) is a quadratic extension over \( \mathbb{Q}_q \). Now we are in the situation described in Theorem A.5 part (a). Since \( q \) divides the square free part of the discriminant of \( N \), \( \text{ord}_q(dM) \) is odd. But according to Theorem A.5 part (a), this means \( \theta(O^+(M_q)) \not\subseteq \mathfrak{N}_q(E) \). Therefore, if \( q \equiv 3 \mod 4 \) divides \( \text{sf}(dN) \), then \( \text{gen}(M) \) cannot have any odd primitive spinor exceptions. Consequently, if 2(c) holds, then \( \epsilon + 4n \) is not primitive spinor exception of \( \text{gen}(M) \).

Now we show that when parts (1) and (2) fail, then \( \text{sf}(dN) \) is a primitive spinor exception of \( \text{gen}(M) \). It is clear from Lemma 1.4 part (1) that \( \text{sf}(dN) \) is represented primitively by \( M_q \) at all odd primes. From the failure of (1) and (2), we know that \( \text{sf}(dN) \equiv 1 \mod 4 \). When \( B(\nu, N_2) = \mathbb{Z}_2 \), then, \( \epsilon \equiv 1 \mod 4 \), as discussed in a previous paragraph. When \( B(\nu, N_2) = 2\mathbb{Z}_2 \), then \( \epsilon \beta \gamma \equiv 1 \mod 4 \), from the failure of part 2(c), and the underlying quadratic space is \([\epsilon, \beta, \gamma]\). Since \( M_2 \) is anisotropic by Lemma 1.4 part (1), therefore \( \beta \gamma \equiv 1 \mod 4 \), and hence \( \epsilon \equiv 1 \mod 4 \). In either case, \( \epsilon \equiv \text{sf}(dN) \equiv 1 \mod 4 \), and therefore \( \text{sf}(dN) \) is represented primitively by \( M_2 \) by Lemma 3.7. Therefore, \( \text{sf}(dN) \) is represented primitively by the genus of \( M \).

Now we need to show that

\[
\theta(O^+(M_q)) \subseteq \mathfrak{N}_q(E) = \theta^*(M_q, \text{sf}(dN))
\]

at every prime \( q \), where \( E = \mathbb{Q}(\sqrt{-\text{sf}(dN)dN}) = \mathbb{Q}(\sqrt{-1}) \). At an odd prime \( q \equiv 1 \mod 4 \), we know that \( \mathbb{Q}_q(\sqrt{-1}) = \mathbb{Q}_q \), and therefore, \( \mathfrak{N}_q(E) = \mathbb{Q}_q^\times \). Since
it is always true that
\[ \mathfrak{N}_q(E) \subseteq \theta^+(M_q, \text{sf}(dN)) \subseteq \mathbb{Q}_q^\times, \]

we get
\[ \theta(O^+(M_q)) \subseteq \mathfrak{N}_q(E) = \theta^+(M_q, \text{sf}(dN)) = \mathbb{Q}_q^\times. \]

At a prime \( q \equiv 3 \mod 4 \), we know from the failure of (1) and (2) that such a prime does not divide \( \text{sf}(dN) \). Furthermore,

\[ M_q \cong (1, -1, -dN), \]

by Lemma 1.4 part (1). Therefore, \( \theta(O^+(M_q)) \subseteq \mathfrak{N}_q(E) \), by Theorem A.5 part (a).

Also, since \( q \) does not divide \( \text{sf}(dN) \), we know that \( \text{sf}(dN) \) will not be contained in any ideal of the form \( (q^{2r+1}) \) for \( r \geq 0 \), and therefore, \( \mathfrak{N}_q(E) = \theta^*(M_q, \text{sf}(dN)) \).

At the prime 2, we will consider cases for \( B(\nu, N_2) = \mathbb{Z}_2 \) or \( 2\mathbb{Z}_2 \) separately. Let us first consider the case \( B(\nu, N_2) = \mathbb{Z}_2 \). From our previous discussion,

\[ M_2 \cong (1, 1, 2^j \epsilon \gamma), \]

with \( j \geq 4 \) even, and \( \epsilon \gamma \equiv 1 \mod 4 \). Again, we are in the situation of Theorem A.4 part (2), with a binary component which is neither odd nor even. So,

\[ \theta(O^+(M_2)) = \{ \gamma \in \mathbb{Q}_2^\times : (\gamma, -1) = 1 \} = \{ 1, 2, 5, 10 \} \mathbb{Q}_2^\times = \mathfrak{N}_2(E). \]

Now, we can use Theorem A.6 part (a) to compute the relative spinor norm,
\( \theta^*(M_2, \text{sf}(dN)) \). The \( K \) and \( K' \) in that theorem are

\[
K \cong \langle 2^{-2\epsilon}, 1\epsilon, 2^j\gamma \rangle,
\]

and \( K' = M_2 \), respectively. Since \( j \) is even, and clearly \( \theta(O^+(K')) = \theta(O^+(M_2)) \subseteq \mathfrak{n}_2(E) \), parts (ii) and (iv) fail immediately. Moreover, part (iii) fails, since \( j \geq 4 \).

Also, since \( \text{sf}(dN) \) is odd, it is not contained in any \( \mathbb{Z}_2 \)-ideal generated by \( 2^j \). For the spinor norm for \( O^+(K) \), we may scale by \( 4\epsilon \), and consider

\[
\langle 1, 4, 2^{j+2}\epsilon\gamma \rangle.
\]

Since \( j \) is even, this lattice is not of type \( E \), as defined in Appendix A. From [5, Theorem 2.7], we know that \( \theta(O^+(K)) = Q(P(U))Q(P(W))Q_2^\times \), where

\[
U \cong \langle 1, 4 \rangle \text{ and } W \cong 4\langle 1, 2^j\epsilon\gamma \rangle.
\]

Since \( U \) represents 1,

\[
Q(P(U))Q_2^\times = \theta(O^+(U)) = \{\mu \in \mathbb{Z}_2Q_2^\times : (\mu, -1) = 1\} = \{1, 5\}Q_2^\times,
\]

by Theorem A.2. Similarly, \( \theta(O^+(\langle 1, 2^j\epsilon\gamma \rangle)) = Q(P(W))Q_2^\times \), and using Theorem A.2,

\[
\theta(O^+(\langle 1, 2^j\epsilon\gamma \rangle)) = \begin{cases} 
Q_2^\times \cup 5Q_2^\times & \text{if } j = 4; \\
Q_2^\times \cup \epsilon\gamma Q_2^\times & \text{if } j \geq 4
\end{cases}
\]

\( \subseteq \{1, 5\}Q_2^\times \).
So, \( \theta(O^+(K)) \subseteq \{1, 5\}Q_2^x \subseteq \mathfrak{N}_2(E) \), and therefore (i) of Theorem A.6 part (a) fails. Therefore, \( \theta^*(M_2, sf(dN)) = \mathfrak{N}_2(E) \) when \( B(\nu, N_2) = \mathbb{Z}_2 \).

Now we deal with the case where \( B(\nu, N_2) = 2\mathbb{Z}_2 \). From the failure of (2) and from Lemma 3.6, we know that

\[
M_2 \cong \langle \epsilon, 4\beta, 2^j\gamma \rangle,
\]

where \( j \geq 4 \) is even, and \( \epsilon\beta\gamma \equiv 1 \mod 4 \). Note that \( M_2 \) is anisotropic, so immediately \( \epsilon\beta \equiv 1 \mod 4 \). Since \( j \) is even, \( M_2 \) is not of Type E. Letting \( L_2 := M_2^\epsilon \), we have

\[
L_2 \cong \langle 1, 4\epsilon\beta, 2^j\epsilon\gamma \rangle.
\]

From [5, Theorem 2.7], we know that \( \theta(O^+(M_2)) = Q(P(U))Q(P(W))Q_2^x \), where

\[
U \cong \langle 1, 4\epsilon\beta \rangle \text{ and } W \cong 4\epsilon\beta\langle 1, 2^{j-2}\epsilon\beta\gamma \rangle.
\]

Now, using Theorem A.2,

\[
\theta(O^+(U)) = \{ \mu \in Z_2Q_2^x : (\mu, -\epsilon\beta) = 1 \} = \{1, 5\}Q_2^x,
\]

and

\[
\epsilon\beta\theta(O^+(\langle 1, 2^{j-2}\epsilon\beta\gamma \rangle)) = \epsilon\beta\begin{cases}
\{ \mu \in Z_2Q_2^x : (\mu, -\epsilon\beta\gamma) = 1 \} & \text{if } j = 4 \\
Q_2^x \cup 5Q_2^x & \text{if } j = 6 \\
Q_2^x \cup \epsilon\beta\gamma Q_2^x & \text{if } j \geq 8
\end{cases}
\subseteq \{1, 5\}Q_2^x.
\]
Therefore, \( \theta(O^+(M_2)) \subseteq \mathfrak{N}_2(E) \). Now we will use Theorem A.6 part (a) to compute \( \theta^*(M_2, \text{sf}(dN)) \). The \( K \) and \( K' \) in that theorem are \( K = M_2 \) and

\[ K' \cong \langle 4\epsilon, 4\beta, 2^j\gamma \rangle, \]

respectively. We have “\( r \)” = 2 here, which is even, and \( \theta(O^+(K)) \subseteq \mathfrak{N}_2(E) \), so parts (i) and (iv) fail. Furthermore, \( \text{sf}(dN) \) is odd, and is therefore not contained in any \( \mathbb{Z}_2 \)-ideal generated by \( 2^2 \) or \( 2^j \). So, we may conclude that \( \theta^*(M_2, \text{sf}(dN)) = \mathfrak{N}_2(E) \).

We have shown that \( \text{sf}(dN) \) is primitively represented by \( \text{gen}(M) \), and

\[ \theta(O^+(M_q)) \subseteq \mathfrak{N}_q(E) = \theta^*(M_q, \text{sf}(dN)) \]

for every prime \( q \), and therefore \( \text{sf}(dN) \) is a primitive spinor exception of \( \text{gen}(M) \).

Supposing that (1) and (2) fail, we will now show that \( H(x) \) is almost universal if and only if (3) holds.

Suppose that (3) holds, so \( \frac{\text{sf}(dN) - \epsilon}{4} \) is represented by \( H(x) \). Then, \( \text{sf}(dN) \) is represented by the coset \( \nu + N \), and therefore by the lattice \( M \) itself. If \( \epsilon + 4n \) is not a primitive spinor exception of \( \text{gen}(M) \), then it is represented by \( M \) for sufficiently large \( n \), by Theorem 1.2. So in this case, all sufficiently large \( n \) are represented by \( H(x) \). On the other hand, if \( \epsilon + 4n \) is a primitive spinor exception of \( \text{gen}(M) \), then \( \epsilon + 4n = m^2 \text{sf}(dN) \), for some integer \( m \), since \( \text{sf}(dN) \) is a primitive spinor exception as well, by the failure of (1) and (2). In this case, \( \epsilon + 4n \) will be represented by the lattice \( M \), and therefore by \( \nu + N \). So in this case, \( n \) is represented by \( H(x) \). Therefore, if (3) holds, then \( H(x) \) is almost universal.

Suppose that (3) fails, so \( \frac{\text{sf}(dN) - \epsilon}{4} \) is not represented by \( H(x) \). Then, \( \text{sf}(dN) \)
is not represented by $\nu + N$, and therefore it is not represented by the lattice $M$ itself. Now there are, as shown in [22], infinitely many primes $q$ so that $\text{sf}(dN)q^2$ is not represented by $M$. But then,

$$n := \frac{\text{sf}(dN)q^2 - \epsilon}{4}$$

is an integer, since $\text{sf}(dN)q^2 \equiv \epsilon \mod 4$, and $n$ is not represented by $H(x)$. Therefore, we have shown that when (3) fails, there are infinitely many integers which are not represented by $H(x)$. Thus, $H(x)$ is not almost universal. \qed

### 3.2 $Q(\nu) \in 2\mathbb{Z}_2^\times$

In this section, we assume that $Q(\nu) \in 2\mathbb{Z}_2^\times$. Without confusion, let $2\epsilon := Q(\nu)$, where $\epsilon \in \mathbb{Z}_2^\times$. According to Corollary 1.5, if $H(x)$ is almost universal, then $\alpha > 1$. Therefore, in this section, we assume that $2 \leq \alpha \leq 3$. Note that now we have $4\mathbb{Z} \subseteq B(\nu, N) \subseteq \mathbb{Z}$.

In the previous section, it was immediate that any representation of $Q(\nu) + 2^{\alpha}n$ would come from the coset $\nu + N$ of $M/N$. Since we are now assuming that $Q(\nu)$ is not a unit, this conclusion no longer holds. In the following theorems, it will be necessary to introduce a sublattice $R \subseteq M$ which represents $Q(\nu) + 2^{\alpha}n$, and which satisfies certain arithmetic properties. This will be done explicitly as needed.

**Theorem 3.10.** Let $\alpha = 2, 3$, $Q(\nu) \in 2\mathbb{Z}_2^\times$, and $B(\nu, N_2) = \mathbb{Z}_2$. Then $H(x)$ is almost universal if and only if $N_q$ represents all $q$-adic integers for every odd prime $q$ and $N_2$ is diagonalizable.

**Proof.** Suppose that $\alpha = 2, 3$, $Q(\nu) \in 2\mathbb{Z}_2^\times$, $B(\nu, N_2) = \mathbb{Z}_2$, and $N_q$ represents all $q$-adic integers for every odd prime $q$.
Suppose that $N_2$ is not diagonalizable. Then, $n(N) = 2\mathbb{Z}$, and

$$N_2 \cong 2^i\langle \eta \rangle \perp 2^jA$$

in a basis $\{e_1, e_2, e_3\}$, with $i \neq j$. If $i < j$, then $i = 1$ and $j > 1$. Let $\nu = \frac{ae_1 + be_2 + ce_3}{2}$. Then

$$Q(\nu) = \frac{2\eta a^2 + 2^i(b^2 + bc + c^2)}{4},$$

and hence $a \in 2\mathbb{Z}$. Therefore, $B(\nu, e_1) \in 2\mathbb{Z}$, $B(\nu, e_2) = \frac{2^i(2b + c)}{2} \in 2\mathbb{Z}$ and $B(\nu, e_3) = \frac{2^i(2c + b)}{2} \in 2\mathbb{Z}$, contradicting our initial assumption. If $j < i$, then $j = 0$, and

$$N_2 \cong \langle 2^j\eta \rangle \perp A,$$

with $i > 0$. Since $\alpha = 2, 3$, $n(\nu, N_2) \subseteq 4\mathbb{Z}_2$, and consequently $Q(e_2) + 2B(\nu, e_2)$ and $Q(e_3) + 2B(\nu, e_3)$ are elements in $4\mathbb{Z}_2$. But consequently, we have $Q(e_2 + e_3) = 2 + Q(e_2) + 2B(\nu, e_2) + Q(e_3) + 2B(\nu, e_3)$ which is clearly not an element $4\mathbb{Z}_2$, a contradiction. Therefore, this case cannot occur.

Suppose that $N_2$ is diagonalizable. Then

$$N_2 \cong \langle 2^{i}\eta, 2^{i}\beta, 2^{j}\gamma \rangle$$

in a basis $\{e_1, e_2, e_3\}$, and

$$Q(\nu) = \frac{2\eta a^2 + 2^i\beta b^2 + 2^j\gamma c^2}{4} = 2\epsilon,$$

with $i \leq j$, and $\nu = \frac{ae_1 + be_2 + ce_3}{2}$ with $0 \leq a, b, c \leq 1$ by Remark 1.8 and Lemma 3.2. If $a = 1$, then $i = 1$ and $b = 1$. If $a = 0$, then by our initial assumptions, $B(\nu, e_2) = 2^{i-1}\beta b$ must be in $\mathbb{Z}_2^X$, so $i = 1$ and $b = 1$. Therefore, in either case
\{\nu, e_1, e_3\} is a basis for \(M_2\). Consider the sublattice \(R := \mathbb{Z}[\nu, 2e_1, 2e_3]\) of \(M\). Then, \(R_q = M_q\) for every odd prime \(q\), since \(2 \in \mathbb{Z}_q^\times\) for every odd prime \(q\). For this choice of \(R\), we have \(\mathbb{Z}_q^\times \subseteq \theta(O^+(R_q))\) for every odd \(q\), by Lemma 1.4, and

\[
n(R_2 \cap N_2) = n(\mathbb{Z}_2[2\nu, 2e_1, 2e_3]) = 8\mathbb{Z}_2.
\]

We will consider the two possible cases for \(a\), namely \(a = 0\) and \(a = 1\).

If \(a = 0\), then \(Q(\nu) = \frac{2\beta + 2\gamma c^2}{4}\), so \(j = 1\) and \(c = 1\). Therefore,

\[
R_2 \cong \begin{bmatrix} 2\epsilon & 0 & 2\gamma \\ 0 & 8\eta & 0 \\ 2\gamma & 0 & 8\gamma \end{bmatrix}
\]

and

\[
R_2^1 \cong \begin{bmatrix} \epsilon & \gamma \\ \gamma & 4\gamma \end{bmatrix} \perp \langle 4\eta \rangle \cong \langle \epsilon, \epsilon(4\epsilon\gamma - \gamma^2), 4\eta \rangle,
\]

which contains the binary sublattice \(\epsilon(1, -1+4\epsilon\gamma)\). So \(R_2^1\) represents every unit in \(\mathbb{Z}_q^\times\), and hence \(R_2\) represents every element in \(2\mathbb{Z}_q^\times\). Therefore, \(\mathbb{Z}_2^\times \subseteq \theta(O^+(R_2))\), and hence \(\mathbb{Z}_q^\times \subseteq \theta(O^+(R_q))\) for every prime \(q\). Since \(2\epsilon + 2^\alpha n\) is represented primitively by \(\text{gen}(R)\) and \(\text{gen}(R)\) has only one spinor genus, \(2\epsilon + 2^\alpha n\) is represented by \(R\) itself if \(n\) sufficiently large, by Theorem 1.2. Since \(n(R_2 \cap N_2) = 8\mathbb{Z}_2\), any representation of \(2\epsilon + 2^\alpha n\) by \(R\) must come from the coset \(\nu + N\). Therefore, we may conclude that \(H(x)\) is almost universal.
If \( a = 1 \), then \( i = 1 \) and \( b = 1 \). Therefore,

\[
R_2 \cong \begin{bmatrix}
2\epsilon & 2\eta & B(\nu, e_3) \\
2\eta & 8\eta & 0 \\
B(\nu, e_3) & 0 & 2^{i+2}\gamma
\end{bmatrix},
\]

and \( R_2^1 \) contains the sublattice

\[
\begin{bmatrix}
\epsilon & \eta \\
\eta & 4\eta
\end{bmatrix} \cong \langle \epsilon, \epsilon(4\eta\epsilon - \eta^2) \rangle,
\]

which clearly represents every unit in \( \mathbb{Z}_2^\times \). Now the proof proceeds as above. \( \square \)

In what follows, define \( \bar{Q}(x) := \frac{1}{2}Q(x) \). Then, \( \bar{Q}(\nu) = \epsilon \in \mathbb{Z}_2^\times \). Furthermore, note that

\[
\bar{B}(\nu, x) = \frac{Q(\nu + x) - Q(\nu) - Q(x)}{2} = \frac{Q(\nu, x) - Q(\nu) - Q(x)}{4} = \frac{B(\nu, x)}{2},
\]

for any \( x \in \mathbb{N} \). We will let \( N \) denote the \( \mathbb{Z} \)-lattice with associated quadratic map \( Q \). Now, \( Q(x) = 2\bar{Q}(x) \) and \( B(\nu, x) = 2\bar{B}(\nu, x) \), so

\[
H(x) = \frac{2\bar{Q}(x) + 2(2\bar{B}(\nu, x))}{2^\alpha} = \frac{\bar{Q}(x) + 2\bar{B}(\nu, x)}{2^{\alpha-1}}.
\]

**Theorem 3.11.** Suppose that \( \alpha = 2 \), \( Q(\nu) \in 2\mathbb{Z}_2^\times \), and \( \mathbb{Z}_2 \neq B(\nu, N_2) \). Then, \( H(x) \) is almost universal if and only if \( N_q \) represents all \( q \)-adic integers whenever \( q \) is odd, and one of the following holds:

(1) \( 2s(N) = n(N) = 8\mathbb{Z} \);

(2) \( N_2 \) is diagonalizable, and \( \text{ord}_2(dN) = 6 \); or,
(3) $N_2$ is diagonalizable, $\text{ord}_2(dN) = 8$, and $B(\nu, N_2) = 4\mathbb{Z}_2$.

Proof. Let $\alpha = 2$, $Q(\nu) \in 2\mathbb{Z}_2^\times$, and suppose that $\mathbb{Z}_2 \neq B(\nu, N_2)$. Then $B(\nu, N_2) = 2\mathbb{Z}_2$ or $4\mathbb{Z}_2$. If $B(\nu, N_2) = 2\mathbb{Z}_2$, then $n(N) = 4\mathbb{Z}$ or $8\mathbb{Z}$, and if $B(\nu, N_2) = 4\mathbb{Z}_2$, then $n(N) = 4\mathbb{Z}$. In either case, $n(N) \subseteq 4\mathbb{Z}$, so any representation of $2\epsilon + 8n$ by $M$ is guaranteed to be from the coset $\nu + N$. Suppose that $N_q$ represents all $q$-adic integers for all odd primes $q$. Then, $\bar{N}_q$ represents all $q$-adic integers for every odd prime $q$. Under the given assumptions, $\bar{B}(\nu, \bar{N}_2) = \mathbb{Z}_2$ and $n(\bar{N}) = 2\mathbb{Z}$ or $4\mathbb{Z}$; or, $\bar{B}(\nu, \bar{N}_2) = 2\mathbb{Z}_2$ and $n(\bar{N}) = 2\mathbb{Z}$. Therefore, we are precisely in the situation described in Theorem 3.3. If (1) holds, then $2s(\bar{N}) = n(\bar{N}) = 4\mathbb{Z}$, so part (1) of Theorem 3.3 holds. If (2) holds, then $\bar{N}_2$ is diagonalizable, and $\text{ord}_2(d\bar{N}) = 3$, so part (2) of Theorem 3.3 holds. If (3) holds, then $\bar{N}_2$ is diagonalizable, $\text{ord}_2(d\bar{N}) = 5$, and $\bar{B}(\nu, \bar{N}_2) = 2\mathbb{Z}_2$, so part (3) of Theorem 3.3. Therefore, if (1), (2) or (3) holds, then $H(x)$ is almost universal by Theorem 3.3.

If (1), (2) and (3) all fail, then parts (1), (2) and (3) of Theorem 3.3 all fail, and therefore $H(x)$ is not almost universal.

For the following theorem, we define an integer $\lambda$ by

$$\lambda := \begin{cases} 1 & \text{if } \text{ord}_2(dN) \text{ is odd,} \\ 2 & \text{if } \text{ord}_2(dN) \text{ is even,} \end{cases}$$

and as usual, let $d\bar{N}'$ denote the odd part of the discriminant of $\bar{N}$.

**Theorem 3.12.** Suppose that $\alpha = 3$, $Q(\nu) \in 2\mathbb{Z}_2^\times$, and $\mathbb{Z}_2 \neq B(\nu, N_2)$. Then $H(x)$ is almost universal if and only if $N_q$ represents every $q$-adic integer in $\mathbb{Z}_q$ for all odd primes $q$, $N_2$ is diagonalizable, and one of the following holds:

1. $B(\nu, N_2) = 2\mathbb{Z}_2$, and
(a) \(\text{ord}_2(dN)\) is even; or,
(b) \(\text{ord}_2(dN) = 7\); or,
(c) \(\text{sf}(dN)\) is divisible by a prime congruent to \(3\) \(\text{mod}\) \(4\); or,
(d) \(N_2\) has a binary Jordan component with the square free part of its discriminant congruent to \(5\) \(\text{mod}\) \(8\).

(2) \(B(\nu, N_2) = 4\mathbb{Z}_2, \ n(G_2) = 8\mathbb{Z}_2\), where \(G_2\) is the orthogonal complement of \(\nu\) in \(N_2\), and

(a) \(\text{ord}_2(dN)\) is even; or,
(b) \(\text{ord}_2(dN) = 9\); or,
(c) \(\text{sf}(dN)\) is divisible by a prime congruent to \(3\) \(\text{mod}\) \(4\).

(3) \(\frac{\text{sf}(dN) - Q(\nu)}{8}\) is represented by \(H(x)\).

Proof. Let \(\alpha = 3\) and \(Q(\nu) \in \mathbb{Z}_2^\times\). Suppose that \(N_2\) is diagonalizable, and that \(N_q\) represents every \(q\)-adic integer in \(\mathbb{Z}_q\) for all odd primes \(q\). Then, \(\bar{N}_q\) represents all \(q\)-adic integers for all odd primes \(q\). By our initial assumptions, \(n(\nu, N) = 8\mathbb{Z}\) and \(B(\nu, N_2) = 2\mathbb{Z}_2\) or \(4\mathbb{Z}_2\). This implies that

\[
H(x) = \bar{Q}(x) + 2\bar{B}(\nu, x),
\]

where either \(\bar{B}(\nu, \bar{N}_2) = \mathbb{Z}_2\) and \(n(\bar{N}) = 2\mathbb{Z}\); or, \(\bar{B}(\nu, \bar{N}_2) = 2\mathbb{Z}_2\) and \(n(\bar{N}) = 4\mathbb{Z}\).

Therefore we are in the situation described in Theorem 3.9. Since \(N_q\) represents all \(q\)-adic integers for every odd prime \(q\), it follows that \(\mathbb{Q}(\sqrt{-tdN}) = \mathbb{Q}(\sqrt{-\lambda})\), by Lemma 1.6.

First, suppose that \(B(\nu, N_2) = 2\mathbb{Z}_2\) and hence \(\bar{B}(\nu, \bar{N}_2) = \mathbb{Z}_2\). If 1(a) holds, then \(\text{ord}_2(d\bar{N})\) is odd, and part 1(a) of Theorem 3.9 holds. If 1(b) holds, then
ord₂(dN) = 4, and part 1(b) of Theorem 3.9 holds. Suppose that 1(a) fails, and 1(c) holds. Then ord₂(dN) is odd, so \( \mathbb{Q}(\sqrt{-tdN}) = \mathbb{Q}(\sqrt{-1}) \) for any odd integer \( t \), and \( \text{sf}(dN) = 2\text{sf}(d\tilde{N}) \) is divisible by a prime congruent to 3 mod 4. Therefore, part 1(c) of Theorem 3.9 holds. If 1(d) holds, then \( \tilde{N}_2 \) has a binary Jordan component with the square free part of its discriminant congruent to 5 mod 8, and hence part 1(d) of Theorem 3.9 holds. Therefore, when any of these hold, \( H(x) \) is almost universal by Theorem 3.9.

Suppose that \( B(\nu, N_2) = 4\mathbb{Z}_2 \), and hence \( n(N) = 8\mathbb{Z} \). Then, \( \mathbb{Z}[2\nu] \cong (8\epsilon) \) splits \( N_2 \) as an orthogonal complement. Therefore,

\[ N_2 \cong (8\epsilon) \perp G_2, \]

and hence

\[ \tilde{N}_2 \cong (4\epsilon) \perp \tilde{G}_2, \]

where \( \tilde{G}_2 := G_2^{\frac{1}{2}} \). Suppose that \( n(G_2) = 8\mathbb{Z}_2 \), meaning that \( n(G_2) = 4\mathbb{Z} \). If 2(a) holds, then ord₂(dN) is odd, and hence part 2(a) of Theorem 3.9 holds. If 2(b) holds, then ord₂(dN) = 6, implying that part 2(b) of Theorem 3.9 holds. If 2(a) fails and 2(c) holds, then \( \mathbb{Q}(\sqrt{-tdN}) = \mathbb{Q}(\sqrt{-1}) \), and \( \text{sf}(dN) \) is divisible by a prime congruent to 3 mod 4, and consequently part 2(c) of Theorem 3.9 holds. Therefore, when any of these hold, \( H(x) \) is almost universal by Theorem 3.9.

Suppose that (1) and (2) fail. Then ord₂(dN) is odd, and therefore

\[ \frac{\text{sf}(dN) - Q(\nu)}{8} = \frac{2\text{sf}(d\tilde{N}) - 2\epsilon}{8} = \frac{\text{sf}(d\tilde{N}) - \epsilon}{4}. \]

If (3) holds, then \( \frac{\text{sf}(d\tilde{N}) - \epsilon}{4} \) is represented by \( H(x) \) and therefore \( H(x) \) is almost universal, by Theorem 3.9 part (3).
For the other direction, suppose that $N_2$ is not diagonalizable. Then $\bar{N}_2$ is not diagonalizable, and therefore $H(x)$ is not almost universal by Theorem 3.5. Furthermore, if (1)-(3) all fail, then $H(x)$ is not almost universal by Theorem 3.9 part (3).

\[3.3\quad Q(\nu) \in 4\mathbb{Z}_2^\times\]

Finally, we treat the case $Q(\nu) \in 4\mathbb{Z}_2^\times$, and without confusion, let $Q(\nu) := 4\epsilon$ where $\epsilon \in \mathbb{Z}_2^\times$. According to Corollary 1.5, we must assume that $\alpha = 3$, or else $H(x)$ is not almost universal. In principle, we need to consider cases for $8\mathbb{Z}_2 \subseteq B(\nu, N_2) \subseteq \mathbb{Z}_2$, but in the following lemma, we establish that the case $B(\nu, N_2) = \mathbb{Z}_2$ will not occur when $N_2$ is anisotropic.

**Lemma 3.13.** Let $\alpha = 3, Q(\nu) \in 4\mathbb{Z}_2^\times$. If $N_2$ is anisotropic, then $B(\nu, N_2) \subseteq 2\mathbb{Z}_2$.

**Proof.** Let $\alpha = 3, Q(\nu) \in 4\mathbb{Z}_2^\times$, and suppose that $N_2$ is anisotropic. For the sake of contradiction, suppose that $B(\nu, N_2) = \mathbb{Z}_2$. Since $Q(x) + 2B(\nu, x) \equiv 0 \mod 8$ for every $x \in N$, this implies that $n(N_2) = 2\mathbb{Z}_2$. If $N_2$ is not diagonalizable, then

\[N_2 \cong 2^i\langle \eta \rangle \perp 2^jA\]

in a basis $\{e_1, e_2, e_3\}$, with $i \neq j$. If $i < j$, then $i = 1$ and $j > 1$. Letting $\nu = \frac{ae_1^2 + be_2^2 + ce_3^2}{2}$, we have

\[Q(\nu) = \frac{2\eta a^2 + 2^{i+1}(b^2 + bc + c^2)}{4},\]

and hence $a \in 2\mathbb{Z}$. Therefore, $B(\nu, e_1) \in 2\mathbb{Z}$, $B(\nu, e_2) = \frac{2^{i}(2b+c)}{2} \in 2\mathbb{Z}$ and
$B(\nu,e_3) = \frac{2^{l(2c+b)}}{2} \in 2\mathbb{Z}$. If $j < i$, then $j = 0$, and

$$N_2 \cong \langle 2^i \eta \rangle \perp A.$$ 

Since $n(\nu,N) = 8\mathbb{Z}$, we know that $Q(e_2) + 2B(\nu,e_2)$ and $Q(e_3) + 2B(\nu,e_3)$ are elements in $8\mathbb{Z}$. But clearly $Q(e_2 + e_3) + 2B(\nu,e_2 + e_3)$ is not in $8\mathbb{Z}$, which is a contradiction. Therefore, if $N_2$ is anisotropic and not diagonalizable, then $B(\nu,N_2) \subset 2\mathbb{Z}_2$.

Suppose that $N_2$ is diagonalizable. Then

$$N_2 \cong \langle 2\eta, 2^i \beta, 2^j \gamma \rangle$$

in a basis $\{e_1,e_2,e_3\}$. Let $\nu = \frac{ae_1 + be_2 + ce_3}{2}$ with $0 \leq a, b, c \leq 1$ by Remark 1.8 and Lemma 3.2. Then,

$$Q(\nu) = \frac{2\eta a^2 + 2^i \beta b^2 + 2^j \gamma c^2}{4},$$

with $i \leq j$. If $a = 1$, then $i = 1$ and $b = 1$. Therefore

$$Q(\nu) = \frac{2\eta + 2\beta + 2^j \gamma}{4} = 4\epsilon$$

and so

$$\eta + \beta + 2^{j-1} \gamma = 8\epsilon.$$ 

If $j > 3$, then $\eta + \beta \equiv 0 \mod 8$, and then $N_2$ is isotropic which is a contradiction. If $j = 3$, then $\eta + \beta \equiv 4 \mod 8$, and therefore

$$N_2 \equiv 2\langle \eta, \beta, 4\gamma \rangle,$$
which is also isotropic. If $a = 0$, then $B(\nu, e_1) = 0$, and

$$Q(\nu) = \frac{2^i\beta b + 2^i\gamma c}{4}.$$ 

If $b = 0$, then $c = 1$ and $j = 4$. In this case, $B(\nu, e_2) = 0$ and $B(\nu, e_3) = 4\gamma$. If $b = 1$, then $j = 1$, $c = 1$, and $\beta + \gamma \equiv 0 \mod 8$. But now $N_2$ is isotropic, which is a contradiction. □

In what follows, $\bar{Q}$ is defined as in the previous section. Additionally, we define $\bar{\bar{Q}} := \frac{1}{4}Q$, and let $\bar{\bar{N}}$ be the quadratic $\mathbb{Z}$-lattice with associated quadratic map $\bar{\bar{Q}}$ and bilinear map $\bar{\bar{B}}$. Then,

$$H(x) = \frac{Q(x) + 2B(\nu, x)}{2^\alpha} = \frac{\bar{\bar{Q}}(x) + 2\bar{\bar{B}}(\nu, x)}{2^{\alpha-1}} = \frac{\bar{\bar{Q}}(x) + 2\bar{\bar{B}}(\nu, x)}{2^{\alpha-2}}$$

**Theorem 3.14.** Suppose that $\alpha = 3$, $Q(\nu) \in 4\mathbb{Z}_2^\times$, and $\mathbb{Z}_2 \neq B(\nu, N_2)$. Then $H(x)$ is almost universal if and only if $N_q$ represents every $q$-adic integer in $\mathbb{Z}_q$ for all odd primes $q$, and one of the following holds:

(1) $B(\nu, N_2) = 2\mathbb{Z}_2$, and $N_2$ is diagonalizable,

(2) $B(\nu, N_2) \subseteq 4\mathbb{Z}_2$, and

(a) $2s(N) = n(N) = 16\mathbb{Z}$; or,

(b) $N_2$ is diagonalizable and $\text{ord}_2(dN) = 9$; or,

(c) $N_2$ is diagonalizable, $\text{ord}_2(dN) = 11$, and $B(\nu, N_2) = 8\mathbb{Z}$.

**Proof.** Let $\alpha = 3$, $Q(\nu) \in 4\mathbb{Z}_2^\times$. Suppose that $B(\nu, N_2) = 2\mathbb{Z}_2$, and hence $n(N) = 4\mathbb{Z}$. Then, $\bar{B}(\nu, \bar{N}_2) = \mathbb{Z}_2$ and $n(\bar{N}) = 2\mathbb{Z}$. Furthermore, $\bar{Q}(\nu) = 2\bar{e}$ and $n(Q(x) + 2B(\nu, x)) = 4\mathbb{Z}$. Therefore, we are in the situation described in Theorem 3.10, and $H(x)$ is almost universal if and only if $N_2$ is diagonalizable.
Suppose that $B(\nu, N_2) = 4\mathbb{Z}_2$, and therefore $n(N) = 8\mathbb{Z}$ or $16\mathbb{Z}$. Then, $$\tilde{B}(\nu, \tilde{N}_2) = \mathbb{Z}_2, \quad n(\tilde{N}) = 2\mathbb{Z} \text{ or } 4\mathbb{Z},$$ and $\tilde{Q}(\nu) = \epsilon$. Suppose that $B(\nu, N_2) = 8\mathbb{Z}_2$, and therefore $n(N) = 8\mathbb{Z}$. Then, $$\tilde{B}(\nu, \tilde{N}_2) = 4\mathbb{Z}_2, \quad n(\tilde{N}) = 4\mathbb{Z},$$ and $\tilde{Q}(\nu) = \epsilon$. In both cases we are in the situation described in Theorem 3.3, and $H(x)$ is almost universal if and only if part (2) holds.

\qed
Chapter 4

Mixed Sums of Squares and Triangular Numbers

In this chapter, we discuss how the results of this thesis imply the results of [1], [2], and hence Kane and Sun’s Conjecture [14, Conjecture 1.19]. Throughout this chapter, \(a, b, c\) are positive odd integers, and \(m, r, s\) are non-negative integers.

In what follows, let \(N \cong \langle 2^{m+3}a, 2^{r+2}b, 2^{s+2}c \rangle\) in a basis \(\{e_1, e_2, e_3\}\). Let \(\nu = \frac{e_2 + e_3}{2}\), and for any vector \(\vec{x} = xe_1 + ye_2 + ze_3\), define

\[
H(\vec{x}) := \frac{Q(\vec{x}) + 2B(\nu, \vec{x})}{8} = 2^m ax^2 + 2^r b(y + 1) + 2^s c(z + 1);
\]

or equivalently, \(H(\vec{x}) = 2^m ax^2 + 2^r bTy + 2^s cTz\), a mixed sum of one square and two triangular numbers. The results in [1] give necessary and sufficient conditions for mixed sums of this form to be almost universal, addressing all possibilities for \(a, b, c, m, r,\) and \(s\), case-by-case. In the following theorem, which is Theorem 3.3 of [1], we treat the case where \(r = 0\) and \(m, s > 0\).

**Theorem 4.1.** Suppose that \(m > 0\) and \(s > 0\). Then \(2^m ax^2 + bTy + 2^s cTz\) is
almost universal if and only if $2^{m+1}ax^2 + by^2 + 2^s cz^2$ represents all $q$-adic integers over $\mathbb{Z}_q$ for every odd $q$, and one of the following holds:

(a) $m$ is even and $s = 1$ or 2; or, $m = 1$ and $s$ is odd;

(b) $sf(abc)$ is divisible by a prime $q$ for which $\left(\frac{-\kappa}{q}\right) = -1$, where $\kappa = 1$ or 2 when $s + m$ is odd or even accordingly;

(c) $(b + 2^s c)sf(abc) \not\equiv 1 \mod 8$;

(d) $\frac{sf(abc) - (b + 2^s c)}{8}$ is represented by $2^m ax^2 + bTy + 2^s Tz$.

Proof. Suppose that $2^{m+1}ax^2 + by^2 + 2^s cz^2$ represents all $q$-adic integers over $\mathbb{Z}_q$ for every odd $q$. Then $a, b$ and $c$ are pairwise coprime. Since $q$ is odd, this is the same as supposing that $N_q$ represents all $q$-adic integers over $\mathbb{Z}_q$. Now, $Q(\nu) = b + 2^s c \in \mathbb{Z}_2^\times$ and

$$N_2 \cong \langle 4b, 2^{s+2} c, 2^{m+3} a \rangle,$$

with $\text{ord}_2(dN) = m + s + 7$. Therefore, we are precisely in the setting of Theorem 3.8. The orthogonal compliment of $\nu$ in $N_2$ is

$$G_2 \cong \langle (b + 2^s c)2^{s+2}bc, 2^{m+3}a \rangle,$$

with $dG_2 = 2^{m+s+5}abc$, and $n(G_2) = 8\mathbb{Z}_2$ if and only if $s = 1$.

Suppose that part (a) holds. First, suppose that $m$ is even, and either $s = 1$ or $s = 2$. In the first case $n(G_2) = 8\mathbb{Z}_2$ and $\text{ord}_2(dN)$ is even, implying that part (1) of Theorem 3.8 holds. In the second case case $n(G_2) = 16\mathbb{Z}_2$ and $\text{ord}_2(dN)$ odd, implying that part (2) of Theorem 3.8 holds. Next, suppose that $m = 1$ and $s$ is odd. If $s = 1$, then $n(G_2) = 8\mathbb{Z}_2$, and $\text{ord}_2(dN) = 9$, implying that part (1)
of Theorem 3.8 holds. If \( s > 1 \) is odd, then \( n(G_2) = 16\mathbb{Z}_2 \), and \( \text{ord}_2(dN) \) is odd, which implies that part (2) of theorem 3.8 holds. Therefore, when (a) holds, then either (1) or (2) from Theorem 3.8 holds.

Define an integer \( \kappa \) by

\[
\kappa := \begin{cases} 
1 & \text{if } s + m \text{ is odd,} \\
2 & \text{if } s + m \text{ is even.}
\end{cases}
\]

Immediately, we observe that \( s + m \) is odd if and only if \( \text{ord}_2(dN) \) is even, and therefore \( \kappa \) is precisely the integer \( \delta \) defined in Section 3.1. From here, it is clear that part (b) is an equivalent condition to part (3) in Theorem 3.8. Moreover, it is clear that parts (c) and (d) are equivalent to parts (4) and (6) of Theorem 3.8, respectively.

Therefore, if any one of (a) – (d) holds, then one of (1), (2), (3), (5) or (6) of Theorem 3.8 holds, which implies that \( 2^m a x^2 + b T y + 2^s c T_z \) is almost universal.

Conversely, suppose that parts (a) – (d) all fail. If (a) fails and \( n(G_2) = 8\mathbb{Z}_2 \), then \( s = 1 \), implying that \( m > 1 \) is odd, which means that \( \text{ord}_2(dN) > 9 \) is odd, and hence part (1) of Theorem 3.8 fails. If part (a) fails and \( n(G_2) = 16\mathbb{Z}_2 \), then \( s = 2 \), implying that \( m \) is odd, and therefore \( \text{ord}_2(dN) \) is even and consequently part (2) of Theorem 3.8 fails. It is clear that if parts (b), (c) or (d) fail, then parts (3), (4) and (6) of Theorem 3.8 fail, respectively. Furthermore, when \( s = 1 \), the failure of (c) implies that \( abc \equiv b + 2c \mod 8 \), and the failure of (b) implies that \( \text{sf}(abc) \) is only divisible by primes which are congruent to 1, 3 \( \mod 8 \), and hence \( a, b, c \equiv 1, 3 \mod 8 \). Recall also that \( a, b \) and \( c \) are pairwise coprime. Therefore, \( b + 2c \equiv 1, 3 \mod 8 \), with \( b, c \equiv 1, 3 \mod 8 \). From this, we conclude that \( b \equiv c \mod 8 \), since \( 1 + 2(3) \equiv 7 \mod 8 \) and \( 3 + 2(1) \equiv 5 \mod 8 \). So, when (c) fails,
and $s = 1$, then $8(b + 2c)$ is represented by

$$G_2 \cong \langle 8(b + 2c)bc, 2^{m+3}a \rangle \equiv \langle 8(b + 2c), 2^{m+3}c \rangle,$$

implying that (5) fails. Therefore, when (a) – (d) all fail, then (1) – (6) all fail, and consequently $2^m ax^2 + bT_y + 2^s cT_z$ is not almost universal.

In the situation above, we observe that condition (5) of Theorem 3.8 is always false. This is not true in general, as we will illustrate in the next theorem, which is Theorem 3.1 from [1]. In what follows, $N$ and $\nu$ are as above, but now $m = 0$.

**Theorem 4.2.** Suppose that $s \geq 1$. Then $ax^2 + bT_y + 2^s cT_z$ is almost universal if and only if $2^m ax^2 + bT_y + 2^s cT_z$ represents all $q$-adic integers over $\mathbb{Z}_q$ for every odd $q$, and one of the following holds:

(a) $s$ is odd, or $s = 2$;

(b) $sf(abc)$ is divisible by a prime $q \equiv 5, 7 \pmod{8}$;

(c) $bc \not\equiv 1 \pmod{8}$;

(d) $\frac{sf(abc) - (b + 2^s c)}{8}$ is represented by $ax^2 + bT_y + 2^s cT_z$.

**Proof.** Suppose that $N_q$ represents all $q$-adic integers over $\mathbb{Z}_q$ for every odd $q$. Since $Q(\nu) = b + 2^s c \in \mathbb{Z}_2^x$, we are in the situation of Theorem 3.8, and

$$G_2 \cong \langle 2^{s+2}(b + 2^s c)bc, 8a \rangle.$$

Therefore, $n(G_2) = 8\mathbb{Z}_2$, and consequently part (2) of Theorem 3.8 will always fail.
Suppose that part (a) holds, so either $s$ is odd, or $s = 2$. If $s$ is odd then $\text{ord}_2(dN) = s + 7$ is even, and if $s = 2$ then $\text{ord}_s(dN) = 9$. Therefore, when part (a) holds, then part (1) of Theorem 3.8 holds.

Suppose that (a) fails, then $s > 2$ is even and $Q(\nu) \equiv b \mod 8$. If (b) holds, then $\text{sf}(abc)$ is divisible by a prime $q \equiv 5, 7 \mod 8$. Equivalently, $\text{sf}(abc)$ is divisible by a prime $q$ such that $\left(\frac{-2}{q}\right) = -1$. Therefore (b) implies part (3) of Theorem 3.8.

Suppose that (a) and (b) fail, and (c) holds. Then, $s > 2$ is even, and

$$G_2 \cong \langle 2^{s+2}(b + 2^s c)bc, 8a \rangle \cong \langle 2^{s+2}c, 8a \rangle.$$  

It is clear that $b$ is represented by $\langle a, 2^{s-1}c \rangle$, if and only if $a \equiv b \mod 8$. If (c) holds and $a \not\equiv b \mod 8$, then $8b$ is not represented by $G_2$, implying that part (5) of Theorem 3.8 holds. If $a \equiv b \mod 8$, then $abc \not\equiv b \mod 8$ (since $bc \equiv 3 \mod 8$), implying that part (4) from Theorem 3.8 holds.

Clearly part (d) is an equivalent condition to part (6) of Theorem 3.8. Therefore, when one of (a) – (d) holds, then one of (1), (3), (4), (5) or (6) holds, which implies that $ax^2 + bT_y + 2^s cT_z$ is almost universal.

Conversely, suppose that (a), (b), (c) and (d) all fail. If (a) fails, then $s > 2$ is even, implying that $\text{ord}_2(dN) > 9$ is odd, and hence part (1) of Theorem 3.8 fails. The failure of (b) implies that any prime $q$ dividing $\text{sf}(abc)$ is congruent to 1, 3 mod 8, which means that (3) fails. Furthermore, from the failure of (b), we know that $a, b, c \equiv 1, 3 \mod 8$, and it is a consequence of our initial assumption that $a, b$ and $c$ are pairwise coprime. If (c) fails, then $bc \equiv 1 \mod 8$, and consequently $Q_2N_2 \cong [2a, b, b]$. If $a \not\equiv b \mod 8$, then $Q_2N_2$ is isotropic, which cannot happen, this means that $a \equiv b \mod 8$. But now, $8b$ is represented by $G_2$ by the argument.
above, and hence part (5) of Theorem 3.8 fails. Moreover, in this case \( ab \equiv b \mod 8 \), which implies that part (4) of Theorem 3.8 fails. It is clear that the failure of part \((d)\) implies the failure of part \((6)\) of Theorem 3.8. Therefore, when \((a) - (d)\) all fail, then \((1) - (6)\) of Theorem 3.8 all fail, and hence \( ax^2 + bTy + 2^s cTz \) is not almost universal.

Allowing \( r = 1 \), we get \( Q(\nu) = 2b + 2^s c \), which is no longer a 2-adic unit. By arguments similar to those above, we see that Theorem 3.12 implies the following result, which is Theorem 3.2 from [1].

**Theorem 4.3.** Suppose \( s \geq 1 \) and \( \text{ord}_2(b+c) = 1 \) if \( s = 1 \). Then \( ax^2 + 2bTy + 2^s cTz \) is almost universal if and only if \( 2ax^2 + 2by^2 + 2^s cz^2 \) represents all \( q \)-adic integers over \( \mathbb{Z}_q \) for every odd \( q \), and one of the following holds:

(a) \( s \) is even or \( s = 1 \);

(b) \( sf(abc) \) is divisible by a prime \( p \equiv 3 \mod 4 \);

(c) \( \frac{sf(abc) - (b+2^{s-1}c)}{4} \) is represented by \( ax^2 + 2bTy + 2^s cTz \).

Without giving a detailed proof, we observe that in this situation

\[
N_2 \cong \langle 8a, 8b, 2^{s+2}c \rangle,
\]

so in particular, \( B(\nu, N_2) = 4\mathbb{Z}_2 \), and \( n(G_2) = 8\mathbb{Z}_2 \), since \( m = 0 \). It is left to the reader to verify that Theorem 4.3 is a consequence of Theorem 3.12.

Lastly, when \( m \geq 0 \), and \( r = s = 0 \), we have the following result, which is Theorem 3.4 from [1].
Theorem 4.4. Suppose that \( m \geq 0 \) and \( \text{ord}_2(b + c) \leq 2 \). Then \( 2^m a x^2 + b T_y + c T_z \) is almost universal if and only if \( 2^{m+1} a x^2 + by^2 + cz^2 \) represents all \( q \)-adic integers over \( \mathbb{Z}_q \) for every odd \( q \), and one of the following holds:

1. \( \text{ord}_2(b + c) = 2 \);
2. \( m \) is odd or \( m = 0 \);
3. \( \text{sf}(abc) \) is divisible by a prime \( p \equiv 3 \mod 4 \);
4. \( \frac{(b+c)}{2} \text{sf}(abc) \not\equiv 1 \mod 4 \);
5. \( \frac{2\text{sf}(abc)-(b+c)}{8} \) is represented by \( ax^2 + 2bT_y + 2cT_z \).

When \( \text{ord}_2(b + c) = 1 \), Theorem 4.4 is implied by Theorem 3.12. If \( \text{ord}_2(b + c) = 2 \), then Theorem 4.4 is implied by Theorem 3.14. Once again we leave all the details to the reader, noting that they are similar to those in the preceding paragraphs.

Letting \( N \cong \langle 4a, 2^{r+2}b, 2^{s+2}c \rangle \) in the basis \( \{e_1, e_2, e_3\} \), and \( \nu = \frac{e_1+e_2+e_3}{2} \), the results from [2] are obtained in a similar manner. For any vector \( \bar{x} = xe_1 + ye_2 + ze_3 \) in \( N \), define

\[
H(\bar{x}) := Q(\bar{x}) + 2B(\nu, \bar{x}) = aT_x + 2bT_y + 2cT_z,
\]

which is a weighted sum of three triangular numbers. Theorems 3.1-3.6 can now be obtained from Theorems 3.12, 3.14, 3.8 using the same argument as above.

Letting \( N \cong \langle 2^{m+3}a, 2^{r+3}b, 2^{s+2}c \rangle \) in the basis \( \{e_1, e_2, e_3\} \), and \( \nu = \frac{e_1}{2} \), the results from [14] can be obtained. For any vector \( \bar{x} = xe_1 + ye_2 + ze_3 \) in \( N \), define

\[
H(\bar{x}) := \frac{Q(\bar{x}) + 2B(\nu, \bar{x})}{8} = 2^m a x^2 + 2b y^2 + 2c T_z,
\]
a weighted sum of two squares and one triangular number. In particular, com-
bining the arguments above with Corollary 3.5 of [1] and Corollary 3.7 of [2],
Conjecture 1.19 can be confirmed affirmatively by the results in this thesis.
Appendix A

Computing Spinor Norms

A.1 Non-dyadic Case

Suppose that a lattice $M$ has the following Jordan splitting into 1-dimensional components at an odd prime $p$,

$$M_p = M_{(1)} \perp \cdots \perp M_{(n)}.$$

Recall from chapter 2, that $n(M_{(k)})$ is the $\mathbb{Z}_p$-ideal generate by $Q(M_{(k)})$, where $Q$ is the quadratic map associated to $M_{(k)}$. We use the following result to compute the spinor norm of $M_p$ for odd primes $p$ (see [16, Satz 3]).

Theorem A.1. Let $m_k$ be the set of numbers $Q(a_k)$ with $a_k$ from $M_{(k)}$, that are not divisible by $pn(M_{(k)})$, and let $m(M_p)$ be the set of numbers that can be written as an even number of factors from $\cup_k m_k$. Then, $\theta(O^+(M_p)) = m(M_p)\mathbb{Q}_p^\times$. 

83
A.2 Dyadic Case

The following sections state the results necessary to compute spinor norms for non-modular lattices. The results regarding modular lattices, which are not essential to this dissertation, can be found in [10].

A.2.1 Binary Case

Suppose that $M$ is a binary lattice with the following Jordan splitting at the prime 2,

$$M_2 \cong \langle 1 \rangle \perp \langle 2^r \alpha \rangle$$

where $r \geq 1$, and $\alpha \in \mathbb{Z}_2^\times$. Then we use the following result to compute the spinor norm of $M_2$ (see [5, 1.9]).

Theorem A.2. For a binary lattice $M_2$,

$$\theta(O^+(M_2)) = \begin{cases} 
\{ \gamma \in \mathbb{Q}_2^\times : (\gamma, 2\alpha) = +1 \} & \text{if } r = 1, 3 \\
\{ \gamma \in \mathbb{Z}_2^\times \mathbb{Q}_2^{x_2} : (\gamma, -\alpha) = +1 \} & \text{if } r = 2 \\
\{ \mathbb{Q}_2^\times \cup \alpha \mathbb{Q}_2^{x_2} \cup 5 \mathbb{Q}_2^{x_2} \cup 5\alpha \mathbb{Q}_2^{x_2} \} & \text{if } r = 4 \\
\{ \mathbb{Q}_2^\times \cup 2^r \alpha \mathbb{Q}_2^{x_2} \} & \text{if } r \geq 5.
\end{cases}$$

A.2.2 Higher Dimensional Case

1-Dimensional Components

Suppose that a lattice $M$ has the following Jordan splitting into 1-dimensional components at the prime 2,

$$M_2 \cong \langle 1 \rangle \perp \langle 2^{r_2} \alpha_2 \rangle \perp \cdots \perp \langle 2^{r_n} \alpha_n \rangle$$
where \( r_i \in \mathbb{Z} \) and \( \alpha_2, ..., \alpha_n \in \mathbb{Z}_2^\times \), and \( r_k < r_{k+1} \) for \( k = 1, ..., n+1 \), \( r_0 = 0 \). A lattice is said to be of Type E if there is at least one index \( k \) for which \( r_{k+1} - r_k = 1 \) or 3, and at the same time there are some \( s, t \) with \( r_s - r_t = 2 \) or 4. Then we have the following theorem regarding lattices of Type E (see [6, 1.1]).

**Theorem A.3.** If \( M_2 \) is of Type E, then \( \theta(O^+(M_2)) = \mathbb{Q}_2^\times \).

When \( M_2 \) is not of Type E, then we obtain a (non-Jordan) splitting of \( M_2 \), as described in [6], and use [6, Theorem 2.7] to compute the spinor norm of \( M_2 \).

**Arbitrary Components**

Suppose that a lattice \( M \) has the following Jordan splitting at the prime 2,

\[
M_2 \cong M_{(1)} \perp 2^{r_2}M_{(2)} \perp \cdots \perp 2^{r_n}M_{(n)},
\]

where each \( M_{(i)} \) is unimodular of arbitrary rank, and \( 0 = r_1 < r_2 < \cdots < r_n \) are natural numbers. A lattice \( M \) has even order if \( \text{ord}_2(Q(x)) \) is even for every primitive vector \( x \in M \) which gives rise to an integral symmetry of \( M \), and \( M \) is of odd order when such \( x \)'s have \( \text{ord}_2(Q(x)) \) odd.

We use the following theorem to compute spinor norms for lattices of this form (see [6, 1.2]).

**Theorem A.4.**

1. If at least one Jordan component has rank \( \geq 3 \), then \( \theta(O^+(M_2)) \neq \mathbb{Q}_2^\times \) (in fact, \( = \mathbb{Z}_2^\times \mathbb{Q}_2^\times \)) if and only if \( 2^{r_k}M_{(k)} \) has odd order for \( k = 1, ..., n \).

2. If \( \text{rank}(M_{(k)}) \leq 2 \) for every \( k \) of which at least one component, say \( M_{(k_0)} \) is binary, then \( \theta(O^+(M_2)) \neq \mathbb{Q}_2^\times \) if and only if any one of the following three cases occur:
(a) all Jordan components have odd order;

(b) all Jordan components have even order;

(c) whenever \( \text{rank}(M_{(k)}) = 2 \), then \( M_{(k)} = \begin{bmatrix} \alpha & 1 \\ 1 & 2\beta \end{bmatrix} \) where \( \alpha, \beta \in \mathbb{Z}_2^\times \);

moreover

i. the associated spaces of all binary components are isometric,

ii. for any unary component, say \( M_{(j)} \cong \langle c_j \rangle \) where \( c_j \in \mathbb{Z}_2^\times \), the Hilbert symbol \( (2^{r_j-r_k}, \alpha c_j, -\det(M_{(j)})) = +1 \), and

iii. \( r_{k+1} - r_k \geq 4 \) for \( k = 1, \ldots, n-1 \).

Finally, in cases (a) and (b), \( \theta(O^+(M_2)) = \mathbb{Z}_2^\times \mathbb{Q}_2^{\times 2} \); and in case (c) it is equal to \( \theta(O^+(M_{(k_0)})) = \{ \gamma \in \mathbb{Q}_2^\times : (\gamma, -\det(M_{(k_0)})) = +1 \} \).

A.3 Computing Relative Spinor Norms

Following the notation given in chapter 2, we let \( E \) denote the quadratic extension \( \mathbb{Q}(\sqrt{-tdM}) \). Let \( p \) denote the extension of \( p \) to \( E \), and let \( \pi \) be a prime element in \( \mathbb{Z}_p \). As usual, \( \mathfrak{N}_p(E) \) denotes the group of local norms from \( E_p \) to \( \mathbb{Q}_p \). To compute the relative spinor norm at an odd prime in the ternary case, we rely on the following theorem (see [7, Theorem 1]).

Theorem A.5. Let \( p \) be nondyadic and \( -tdM \notin \mathbb{Q}_p^{\times 2} \).

(a) Suppose that \( E_p/\mathbb{Q}_p \) is unramified. Then \( \theta(O^+(M_p)) \subseteq \mathfrak{N}_p(E) \) if and only if \( M_p \cong \langle b_1, \pi^{2r}b_2, \pi^{2s}b_3 \rangle \) with \( b_i \in \mathbb{Z}_p^\times \) for \( 0 \leq r \leq s \). In this case \( \theta^*(M_p, t) \neq \mathfrak{N}_p(E) \) if and only if \( r \neq s \), \( -b_1b_2 \in \mathbb{Q}_p^{\times 2} \) and \( t \in p^{2r+1} \).

(b) Suppose that \( E_p/\mathbb{Q}_p \) is ramified. If \( \theta(O^+(M_p)) \subseteq \mathfrak{N}_p(E) \) then \( M_p \cong \langle b_1, \pi^r b_2, \pi^s b_3 \rangle \).
with \( b_i \in \mathbb{Z}_p^\times \) and \( 0 < r < s \). In this case, \( \theta^*(M_p, t) \neq \mathfrak{N}_p(E) \) if and only if one of the following holds:

(i) \( r \) is even and \( t \in p^r \),
(ii) \( r \) is odd and \( t \in p^s \), except when \( t \in \pi^s b_3 \mathbb{Z}_p^\times \) and \( p \leq 5 \), in which case \( \theta^*(M_p, t) = \mathfrak{N}_p(E) \).

At the prime 2 we rely on the following theorem (see [7, Theorem 2]). Although the case where \( E/\mathbb{Q} \) is unramified at the prime 2 is included in the original theorem, it has been omitted here since it is not pertinent to the discussions in this thesis.

**Theorem A.6.** Let \( p \) be 2-adic and \(-tdM \notin \mathbb{Q}_p^\times\).

(a) Suppose that \( E_p/\mathbb{Q}_p \) is ramified and \( \text{ord}_p(-tdM) \) is even. Then \( \theta(O^+(M_p)) \subseteq \mathfrak{N}_p(E) \) implies that \( M_p \cong \langle b_1, 2^r b_2, 2^s b_3 \rangle \) with \( b_i \in \mathbb{Z}_p^\times \) for \( 0 \leq r \leq s \). When \( \theta(O^+(M_p)) \subseteq \mathfrak{N}_p(E) \) let \( K = \langle 2^r b_1, 2^r b_2, 2^s b_3 \rangle \) and \( K' = \langle 2^r b_1, 2^r b_2, 2^s b_3 \rangle \). Then, \( \theta^*(M_p, t) \neq \mathfrak{N}_p(E) \) if and only if one of the following holds:

(i) \( r \) is even \( \theta(O^+(K)) \notin \mathfrak{N}_p(E) \) and
   (1) \( r \neq s \), \( t \in p^{r-2} \),
   (2) \( r = s \), \( t \in p^r \);

(ii) \( r \) is even, \( \theta(O^+(K)) \subseteq \mathfrak{N}_p(E) \), \( \theta(O^+(K')) \nsubseteq \mathfrak{N}_p(E) \), and \( t \in p^r \);

(iii) \( r \) is even \( \theta(O^+(K)) \subseteq \mathfrak{N}_p(E) \), \( \theta(O^+(K')) \subseteq \mathfrak{N}_p(E) \) and \( t \in p^s \);

(iv) \( r \) is odd, and \( t \in p^{r-3} \).

(b) Suppose that \( E_p/\mathbb{Q}_p \) is ramified and \( \text{ord}_p(-tdM) \) is odd. Then \( \theta(O^+(M_p)) \subseteq \mathfrak{N}_p(E) \) implies that \( M_p \cong \langle b_1, 2^r b_2, 2^s b_3 \rangle \) with \( b_i \in \mathbb{Z}_p^\times \) for \( 0 < r < s \). When
\( \theta(O^+(M_p)) \subseteq \mathfrak{N}_p(E) \) let \( K = \langle 2^{r-3}b_1, 2^rb_2, 2^s b_3 \rangle \). Then, \( \theta^*(M_p, t) \neq \mathfrak{N}_p(E) \) if and only if one of the following holds:

(i) \( r \) is even, \( t \in p^{r-4} \),

(ii) \( r \) is odd, \( \theta(O^+(K)) \not\subseteq \mathfrak{N}_p(E), t \in p^{r-3} \),

(iii) \( r \) is odd, \( \theta(O^+(K)) \subseteq \mathfrak{N}_p(E), \) and \( t \in p^{s-2} \).


