Lord Kelvin’s Error? An Investigation into the Isotropic Helicoid

by

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Abstract

In a publication in 1871, Lord Kelvin, a notable 19th century scientist, hypothesized the existence of an isotropic helicoid. He predicted that such a particle would be isotropic in drag and rotation translation coupling, and also have a handedness that causes it to rotate. Since this work was published, theorists have made predictions about the motion of isotropic helicoids in complex flows. Until now, no one has built such a particle or quantified its rotation translation coupling to confirm whether the particle has the properties that Lord Kelvin predicted.

In this thesis, we show experimental, theoretical, and computational evidence that all conclude that Lord Kelvin’s geometry of an isotropic helicoid does not couple rotation and translation. Even in both the high and low Reynolds number regimes, Lord Kelvin’s model did not rotate through fluid. While it is possible there may be a chiral particle that is isotropic in drag and rotation translation coupling, this thesis presents compelling evidence that the geometry Lord Kelvin proposed is not one. Our evidence leads us to hypothesize that an isotropic helicoid does not exist.
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1.1 What is an Isotropic Helicoid?

In an 1871 essay titled “Hydrokinetic solutions and observations” published in *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, William Thomson, also known as Lord Kelvin (and henceforth referred to by this name), theorized the existence of an isotropic helicoid, a particle that is invariant in rotation but not in reflection:

“An isotropic helicoid may be made by attaching projecting vanes to the surface of a globe in proper positions; for instance, cutting at 45° each, at the middles of the twelve quadrants of any of the three great circles dividing the globe into eight quadrantal triangles. By making the globe and the vanes of light paper, a body is obtained ridged enough and light enough to illustrate by its motions through air the motions of an isotropic helicoid through an incompressible liquid. But curious phenomena, not deductible from the present investigation, will, no doubt, on account of viscosity, be observed.” [3]

A visualization of Lord Kelvin’s isotropic helicoid created by Luca Bifarale and Kristian Gustavsson is shown in Figure 1.1 [4].
1.1.1 Lord Kelvin’s Motivation

Lord Kelvin hypothesized the isotropic helicoid while studying the vortex theory of the atom. This theory focused on explaining the universe with equations of fluid mechanics. In the early 1850s, Kelvin had an influential encounter with Helmholtz’ work. Helmholtz was studying the forms of motion that friction produces in fluid, and found that vortex filaments or tubes placed in a frictionless fluid will form closed rings or may terminate in the bounding surface. In either case, vortex filaments or tubes cannot be either created or annihilated. This work exposed Lord Kelvin to the idea that the discreteness of matter could be accounted for as part of a continuum theory.

The vortex theory pictured atoms as small vorticies in an ideal or frictionless fluid, which Kelvin identified as the ether. To Kelvin, this accounted for natural phenomena such as elasticity. The beginning of Kelvin’s work, especially in his paper “On Vortex Motion”, he investigated the more technical solutions to hydrodynamic problems. Lord Kelvin needed hydrodynamics to quantitatively explain what a vortex is and how it could be responsible for what composed atoms. Since the vortex theory of atoms failed to explain gasses, gravity, electromagnetism, or internal constraints of atoms, it was abandoned by 1890. While abandoned, the theory has had lasting effects on the development of hydrodynamics [6], including Lord Kelvin’s circulation theorem. According to Kelvin’s circulation theorem, if a fluid is inviscid flow, meaning the
viscosity of the flow is equal to zero, the circulation, \( \Gamma \), along any fluid ring is a constant \[7\]. Mathematically, this means that \( \frac{d\Gamma(t)}{dt} = 0 \) \[8\]. While Lord Kelvin was studying hydrodynamics, he was trying to understand the motion of free solids through a liquid. In his analysis, Kelvin conceived of the simplest particle that showed rotation and translational coupling. He thought such a particle would be isotropic in drag and rotation translation coupling, and also have a handedness that causes it to rotate through fluid. \[3\]

### 1.1.2 Mentions of an Isotropic Helicoid in Literature

Since Kelvin’s first publication, there is no published record of a fabricated isotropic helicoid, but there are multiple references to Lord Kelvin’s work. In this section, we give an overview of a sample of the different mentions of an isotropic helicoid in major fluid mechanics textbooks and journal articles. The purpose of this section is to show that the isotropic helicoid has not been abandoned, after Lord Kelvin first hypothesized it.

In *Low Reynolds Number Hydrodynamics*, John Happel and Howard Brenner mention another geometry of the isotropic helicoid that is a tetrahedron with the edges as screws: “According to Larmor if we take a regular tetrahedron (or other regular solid) and replace the edges by skew bevel faces placed in such ways that when looked at from any corner they all slope the same way, we have an example of an isotropic helicoid. This would be the result if three plagiedral faces sloping the same way were imposed on each vertex of the tetrahedron. The first process probably gives the simplest form that a solid of this class can have. A form equivalent to the second is obtained by fixing four equal symmetrical screw-propellers on the surface of a sphere at the corners of an inscribed regular tetrahedron \[9\].

In an article in *The Quarterly Journal of Pure and Applied Mathematics*, James Joseph Sylvester, James Witbread and Lee Glaisher reference an isotropic helicoid it follows, therefore, that if a solid possesses the character of helicoidal symmetry about any two intersecting axes, it is an isotropic helicoid; and as a particular case, if it possesses the character of perfect symmetry about two intersecting axes, it possesses the perfect isotropic character of a sphere.” \[10\]

Most recently, Gustavsson and Biferale theoretically investigate the properties of an isotropic helicoid in a paper titled “Preferential sampling of helicity by isotropic helicoids” Their theoretical research shows that isotropic helicoids preferentially sample different helical regions in
laminar or chaotic advecting flows [1]. Gustavsson and Biferale chose to study the isotropic helicoid because “their internal orientation does not affect their translational motion, isotropic helicoids can be described in a lower phase-space dimension than spheroidal particles. This makes the dynamics of isotropic helicoids the simplest extension of the much studied case of small spherical particles.”

1.2 Motivations

We decided to try and build the theorized isotropic helicoid particle to see how Biferale and Gustavsson’s theory matched with an experimental result. Through our investigation, which began with a 3D printed version of the particle, shown below in Figure 1.1, we found conflicting results about the ability of the particle to couple rotation and translation. We then tried to build a model based on Lord Kelvin’s design, but since it was not sturdy enough, we modified the design to the cubic one described in Chapter 2. Once we studied the isotropic helicoid and observed no rotation, we wanted to further investigate the particle at low Reynolds number to understand if there was no rotation for Lord Kelvin’s exact specifications.

Another reason we want to study the isotropic helicoid is because it will help us build our foundation of what we know about non-spherical particles. We chose the type of particle because it is a rotating non-spherical particle. There has been a significant amount of research dedicated to low Reynolds number spherical particles. Extending knowledge of fluid mechanics to include non-spherical dynamics is important for many real world application, such as bio-locomotion, aerosol particle and icy clouds. In these cases, scientists usually study axisymmetric particles such as rods and discs. [11] By modeling the isotropic helicoid in laminar flow, we can better understand the dynamics of a complex system, while using the information about discs that we have.

1.3 Reynolds Number

The Reynolds number is an important dimensionless quantity in fluid mechanics that measures the ratio of fluid inertial forces to viscous force. It indicates whether a flow is laminar, turbulent, or transitional. Laminar flow is characterized by the absence of cross currents and swirls, and
represented by lower Reynolds number. Turbulent flow is characterized by chaotic changes in pressure and flow velocity and the presence of swirls. [12] We are interested in laminar flow, not turbulent flow, since we want to understand the properties of rotation of the object independent of outside forces. The turbulent fluid would be an external factor causing the object to rotate, one that is not inherent to the isotropic helicoid. Reynolds number is calculated by the following equation.

\[ Re := \frac{ud}{v} \]  

(1.1)

\( u \) is the free stream velocity
\( d \) is the diameter
\( v \) is the kinematic viscosity, which is \( 1.00 \times 10^{-6} \text{m}^2/\text{s} \) for water at 293K.

The Reynolds number is related to both diameter and velocity. In a medium with a specific kinematic viscosity, an object can have the same Reynolds number if the product of the velocity and the diameter remains constant.

### 1.4 Fundamentals of Rotation and Translation Coupling

Lord Kelvin’s proposed geometry of the isotropic helicoid has vanes attached around a sphere causing the helicoid to couple rotation and translation. To understand this coupling, we will discuss the mathematics behind quantifying the rotation and translation coupling tensor.

In the theoretical section, we will construct a mathematical model of an isotropic helicoid using twelve discs. To understand the dynamics of discs moving through low Reynolds number fluid, we must look at the resistance tensors. A resistance tensor is second rank tensor that relates velocity and force.

For a solid disc moving with translational velocity of \( \mathbf{U}(t) \) through an unbounded, quiescent viscous fluid in the creeping flow limit \( (Re << 1) \), we can find the resistance tensor from the relationship between the force and the velocity. The force, \( \mathbf{F} \) acting on the particle is related to the translational velocity, \( \mathbf{U} \), by a resistance tensor, \( \hat{A} \).

\[ \mathbf{F} = \hat{A} \cdot \mathbf{U} \]  

(1.2)
Since we want to understand what the difference is between the rotation of the great circles and the rotation of the isotropic helicoid, we are interested in the torque. To find the torque, $\mathbf{G}$, we take the cross product of the distance with the force \[7\].

\[
\mathbf{G} = \mathbf{r} \times (\hat{A} \mathbf{U}) \tag{1.3}
\]

\[
\mathbf{G} = \hat{C} \mathbf{U} \tag{1.4}
\]

Where $\hat{C}$ is the product of $\mathbf{r}$ and the second rank tensor, $\hat{A}$. We will call $\hat{C}$ the rotation translation coupling tensor. \[7\]

1.5 Overview

This thesis aims to present evidence of a video experiment, a computer simulation, and a mathematical model on how the proposed isotropic helicoid models do not couple rotation and translation. Although there is still work to be done in all the sections, the primary conclusion is consistent throughout. All the chapters show that there is no rotation translation coupling for Lord Kelvin’s isotropic helicoid geometry.

Chapter 2 provides video documentation of a high Reynolds number isotropic helicoid falling through air. We drop a cubic model of an isotropic helicoid made of six plastic, right handed propellers from multiple angles to quantify the rotation translation coupling of an isotropic helicoid. We compare the rotation translation coupling of the isotropic helicoid to the rotation translation coupling of a single propeller blade to understand how significant the difference in the coupling is.

Chapter 3 presents the results from a computer simulation of the torque on the isotropic helicoid in a low Reynolds number regime. We first measure the drag force on a sphere to understand the limitations of the simulation. Then, we construct Lord Kelvin’s model of the isotropic helicoid by orienting twelve discs around a the great circles of a sphere. We measure the torque acting on the particle in order to quantify the rotation translation coupling of the isotropic helicoid. We compare the rotation translation coupling of the isotropic helicoid to the rotation translation coupling of four discs around one great circle and eight discs around two great circles.
Chapter 4 presents a simplified mathematical model of the isotropic helicoid. The model measures the individual resistance tensors of twelve discs placed along the great circles of a sphere. We calculate the rotation translation coupling tensor from each rotated resistance tensor. Then, we add the rotation translation coupling tensors together to find the rotation translation coupling tensor of the isotropic helicoid.
Experiment

In this chapter, we present the model of the isotropic helicoid we constructed from plastic propellers and quantify the translation rotation coupling of both the isotropic helicoid and a single right handed propeller.

2.1 Experimental design

We constructed the isotropic helicoid by connecting six right-handed plastic propellers to form cubic shape as seen in Figure 2.1. We modified Lord Kelvin’s original design of twelve vanes positioned around a sphere. However, each blade is helical and also isotropic, since it is a cube. We dropped the isotropic helicoid, a single horizontal and single vertical propeller 5.47 meters in order to compare their rotation rates. We dropped the isotropic helicoid from multiple different angles to understand how the angle of its initial orientation affects the rotation. We filmed the drops with two different cameras: a Nikon DSLR camera positioned above the dropping point and, a high speed camera positioned at the side.
2.2 Horizontal and Vertical Blade Drops

We recorded the rotational velocity of the horizontal single propeller shown in Figure 2.2. The rotational velocity increases approximately linearly with time, after 0.8 seconds. The single horizontal propeller blade rotated 4.25 times in 1.485 seconds. The final rotational velocity directly before the propeller hit the ground was 46.2 rad/s, and the final translational velocity was approximately 4.3 m/s.

Figure 2.1: Cubic model of the isotropic helicoid made with six plastic, right handed propellers
Figure 2.2: Rotational Velocity vs Time for one horizontally dropped propeller blade

The single vertical propeller irregularly rotated 2.2 times in 1.117 seconds. Shown in Figure 2.3 is the rotational velocity versus time. The vertical propeller also rotated approximately linearly after 0.7 seconds. The rotational velocity right before it hit the ground was 25 rad/s. The translational velocity right before it hit the ground was approximately 13 m/s.
2.3 Isotropic Helicoid Drops

We released the isotropic helicoid from three different angles. One of the drops had the isotropic helicoid oriented with the propeller face upward. In particle coordinates, the gravitational vector for this drop was \( \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \).
Figure 2.4: Isotropic helicoid drop with one propeller face upward viewed from the side and above at various points during the drop
Figure 2.5 displays the drop from one edge. In particle coordinates, the gravitational vector for this drop is \( \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \).

Figure 2.5: Isotropic Helicoid drop from one edge viewed from the side

Figure 2.6 shows the coordinate system we are using to analyze the axis of rotation for the two drops from the edges. For the drop from one edge, the isotropic helicoid rotated around a vector that was rotated 45° from the x-axis in quadrant III in the x-y plane. As it fell, the helicoid rotated about 70° in 1.235 seconds. The translational velocity at the bottom of the drop was approximately 10 m/s.
Lastly, we then dropped the isotropic helicoid from two edges. In particle coordinates, the gravitational vector for this drop is \( \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \). Figure 2.6 displays the drop for this orientation. The isotropic helicoid rotated 80° irregularly about a vector rotated 45° from the x-axis in quadrant IV in the x-y plane in 1.182 seconds, producing a rotational velocity of 1.18 rad/s. The translational velocity towards the bottom of the drop was approximately 12 m/s.
2.3.1 Video Links

Video footage of the drops may be found at the following links:

Isotropic Helicoid drop face up aerial view: https://www.youtube.com/watch?v=ryQ0PEAz874

Isotropic Helicoid drop from one edge aerial view: https://www.youtube.com/watch?v=nPQ6xX1Cu9kfeature=youtu.be

Isotropic Helicoid drop from two edges aerial view: https://www.youtube.com/watch?v=p3XF8E3xq2Y

Horizontal Propeller Drop aerial view: https://www.youtube.com/watch?v=jYMq-NULxRYfeature=youtu.be
CHAPTER 2. EXPERIMENT

Vertical Propeller Drop aerial view: https://www.youtube.com/watch?v=jYMq-NULxRYfeature=youtu.be

2.4 Analysis of Rotation Translation Coupling

To compare the rotation translation coupling of the isotropic helicoid with that of the single propeller blade, we want to look at each object’s ratio of the rotational velocity to the translational velocity. In the limit where rotational acceleration goes to zero, rotational drag balances torque. Thus, we can assume that the rotational velocity, or $\omega$, is proportional to torque. Torque is equal to $\hat{C}v$ where $\hat{C}$ is the rotation translation coupling tensor and $v$ is the translational velocity. Therefore, the magnitude of the rotation translational coupling tensor is proportional to the ratio of the rotational velocity to the translational velocity.

We expected the isotropic helicoid to have a smaller rotation translation coupling tensor than the single horizontal propeller blade. However, the rotation from the drops both edges was perpendicular, rather than parallel, to the translational velocity, as would be expected for the isotropic helicoid. Therefore, we believe the rotation could have been related to the experimental design. The blades were attached using duct tape, making it possible that an uneven placement could have affected the rotational dynamics. Additionally, the isotropic helicoid could have had a small initial velocity that affected its rotation.

For the single propeller blade, $\frac{\omega}{v} = 11\text{rad/m}$ For the isotropic helicoid oriented in particle coordinates with the gravitational vector $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}$ the ratio of the angular velocity over the translational velocity is $\frac{\omega}{v} = 0.10\text{rad/m}$. The coupling for the single propeller is 110x smaller than that of the single propeller. For the isotropic helicoid oriented in particle coordinates with the gravitational vector $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, $\frac{\omega}{v} = 0.10\text{rad/m}$, a ratio 110x smaller than that of the single propeller. For the isotropic helicoid oriented in particle coordinates with the gravitational vector $\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$ has $\frac{\omega}{v} = 0$.

Even in a much higher Reynolds number regime than Lord Kelvin discussed, the largest coefficient of rotation that we saw was a factor of 110x smaller than the single propeller blade, and it was perpendicular to the direction of the velocity. This calls into question whether this particle can couple rotation and translation. Since this model has a different geometry than Lord Kelvin’s design, we want to explore how the Lord Lord Kelvin’s behaves. In the next chapter, we use a simulation to understand Kelvin’s geometry in a low Reynolds number regime.
Simulation

In order to investigate how the geometry Lord Kelvin described behaves in a low Reynolds number, we use a computer simulation to explore a situation that we cannot construct experimentally. In this chapter, we first demonstrate the capabilities and limitations of the SolidWorks flow simulation by measuring the drag coefficient of a sphere in laminar fluid flow. Then, we measure the coefficient of rotation of the isotropic helicoid to quantify its rotation. We compare the value of the coefficient of rotation for the isotropic helicoid’s rotation to the rotation of the isotropic helicoid’s component.

3.1 Sphere Simulation

SolidWorks flow simulation is a general parametric flow simulation tool that uses the finite volume method with refined grid near the particle. Using SolidWorks to run a sphere simulation, we measure the drag force acting on a sphere using parameters appropriate for water at room temperature. While we use the SolidWorks simulation to measure the isotropic helicoid’s torque, we measure the sphere’s drag coefficient since we have experimental values for the drag coefficient of a sphere. At two different Reynolds numbers, we compare the simulated drag coefficient on a sphere to the experimentally derived drag coefficient values. We want to test the two different Reynolds numbers in order to understand how the simulation accuracy varies by Reynolds number. Reynolds number of 10 and 0.2 are both laminar flow, but only 0.2 is close to the
creeping flow limit. Generally, as Reynolds number increases the drag coefficient decreases. Figure 3.1 displays the graph of the relationship between drag coefficient and Reynolds number for different theoretical models.

\[ C_d := \frac{24}{Re} \left( 1 + \frac{3}{16} Re \right) \]  

\[ C_d := \frac{24}{Re} \left( 1 + \frac{3}{16} \frac{Re}{1 + \frac{19}{240} Re} \right) \]  

The simulation measures the drag force so the only unknown is the drag coefficient. The drag coefficient is calculated from the drag force equation displayed in Equation 3.3.

\[ F_d := \frac{1}{2} C_d A v^2 \rho \]
$A$ is the frontal area. The frontal area of a sphere is a circle with an area that is equal to $\pi r^2$, with the radius of our sphere equal to 0.1m.

$F_d$ is the drag force measured in the direction of the velocity. The velocity in this simulation is in the x direction.

$v$ is the free stream velocity which is $5.00 \times 10^{-5}$m/s for Reynolds number 10 and $1.00 \times 10^{-6}$ for Reynolds number 0.2.

$\rho$ is the density of water, which is equal to $998 \frac{kg}{m^3}$ at 293 K.

The sphere simulation for Reynolds number 10 found a drag coefficient value of 4.38, which is a 10.7% difference from the expected value of 4.91. The volume of the sphere was 0.119 % of the computational domain. In the direction that fluid was flowing, the domain extended from -0.7m to 1.1m. In the y and z directions, the domain extended from -0.7m to 0.7m. We ran the simulation at multiple different resolutions and domains to compare how the different number of computational points affects the drag coefficient. The values for the coefficient ranged from 4.34 for a smaller domain size, to 4.33 for a larger domain size, to 4.30 for a smaller mesh resolution. Therefore, the domain size and the resolution had little affect on the accuracy of the drag coefficient.

The simulation for Reynolds number 0.2 found a drag coefficient gave 160.8 which is 33% greater than the expected value of 124.5.

While the simulation has a higher accuracy in a high Reynolds number regime, we want to see how the isotropic helicoid behaves in a low Reynolds number regime. This computational limitation prevented us from testing the sphere at a lower Reynolds number, where it would truly be in the creeping flow limit.

These inaccuracies at low Reynolds numbers make us skeptical about the precise numbers we get from the simulation. Despite the quantitative limitations, we conclude that the SolidWorks simulations can be trusted for qualitative results.
3.2 Building the Isotropic Helicoid

To model the isotropic helicoid shown in Figure 3.2, we begin by constructing its component parts. We want to measure the torque around the component parts to compare their torque to the torque of the isotropic helicoid. The isotropic helicoid is composed of twelve ellipsoids, or discs, located around the x, y, and z great circles. The great circle refers to the circle around the circumference of the sphere in the x-y, y-z, and x-z planes. The normal vector to the great circle in the y-z plane is in the x direction, so we refer to it as the x great circle (the same applies for the great circles in the y and z direction, which we refer to by the vector pointing normal to that circle). There are three great circles: the x, y and z. Each great circle has four ellipsoids rotated 45° at different locations around the circle. These four ellipsoids are oriented similar to a four-bladed propeller. Therefore, we refer to the ellipsoids around the great circle as the great circle propeller. Figure 3.3 shows an image of the x great circle propeller.

![Image of the isotropic helicoid constructed in SolidWorks viewed along an arbitrary vector](image)

**Figure 3.2:** Image of the isotropic helicoid constructed in SolidWorks viewed along an arbitrary vector

In this section, we measure the rotation of the x great circle propeller in order to understand
how one of the great circle propellers rotates compared to the isotropic helicoid. The diameter of
the great circle, measured from the center of one of the vanes to the center of the vane opposite
it, is 0.2m. With a velocity of $1.00 \times 10^{-6}$ m/s, the great circle propeller has a Reynolds number
of 0.2.

![Figure 3.3: The discs around the x great circle viewed along the x direction](image)

3.2.1 Construction of the Great Circle Propeller

We created an ellipsoid of aspect ratio 5 with a width to the height ratio of 5:1. The width
is 0.1m, and the height is 0.02m. We then copied the one ellipse, and rotated it by 45° about
the z axis, and translated it 0.1m so that the distance between the center of the two ellipsoids
opposite each other was 0.2m. After it was translated, we rotated it $\frac{\pi}{4} + \frac{2\pi}{2}$ radians (where n
ranges from 0 to 3), around the x axis until the four ellipsoids were placed at the appropriate
place on the great circle as shown in Figure 3.3 and Figure 3.4.
We oriented the ellipsoids to form a left handed propeller. The handedness is defined by the relationship between the particle’s solid body rotation rate vector and the fluid velocity. We define the handedness of the particles to be the direction of the velocity of the particle with respect to the fluid. In this case, the particle's rotation vector is anti-parallel to the fluid moving past the particle. Using Gustavsson and Biferale’s convention [1] shown in Figure 3.5, we call this a left handed propeller.

**Figure 3.4:** The ellipsoids around the x great circle viewed along the y axis

![Image of ellipsoids around an x great circle viewed along the y axis](image)
3.2.2 Great Circle simulation

In this section, we measure the torque of the x great circle propeller. The domain ranged from -1.5m to 1.5m in the x direction and -1m to 1m in the y and z directions, producing a total domain of 12m$^3$. The mesh was set to the maximum resolution, so there were 362,820 computation points. The thicker boundary layer of lower Reynolds number regime requires a larger resolution and domain size.

We calculated the force and the torque on the x great circle propeller in the x, y, and z directions. The results can be shown in Figure 3.6. In the x direction, the force is $2.3 \times 10^{-9}$N and the torque is $-2.9 \times 10^{-11}$Nm.

Torque is $\mathbf{G} = \mathbf{r} \times \mathbf{F}$, so we expect the magnitude of the force to be $\frac{G}{r}$. With a radius of 0.1m, we expect the force to be 10x the torque. The simulation gave us a force 82x greater than the torque. This result is on the the same order of magnitude, but has a different value due to the
specific geometry of our particle. The force and torque in the y and z directions should be zero but are limited by the simulation accuracy.

![Table]

<table>
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<th>Value</th>
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<td>[N]</td>
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</tr>
<tr>
<td>GG Force (Y) 1</td>
<td>[N]</td>
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<tr>
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<td>GG Torque (Z) 1</td>
<td>[N\text{m}]</td>
<td>-2.958e-013</td>
</tr>
</tbody>
</table>

**Figure 3.6:** The results for the force and torque of the x great circle in the x,y and z directions

### 3.3 Two Great Circles

In this section, we measure the torque on the y and z great circle propellers with the fluid velocity in the same direction. The y and z great circle propellers are each left handed propellers viewed along their symmetry axes. Figure 3.7 shows an images of the y and z great circle propellers viewed along the x-axis. The results for the torque and the force are displayed in Figure 3.8. In the x direction, the force is $2.7 \times 10^{-9}$N and the torque is $1.9 \times 10^{-11}$Nm. The magnitude of the torque for the two great circle propellers is slightly smaller than the one great circle propeller but in the opposite direction. The y and z great circle propellers together had a positive torque, but the x great circle propeller had a negative torque.
Figure 3.7: The y and z great circle propellers viewed along the x axis

Figure 3.8: The simulation results for the torque and force of the two great circle propellers in the x,y and z directions
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3.4 Isotropic Helicoid

In our final SolidWorks analysis, we combined the three great circle propellers together to form the isotropic helicoid. A view of the isotropic helicoids can be seen in Figure 3.11. The results for the simulations is displayed in Figure 3.9. The force for the isotropic helicoid is slightly bigger than the force for the one great circle propeller and two great circle propellers, but the torque in the x direction is significantly less. The number is even less than what we consider our noise in the y and z directions so it is essentially zero.

![Figure 3.9](image-url)

*Figure 3.9:* The simulation results for the force and torque of the isotropic helicoid in the x,y and z directions without the sphere
To be consistent with Lord Kelvin’s design, we added a sphere between the twelve ellipsoids as shown in Figure 3.12. We found that the addition of the sphere did not significantly affect the results. The torque of the isotropic helicoid with the sphere is only 1.7x greater than the torque of the isotropic helicoid without the sphere.

![Figure 3.10: Image of the isotropic helicoid without a sphere in SolidWorks viewed along an arbitrary vector](image)

<table>
<thead>
<tr>
<th>Goal Name</th>
<th>Unit</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
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<td>[N]</td>
<td>3.1111E-09</td>
</tr>
<tr>
<td>GG Force (Y) 1</td>
<td>[N]</td>
<td>-2.80049E-13</td>
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<tr>
<td>GG Force (Z) 1</td>
<td>[N]</td>
<td>-5.83757E-13</td>
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<td>GG Torque (X) 1</td>
<td>[N*m]</td>
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<td>[N*m]</td>
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<tr>
<td>GG Torque (Z) 1</td>
<td>[N*m]</td>
<td>-3.7265E-13</td>
</tr>
</tbody>
</table>

*Figure 3.11: The simulation results for the force and torque of the isotropic helicoid in the x, y, and z directions with the sphere*
3.5 Analysis

In order to compare the magnitude of difference of the torques between the isotropic helicoid, the one great circle propeller, and the two great circle propellers, we compare their respective coefficients of rotation. Just as the drag coefficient is a non-dimensional number that quantifies the resistance or drag of an object in a fluid, the coefficient of rotation is a non-dimensional number that quantifies the torque of an object in a fluid. The coefficient of rotation explains how the shape affects the torque (just as the drag coefficient explains how the shape affects the drag).

The torque is related to the coefficient of rotation by

\[ G = \frac{1}{2} \rho r A c_r v^2 \]

where \( G \) is the torque, \( \rho \) is the density, \( A \) is the frontal area, \( c_r \) is the coefficient of rotation, and \( v \) is the velocity of the fluid. We want to solve for \( c_r \) for each propeller using the torque and velocity obtained from the simulation.
The velocity is only in the x-direction, as such the velocity vector with units of m/s is \( \begin{pmatrix} 1.00 \times 10^{-6} & 0 & 0 \end{pmatrix} \).

For the isotropic helicoid the torque vector with units of Nm is \( \begin{pmatrix} -6.6 \times 10^{-14} & 0 & 0 \end{pmatrix} \).

For the two great circle propellers the torque vector with units of Nm is \( \begin{pmatrix} 1.9 \times 10^{-11} & 0 & 0 \end{pmatrix} \).

For the one great circle propeller the torque vector with units of Nm is \( \begin{pmatrix} -2.9 \times 10^{-11} & 0 & 0 \end{pmatrix} \).

The coefficient of rotation values are: for one great circle \( c_r = -18.0 \), for two great circles \( c_r = 12.1 \), for the isotropic helicoid \( c_r = -0.07 \).

The x great circle propeller has a coefficient of rotation that is 261x greater than the isotropic helicoid, and the two great circle propellers have a has a coefficient of rotation that is 175x greater than the isotropic helicoid.

As we saw in the experimental section, the vertical propeller blade had an opposite handedness than that of the horizontal propeller blade. This was the key piece Lord Kelvin missed in his analysis. He assumed that since all the propellers were right handed, they would all rotate in the same way. However, handedness depends on the orientation of the particle. [13]. The left handed x great circle propeller had a positive torque, but the two propellers in the y and z direction had a negative torque, which prevented the isotropic helicoid from rotating. Consistent with the results of the experiment, the simulation found that the components of the isotropic helicoid give torques in the opposite direction. Although there are quantitative limitation in the SolidWorks simulation, qualitatively we see that the coefficient of rotation is reduced to essentially zero for the isotropic helicoid, while it is significantly greater for the one and two propellers. This is evidence that, even in the low Reynolds number regime, the isotropic helicoid does not couple rotation and translation.
Theoretical Modeling

In this chapter, we examine a mathematical model of the isotropic helicoid. The process of building the isotropic helicoid involves rotating each of twelve disc to the a specific location around an sphere. The objective of this chapter is to obtain a value for the rotation translation coupling tensor for the isotropic helicoid in the creeping flow limit, so that we can compare it to the rotation translation values produced by the experiment and simulation sections.

4.1 Building the Isotropic Helicoid Model

In order to understand why the isotropic helicoid was giving us a rotation translation coupling coefficient significantly smaller than expected, we need to look at the rotation translation coupling tensors of each disc. The relationship between force and velocity is $F = \cdot U$. Since we are interested in the rotation translation coupling tensor, we look at the torque, $G = \hat{C}U$, to calculate the rotation translation coupling tensor, $\hat{C}$. In cases where the particle is offset from the origin, we can take the cross product of the position with the resistance tensor to relate torque to the velocity $G = r \times (\hat{A}U)$.

For a disc of revolution with the origin of the coordinate system at the geometric center and coordinate axes parallel and perpendicular to the principal axes of the disc, the resistance tensor $A$ is
Figure 4.1: Display of a vector pointing from the origin to the center of one of the discs in the x great circle

\[ A = a_p \begin{pmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (4.1)

Where \( a_p \) is the drag force on an object oriented perpendicular to the velocity and \( k a_p \) is the drag force on an object oriented parallel to the velocity [7].

For a disc about its center of mass, the rotation translation coupling is zero. If we move a disc away from the origin, it will have a non-zero rotation translation coupling about the origin.

First, we will calculate rotation translation coupling tensors for the four discs in the x great circle, as shown in Figure 4.1.

Listed below are the positions of the centers of the discs for in the x great circle.
In order to orient each disc along the x great circle, we use a rotation matrix.

We start with the thin principle axis of the disc in the x direction. We rotate each disc about the y axis by $\frac{\pi}{4}$ so that each disc is tilted by $\frac{\pi}{4}$. Then, we rotate the disc around the x axis to the correct orientation for that particle by $\frac{\pi}{4} + \frac{n\pi}{2}$ where n ranges from 1 to 4.

The rotation matrix for the disc at the position $r_1$ is:

$$
R_1 = R_x\left(\frac{\pi}{4}\right)R_y\left(\frac{\pi}{4}\right)
$$

$$
R_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\end{pmatrix}
$$

The other rotation matrices for the x great circle are:
\[R_2 = R_x \left( \frac{3\pi}{4} \right) R_y \left( \frac{\pi}{4} \right)\]

\[R_3 = R_x \left( \frac{5\pi}{4} \right) R_y \left( \frac{\pi}{4} \right)\]

\[R_4 = R_x \left( \frac{7\pi}{4} \right) R_y \left( \frac{\pi}{4} \right)\]

Since the analytic expressions for the rotational translation coupling tensors becomes cumbersome, we numerically evaluate the expressions in MatLab. We use as 3/2 for \(k_a p\) since that is the value for a disc in the creeping flow limit \[14\]. Later, we will show that the rotational translational coupling tensor is identically zero, which means that the result is independent of the value of \(k_a p\).

The resistance matrix becomes
\[
\begin{pmatrix}
3/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

We will rotate the resistance tensors with the respective rotation matrices.

For the particle at the position \(r_1\), the resistance tensor is
\[
A_1 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\
\end{pmatrix}
\begin{pmatrix}
3/2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\
0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \\
\end{pmatrix}
\]

\[
A_1 = \begin{pmatrix}
1.250 & 0.1768 & -0.1768 \\
0.1768 & 1.1250 & -1.1250 \\
-0.1768 & -0.1250 & 1.1250 \\
\end{pmatrix}
\]

Using this resistance matrices for the oriented particles, we can take the cross product of the position of the disc with the resistance matrix to find the rotation translation coupling matrix.

The torque about the origin is \(G_1 = \mathbf{r} \times (\hat{\mathbf{A}} \mathbf{U})\). Since these are linear operations, this can be written as a rotation translation coupling tensor, which for the particle at \(r_1\) numerically evaluates to
For the $x$ great circle, when we add the rotational translation coupling tensors for all four particles together and obtain

$$C_{xt} = C_1 + C_2 + C_3 + C_4$$

$$C_{xt} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

For the $y$ and $z$ great circles, we follow a similar process, but choose the thin axis in the $y$ and $z$ direction respectively. This means we have to rotate the reference resistance tensor in Equation 4.1.

The $A_y$ reference resistance tensor is rotated $\frac{\pi}{2}$ about the $z$ axis. The $A_z$ reference resistance tensor is rotated $-\frac{\pi}{2}$ about the $y$ axis. The $A_y$ and $A_z$ reference resistance tensors are:

$$A_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}$$

For the $y$ great circle, we rotate about the $z$-axis by $\frac{\pi}{4}$, so that each disc is tilted by $\frac{\pi}{4}$. Then we rotate the disc about the $y$-axis to the correct orientation for that particle by $\frac{\pi}{4} + \frac{n\pi}{2}$ where $n$ ranges from 1 to 4. So the rotation matrices are

$$R_3 = R_y(\pi)R_z(\pi)$$
CHAPTER 4. THEORETICAL MODELING

\[ R_6 = R_y(\frac{3\pi}{4})R_z(\frac{\pi}{4}) \]

\[ R_7 = R_y(\frac{5\pi}{4})R_z(\frac{\pi}{4}) \]

\[ R_8 = R_y(\frac{7\pi}{4})R_z(\frac{\pi}{4}) \]

For the \( z \) great circle, we rotate about the \( x \)-axis by \( \frac{\pi}{4} \) so each disc is tilted by \( \frac{\pi}{4} \). Then we rotate the disc about the \( z \)-axis to the correct orientation for that particle by \( \frac{\pi}{4} + \frac{n\pi}{2} \) about the \( z \)-axis.

So the rotation matrices are

\[ R_9 = R_z(\frac{\pi}{4})R_x(\frac{\pi}{4}) \]

\[ R_{10} = R_z(\frac{3\pi}{4})R_x(\frac{\pi}{4}) \]

\[ R_{11} = R_z(\frac{5\pi}{4})R_x(\frac{\pi}{4}) \]

\[ R_{12} = R_z(\frac{7\pi}{4})R_x(\frac{\pi}{4}) \]

Once we have the rotation matrices, we follow the same process as with \( x \) great circle, using the correct reference resistance tensor and positions of the discs.

The rotation translation coupling tensors for the \( y \) and \( z \) great circles are

\[ C_{yt} = \begin{pmatrix}
\frac{1}{2} & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & \frac{1}{2}
\end{pmatrix} \]
The total summation of all twelve torques is

\[ C_T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

This model shows that each great circle propeller has a rotation translation coupling that is non zero independently. The addition of the tensors of all three great circle propellers cause the rotation translation coupling tensor for the isotropic helicoid to cancel. This particle is isotropic but has no rotation translation coupling. Our model is consistent with the results of the simulation and the experiment, but without the limitations of both.

In constructing this mathematical model, we built Kelvin’s design for the isotropic helicoid particle, and saw that it did not couple rotation and translation. We believe that Lord Kelvin’s design does not rotate in fluid, which calls into question if Lord Kelvin’s geometry for an isotropic helicoid is truly an isotropic helicoid.
Conclusion

5.1 Summary

In this thesis, we presented our experiment, simulation, and theoretical evidence about the existence of an isotropic helicoid. Through the experimental section, we presented our physical model of the isotropic helicoid that had no rotation or only a small rotation perpendicular to the translational velocity, depending on the orientation of the particle. Even in a high Reynolds number regime, the isotropic helicoid had a very small rotation translation coupling. Furthermore, the simulation showed us how Kelvin’s isotropic helicoid behaves in a low Reynolds number regime. We calculated the coefficient of rotation from the results and found that the coefficient of rotation for the one great circle propeller and the two great circle propellers were significantly greater than the rotation for the isotropic helicoid. The simulation produced a coefficient of rotation with the opposite sign for one propeller versus two propellers. This told us that the handedness of the particle is dependent upon the orientation of the particle. Even when we rotated two left handed propellers, since they were perpendicular to the velocity we saw that their torque was in the opposite direction as the single propeller. Lastly, we evaluated a mathematical model of the isotropic helicoid by orienting twelve discs around the great circles of a sphere and rotating them by 45° as Lord Kelvin specified. We then measured the rotation translation coupling tensor and found that the tensor cancels when the great circles are together, thus preventing the design from coupling rotation and translation.
The models proposed for isotropic helicoids thus far have shown no rotation translation coupling. While it is possible there may be a chiral particle that is isotropic in drag and rotation translation coupling, this thesis presents compelling evidence that the geometry Lord Kelvin proposed is not truly an isotropic helicoid. If indeed there is not such a particle, there must be a symmetry preventing the particle from coupling rotation and translation. The next step in proving that the isotropic helicoid does not couple rotation and translation is identifying that symmetry.
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First and foremost, I would like to thank my research advisor, Dr. Greg Voth, who has encouraged and advised me, even through all my frustration and misunderstandings. I would not be the physicist I am today without him and a simple thank you does not suffice for the endless support he has given me.

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Bibliography


