The Alignment of Rods and Disks in Turbulence

by

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Dedication

To my parents, Paul and Bernie, for always having faith in my abilities.
Abstract

We study the orientation and rotational dynamics of anisotropic particles in homogenous isotropic turbulence. By analyzing direct numerical simulation (DNS) data at Taylor-microscale Reynolds number of $R_\lambda = 180$, we quantify the preferential alignment between particle orientations and vorticity, as well as alignment with principal stretching directions defined by the Cauchy-Green strain tensor. This tensor quantifies stretching experienced by material elements in turbulence and provides a natural basis for studying particle alignment in turbulence. While previous work has focused primarily on thin rods, we extend the study to oblate disks. Both rods and disks are a specific class of anisotropic particles known as axisymmetric ellipsoids. These particles are defined by their aspect ratio ($\alpha$), the ratio of their length ($L$) to their diameter, ($d$). Rods have an aspect ratio of $\alpha > 1$ while disks have an aspect ratio of $\alpha < 1$. The case of $\alpha = 1$ is a sphere.

In this thesis, we compare the preferential alignments of rods with disks in turbulence. Rods preferentially align with vorticity as a result of both quantities independently aligning with the strongest extensional stretching direction, as defined by the maximum Cauchy-Green eigenvector, $\hat{e}_{L1}$. In contrast, disks orient perpendicular to vorticity and preferentially align with the strongest compressional stretching direction, as defined by the smallest Cauchy-Green eigenvector, $\hat{e}_{L3}$. Furthermore, we study the relationship between the principle stretching eigenframe defined by the eigenvectors of the Cauchy-Green strain tensor and the principle rate of stretching eigenframe defined by the eigenvectors of the strain-rate tensor, the symmetric part of the velocity gradient tensor.
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Chapter 1

Introduction

1.1 Motivations and Applications

Anisotropic particles are involved in a wide range of environmental, industrial, and biological turbulent fluid flows. A natural expectation is that turbulence randomizes particle orientation, but it is found that particles align with fluid velocity gradients. This thesis focuses on axisymmetric ellipsoids, a subclass of anisotropic particles, and their alignments in turbulence. These particles are small compared to the characteristic length scales of the fluid. Axisymmetric ellipsoids occur in a wide variety of shapes, ranging from a prolate spheroid, or rod, to an oblate spheroid, or disk. While these particles translate with the fluid, their geometry ultimately determines rotational motion and degree of alignment in turbulence. Until recently, research efforts have primarily focused on understanding the dynamics of rods in turbulence as a result of their industrial applications and experimental convenience of imaging one-dimensional-like objects [1]. Some specific applications of rods in turbulence include ice crystals in clouds [2–4], grain dynamics in accretion disks [5], and fibre suspensions used in paper making [6]. While disks have previously received limited study, they have slowly demanded more attention due to their relevance in understanding the deformability of red blood cells in circulatory regulation [7] and the rotational dynamics of plankton [8, 9]. Through a direct numerical simulation of Navier-Stokes equations, this thesis will build off of an extensive study of rods in turbulence to provide a phenomenological description of the alignment of disks in homogenous isotropic
1.2 Characterizing Fluid Flows

The study of fluid dynamics relies on the characterization of a fluid flow. A fluid flow is the continuous movement of a fluid or gas from one location to another. There are two types of flows that we will refer to in this thesis: laminar and turbulent. An example of a laminar flow is the plane channel flow shown in figure 1.1. This flow occurs when fluid elements move in parallel layers with no disruption between the layers [10]. There are no cross currents, swirls, or eddies. A common everyday example of a laminar flow is the smooth flow of a viscous fluid through a pipe or faucet whose flow velocity is given by the average speed of the molecules that comprise the fluid elements. On the other hand, a turbulent flow is characterized by the chaotic motion of fluid through a given region. Unlike laminar flows, cross currents, swirls, and eddies are significant features of turbulent flows. These flows are very common in nature and include fast flowing rivers, air currents in our atmosphere, and the convection zone of our Sun.

Before providing a quantitative description of turbulence and how it is generated, it is useful to consider qualitative pictures and examples common to our lives. How easily a fluid becomes turbulent depends on its viscosity. Viscosity can be understood as a medium’s resistance to motion and describes interactions between fluid layers that allow for momentum transfer across or between them. Highly viscous fluids are more likely to be laminar than turbulent. For example, water and air have low viscosities, and, as a result, can become turbulent quite easily. On the other hand, viscous fluids, such as honey or syrup, tend not to be turbulent.

Fluids can become turbulent through a number of means. One example includes heating. Consider what happens when you boil a pot of water. The addition of heat causes turbulent mixing by means of convection. Similarly, increasing flow pressure contributes to a fluid’s degree of turbulence. Water streams that flow from a jar faucets are laminar, but these flows can become turbulent as one opens the faucet to its full extent. The water will flow out with greater speed and no longer in a uniform direction. The distinction between a laminar and turbulent flow can be better understood by considering a quantity known as the Reynolds number. It is formally defined as the ratio of momentum forces to viscous forces and is given by the following:
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Figure 1.1: A laminar flow is characterized by fluid moving in parallel layers without lateral mixing. On the other hand, a turbulent flow is characterized by cross currents and eddies. Laminar flows tend to occur at lower velocities, below a certain threshold. Similarly, laminar flows occur at high viscosities. This threshold is ultimately quantified by the Reynolds number, $Re$ (Image courtesy of www.periodni.com).

$$Re = \frac{\rho v L}{\mu} = \frac{vL}{\nu}$$

Qualitatively, the Reynolds number can be understood as the ratio, or balance, between energy input and energy dissipation. The specific parameters that define the Reynolds number depend on the physical situation in question but generally depend on fluid density, $\rho$, and dynamic viscosity, $\mu$, in addition to a bulk fluid velocity, $v$, and a characteristic length, $L$. For a laminar flow through a pipe, the length, $L$, is the distance the fluid travels through the pipe, and the velocity, $v$, is the fluid’s mean velocity. As shown, equation 1.1 can also be expressed in terms of the kinematic viscosity, $\nu = \mu/\rho$, instead of $\mu$. Since laminar flows are characterized by high viscosity and small fluid velocities, they are specified by a small Reynolds number. On the other hand, turbulent flows are characterized by low viscosity and large fluid velocities which corresponds to a greater Reynolds number. The transition between laminar and turbulent flows is described by a critical Reynolds number which is highly dependent on flow geometry. In 1883, Osbourne Reynolds sought to determine this critical value by studying the critical velocity at
which existing eddies would die out in a 16 ft pipe. He measured the pressure drop in a 5 ft section at the end of the pipe and found that the pressure loss was proportional to the first power of the velocity at low speeds but proportional to some higher power above a critical velocity. He quoted these critical values as corresponding to a Reynolds number between 1900 and 2000 [11]. Reynolds provided a range since he found it practically impossible to specify an exact number due to the intermittency of the flow. For pipes flows with a diameter, $D$, the critical value quoted in most fluid mechanics textbooks is $Re = 2300$ [11]. Recent efforts by Bjorn Hof have focused on turbulent transitions in pipe flows in order to provide a more reliable critical Reynolds number [12]. The understudying of these systems have emerged from considering state space, the space of all velocity fields that are either prepared as initial conditions or obtained in the time evolution of the flow. This analysis has lead to a critical pipe flow Reynolds number of $Re = 2250$.

### 1.3 Fundamentals of Turbulence

This distinction between laminar and turbulent flows begs a larger conceptual question about energy transport and flow structure. The eddies that characterizes a turbulent flow can exist in different scales and sizes, and, as a result, suggest a hierarchy of scales through which energy from the large scale is passed down to the smaller scales. In 1941, the Russian physicist, Andrey Kolmogorov, provided an extensive study on this hierarchy of scale. Kolmogorov asserted that, while the large scales are highly dependent on the geometry of the boundaries, the small scales are statistically isotropic and depend only on the viscosity, $\nu$, and the energy dissipation rate, $\epsilon$ [13]. From this principle, he defined a characteristic length, $\eta$, given by $\eta = (\nu^3/\epsilon)^{1/4}$, and a characteristic time scale, $\tau_\eta$, given by $\tau_\eta = (\nu/\epsilon)^{1/2}$. Collectively known as Kolmogorov microscales, these quantities characterize the small scales of turbulence where viscosity dominates and turbulent kinetic energy is dissipated into heat [14]. In addition, the process through which this energy is dissipated into heat is given by the local velocity gradient and the kinematic viscosity:

$$\epsilon = 2\nu \langle S_{ij} S_{ij} \rangle$$  \hspace{1cm} (1.2)
Here, $S_{ij}$ is the strain-rate tensor, the symmetric component of the full velocity gradient tensor, $A_{ij} = \frac{\partial u_i}{\partial x_j}$, where $u$ is the fluid velocity field. The velocity gradient tensor is at the heart of our calculations, and it is used to map spatial changes in the velocity field of a fluid. We can derive many key features from the velocity gradient, including the strain-rate, $S_{ij}$, which is responsible for deformations in a flow. The strain-rate tensor can be used to define principle axes of stretching which serve as a means for determining preferential alignments of axisymmetric ellipsoids with velocity gradients (for further detail, see section 2.2). In addition to the energy dissipation rate, we can express the Kolmogorov time, $\tau_\eta$, in terms of the strain-rate.

$$\tau_\eta = \left(2\langle S_{ij}S_{ij} \rangle\right)^{-1/2}$$

Together, the Kolmogorov length and time determine how turbulent a given flow is. A steep velocity gradient requires a large strain-rate, which produces large flow deformations and, consequently, a large rate of energy dissipation. On the other hand, steep velocity gradients result in smaller times for which the flow is statistically isotropic. We will frequently discuss how axisymmetric ellipsoids behave over a small and large range of Kolmogorov times. In addition to the Kolmogorov microscales, we sometimes make use of the Taylor-microscales, an intermediate length scale that characterizes viscosity-dominated fluid regions. For our purposes, we make use of the Taylor-microscales Reynolds number, $R_\lambda$, which can we relate to the Reynolds number with $R_\lambda = \sqrt{15Re}$ [14].

### 1.4 Jeffery’s equation and Rotational Dynamics in Turbulence

One of the major goals of this thesis is to describe the rotational dynamics of an ensemble of small, non-inertial particles over a range of Kolmogorov times. In order to do so, we must consider how geometry affects particle motion. Axisymmetric ellipsoid particles are defined by their aspect ratio, $\alpha$. This quantity is the ratio of the dimension along the symmetry axis, $L$, to the perpendicular dimension, $d$, such that $\alpha = \frac{L}{d}$. This quantity is greater than 1 for rods, less than 1 for disks, and equal to 1 for spheres. For a given axisymmetric ellipsoid, we can define an orientation, $\hat{p}$, along the symmetry axis. It is standard convention to treat $\hat{p}$ as a unit vector.
For rods, \( \mathbf{p} \) lies along the length, and for disks, it is perpendicular to the surface. Additionally, the orientation of a perfect sphere is undefined. We demonstrate this in figure 1.2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Axisymmetric ellipsoids are defined by their aspect ratio, \( \alpha \). When \( \alpha < 1 \), we have an oblate spheroid, or disk, as shown by the left figure. For the case of \( \alpha = 1 \), we have a perfect sphere as shown in the middle, and for the case of \( \alpha > 1 \), we have a prolate spheroid, or rod, as shown by the right figure. The orientation vector \( \mathbf{p} \) is distinctly defined for rods and disks but not for spheres.}
\end{figure}

When axisymmetric ellipsoid particles are small compared to the Kolmogorov length, \( \eta \), their rotation rate and orientation is determined by the velocity gradients through Jeffery’s equation [15]:

\[
\dot{p}_i = W_{ij}p_j + \frac{\alpha^2 - 1}{\alpha^2 + 1}[S_{ij}p_j - p_i p_j S_{jk}p_k] \tag{1.4}
\]

Here, \( S_{ij} \) is again the strain-rate tensor and \( W_{ij} \) is the rate-of-rotation tensor, the anti-symmetric part of the velocity gradient tensor. The rate-of-rotation tensor characterizes the local spinning of a fluid flow and plays a prominent role in determining the rotational motion of axisymmetric ellipsoids in turbulence. Particle dynamics become much more complicated when particles are large [16] or when high particle concentrations produce particle interactions and two-way coupling [17, 18]. Using our direct numerical simulation (DNS) data, we can access the entire range of aspect ratios, \( \alpha \), in order to determine an axisymmetric ellipsoids’s tumbling rate, \( \dot{p}_i \) [19]. The dynamical differences between rods and disks are a direct result of tumbling rate’s functional dependence on aspect ratio. In order to show this clearly, we can rewrite Jeffery’s
equation as a contribution due to rotation and strain. Namely, the tumbling due to rotation is 
\[ \dot{p}_i^W = W_{ij}p_j \]
and the tumbling due to strain is 
\[ \dot{p}_i^S = \frac{\alpha^2 - 1}{\alpha^2 + 1} \{ S_{ij}p_j - p_i p_j S_{jk}p_k \} \]
Together, these expressions give us Jeffery’s equation 
\[ \dot{p}_i = \dot{p}_i^W + \dot{p}_i^S \]
In the limit that \( \alpha \) is very large, we have a long rod whose tumbling due to strain simplifies to 
\[ \dot{p}_i^S = S_{ij}p_j - p_i p_j S_{jk}p_k. \]
On the other hand, in the limit that \( \alpha \) is very small, we have a disk whose tumbling due to strain simplifies to 
\[ \dot{p}_i^S = -S_{ij}p_j + p_i p_j S_{jk}p_k. \]
This mathematical difference in the equation of motion is ultimately responsible for the different dynamical picture between rods and disks. We will primarily explore this difference by examining a large ensemble of particles but will also look at a few single particle trajectories to help iron out subtleties associated with their rotational motions.

In addition to tumbling rate, the rotational dynamics of rods and disks exhibit differences in their total angular velocity, \( \Omega \). We can obtain this quantity directly from Jeffery’s equation. For the case of a perfect sphere (\( \alpha = 1 \)), the total tumbling rate is due entirely to rotation and described by 
\[ \dot{p}_i = W_{ij}p_j \]
We use the definition that 
\[ \dot{\hat{p}} = \Omega \times \hat{p} \]
to determine that the angular velocity, \( \Omega \), of a sphere is given by half the fluid vorticity, \( \omega \). The vorticity, \( \omega \), is a pseudovector which describes the local spinning and it is given by 
\[ \omega = \nabla \times u \]
where \( u \) is the fluid velocity field. We will explore this quantity in further detail starting in section 2.2. Furthermore, to obtain the angular velocity for any axisymmetric ellipsoid, we can decompose \( \Omega \) into components that are parallel and orthogonal to its orientation, \( \hat{p} \).

\[ |\Omega|^2 = |\dot{\hat{p}}|^2 + |\hat{p} \cdot \Omega|^2 \]  

The magnitude of the orthogonal component is the tumbling rate, \( |\dot{\hat{p}}| \), given by Jeffery’s equation, and the magnitude of the parallel component, \( |\hat{p} \cdot \Omega| \), describes the rate at which the particle rotates, or spins, about its symmetry axis, \( \hat{p} \). Again, we obtain \( \hat{p} \) from our DNS data by numerically integrating Jeffery’s equation for every particle trajectory over the a range of Kolmogorov times. In figure 1.2, we show how the total angular velocity, tumbling rate, and spinning rate changes as a function of aspect ratio. The effect of aspect ratio on spinning and tumbling is strong for small departures from \( \alpha = 1 \). Outside the range of \( 0.1 < \alpha < 10 \), axisymmetric ellipsoids rotate at a constant value and are insensitive to changes in aspect ratio. For disk-like objects with \( 0.1 < \alpha < 1 \), the spinning rate is small, and the total angular velocity, \( \Omega \),
is almost entirely dominated by tumbling. On the other hand, the total angular velocity for a rod with $1 < \alpha < 10$, is due to contributions from both spinning and tumbling. While true, the spinning contribution is more prominent.

![Figure 1.3](image)

**Figure 1.3:** Direct numerical simulation results for the variance of angular velocity, spinning rate, and tumbling rate as a function of aspect ratio, $\alpha$, for $0.1 < \alpha < 10$ for axisymmetric ellipsoids in isotropic turbulence. The quantities are scaled by the Kolmogorov time squared, $\eta^2$.

### 1.5 Overview

In addition to understanding the rotational dynamics of rods and disks, this thesis aims to quantify the alignment of rods and disks with fluid stretching. Previous studies have extensively studied this phenomenon by using the strain-rate tensor, $\mathcal{S}$. Using this quantity, we can construct an eigensystem defined by the strain-rate eigenvectors, $\hat{e}_i$ ($i = 1, 2, 3$). Similar to the spinning rate, the alignment of an axisymmetric ellipsoid with these eigenvectors is quantified by the magnitude of the dot product between $\hat{p}$ and $\hat{e}_i$. While the eigenvectors of the strain-rate tensor have served as an effective means for quantifying the alignment of axisymmetric ellipsoids with stretching in turbulence, this thesis will assert that particle alignment with stretching is more clearly described by the eigenvectors of the Cauchy-Green strain tensor. We will proceed as follows:
Chapter 1 - Introduction

- Chapter 2 will provide further theoretical descriptions for how we model fluid flows and the motions of axisymmetric ellipsoids in these flows. We will start with an introduction to Navier-Stokes equations, the governing equations in fluid dynamics. The solution to Navier-Stokes equation is a flow velocity field. From here, we will discuss velocity gradients and their decomposition into strain-rate and rate-of-rotation. Once this groundwork is in place, we will consider a simple flow, known as the channel Couette flow, for which we will solve Navier-Stokes equations. While this flow is laminar, it will provide us with a starting example for how to model the motion and rotational dynamics of axisymmetric ellipsoids in any given fluid flow. The techniques developed in this example are ultimately applied to our DNS data in order to model particle motion in turbulence. We will also provide a further qualitative and quantitative description of fluid stretching as defined by the eigenvectors of the strain-rate tensor and the Cauchy-Green strain tensor. While these eigenvectors provide two different means for quantifying stretching, we will discuss how we can relate these eigenframes by examining how the velocity gradient tensor evolves in time.

- Chapter 3 will provide a short description of our direct numerical simulation data and the algorithms used to analyze particle motion along trajectories in isotropic turbulence. We use a 4th order Runge-Kutta scheme to numerically integrate the velocity gradient tensor to obtain deformation tensor, $F$, from which we can construct the Cauchy-Green strain tensor. Similarly, we must numerically integrate Jeffery’s equation to obtain $\hat{p}$ for our ensemble of particles. In addition, we provide a description of some codes used to analyze data.

- Chapter 4 will present our work comparing the rotational dynamics and alignment of rods and disks in turbulence. We will first summarize previous study on rods and demonstrate the relative effectiveness of using the Cauchy-Green strain tensor to describe alignment with fluid stretching. We will subsequently present our work on disks and further assert that the Cauchy-Green eigenframe is the natural eigenframe for studying the alignment of disks in turbulence. Lastly, we will provide a phenomenological description of the time evolution of the velocity gradient tensor in order to quantify the relationship between the strain-rate tensor and the Cauchy-Green strain tensor.
Chapter 2

Theory

In this chapter, we provide a brief introduction to some governing equations of our work. We explore the concept of a velocity gradient and how axisymmetric ellipsoid particles respond to these spatial changes. In addition, we explore how to model axisymmetric ellipsoid motion in a laminar flow in order to lay the foundation for modeling motion in turbulence.

2.1 Navier-Stokes Equations

Navier-Stokes equations are the governing equations in all of fluid dynamics. They are a consequence of conservation of mass and momentum and a direct application of Newton’s second law, \( \mathbf{F} = m \mathbf{a} \), which relates the temporal and spatial changes in fluid velocity to the sum of all external forces. For an incompressible fluid flow with a constant density, we can write Navier-Stokes equations in vector notation as

\[
\rho \left[ \frac{du}{dt} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] = -\nabla P + \mu \nabla^2 \mathbf{u} + \mathbf{f} \tag{2.1}
\]

where \( \rho \) is the fluid density and \( \mu \) is the dynamic viscosity (as introduced in section 1.2). These forces include a pressure term \( (\nabla P) \), a viscous term \( (\mu \nabla^2 \mathbf{u}) \), and a general term, \( \mathbf{f} \), which accounts for other external forces imposed upon the given system. For example, one might need
to consider the gravitational field in which the fluid flows [20]. It is worth noting that this body force term has dimensions of Newtons per length cubed. Furthermore, the forces due to pressure and viscosity are both intimately related to the velocity field. The pressure term accounts for the diffusion of fluid from areas of higher pressure to lower pressure, while the viscosity opposes the relative motions of neighboring fluid particles. In addition, the viscosity ultimately characterizes the random motions of fluid resulting in the transfer of momentum across a local region.

A key feature of the pressure and viscosity is that they work together to generate stress across an arbitrary fluid surface. As a result, we can express the total net stress, $\sigma_{ij}$, acting on a fluid as a second-rank tensor in the following manner:

$$\sigma_{ij} = -P\delta_{ij} + \tau_{ij}$$

(2.2)

Here, $P$ denotes the pressure and $\tau_{ij}$ is the shear stress. In the general case, the stress is a quantity with a magnitude and two directions. If we consider a three-dimensional cube (as shown in figure 2.1) the index $i$ specifies the surface on which the stress acts, and the index $j$ specifies the direction in which the stress acts on the given surface. As a result, $\sigma_{ij}$ quantifies the net stresses acting in all directions, on all surfaces. The inclusion of the Kronecker delta, $\delta_{ij}$, next to the pressure term specifies that the non-zero pressure contributions occur along the diagonal of the matrix when $i=j$. Physically, the pressure contributions occur along the directions normal to the faces of the cube. On the other hand, the vicious contributions, quantified by the shear stresses, $\tau_{ij}$, occur as off-diagonal terms and are physically responsible for deformations of the rectangular faces. It is also important to note that stresses on opposite sides of the cube are exactly equal and opposite. The net force on a parcel of fluid is ultimately given by the differences in stresses acting across and along opposite faces. For example, a net force along perfectly along the $X$ direction occurs when there is an imbalance between the two values of $\sigma_{xx}$. In addition, a force in the $X$ direction also occurs due to variations in the $\sigma_{yx}$ and $\sigma_{zx}$ directions.

The shear stress, $\tau_{ij}$, acting on a fluid is ultimately characterized by velocity differences that allow for momentum transfer across a specified region of fluid. The presence of these velocity differences results in the deformation of the fluid, and the resistance to this deformation, as mentioned before, is quantified by the viscosity. We can account for all these factors by expressing the shear stress as the following:
Figure 2.1: The stress tensor is a second rank tensor that specifies all stress forces, \( \sigma_{ij} \), present on a given continuum (Image courtesy of www.pasqualerobustini.com).

\[
\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.3)
\]

The velocity gradients, \( \frac{\partial u_i}{\partial x_j} \), play a significant role in determining the total stress exerted on a fluid. If we rewrite Navier-Stokes equation using this index notation, we can see how the velocity gradients are intimately involved in determining the motion and behavior of a given flow.

\[
\rho \left[ \frac{\partial (u_j)}{\partial t} + u_i \left( \frac{\partial u_j}{\partial x_i} \right) \right] = -\frac{\partial P}{\partial x_j} + \mu \left( \frac{\partial^2 u_j}{\partial x_i^2} \right) + f_j \quad (2.4)
\]

The velocity gradient appears once on both sides of the expression. On the right hand side, it appears in the viscous term, and on the left hand side, it appears in the non-linear term, \( u_i \left( \frac{\partial u_j}{\partial x_i} \right) \). This term is responsible for a great deal of the mathematical difficulty in fluid dynamics, and it is the most significant reason for why most of what we know about the most complicated flows is done through experimental and computational work [21]. With this outline of our governing equation, we proceed by highlighting the important features of velocity gradients.

### 2.2 Velocity Gradients

The velocity gradient tensor is a second rank tensor that quantifies spatial variations of a fluid velocity, \( \mathbf{u} \), and characterizes local fluid behavior. It contains information about a flow’s defor-
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formation, rotation, and energy dissipation rate. In 3 dimensions, the velocity gradient tensor is given by:

\[ A_{ij} = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} & \frac{\partial u_2}{\partial x_3} & \frac{\partial u_3}{\partial x_3} \end{pmatrix} \] (2.5)

For the purpose of this thesis, the subscripts \( i, j = 1, 2, 3 \) will correspond to \( x, y, \) and \( z \) components unless otherwise stated. A natural decomposition of the flow is to separate \( A_{ij} \) into its symmetric (\( S_{ij} \)) and antisymmetric (\( W_{ij} \)) parts.

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \] (2.6)

\[ W_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \] (2.7)

The symmetric strain-rate tensor, \( S_{ij} \), characterizes the rate at which the fluid is being deformed or stretched in the neighborhood of a given point. This dynamic process is characterized by three eigenvectors, \( \hat{e}_i (i = 1, 2, 3) \), which correspond to three real eigenvalues, \( \lambda_i \), with \( \lambda_1 > \lambda_2 > \lambda_3 \). In the nondegenerate case of unequal eigenvalues, these three eigenvectors are orthogonal and define three principle axes of strain-rate. The eigenvalues correspond to the directions of the maximum rate of extension (\( \lambda_1 > 0 \)) and compression (\( \lambda_3 < 0 \)). The intermediate eigenvalue, \( \lambda_2 \), corresponds to a class of objects that is either extensional or compressional. Our research primarily focuses on incompressible flows for which the fluid density is constant and the velocity gradient is divergence free. As a result, we require that \( A_{ii} = S_{ii} = \lambda_1 + \lambda_2 + \lambda_3 = 0 \) everywhere in our simulation.

The antisymmetric rate-of-rotation tensor, \( W_{ij} \), describes the magnitude and direction of the fluid’s rigid body rotation.

\[ W_{ij} = \begin{pmatrix} 0 & -W_{12} & -W_{13} \\ W_{12} & 0 & -W_{23} \\ W_{13} & W_{23} & 0 \end{pmatrix} \] (2.8)
\( W_{ij} \) only has three independent components and can be expressed as a vector by considering \( \omega \), the curl of the flow velocity, \( u \).

\[
\omega = \nabla \times u = \left( \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \hat{e}_1 + \left( \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \right) \hat{e}_2 + \left( \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \hat{e}_3
\]  

(2.9)

This quantity, \( \omega \), is the vorticity, as introduced in section 1.4. It is a pseudovector that describes the local spinning of a fluid in a given region. Comparing equation 2.7 with 2.9, we can see that \( 2W_{ij} = \omega_k \epsilon_{ijk} \), and we are left with the following:

\[
W_{ij} = \frac{1}{2} \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-\omega_3 & 0 & \omega_1 \\
\omega_2 & -\omega_1 & 0
\end{pmatrix}
\]  

(2.10)

In summary, if the velocity gradient tensor is specified for a given flow, we can ultimately determine the principle axes, \( \hat{e}_i \), in which the fluid is stretched and the rate, \( \omega \), at which it rotates in a given region.

### 2.3 Planar Couette Flow

To highlight some important features of the velocity gradient tensor, let’s consider the planar Couette flow, a canonical flow in fluid dynamics. The planar Couette flow is a two-dimensional laminar flow of a viscous fluid between two parallel boundary plates. The top plate moves relative to the bottom plate, and this motion induces simple shear, a special case of fluid deformation in which only one component of the velocity vector is non-zero. In this example, we set \( u(x) = u(z) = 0 \) and \( u(y) = f(y) \) which we can determine by integrating Navier-Stokes equation. If we neglect pressure gradients, Navier-Stokes equation for this simple flow reduces to \( \frac{d^2 u(y)}{dy^2} = 0 \). If we integrate with respect to \( y \) and impose that \( u(0) = 0 \) and \( u(h) = V \), we are left with \( u(y) = \left( \frac{y}{h} \right) V \). By taking derivatives with respect to \( x \), \( y \), and \( z \), we can determine the velocity gradient tensor for our flow.
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**Figure 2.2:** Planar Couette flow configuration. Two parallel boundary plates, separated by a distance $h$, are used to induce shear driven fluid motion. Image courtesy of www.cfd-online.com.

![Planar Couette flow configuration](image)

As shown before, we can decompose any second rank tensor into a symmetric and antisymmetric part. For our simple planar Couette flow, we specify a contribution due to pure strain and pure rotation in equation 2.11:

$$A_{i j} = \begin{pmatrix} 0 & \frac{V}{2h} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{V}{2h} & 0 \\ \frac{V}{2h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{V}{2h} & 0 \\ -\frac{V}{2h} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (2.11)

As shown before, we can decompose any second rank tensor into a symmetric and antisymmetric part. For our simple planar Couette flow, we specify a contribution due to pure strain and pure rotation in equation 2.11.

**Figure 2.3:** A simple shear flow is the result of a contribution due to pure strain and pure rotation.

![Shear Flow](image)  \hspace{1cm} ![Pure Strain](image)  \hspace{1cm} ![Pure Rotation](image)

(a) Shear Flow  \hspace{1cm} (b) Pure Strain  \hspace{1cm} (c) Pure Rotation

The strain-rate of the planar Couette flow is characterized by two eigenvectors, $\hat{e}_i$, with two real eigenvalues, $\lambda_i$, given by the expression $S\hat{e}_i = \lambda_i\hat{e}_i$. Solving the characteristic eigenvalue equation, we determine that $\lambda_1 = \frac{V}{2h}$ and $\lambda_2 = -\frac{V}{2h}$. Physically, these eigenvalues represent the rate at which the continuum is both extended and compressed between the two boundary layers. Extensional or compressional strain-rate is ultimately determined by our rate of shear,
$R = \frac{V}{\pi}$, given by the physical parameters of our flow. If we increase the speed, $V$, at which the top boundary moves relative to the bottom, we increase our strain-rate. We can achieve the same goal by moving the boundary plates closer together, thereby decreasing their separation, $h$. Similarly, larger shear rate results in a larger vorticity, $\omega$, as given by the non-zero components of $W_{ij}$.

\section*{2.4 Characterizing Axisymmetric Ellipsoids in Simple Shear}

Axisymmetric ellipsoid particles in simple shear flows rotate periodically in what are known as Jeffery orbits. There are infinitely many initial orientations, and, as a result, infinitely many stable Jeffery orbits. Despite this fact, the rotation period of any Jeffery orbit is given by the following \cite{22}:

$$T = \frac{2\pi}{R} \left( \alpha + \frac{1}{\alpha} \right)$$  \hfill (2.12)

We will investigate the rotational periods of Jeffery orbits by considering 3 limiting cases of aspect ratio: $\alpha = 1$, $\alpha \gg 1$, and $\alpha \ll 1$. Jeffery’s equation, given by equation 1.4, states that an axisymmetric ellipsoid’s tumbling rate is the result of tumbling due to both strain and vorticity. In the case of a sphere ($\alpha = 1$), tumbling is entirely due to vorticity, and Jeffery’s equation reduces to $\dot{p}_i = W_{ij}p_j$ which we can solve analytically. In matrix notation, we can write:

$$\begin{pmatrix} \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{pmatrix} = \begin{pmatrix} 0 & \frac{V}{\pi} & 0 \\ -\frac{V}{\pi} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$ \hfill (2.13)

From these expressions, we can formulate two coupled, first-order differential equations for the orientation, $\hat{p}$, of our sphere. In order to uncouple our equations, we take second derivatives with respect to time and substitute using our original coupled expressions. Noting that $R = \frac{V}{\pi}$, we are left with the following:
\[ p_x = -\frac{R^2}{4} p_x \] (2.14)
\[ p_y = -\frac{R^2}{4} p_y \] (2.15)

Both expressions have sinusoidal solutions whose angular frequency is determined by the flow’s shear rate and whose phase angle is determined by the particle’s initial orientation in the flow. If we arbitrarily orient the spheroid in the streamwise direction, along the x axis, we can represent our initial orientation as \( \hat{p} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \). Imposing this initial condition, we are left with \( p_x = \cos(\frac{R^2}{4} t) \) and \( p_y = \sin(\frac{R^2}{4} t) \). In addition, since our simple shear flow is two dimensional, \( p_z = 0 \) throughout the entire flow.

**Figure 2.4:** Alignment of a sphere, long rod, and oblate disk in a planar Couette flow with \( R=10^{-2} s^{-1} \). The sphere’s orientation in the flow varies periodically while both the rod and disk have a stable orientation which only changes after very long times. The rod is preferentially aligned in the direction of the flow while the disk is preferentially perpendicular.

For \( \alpha > 1 \) and \( \alpha < 1 \), tumbling is due to both vorticity and strain. Determining analytical solutions for these cases is quite difficult and beyond the scope of this thesis. Alternatively, we can solve Jeffery’s equation numerically and examine our results qualitatively. In the case of \( \alpha \gg 1 \), we have a long thin rod. If we orient this rod along the streamwise direction of flow, we expect the rod’s rotational period to approach infinity as shown in equation 2.12. Physically, this means that the rod’s orientation does not change. Similarly, for the case where \( \alpha \ll 1 \), we have an oblate disk whose rotational period also approaches infinity. If the disk is oriented in the streamwise direction, it will quickly rotate about the plane of the flow and become aligned preferentially perpendicular to the stream.

The more interesting case of preferential alignment involves studying small deviations from
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Figure 2.5: Alignment of a rod and disk with a planar Couette flow for $R=10^{-2} s^{-1}$. Their aspect ratios deviate from $\alpha = 1$ by one order of magnitude, and these deviations result in preferential orientations of $\hat{p}$ with the flow.

$\alpha=1$, on the order of unity. Figure 2.5 demonstrates these orientations as a function of time for $R = 10^{-2} s^{-1}$. An interesting feature is the periodic turnover from perfect alignment to anti-alignment for both rods and disks. This is the direct result of the time dependent contribution of pure strain and rotation to the solid body rotation of the particle. When the rod is perfectly aligned and stationary along the direction of the stream, the tumbling contributions due to strain and vorticity are in direct opposition to each other. But, at some later time, the tumbling due to strain and vorticity will work together to cause the rod to instantaneously rotate, or tumble, about the plane and become anti-aligned with the flow field. The same picture is true for a disk. The only difference is that the disk tends to be perpendicular to the direction of the stream.

2.5 Deformation and Strain

As mentioned previously, the symmetric strain-rate tensor, $S_{ij}$, is characterized by three eigenvectors, $\hat{e}_i$, which define principle directions of strain-rate. In other words, they provide a basis for studying stretching phenomenon and the rate at which is fluid is being compressed or extended. While previous work has used the strain-rate eigenvectors to characterize the preferential
alignments of axisymmetric ellipsoids with principle axes of stretching, R. Ni. et al. (2014) claim that alignment in turbulence is more easily understood using the deformation tenor, $F_{ij}$ [23]. This second rank tensor quantifies the total integrated deformation of a continuum over a given region. For a two-dimensional flow, this process can be visualized by considering a infinitesimal circular fluid element that is deformed into a bi-axial ellipsoid as it translates in space.

**Figure 2.6:** A circular fluid element at an initial position, $X$, at an initial time, $t_0$, experiences a bi-axial deformation as it moves to a final position, $x$, at a later time, $t_0 + \Delta t$. The vectors $\hat{e}_{R1}$ and $\hat{e}_{R2}$ are the eigenvectors of the right Cauchy-Green strain tensor which defines the principle stretching directions at an initial time, $t_0$, while $\hat{e}_{L1}$ and $\hat{e}_{L2}$ are the eigenvectors of the left Cauchy Green strain tensor which define principle stretching at a later time, $t + \Delta t$.

The deformation tensor is formally defined as $F_{ij} = \frac{\partial x_i}{\partial X_j}$, where $X$ is an initial position and $x$ is a final position. It is not necessarily a symmetric tensor so its eigenvalues might not be real. As a result, we use the right and left Cauchy-Green strain tensors which are given by the inner products of the deformation tensor, $F$.

\[
\mathcal{C}^{(R)} = F^T F \quad (2.16) \\
\mathcal{C}^{(L)} = FF^T \quad (2.17)
\]

These two tensors have the same eigenvalues, $\Lambda_i$, but different eigenvectors. The eigenvectors of the right Cauchy-Green strain tensor, $\hat{e}_{Ri}$, define the principle stretching directions for a fluid element at the initial time, $t_0$, while the eigenvectors of the left Cauchy Green strain tensor, $\hat{e}_{Li}$.
\( \hat{e}_{Li} \), define the principle stretching directions at a later time, \( t_0 + \Delta t \). Similar to the strain-rate eigenvalues, the largest eigenvalue, \( \Lambda_1 \), represents extension while the smallest eigenvalue, \( \Lambda_3 \), represents compression. In addition, the intermediate eigenvalue, \( \Lambda_2 \), represents a class of objects that is either extensional or compressional. Together, these eigenvalues must satisfy the relation \( \Lambda_1 \Lambda_2 \Lambda_3 = 1 \).

To further highlight the physical meaning of the eigenvectors, \( \hat{e}_{R1} \) and \( \hat{e}_{L1} \), we will use an effective example shown by R. Ni et al. in 2014 [23]. Let’s consider a material line segment initially aligned in the direction \( \hat{e}_{R1} \), the eigenvector which corresponds to the most extensional eigenvalue, \( \Lambda_1 \). This can be represented by \( \textbf{l}(t_0) = \hat{e}_{R1} \) where \( \textbf{l}(t_0) \) is a deformable material line at an initial time. After some time \( \Delta t \), the material line will be deformed into \( \textbf{l}(t) = F \hat{e}_{R1} \). If we multiply both sides of this equation by \( \mathbf{C}(L) \) and substitute using equations 2.16 and 2.17, we will have the following:

\[
\mathbf{C}^{(L)} \textbf{l}(t) = \mathbf{C}^{(L)} [F \hat{e}_{R1}] = FF^T [F \hat{e}_{R1}] = FC^{(R)} \hat{e}_{R1} = FA \hat{e}_{R1} = \Lambda_1 \textbf{l}(t) \quad (2.18)
\]

As a result, the final direction of this material line is the eigenvector of the left Cauchy-Green strain tensor that corresponds to the maximum eigenvalue, \( \Lambda_1 \). Material lines initially aligned along an eigenvector of the right Cauchy-Green strain tensor will become aligned with the corresponding eigenvector of the left Cauchy-Green strain tensor at a later time. In addition, this material line will be stretched along this direction by a factor of \( \Lambda_1 \). This same derivation applies to the other two pairs of eigenvectors. In the case of \( \hat{e}_{R3} \), the material line will become aligned and compressed along \( \hat{e}_{L3} \) at a later time.

In this thesis, we will demonstrate that the eigenvectors of the Cauchy-Green strain tensor provide a natural basis for quantifying preferential alignments of axisymmetric ellipsoids in turbulence. We will compare their effectiveness to the effectiveness of the eigenvectors defined by the strain-rate tensor. The fundamental difference is that strain-rate eigenvectors define principles axes of a fluid’s instantaneous deformation rate while the Cauchy-Green eigenvectors quantify axes of total integrated deformation. We will use both sets of principle axes to quantify the preferential alignments of rods, disks, and vorticity. While we will be comparing the relative effectiveness of the strain-rate eigenvectors to the Cauchy-Green eigenvectors, we ultimately quantify their relationship as a simple coordinate transformation by studying the evolution of
the velocity gradient tensor. We will proceed by examining the time dependence of the velocity gradient in addition to the evolution of the strain-rate and rate-of-rotation tensor.

### 2.6 The Evolution of the Velocity Gradient Tensor

The evolution equation of the velocity gradient tensor is obtained by taking the spatial derivative of the incompressible three-dimensional Navier-Stokes equation. This operation yields

\[
\frac{\partial}{\partial t} \mathbf{A} + \mathbf{u} \cdot \nabla \mathbf{A} = -\mathbf{A}^2 - \mathbf{H} + \nu \nabla^2 \mathbf{A} + \mathbf{G} \tag{2.19}
\]

where \( \mathbf{H} = \nabla^2 P \) is the Hessian of the kinematic pressure field, and \( \mathbf{G} \) is the gradient of an arbitrary external force present on the fluid. This expression states that the time evolution of the velocity gradient is subject to self-amplification and attenuation, as well as nonlocal contributions due to pressure, viscosity, and external forcing [24]. If we decompose the velocity gradient tensor into its symmetric and antisymmetric parts, we can write the evolution of the strain-rate and rate-of-rotation tensors. We first consider the symmetric strain-rate tensor.

\[
\frac{\partial}{\partial t} \mathbf{S} + \mathbf{u} \cdot \nabla \mathbf{S} = -\left[ \mathbf{S}^2 - \frac{1}{3} \text{Tr}(\mathbf{S}^2) \mathbf{I} \right] - \frac{1}{4} \left[ \mathbf{W} \mathbf{W}^T - \frac{1}{3} \text{Tr}(\mathbf{W}^2) \mathbf{I} \right] - \mathbf{H} + \nu \nabla^2 \mathbf{S} \tag{2.20}
\]

In addition to the pressure and viscous contributions, the evolution of the strain-rate is due to self-amplification and attenuation as demonstrated by the first bracket on the right hand side of the equation as well as a contribution due to the rate-of-rotation shown in the second bracket. To develop a better understanding of this expression, we rewrite equation 2.20 in terms of the vorticity vector, \( \omega \), and discuss the components of \( \mathbf{S}^2 \), the square magnitude of the strain-rate.

\[
\frac{\partial}{\partial t} \mathbf{S} + \mathbf{u} \cdot \nabla \mathbf{S} = -\left[ \mathbf{S}^2 - \frac{1}{3} \text{Tr}(\mathbf{S}^2) \mathbf{I} \right] - \frac{1}{4} \left[ \omega \omega^T - \frac{1}{3} \text{Tr}(\omega^2) \mathbf{I} \right] - \mathbf{H} + \nu \nabla^2 \mathbf{S} \tag{2.21}
\]

If we work in the eigenframe of the strain-rate tensor, the first bracket on the right is diagonal and only dependent on the eigenvalues of the strain-rate tensor.
If we consider a counterclockwise rotation about the xy plane, we can write the vorticity vector as 

\[ \omega = \begin{pmatrix} -\omega_1 & \omega_2 & 0 \end{pmatrix} \]

and subsequently express \( \omega \omega^T \) as a second rank tensor in the following manner:

\[
\omega \omega^T = \begin{pmatrix}
\omega_1^2 & -\omega_1 \omega_2 & 0 \\
-\omega_2 \omega_1 & \omega_2^2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The inclusion of this vorticity term results in off-diagonal terms in strain-rate tensor at later times which physically corresponds to a rotation about the xy plane of the strain-rate eigenframe. While the nonlocal pressure and viscous terms contribute to this eigenframe rotation, there is an extensive literature for the case where \( \nu = 0 \) and the pressure Hessian term is ignored [25–27]. Neglecting this pressure contribution amounts to assuming that the pressure gradient is linear throughout space. These restriction leads to the restricted Euler equations where the evolution of the velocity gradient tensor is solely dependent on the initial conditions, \( \Lambda(t_0) \). Likewise, the evolution of the strain-rate tensor is only due to self-amplification, self-attenuation, and rotation due to vorticity. Additionally, we can write down an evolution equation for the rate-of-rotation tensor and the vorticity vector:

\[
\frac{\partial}{\partial t} \mathbb{W} + \mathbf{u} \cdot \nabla \mathbb{W} = -S \mathbb{W} - \mathbb{W} S + \nu \nabla^2 \mathbb{W} \tag{2.22}
\]
\[
\frac{\partial}{\partial t} \omega + \mathbf{u} \cdot \nabla \omega = -S \omega + \nu \nabla^2 \omega \tag{2.23}
\]

The vorticity is an active vector that is amplified and attenuated by stretching and affected by viscosity. This viscous contribution plays an important role in the alignment of vorticity with the principle stretching directions defined by eigenvectors of the strain-rate and Cauchy-Green strain tensors. While the vorticity exhibits a preferential alignment with fluid stretching,
the vorticity will not stay in perfect alignment for more than a few Kolmogorov times as a result of this viscous contribution. We will demonstrate this dynamical picture with real time movies of single particle trajectories and contrast the time dependent alignment of the active vorticity with the more passive vector, \( \hat{p} \), which defines axisymmetric ellipsoid orientation.
Computational Algorithms

In this section, we describe our direct numerical simulation data and the basic parameters of our flow. Our analysis requires the use of numerical integration, so here we describe the algorithms used. Additionally, we describe our analysis codes written in Matlab and acknowledge numerical inefficiencies.

3.1 Direct Numerical Simulation Data

Our project works with a direct numerical simulation of homogenous isotopic turbulence provided to us by Federico Toshi and Enrico Calzavarini. The data was generated from a simulation with $N^3 = 512^3$ collocation points with a corresponding Taylor-microscales Reynolds number of $R_\lambda = 180$, which corresponds to a Reynolds number of $Re = 2160$ as given by $R_\lambda = \sqrt{15Re}$. A total of $1.2 \times 10^4$ particles trajectories were followed for $O(1)$ large-eddy turnover times. These particle trajectories are stored in 10 different .h5 files, 316 MB in size. Each particle trajectory evolves for a total of 330 $\tau_\eta$, and the velocity vector and gradient is stored at every tracer position across the grid. The ratio of the energy dissipation rate to the kinematic viscosity is $\epsilon/2\nu = 227s^{-2}$, as given by equation 1.2, and the Kolmogorov time is $\tau = 0.04s$, as given by equation 1.3. The time step used to integrate Navier-Stokes equations is $O(10^{-2}\tau_\eta)$ and the data along each trajectory is stored every $0.1\tau_\eta$. The orientations of rods and disks is obtained by integrating Jeffery’s equation (equation 1.4) along each particle trajectory. For all of our
calculations, disks are defined by $\alpha = 0.1$ and rods by $\alpha = 10$.

### 3.2 Numerical Integration

We use numerical integration to obtain $p$ from Jeffery’s equation and the deformation gradient tensor, $F$, from the velocity gradient tensor. For both cases, we use a fourth order Runge-Kutta scheme to integrate along particle trajectories. We proceed to briefly outline the general problem for which we use Runge-Kutta.

We approximate the solution to a first order differential equation given by

$$\frac{dy(t)}{dt} = f(y, t)$$  \hspace{1cm} (3.1)

with an appropriate initial condition, $y(t_0) = y_0$. We can approximate the slope at $t_0$ using a time step, $h$, and defining the following quantities:

$$k_1 = f(y(t_0), t_0)$$

$$k_2 = f(y(t_0) + k_1 \frac{h}{2}, t_0 + \frac{h}{2})$$

$$k_3 = f(y(t_0) + k_2 \frac{h}{2}, t_0 + \frac{h}{2})$$

$$k_4 = f(y(t_0) + k_3 h, t_0 + h)$$

We can describe all quantities physically. For example, $k_1$ is the slope at the beginning of the time step. If we use this $k_1$ to step halfway through the next time step, then $k_2$ is an estimate of the slope at the midpoint. Likewise, if we use $k_2$ to step halfway through the next time step, then we obtain $k_3$, an additional estimate of the slope at the midpoint. Lastly, we use $k_3$ to step forward one full time step and obtain $k_4$, an estimate of the slope at the endpoint. We can now determine a weighted sum of these slopes, $m$, in order to determine our estimate of $y(t + \Delta t)$.

$$y(t_0 + \Delta t) = y(t_0) + m \Delta t = y(t_0) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\Delta t$$
When we integrate Jeffery’s equation, we must specify an appropriate initial condition, \( \hat{\mathbf{p}}(t_0) \). Since we are considering homogenous isotropic turbulence, our choice of initial orientation is arbitrary. In our code, we use the Matlab function, \texttt{rand()}, to define a random initial orientation. On the other hand, when we integrate the deformation gradient, \( \mathbf{F}(t) \), our initial condition, \( \mathbf{F}(t_0) \), is equal to the identity tensor, \( \mathbb{I} \). In order to accurately determine flow deformations, we must choose an appropriate \( \Delta t \) for which we integrate the deformation tensor. This choice of parameter affects the degree of alignment that we measure. We provide a more complete description of this problem in section 4.2.

### 3.3 Matlab Codes

We perform numerical analysis and generate plots using Matlab. As mentioned previously, each data set is 316 MB. Our analysis was sometimes limited by long run times for our codes. This problem only arose for certain calculations. For example, producing a probability distribution function (PDF) for \( \hat{\mathbf{p}} \) with \( \hat{\mathbf{e}}_i \), the strain-rate eigenvectors, over one full data set takes approximately 9 minutes (figure 4.1). As a result, it only takes an hour and a half to produce these figures over 10 data sets. On the other hand, calculations involving the deformation gradient, \( \mathbf{F} \), require longer run times. While the strain-rate tensor only relies on the instantaneous velocity gradient, the deformation tensor at a given time is the result of the total integrated deformation from an initial time up until that given time. We produce PDF’s for the Cauchy-Green eigenvectors, \( \hat{\mathbf{e}}_L_i \), using \( \Delta t = 20\tau_\eta \) (figure 4.3, figure 4.4b). These plots take approximately 66 minutes to produce which results in over 11 hours in run time for all data sets. While this difference in time between the strain-rate PDF’s and the Cauchy-Green PDF’s is on the order of 10, it was not much of a burden to run these codes over night for 11 hours. On the other hand, this difference in run time played a huge role in our analysis of the alignment of rods and disks with the Cauchy-Green eigenvectors as a function of the integration time (figure 4.2b, figure 4.5). For one full data set, it takes approximately 13 hours to run, and as a result, it would take almost 5 and a half days to produce plots for all data sets. Fortunately for us, we validated our numerical algorithms with one full data set by comparison with the results of Ni et al for rods. We reproduce this in figure 4.2b.
Chapter 4

Results

In this section, we present our work comparing the rotational dynamics and alignment of rods and disks in turbulence. We summarize previous study on rods and demonstrate the relative effectiveness of the Cauchy-Green strain tensors to describe particle alignment with fluid stretching. Rods and vorticity both preferentially align with the strongest stretching direction, as defined by the maximum eigenvector of the Cauchy-Green strain tensor, \( \hat{e}_{L1} \). As a result, both rods and vorticity align with each other. We subsequently present our work on disks and show that the Cauchy-Green eigenframe is the natural eigenframe for studying the alignment of disks due to fluid stretching. While rods and vorticity both tend to align with \( \hat{e}_{L1} \), disks tend to preferentially align with the weakest stretching direction, as defined by the minimum eigenvector of the Cauchy-Green strain tensor, \( \hat{e}_{L3} \). As a result, disks and vorticity are preferentially perpendicular to each other. Lastly, we provide a phenomenological description of the time evolution of the velocity gradient tensor in order to quantify a relationship between the strain-rate tensor and the Cauchy-Green strain tensor. We will demonstrate that further study is needed to determine how pressure and viscosity contributes to the strain-rate evolution.
4.1 The Alignment of Rods and Vorticity with the Strain-Rate Tensor

The alignment of rods and vorticity in fluid turbulence has been an active research topic for the past few decades. Ashurst et al. (1987) first showed the preferential alignment of the vorticity, \( \hat{\omega} \), with the intermediate eigenvector of the strain-rate tensor, \( \hat{e}_2 \) [28]. Pumir and Wilkinson (2011) numerically showed the preferential alignment of rods, \( \hat{p} \), with \( \hat{e}_2 \) [29]. In addition, they demonstrated the strong alignment between rods and vorticity. While these results have already been established, we will reproduce these results using our simulation data.

Figure 4.1: On the left, we have the probability distribution function of the cosine of the angle between the vorticity, \( \hat{\omega} \), and the eigenvectors of the strain-rate tensor, \( \hat{e}_i \). On the right, we have the probability distribution function of the cosine of the angle between rods, \( \hat{p} \), with the vorticity, \( \hat{\omega} \), and the eigenvectors of the strain-rate tensor, \( \hat{e}_i \).

The symmetric strain-rate tensor, \( S_{ij} \), is used to quantify the instantaneous alignment of rods and vorticity with fluid velocity gradients. Figure 4.1a shows the probability distribution function (PDF) of the cosine of the angle, \( \theta \), between \( \hat{\omega} \) and eigenvectors of the strain-rate tensor. We use the cosine of the angle because 3D vectors that are randomly oriented with respect to each other have a uniform distribution of cosines. In addition, when making these PDF’s, we normalize all vectors so that the dot products between them are equal to \( \cos \theta \). As we know from previous work, the vorticity tends to preferentially align most strongly with the intermediate strain-rate eigenvector \( \hat{e}_2 \) [28]. This curious result has caused some confusion since one
intuitively expects the vorticity to align with the most extensional eigenvector, \( \hat{e}_1 \), due to conservation of angular momentum [23]. Furthermore, figure 4.1b shows the alignment of rods, \( \hat{p} \), with vorticity and the eigenvectors of the strain-rate tensor. Rods align most strongly with vorticity. In addition, rods are strongly aligned with \( \hat{e}_2 \), slightly less aligned with \( \hat{e}_1 \), and strongly perpendicular to \( \hat{e}_3 \) which suggests that rods live in the plane defined by \( \hat{e}_1 \) and \( \hat{e}_2 \).

### 4.2 The Alignment of Rods and Vorticity with the Cauchy-Green Strain Tensor

Similar to the strain-rate eigenvectors, the eigenvectors of the Cauchy-Green strain tensor provide three principle axes in which fluid is stretched: \( \hat{e}_{L1}, \hat{e}_{L2}, \) and \( \hat{e}_{L3} \). As previously mentioned, the main difference between the strain-rate and Cauchy-Green eigenframes is that the former characterizes the rate at which a fluid deforms while the later characterizes the total integrated deformation. As a result, while the strain-rate eigenframe allows us to study the instantaneous alignment of rods with velocity gradients, the Cauchy-Green eigenframe quantifies the alignment with stretching directions over a longer range of Kolmogorov times. To understand this distinction more fully, let’s reconsider the circular fluid element introduced in section 2.5. At some initial time, \( t_0 \), we have a circular fluid element which becomes an ellipse at a later time, \( t = t_0 + \Delta t \). The position of any point, \( X \), inside the circular element can be mapped to a different position, \( x \), at this later time by considering the deformation gradient tensor, \( F \). It is useful to consider the evolution equation of the deformation gradient which we express in terms of the instantaneous velocity gradient:

\[
\frac{dF_{ij}(t)}{dt} = A_{ik}F_{kj}(t)
\]  

The initial condition is \( F_{ij}(t_0) = \delta_{ij} \). We use Runge-Kutta to numerically determine \( F_{ij} \) at each successive time. As mentioned in section 3.2, we must choose an appropriate \( \Delta t \) for which we integrate the deformation tensor. In figure 4.2, we show the alignment of rods and vorticity with the Cauchy-Green eigenvectors, \( \hat{e}_{Li} \), as a function of the integration time, \( \Delta t \). The parameter, \( \Delta t \), significantly contributes to the degree of alignment that we measure but ultimately saturates at 0.8 for rods when \( \Delta t > 20\tau_\eta \) and around 0.55 for vorticity when \( \Delta t > 10\tau_\eta \). While figure 4.2b
is insensitive to small changes in aspect ratio above \( \alpha = 10 \), there is an integration accuracy limit. Perfect integration would result in perfect alignment after long times.

Figure 4.2 shows that for an appropriate \( \Delta t \), rods and vorticity align most strongly with the most extensional stretching direction, \( \hat{e}_{L1} \), as opposed to the intermediate direction, \( \hat{e}_2 \), as mapped by the strain-rate eigenframe. The degree of alignment of rods with \( \hat{e}_{L1} \) is stronger than it is for the vorticity since vorticity is an active vector whose evolution in time is dictated by the flow’s strain-rate and viscosity (equation 2.22). The contribution from strain moves \( \hat{\omega} \) towards \( \hat{e}_{L1} \) while the viscous contribution tends to move them apart. This accounts for why the vorticity is not perfectly aligned with \( \hat{e}_{L1} \). Furthermore, since both \( \hat{p} \) and \( \hat{\omega} \) independently align with \( \hat{e}_{L1} \), they both preferentially align with each other.

\[
R = \langle [\hat{e}_{L1} \cdot \hat{\omega}]^2 \rangle
\]

\[
R = \langle [\hat{e}_{L2} \cdot \hat{\omega}]^2 \rangle
\]

\[
R = \langle [\hat{e}_{L3} \cdot \hat{\omega}]^2 \rangle
\]

\[
R = \langle [\hat{e}_{L1} \cdot \hat{p}]^2 \rangle
\]

\[
R = \langle [\hat{e}_{L2} \cdot \hat{p}]^2 \rangle
\]

\[
R = \langle [\hat{e}_{L3} \cdot \hat{p}]^2 \rangle
\]

\( \Delta t/\tau_\eta \)

\( \Delta t/\tau_\eta \)

---

Figure 4.2: The alignment of rods and vorticity with the Cauchy-Green eigenvectors as a function of the time, \( \Delta t \), used to integrate the deformation tensor.

It is worth taking some time to consider the alignment of rods and vorticity with the eigenvectors of the Cauchy-Green strain tensor in the limit that our integration time, \( \Delta t \), is infinitesimally small. Using the initial condition \( F_{ij}(t_0) = \delta_{ij} \), we linearize the deformation gradient at a later time, \( t = t_0 + \Delta t \), using Euler’s method in the following manner:

\[
F_{ij}(t) = \delta_{ij} + A_{ij} \Delta t
\]  

(4.2)

As a result, the left Cauchy-Green strain tensor can be written as
\[ C^{(L)} = \mathbb{F} \mathbb{F}^T = (\delta + A \Delta t)(\delta + A \Delta t)^T \]
\[ C^{(L)} = \mathbb{F} \mathbb{F}^T = \delta + (A + A^T) \Delta t + ... \]

This expression is composed of an isotropic term, \( \delta \), and a contribution from the strain-rate tensor, \( S \equiv (A + A^T) \). As a result, the eigenvectors of \( C^{(L)} \) are the strain-rate eigenvectors for a small infinitesimal \( \Delta t \). At \( \Delta t = 0 \), figure 4.2 tells us that the instantaneous alignment of \( \hat{\omega} \) with \( \hat{e}_i \) is 0.32, 0.52, 0.16 for \( i = (1, 2, 3) \). This demonstrates that the vorticity is most preferentially aligned with \( \hat{e}_2 \), consistent with figure 4.1a. Similarly, the instantaneous alignment of rods with the eigenvectors of the strain-rate tensor, \( \hat{e}_i \), is 0.4, 0.44, and 0.16 for \( i = (1, 2, 3) \). This suggests that rods are more preferentially aligned with \( \hat{e}_2 \) than with \( \hat{e}_1 \) which is also consistent with figure 4.1b.

![Figure 4.3](image-url)

**Figure 4.3:** On the left, we have the probability distribution function of the cosine of the angle between the vorticity, \( \hat{\omega} \), and the eigenvectors of the Cauchy-Green strain tensor, \( \hat{e}_{L_i} \). On the right, we have the probability distribution function of the cosine of the angle between rods, \( \hat{p} \), with the vorticity, \( \hat{\omega} \), and the eigenvectors of the Cauchy-Green strain tensor, \( \hat{e}_{L_i} \).

To conclude our discussion on the alignment of rods in turbulence, figure 4.3 shows the probability distribution function of the cosine of the angle between the vorticity and the Cauchy-Green eigenvectors, as well as rods with the Cauchy-Green eigenvectors. The deformation tensor used to create these figures was calculated with an integration time of \( \Delta t = 20 \). These results are ultimately consistent with what figure 4.2 tells us: both rods and vorticity are preferentially
aligned along the direction of $\hat{e}_{L1}$. As a result, both $\hat{p}$ and $\hat{\omega}$ align strongly with each other, as shown by the turquoise curve in figure 4.3b. While both $\hat{e}_2 \cdot \hat{p}$ and $\hat{e}_2 \cdot \hat{\omega}$ have a high probability of alignment, as demonstrated by figure 4.1, the alignment between $\hat{e}_{L1} \cdot \hat{p}$ and $\hat{e}_{L1} \cdot \hat{\omega}$ is significantly greater. We can understand this qualitatively by just comparing the two figures and noting the relative difference in number of samples that are either perfectly aligned or perpendicular. Alternatively, we can compare maximum values of probability for these corresponding curves in order to analyze these differences more quantitatively. There is probably some inaccuracy in these numbers due to the steep distribution which changes significantly across one bin, but they help provide a better sense for how strongly rods, disks, and vorticity align with principle stretching directions. In the strain-rate eigenframe, the normalized probability for rods is maximum along the direction of $\hat{e}_2$ with a maximum value of PDF = 2.81. The maximum probability for $\hat{e}_1 \cdot \hat{p}$ is PDF = 1.72 which occurs when $\hat{e}_1 \cdot \hat{p} = 1$. As a result, rods are much more likely to be perfectly aligned with $\hat{e}_2$ than with $\hat{e}_1$. Furthermore, the maximum probability for $\hat{e}_3 \cdot \hat{p}$ is PDF = 2.08 which occurs when $\hat{e}_3 \cdot \hat{p} = 0$. On the other hand, the maximum values of probability for rods in the Cauchy-Green eigenframe are greater than the strain-rate probabilities by more than an order of magnitude. We limit the y-axis of figure 4.3b to present these ideas clearly. For the alignment of rods with $\hat{e}_{L1}$, the normalized probability has a maximum of PDF = 157 when $\hat{e}_{L1} \cdot \hat{p} = 1$. The maximum probability of rods with $\hat{e}_{L2}$ is PDF = 9.12 when $\hat{e}_{L2} \cdot \hat{p} = 1$, and the maximum probability with $\hat{e}_{L3}$ is PDF = 64.8 when $\hat{e}_{L3} \cdot \hat{p} = 0$. In this way of looking at the data, it seems that rods have the greatest probability of being perfectly aligned with $\hat{e}_{L1}$ in addition to a significant probability of being perpendicular to $\hat{e}_{L3}$. In section 4.4, we will present a more accurate visualization for understanding the alignment of rods in turbulence.

### 4.3 The Alignment of Disks in Turbulence

In this section, we present our new results for disks. We quantify the alignment of disks in turbulence using the same techniques as described for rods. In figure 4.4, we plot the probability distribution function for disks with the eigenvectors of the strain-rate tensor and the Cauchy-Green strain tensor. Similar to before, the deformation tensor used to created these figures was calculated with an integration time of $\Delta t = 20$. In the strain-rate eigenframe, disks tends to preferentially align with the most compressional eigenvector, $\hat{e}_3$. The maximum normalized probability for disks along the direction of $\hat{e}_3$ is PDF = 3.41 when $\hat{e}_3 \cdot \hat{p} = 1$. In the directions
entirely perpendicular to $\hat{e}_1$ and $\hat{e}_2$, PDF is equal to 1.22 and 1.98 respectively. As a result, disks are more likely to be preferentially perpendicular to $\hat{e}_2$ than $\hat{e}_1$. This physical picture corresponds to disks preferentially aligned in and rotating about the plane defined by $\hat{e}_1$ and $\hat{e}_3$. The reason disks tend to be perpendicular with $\hat{e}_2$ has to do with the fact that the vorticity tends to align with $\hat{e}_2$ (figure 4.1a) and disks orient in directions perpendicular to $\hat{e}_2$, as shown by the turquoise curve in figure 4.4a and 4.4b.

Figure 4.4: On the left, we have the probability distribution function of the cosine of the angle between disks, $\omega$, and the eigenvectors of the strain-rate tensor, $\hat{e}_i$. On the right, we have the probability distribution function of the cosine of the angle between disks, $\hat{p}$, and the eigenvectors of the Cauchy-Green strain tensor, $\hat{e}_{Li}$.

In the Cauchy-Green eigenframe, disks also align most strongly with the most compressional eigenvector, $\hat{e}_{L3}$. Similar to rods, the maximum values of probability for disks in the Cauchy-Green eigenframe are greater than the strain-rate probabilities by more than 1 to 2 orders of magnitude. Again, we limit the y-axis of figure 4.4b to present these ideas clearly. For the alignment of disks along $\hat{e}_{L3}$, the normalized probability has a maximum at PDF= 275. Likewise, the maximum value of probability in a direction perfectly perpendicular to $\hat{e}_{L1}$ and $\hat{e}_{L2}$ is PDF= 63.7 and PDF= 24.3 respectfully. As a result, this data suggests that disks are most likely to be aligned along $\hat{e}_{L3}$, and more likely to be perpendicular to $\hat{e}_{L1}$ than to $\hat{e}_{L2}$. The reason why disks are more likely to be perpendicular to $\hat{e}_{L1}$ follows the same logic used to describe disks in the strain-rate eigenframe. Namely, in the Cauchy-Green eigenframe, the vorticity aligns strongly with $\hat{e}_{L1}$ (figure 4.3a, 4.4a), and disks align in directions preferentially perpendicular to the vorticity (figure 4.4).
We can better understand the dynamics of disks in the Cauchy-Green eigenframe by examining the alignment with $\hat{e}_{Li}$ for different times, $\Delta t$, used to integrate the deformation tensor, $\mathbf{F}$. We summarize these results in figure 4.5. Compared to the alignment of rods, as shown in figure 4.2b, the alignment of disks with the Cauchy-Green strain tensor occurs more rapidly. While rods become almost perfectly perpendicular to $\hat{e}_{L3}$ within $\Delta t \sim 10\tau_\eta$, it takes a much longer time to become perpendicular to $\hat{e}_{L2}$. Rather, rods initially lie in the plane defined by $\hat{e}_{L1}$ and $\hat{e}_{L2}$. In the long time limit, rods tend to align more strongly with $\hat{e}_{L1}$ and become perpendicular to $\hat{e}_{L2}$. On the other hand, disks preferentially align with $\hat{e}_{L3}$ and orient perpendicular to both $\hat{e}_{L1}$ and $\hat{e}_{L2}$ within $\Delta t \sim 15\tau_\eta$. Previous work on the alignment of rods in isotropic turbulence shows that the slow approach of rods to perfect alignment with $\hat{e}_{L1}$ is due to events where the most extensional eigenvalue, $\Lambda_1$, is approximately equal to the intermediate eigenvalue, $\Lambda_2$ [23]. For disks, a similar phenomenon occurs for events where the most compressional eigenvalue, $\Lambda_3$, is approximately equal to $\Lambda_2$. However, events where fluid deforms into a cigar shape ($\Lambda_3 \approx \Lambda_2$) are much less common than events where fluid deforms into a pancake shape ($\Lambda_2 \approx \Lambda_1$). As a result, disks are more likely to experience events where they immediately become perpendicular to both $\hat{e}_{L1}$ and $\hat{e}_{L2}$, as opposed to rods that only become perpendicular to $\hat{e}_{L2}$ after long times.
4.4 An Alternate Means for Visualizing Alignment

In this section, we plot our probability distributions on the unit sphere. While 1D probability distributions gave us a sense for how rods, disks, and vorticity align in turbulence, probability distributions on the unit sphere provide a means for which it is easier to obtain an accurate understanding of alignment. Likewise, we believe these figures more clearly demonstrate how the eigenvectors of the Cauchy-Green strain tensor provide a natural basis for studying the alignment of axisymmetric particles. Using the eigenvectors of the strain-rate tensor and the Cauchy-Green strain tensor, we construct our spherical coordinate system defined by coordinates \((r, \theta, \phi)\) for both eigenframes. The coordinate, \(r\), is a radial distance measured from the origin, the polar angle, \(\theta\), is measured relative to \(\hat{e}_1\) and \(\hat{e}_{L3}\), and the azimuthal angle, \(\phi\), spans the plane defined by \((\hat{e}_1, \hat{e}_2)\) and \((\hat{e}_{L1}, \hat{e}_{L2})\). As always, our \(i=(1,2,3)\) indices correspond to \((x, y, z)\). We present this geometry for the Cauchy-Green eigenframe in the figure below.

![Figure 4.6: Spherical coordinate system as defined by the Cauchy-Green eigenvectors, \(\hat{e}_{L_i}\).](image)

We create spherical histograms by plotting the probability density on the unit sphere. We used a geodesic tessellation of the sphere with 10,242 bins. A geodesic tessellation does not produce exactly uniform bins, and we address this by assigning each sample to a bin if its dot product
with the bin unit vector is larger than a given threshold. Due to the slightly non-uniform bin spacing, there are some samples that fall in no bins and some samples that fall in two bins. As a result, we set the threshold to insure that each sample falls in one bin on average. This method produces a reliable spherical histogram.

In order to represent particle alignment in the strain-rate eigenframe and the Cauchy-Green eigenframe, we must consider the fact that eigenvectors have a sign ambiguity since multiplication by -1 results in the same eigenvector. We could place all particle orientations in the first octant, but we chose to make a full spherical histogram by including all permutations of positive and negative sign conventions for our coordinate axes. We create these figures using 1 full data set corresponding to 1280 particle trajectories over a range of 3300 time steps. As mentioned previously, one data point corresponds to 0.1\(\tau_\eta\) so our 3D histograms were made for each particle trajectory over 330 Kolmogorov times. If we account for the 8 possible coordinate systems, we use \(3.4 \times 10^7\) data points to construct our spherical histograms.

![Figure 4.7: The alignment of rods in the strain-rate eigenframe](image)

In figures 4.7 and 4.8, we show the alignment of rods in the strain-rate and Cauchy-Green eigenframe. As demonstrated before, the rods align most strongly with \(\hat{e}_2\) and less strongly with \(\hat{e}_1\) in the strain-rate eigenframe. The picture is straightforward and consistent with our previous picture. Similarly, the spherical histogram for rods in the Cauchy-Green eigenframe shows that rods have an extremely high probability of aligning directly along \(\hat{e}_{L1}\). This probability
dramatically drops off as we move off axis, from one bin to the next, and as a result, we observe a warm, non-black strip entirely confined to a small range of angles around $\hat{e}_{L1}$. In addition, our figure accurately suggests that there is little probability for our rods to lie along $\hat{e}_{L3}$, consistent with what we have shown in figure 4.2b and 4.3b.

While these results are somewhat consistent with our previous work, the degree to which rods are confined to a very small number of bins along $\hat{e}_{L1}$ is quite curious. We initially expected to see a more gradual drop-off in probability as rods moved away from $\hat{e}_{L1}$ as opposed a drastic drop-off on the order of a factor of 10 as we sweep out a small angle $\phi$ away from $\hat{e}_{L1}$. To challenge this intuition, we revisit the corresponding 1D histogram shown in figure 4.3b. This figure tells us that the probability of rods preferentially aligning with $\hat{e}_{L1}$ (PDF = 157) is much greater than the probability that rods are preferentially perpendicular to $\hat{e}_{L3}$ (PDF = 64.8). On the other hand, our 3D spherical histogram suggests that the probability of being perpendicular to $\hat{e}_{L3}$ is much greater than the probability of aligning with $\hat{e}_{L1}$. This intuitively makes sense since there are more ways for rods to be perfectly perpendicular to $\hat{e}_{L3}$ than perfectly aligned with $\hat{e}_{L1}$.

To quantify this, I calculated how the number of bins changes as a function of $\theta$ on the surface of the sphere. In other words, some bins will span a greater range of solid angles dependent on how far away it is from the pole, along $\hat{e}_{L3}$. In order to convince myself of this
fact, I decided to calculate how bins were concentrated around the pole, along $\hat{e}_{L3}$, and around the equator, along $\hat{e}_{L1}$. The following integral relates the total surface area of the sphere to the total number of bins, $b$, in addition to the starting and ending values of theta, $\theta_1$ and $\theta_2$ around the given axis of interest:

$$\frac{4\pi}{b} = \int_0^{2\pi} \int_{\theta_1}^{\theta_2} \sin \theta d\theta d\phi$$

(4.3)

We will first compute the angle, $d\theta_{L3}$, needed to include one ring of bins off axis from the pole. Our expressions looks as follows:

$$\frac{4\pi}{b} = 2\pi \int_0^{d\theta_{L3}} \sin \theta d\theta = -\cos \theta \bigg|_0^{d\theta_0}$$

Our starting angle, $\theta_1$, is equal to 0 since we measure our angle from the pole. We are now left with an expression for $d\theta_{L3}$ dependent on the total number of bins:

$$d\theta_{L3} = \cos^{-1} \left(1 - \frac{2}{b}\right)$$

Similarly, we can obtain expression for the angle, $d\theta_{L1}$, that includes one ring of bins off axis from $\hat{e}_{L1}$. As a result, $\theta_1 = \frac{\pi}{2} - d\theta_{L1}$ and $\theta_2 = \frac{\pi}{2}$.

$$\frac{4\pi}{b} = 2\pi \int_{\frac{\pi}{2}-d\theta_{L1}}^{\frac{\pi}{2}} \sin \theta d\theta = -\cos \theta \bigg|_{\frac{\pi}{2}-d\theta_{L1}}^{\frac{\pi}{2}}$$

which yields an expression for $d\theta_{L1}$ given by the following:

$$d\theta_{L1} = \sin^{-1} \left(\frac{2}{b}\right)$$

(4.4)

Using $b = 10,242$ to make our spherical histograms, we determine that $d\theta_{L3} = 1.98 \times 10^{-2}$ rad and $d\theta_{L1} = 1.96 \times 10^{-4}$ rad which tells us that there are approximately $10^2$ times as many bins around the pole at $\hat{e}_{L3}$ than along $\hat{e}_{L1}$ on the equator. This accounts for why we only see such
a narrow strip of samples off axis from $\hat{e}_{L1}$. Likewise, this difference in bin number ultimately explains why it appears as though the probability of being perpendicular to $\hat{e}_{L3}$ is much greater than the probability of aligning with $\hat{e}_{L1}$.

![Figure 4.9: The alignment of the vorticity in the (a) strain-rate eigenframe and the (b) Cauchy-Green eigenframe.](image)

We also present the alignment of vorticity in both eigenframes in figure 4.9. In addition, we present the alignment of disks in both eigenframes in figure 4.10 and 4.11. We note the similarity between rods and disks in the Cauchy-Green eigenframe. Namely, the high probability of being perpendicular to the $(\hat{e}_{L2}, \hat{e}_{L3})$ and $(\hat{e}_{L1}, \hat{e}_{L2})$ plane respectively. In addition, the high probability of disks aligning with $\hat{e}_{L3}$ further illustrates how disks are likely to experience events where the fluid deforms into a pancake ($\Lambda_2 \approx \Lambda_1$), as previously discussed in section 4.3.

### 4.5 Rotational Dynamics of Single Trajectories

While the crux of this thesis focuses on ensembles of particle trajectories, we can also learn a great deal by studying how alignment and rotation rates change as a function of time for single particle trajectories. We began this investigation in an attempt to better understand some work performed by our collaborators on the shape dependence of particle rotations in isotropic turbulence [9]. Byron et al. (2015) looked closely at the dynamical differences between rods and disks and how shape significantly affects spinning, tumbling, and angular velocity. This principle investigation involves phenomenologically describing disks that pass through regions
of intense vorticity. In the small particle limit, intense fluid vorticity directly affects a particle’s solid body rotation, $\Omega$, and in this section, we compare the rotational dynamics of rods and disks by presenting these single trajectories over a range of $50\tau_\eta$.

In figure 4.12a, we plot the magnitude of the angular velocity squared, $|\Omega|^2$, for a given rod and disk as a function of time. We show that this quantity for the rod is almost identical to
Figure 4.12: Instantaneous alignment and rotation rates for a rod and disk over a range of $50 \tau_\eta$. In figure (a), we plot half the flow vorticity squared, $|\omega|^2$, alongside the angular velocity, $|\Omega|^2$ for a rod and disk. In figure (b), we show the alignment of the rod and disk with flow vorticity. In figure (c), we plot the alignment of the disk with $\hat{e}_1$ and $\hat{e}_3$. In figure (d), we show the alignment of the flow vorticity with $\hat{p} \times \hat{S} \hat{p}$. In the presence of high fluid vorticity, between $t/\tau_\eta = 60$ and $t/\tau_\eta = 80$, we observe that the disk’s angular velocity fluctuates rapidly. This qualitative difference is accounted for by the fact that rods, in the presence of high vorticity, instantaneously align with fluid velocity gradients while disks orient in perpendicular directions. We demonstrate this behavior in figure 4.12b. In addition to aligning perpendicular to the vorticity, the disk must also be perpendicular to $\hat{e}_2$ since the vorticity aligns strongly with
the intermediate strain-rate eigenvector. As a result, the disk must align with \( \hat{e}_1 \) and \( \hat{e}_3 \), the eigenvectors that define the plane perpendicular to \( \hat{e}_2 \) and \( \hat{\omega} \). While the vorticity tends towards a stable orientation during this short time interval, figure 4.12c shows the disk rapidly fluctuating between alignment with \( \hat{e}_1 \) and \( \hat{e}_3 \). We can better understand this physical picture by considering the tumbling contribution from strain, \( \hat{p} \times \mathbb{S}\hat{p} \), in this high vorticity region. In figure 4.12d, we show that, when the vorticity is strong (between \( t/\tau_\eta = 60 \) and \( t/\tau_\eta = 80 \)), the tumbling contribution from strain both enhances and opposes the rotation of \( \hat{p} \) around the vorticity, aligned along \( \hat{e}_2 \). This cross product is significant in magnitude and is responsible for these rapid fluctuations in the disk’s total angular velocity. For additional investigation on these particular particle trajectories, see the supplementary videos paired with this thesis. Here, we provide movies of these single trajectories in the eigenframe of the lab, the strain-rate tensor, and the Cauchy-Green strain tensor. This serves to provide a clearer picture of the disk’s rotation about the plane spanned by \( \hat{e}_1 \) and \( \hat{e}_3 \). Likewise, these videos further make the case that the Cauchy-Green eigenframe provides an effective means for describing the alignment of disks in turbulence.

While this particular trajectory provides an interesting picture of the rotational dynamics of disks in high vorticity regions, the story is much different in regions of low vorticity. In fact, this single trajectory is not representative of the larger ensemble of particles. We provide a more common example below in figure 4.13. In figure 4.13a, we show that in regions of low vorticity, the total solid body rotation rate of the disk does not fluctuate rapidly. In fact, its total solid body rotation rate is approximately less than \( \frac{1}{2} |\hat{\omega}|^2 \) over this range of time. Similarly, since the rod aligns with fluid velocity gradients, its total solid body rotation rate is also greater than the disk’s. In contrast to this general picture, we also see that the disk’s angular velocity spikes in magnitude when the vorticity rises (\( t/\tau_\eta = 50 \) to \( t/\tau_\eta = 55 \)) which is consistent with figure 4.12a. Furthermore, figures 4.13b demonstrates that rods instantaneously align with the vorticity while disks instantaneously orient in directions perpendicular. Figure 4.13c shows the instantaneous alignment of the disk with \( \hat{e}_3 \). This figure is in contrast to 4.12c where we see much more rapid fluctuation between alignment with \( \hat{e}_1 \) and \( \hat{e}_3 \) as a result of intense vorticity due to strain. For this particular trajectory, the vorticity due to strain, as shown in figure 4.13d, is much weaker in comparison to the single event shown in 4.12d. As a result, the disk does not rotate about the \((\hat{e}_1, \hat{e}_3)\) plane in the fashion that we described previously.
Figure 4.13: Similar to figure 4.12, we plot the instantaneous alignment and rotation rates for a rod and disk over a range of $50 \tau_\eta$ for a more representative single trajectory. In figure (a), we plot half the flow vorticity squared, $|\omega|^2$, alongside the angular velocity, $|\Omega|^2$ for a rod and disk. In figure (b), we show the alignment of the rod and disk with flow vorticity. In figure (c), we plot the alignment of the disk with $\hat{e}_1$ and $\hat{e}_3$. In figure (d), we show the alignment of the flow vorticity with $\hat{p} \times \hat{s} \hat{p}$.

### 4.6 Coordinate Transformations and the Evolution of the Strain-Rate Tensor

In section 4.2, we show that the Cauchy-Green eigenvectors are the same as the strain-rate eigenvectors for an infinitesimal time, $\Delta t$, used to integrate the deformation tensor. After longer times, the Cauchy-Green eigenvectors evolve as the fluid undergoes larger deformations. Here,
we proceed to discuss the time-dependence of the Cauchy-Green eigenframe and its relationship to the time-dependence of the strain-rate, as introduced in section 2.6. Additionally, we do this in an attempt to explain why the vorticity aligns with the intermediate eigenvector, \( \hat{e}_2 \), in the strain-rate eigenframe and the largest eigenvector, \( \hat{e}_{L1} \), in the Cauchy-Green eigenframe.

Figure 4.14: The alignment of \( \hat{e}_1 \) and \( \hat{e}_3 \) with the Cauchy-Green eigenvectors as a function of the time, \( \Delta t \), used to integrate the deformation tensor.

Figure 4.14 shows the alignment of \( \hat{e}_1 \) and \( \hat{e}_3 \) with the Cauchy-Green eigenvectors. At \( \Delta t = 0 \), we note the unsurprising result that \( \hat{e}_{L1} \cdot \hat{e}_1 = 1 \) and \( \hat{e}_{L3} \cdot \hat{e}_3 = 1 \). This just tells us the eigenvectors in both eigenframes are the same at this given time. After this initial time, \( \hat{e}_{L3} \) and \( \hat{e}_1 \) become weakly aligned at \( R = 0.23 \), while \( \hat{e}_{L2} \) and \( \hat{e}_1 \) approach random alignment at \( R = 0.32 \). Similarly, \( \hat{e}_{L2} \) and \( \hat{e}_3 \) are weakly aligned at \( R = 0.26 \), while \( \hat{e}_{L1} \) and \( \hat{e}_3 \) show little to no alignment and approach \( R = 0.12 \) at later times. The most interesting feature of these plots is that \( \hat{e}_{L3} \) and \( \hat{e}_3 \) begin in perfect alignment and stay relatively aligned at later times. This quantity ultimately saturates at \( R = 0.54 \).

The crux of the story is told by the alignment of \( \hat{e}_2 \) with the Cauchy-Green eigenvectors, as shown figure in 4.15. Both \( \hat{e}_{L2} \) and \( \hat{e}_2 \) start out in perfect alignment and approach random alignment at \( R = 0.33 \) at later times. In addition, \( \hat{e}_{L3} \) and \( \hat{e}_2 \) start out perpendicular to each other and move towards weak alignment at \( R = 0.15 \) at later times. The most interesting feature of this figure is the evolution of \( \hat{e}_{L1} \) and \( \hat{e}_2 \) relative to each other. We originally hypothesized that this quantity must show a strong degree of alignment since the vorticity aligns with both
quantities in their individual eigenframes. Figure 4.15 shows us that, for times greater than $8\tau_\eta$, the alignment between $\hat{e}_{L1}$ and $\hat{e}_2$ becomes greater than the alignment between $\hat{e}_{L2}$ and $\hat{e}_2$. The corresponding physical picture is that $\hat{e}_2$ moves towards $\hat{e}_{L1}$ and away from $\hat{e}_{L2}$, where it initially was. Furthermore, as mentioned before, $\hat{e}_3$ tends to stay preferentially aligned to $\hat{e}_{L3}$ and perpendicular to $\hat{e}_{L1}$ and $\hat{e}_{L2}$ for all times. So, while $\hat{e}_3$ remains relatively fixed, $\hat{e}_1$ and $\hat{e}_2$ evolve quite dramatically.

![Figure 4.15](image)

**Figure 4.15:** The alignment of two strain rate eigenvectors with the Cauchy-Green eigenvectors as a function of time, $\Delta t$, used to integrate the deformation tensor: (a) alignment of $\hat{e}_1$ with $\hat{e}_{L1}$ (b) alignment of $\hat{e}_3$ with $\hat{e}_{L1}$

The gradual alignment of the intermediate strain-rate eigenvector, $\hat{e}_2$, with the most extensional Cauchy-Green eigenvector, $\hat{e}_{L1}$, led us to consider how this intermediate eigenvector moves towards or away from the most extensional strain-rate eigenvector, $\hat{e}_1$. As a result, we present some figures describing how the current strain-rate eigenvectors align with where the strain-rate eigenvectors once were. Additionally, we reconsider the evolution equations of the strain-rate tensor introduced in section 2.6 with equations 2.20 and 2.21. For convenience, we reproduce them here:

\[
\frac{\partial}{\partial t} S + \mathbf{u} \cdot \nabla S = - \left[ S^2 - \frac{1}{3} Tr(S^2)I \right] - \left[ W^2 - \frac{1}{3} Tr(W^2)I \right] - H + \nu \nabla^2 S
\]
\[
\frac{\partial}{\partial t} S + \mathbf{u} \cdot \nabla S = - \left( S^2 - \frac{1}{3} Tr(S^2)I \right) - \frac{1}{4} \left[ \omega \omega^T - \frac{1}{3} Tr(\omega^2)I \right] - \mathbb{H} + \nu \nabla^2 S
\]

As mentioned in section 2.6, the strain-rate evolves due to self-attenuation and amplification, a rotation due to vorticity, in addition to pressure and viscous contributions. Both figures 4.14 and 4.15 suggest that there are significant contribution due to rotation, pressure, or viscosity. To obtain a clearer picture of what is happening, we will end this chapter by considering the history of the strain rate eigenvectors in 4 different fluid regions: regions of strong vorticity and strong strain, strong vorticity and weak strain, strong strain and weak vorticity, and weak vorticity and weak strain. We determine high vorticity and strain regions relative to the mean square strain rate, \( \langle S_{ij} S_{ij} \rangle \), and rotation rate, \( \langle W_{ij} W_{ij} \rangle \). Both of these quantities are equal in homogenous isotropic turbulence, and as a result, \( \langle S_{ij} S_{ij} \rangle = \langle W_{ij} W_{ij} \rangle = 227 s^{-2} \) for our DNS data.

In figure 4.16, we plot the alignment of \( \hat{e}_1(t_0) \) with \( \hat{e}_2 \) and \( \hat{e}_3 \) as a function of time. Most of these history curves approach random alignment after long times, but we include the alignment of \( \hat{e}_1(t_0) \) with \( \hat{e}_3 \) as a reference curve for studying the evolution of \( \hat{e}_2 \). In regions of strong strain, the strain-rate eigenvectors are well-defined, so there is significant contribution of the evolution of the strain-rate due to self-amplification and attenuation. In figure 4.16a, we show that in regions of strong strain and strong vorticity, the alignment between \( \hat{e}_1(t_0) \) and \( \hat{e}_2 \) is mostly random, while in figure 4.16c, regions of strong strain, in the absence of strong vorticity, show a more significant degree of alignment between \( \hat{e}_1(t_0) \) and \( \hat{e}_2 \). Furthermore, in the presence of both weak strain and weak vorticity, figure 4.16d demonstrates a considerable peak in alignment between \( \hat{e}_1(t_0) \) and \( \hat{e}_2 \). This suggests that the rotation of \( \hat{e}_2 \) towards where \( \hat{e}_1 \) was previously is the result of pressure and viscous effects. In order to study the phenomenology of the evolution of the strain-rate tensor, simplified models have been used to identity different features, and, as mentioned in section 2.6, a common model is the restricted Euler model which ignores the pressure and viscous terms. However, this model results in a system that evolves to a singular state and its time-evolution on long scales is not comparable to the realistic Navier-Stokes dynamics [30]. Similarly, our new results presented in figure 4.16 also indicate that the contributions due to pressure and viscosity must be included in order to fully understand the dynamics of the velocity gradient over long and short times. These figures open a new door of possibilities for future work in the Voth group concerning the rotation, stretching, and evolution
Figure 4.16: The alignment of $\hat{e}_1(t_0)$ with $\hat{e}_2$ and $\hat{e}_1(t_0)$ with $\hat{e}_3$ as a function of time. We show this figure for (a) strong vorticity and strong strain, (b) strong vorticity and weak strain, (c) strong strain and weak vorticity, and (d) weak vorticity and strain.

of the velocity gradient. A natural next step is to compute the evolution of the velocity gradient using the restricted Euler equations and compare it to evolution as given by our DNS data. This will allow us to determine how much of the evolution is due to both pressure and viscosity at every time step.
Conclusion

5.1 Summary

In this thesis, we presented our work comparing the rotational dynamics and alignment of rods and disks in turbulence. We first summarized previous study on rods and vorticity by using the eigenvectors of the strain-rate tensor to create 1D probability distributions functions and 3D spherical histograms. Likewise, we used the eigenvectors of the Cauchy-Green strain tensor to produce similar figures in order to demonstrate the relative effectiveness of using the Cauchy-Green strain tensor to describe alignment with fluid stretching. Furthermore, we related the two eigenframes by showing the alignment of rods as a function of the time used to integrate the deformation tensor. To summarize our results: rods and vorticity both preferentially align with the intermediate eigenvector of the strain-rate tensor, \( \mathbf{e}_2 \), and the maximum eigenvector of the Cauchy-Green strain tensor, \( \mathbf{e}_{L1} \). We subsequently presented our new work on disks and further asserted that the Cauchy-Green eigenframe is the natural eigenframe for particle alignment in turbulence. While rods and vorticity both tend to align with \( \mathbf{e}_{L1} \), disks preferentially align with the weakest stretching direction, as defined by the minimum eigenvector of the Cauchy-Green strain tensor, \( \mathbf{e}_{L3} \). As a result, disks and vorticity are preferentially perpendicular to each other. In addition to studying ensemble averages of large data sets, we presented some single trajectory plots in order to highlight some differences of the rotational motions of rods and disks. We complemented this written presentation with some real-time movies. Lastly, we
provided a phenomenological description of the time evolution of the velocity gradient tensor in order to quantify a relationship between the strain-rate tensor and the Cauchy-Green strain tensor. Since rods and vorticity are strongly aligned with $\hat{e}_2$ and $\hat{e}_{L1}$, we expected to observe relatively strong alignment between these two vectors. We showed this alignment in figure 4.15 and proceeded to investigate means for describing how these eigenframes move towards each other. We did this by examining the history of the strain-rate and how it evolves in time. While we expected the rotation of the strain-rate tensor to be the result of strong vorticity, figures 4.16 suggests that the rotation is mostly due to pressure and viscous effects.

5.2 Current and Future Work

Our immediate first task is to further investigate viscous and pressure contributions to the evolution of the strain-rate tensor. My lab mate, Rami Hamati, has spent part of the semester working with the Johns Hopkins turbulence database and accessing their direct numerical simulation of isotropic turbulence. These simulations offer direct access to the viscous and pressure terms of Navier-Stokes equations. As a result, we can perform a restricted Euler analysis on their data and compare with analysis that would explicitly include these viscous and pressure terms to determine how significant their effect is on the flow evolution.

There has been significant work already done on the decomposition of the strain-rate into its local (strain and rotation) and nonlocal (viscosity and pressure) parts. Hamlington et. al. (2008) performed a direct Biot-Savart integration to determine the alignment of vorticity with the local and nonlocal parts [31]. This was done in an attempt to explain why the vorticity aligns with intermediate strain-rate eigenvector and not the largest eigenvector. While this issue has been resolved as a result of a Cauchy-Green analysis of stretching, this Biot-Savart analysis could be useful in determining the strength and evolution of this nonlocal contribution. Furthermore, Wilczek et. al. (2014) evaluated the effects of non-local pressure Hessian contributions and viscous diffusion in the framework of a statistical evolution equation for the velocity gradient tensor under the assumption of Gaussian incompressible velocity fields [24]. They treat the viscous term as a linear damping term and the pressure Hessian contributions as a combination of quadratic, traceless, and symmetric expressions of the strain-rate and rate-of-rotation tensors. In addition, this analysis yields a term which induces a rotation of the strain-rate eigenframe.
whose coefficient depends on the velocity two-point correlation. While the assumption of Gaussian closure allowed for the discussion of how the various non-local contributions contribute to the strain-rate evolution, this assumption is ultimately insufficient and inconsistent with DNS results. The nonlocal coefficients also depend on strain skewness and entropy production which must be taken into account in the future.

In addition to understanding the evolution of the strain-rate tensor, the Voth group hopes that the Cauchy-Green analysis will gain more traction in the field. Other members of our group are already using it in their own work. For example, Bardia Hejazi studies singularities in the eigenvector field of the standard map. Singularities in the standard map occur in regions where the deformation gradient tensor is purely a rotation matrix. As a result, there is no stretching in this region. Bardia is doing this study for a two dimensional space with the intention of extending his work to three dimensions.

The Cauchy-Green eigenframe ultimately provides a natural way for understanding particle alignments and can hopefully be extended to larger particles outside the realm of Jeffrey’s equation. These particles require several different considerations to determine their motions. For example, they have inertia in translational and rotational degrees of freedom. In addition, the forces and torques must include the effect of fluid inertia. These effects are made more complicated if the particle and fluid are density-mismatched. These inertial particles have provided a number of cool projects for the members of the Voth group. For example, Brendan Cole works with time-resolved positions and orientations of anisotropic inertial-range particles via experimental measurements. His work focuses on tetrads (four slender rods in tetrahedral symmetry) triads (three rods in triangular planar symmetry). His current works aims to show that small tetrads and triads rotate like spheres and disks, respectively, and that this small-particle prediction can be experimentally extended well into the inertial range. In addition to Brendan’s work on tetrads and triads, Stefan Kramel has done extensive study on the preferential rotation of inertial chiral dipoles in turbulence. Longer terms projects would involve computationally and experimentally determining the alignment of inertial particles with the Cauchy-Green strain tensor.
Appendix

6.1 Derivation of Navier-Stokes Equations

The Navier-Stokes equations are derived from the principles of conservation of mass and momentum. Let’s consider a stationary control volume placed in a fluid with a finite mass, $M$.

![Figure 6.1: Stationary control volume, $V$, through which our fluid flows.](image)

If we assume that the conservation of mass holds, we can relate the rate at which fluid mass
enters or exits the boundary of the control volume to the rate at which the fluid mass changes within the control volume.

$$\frac{d}{dt}(M) = \frac{d}{dt} \int dm = \frac{d}{dt} \int_V \rho dV = - \int_A (\rho u) \cdot dA$$

(6.1)

Here, $\rho u$, is the mass flux density that sweeps out a given surface area of our control volume. The negative sign comes from how we define the orientation of $dA$. We define $dA$ to be normal to the surface and express this vector quantity as $dA = dA\hat{n}$. As a result, fluid flowing out of the volume will make the quantity, $u \cdot dA$, positive. On the other hand, if fluid flows into the volume, then this quantity is negative. In other words, if there is a positive rate of change of mass within the volume, this must mean that fluid is flowing through the boundary and into the volume. Our next step is to express our surface integral in terms of a volume integral via the divergence theorem:

$$- \int_A (\rho u) \cdot dA = - \int_V \nabla \cdot (\rho u) dV$$

(6.2)

As a result, we are left with the following expression:

$$\frac{d}{dt} \int_V \rho dV + \int_V \nabla \cdot (\rho u) dV = 0$$

(6.3)

$$\int_V \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right] dV = 0$$

(6.4)

Since this expression is true for any arbitrary control volume, we can conclude that the expression in the integrand is equal to 0. As a result, we are left with an expression which relates the time rate of change of the fluid density with the divergence of the mass flux density. This expression is commonly referred to as the continuity equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0.$$

(6.5)
For a fluid with constant density, we are left with the incompressibility condition, \( \nabla \cdot \mathbf{u} = 0 \). In tensor notation, we can express the continuity equation in the following manner:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0 \quad (6.6)
\]

Our next task is to derive an expression for the conservation of momentum by relating the momentum change to the sum of external forces acting on the control volume.

\[
\frac{d}{dt}(\mathbf{p}) = \frac{d}{dt}(M \mathbf{u}) = \frac{d}{dt} \int_V (\rho \mathbf{u})dV = - \int_A (\rho \mathbf{u}) \cdot d\mathbf{A} + \int_A \sigma \cdot d\mathbf{A} + \int_V \rho g dV \quad (6.7)
\]

The time rate of change of the momentum, \( \mathbf{p} \), is the result of 3 separate terms: the momentum flux through the boundary (1), the surface forces (2), and the body forces acting on the control volume (3). Both expressions (1) and (2) are second rank tensors, we and refer to \( \sigma \) as the stress tensor. We can again use the divergence theorem to express terms (1) and (2) as volume integrals. As a result, we are left with the following:

\[
\int_V \left[ \frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u}) - \nabla \cdot \sigma - \rho \mathbf{g} \right] dV = 0 \quad (6.8)
\]

Similar to before, the integrand will equal zero for any arbitrary control volume. We proceed to write this expression in tensor notation and expand the derivative terms.

\[
\frac{\partial}{\partial t} (\rho u_j) + \frac{\partial}{\partial x_i} (\rho u_i u_j) = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j \quad (6.9)
\]

\[
\rho \frac{\partial u_j}{\partial t} + u_j \frac{\partial \rho}{\partial x_i} + \rho u_i \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial (\rho u_i)}{\partial x_i} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j \quad (6.10)
\]

\[
\rho \frac{\partial u_j}{\partial t} + \rho u_i \frac{\partial u_j}{\partial x_i} = -u_j \left[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} \right] + \frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j \quad (6.11)
\]
The first expression on the right hand side is the continuity equation. As a result, this entire term drops out, and we are left with Cauchy’s equation of motion:

\[
\rho \frac{\partial u_j}{\partial t} + \rho u_i \frac{\partial u_j}{\partial x_i} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho g_j \tag{6.12}
\]

In order to arrive at Navier-Stokes equation, we must specify the terms of the stress tensor. In section 2.1, we discuss how the stress tensor is composed of contribution from hydrostatic pressure, \(P\), and shear stress, \(\tau_{ij}\).

\[
\sigma_{ij} = -P \delta_{ij} + \tau_{ij} \tag{6.13}
\]

Furthermore, we discussed how the shear stress is directly proportional to fluid viscosity and rate-of-strain.

\[
\tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{6.14}
\]

We proceed to expand the spatial derivative of the stress tensor.

\[
\frac{\partial \sigma_{ij}}{\partial x_i} = -\frac{\partial P}{\partial x_j} + \mu \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{6.15}
\]

Using our incompressibility condition, we can eliminate the first term in parenthesis, and we are left with a final expression for the spatial derivative of the stress tensor.

\[
\frac{\partial \sigma_{ij}}{\partial x_i} = -\frac{\partial P}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i^2} \tag{6.16}
\]

Now, we plug this back into Cauchy’s equation of motion (equation 6.12) and obtain the Navier-Stokes equation shown in equation 6.18.
\[
\rho \frac{\partial u_j}{\partial t} + \rho u_i \frac{\partial u_j}{\partial x_i} = -\frac{\partial P}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i^2} + \rho g_j
\]  \hspace{1cm} (6.17)

\[
\rho \left[ \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i} \right] u_j = -\frac{\partial P}{\partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_i^2} + \rho g_j
\]  \hspace{1cm} (6.18)

This final expression relates the total temporal and spatial changes in the velocity field to the gradient of the pressure field, the fluid viscosity, and the body forces present on the fluid. This is the expression we presented in equation 2.1 and 2.4.
First and foremost, I’d like to thank my research advisor, Greg Voth, for 3 years of support and encouragement. We started working together during a time when I knew absolutely nothing about fluid dynamics and very little about most areas of physics. Despite that, he invested time and effort in helping me develop essential skills as a scientist and teaching me the art of productive struggle. In addition, I’d like to thank all the wonderful members of my lab: Stefan Kramel, Brendan Cole, Lydia Tierney, Wyatt Rees, Michael Krellenstein, Bardia Hejazi, Rami Hamati, and Alexis Braunrot. I’ve spent a fair amount of time working with most of them, and they always make our weekly group meetings vibrant and productive. I’d also love to thank my parents for their unconditional support and all that they’ve sacrificed for me to receive a Wesleyan education. Shout out to Summies and Freeman for always providing me with love, strength, and onion rings. And last, but not least, shout out to my 4 house mates, Michael Glasser, Connor Justice, Rebecca Brand, and Rachael Metz for their support even though they didn’t know I was writing a thesis until December.
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