Wesleyan University

Precoloring Extensions Involving Cliques and the Pairwise Distance Needed Between Them

by

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Abstract

The main focus of this paper is exploring precoloring extensions problems, specifically when the precolored subgraph consists of cliques. We explore this topic by providing more details to a proof by Michael O. Albertson in the paper ‘You Can’t Paint Yourself into a Corner’ [1]. Albertson shows that for a graph $G$ with chromatic number $r$ and $P \subseteq V(G)$, if $P$ induces a set of cliques such that each clique is of order $k \leq r$ and whose pairwise distance is at least $6k - 2$, then any proper $(r + 1)$-coloring of $P$ can be extended to a proper $(r + 1)$-coloring of $G$. Before considering this theorem, we provide background on precoloring extensions, touch upon list-colorings and their relation to precoloring extensions, and discuss an important proof technique involving Kempe chains that we use throughout the paper. We also provide a result by Alexandr Kostochka [2], which is actually an improvement to Albertson’s result mentioned above. Kostochka’s result decreases the pairwise distance needed between cliques of order $k$ to only $4k$. We then provide Albertson and Moore’s more precise result concerning the distance needed between precolored cliques [2]. Finally, building off of the arguments made in Kostochka’s proof, we prove that when the pairwise distance between cliques of order $k$ is $2k + 2$, any proper $(r + 2)$-coloring of $P$ can be extended to a proper $(r + 2)$-coloring of $G$. 
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Chapter 1

Introduction

A precoloring extension problem is a problem in which there is a proper \( k \)-coloring on a certain subgraph of a graph \( G \). A proper \( k \)-coloring is an assignment of \( k \) colors to the vertices of a graph, one color to each vertex, in a way such that no two adjacent vertices receive the same color. Then, the question is: Can we extend the \( k \)-coloring of the subgraph to a \( k \)-coloring of the whole graph \( G \)? There are many variations of this problem including additional conditions on the graph, such as its chromatic number and whether its planar, and additional conditions on the subgraph, such as the minimum distance between vertices in the subgraph.

A main part of this paper is focused on a result in which the precolored subgraph is the union of cliques, which are complete subgraphs. In this result, we also have an additional color in the coloring we are extending from the subgraph to the whole graph, and we have a condition on the minimum distance between any two cliques. The distance between two cliques is the length of the shortest path from any vertex in one clique to any vertex in the other clique. This result is from the paper ‘You Can’t Paint Yourself into a Corner’ by Michael Albertson [1]. Albertson’s result is as follows:

Suppose \( \chi(G) = r \) and \( W \subseteq V(G) \) induces a subgraph of \( G \) such that \( G[W] = W_1 \cup W_2 \cup \cdots \cup W_m \), where each \( W_i \) is a clique. If \( |W_i| \leq k \ \forall \ i \) and \( \text{dist}(x,y) \geq 6k - 2 \) whenever \( x \in W_i, y \in W_j, \) and \( i \neq j \), then any proper \( (r+1) \)-coloring of \( W \) can be
extended to a proper \((r + 1)\)-coloring of \(G\).

The strategy used in this proof is to \(r\)-color the graph \(G\), and then modify the proper \(r\)-coloring so that the colors on the vertices of the cliques agree with their precolored colors. While modifying the \(r\)-coloring, the colors of vertices outside of the clique will need to be changed so that we always have a proper coloring of the graph. We recolor one clique of the subgraph at a time and for each clique, we create barriers surrounding that clique. This is so that the color changes to vertices outside of a specific clique do not affect the colors of vertices in and around other cliques. Since each clique in our subgraph has at most \(k \leq r\) vertices, the precoloring on each clique uses at most \(k\) colors from \(\{1, 2, \ldots, r, r + 1\}\). For simplicity, it is not specified which \(k\) colors from \(\{1, 2, \ldots, r, r + 1\}\) are being used on a clique. By proceeding in this way, we are not shown the differences in the arguments we need to make when \(r + 1\) is or is not a color used by the precoloring on the clique we are considering.

We found that that we have to alter the arguments when \(r + 1\) is and is not by the precoloring on a clique to properly address the situation. We also found that there are many subtleties regarding exactly how we need to go about recoloring a clique, such as the order in which we need to recolor each vertex in clique. To address the issues in the result mentioned above, we provide a revised proof of Albertson’s result in Chapter 4. We use many of the same ideas and arguments found in Albertson’s proof, but we provide the additional details and arguments mentioned above.

To be able to understand all the aspects of this result, we provide basic graph theory definitions in Chapter 2. In this chapter, we also talk in depth about precoloring extensions problems, and we discuss list-colorings and their relation to precoloring extension problems. List-colorings are a generalization of \(k\)-colorings where each vertex in a graph is assigned a list of colors. The graph is list-colorable if each vertex in the graph can be assigned a color from its list such that no two vertices are assigned the same color. In this chapter, we also discuss an important tool that is used in multiple results- Kempe chains. Kempe chains are connected paths of vertices colored two alternating colors.
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Then, in the following chapter, Chapter 3, we provide other results from the paper ‘You Can’t Paint Yourself into a Corner’ [1]. The results we provide have proofs that are easy to follow and the proof techniques used in these proofs will help us gain insight before considering the revised proof in Chapter 4.

After making revisions to Albertson’s proof mentioned above, we came across a result by Alexandr Kostochka [2] that is an improvement of Albertson’s theorem since the distance needed between any two cliques is decreased. For cliques of order \( k \), Kostochka reduces the distance needed between any two cliques to \( 4k \). We provide this result in Chapter 5. This result is only concerning \( k \)-cliques, or cliques of order \( k \), but can be easily extended for cliques of order at most \( k \). We also provide and discuss Albertson and Moore’s more precise result concerning the distance needed between precolored cliques [2].

Then, in Chapter 6, we consider the situation in which a subgraph composed of cliques is precolored with two extra colors, instead of just one. Building off of the methods used in Kostochka’s proof, we prove that for \( G \) and \( P \subset V(G) \), where \( P \) induces \( k \)-cliques, when the pairwise distance between \( k \)-cliques is \( 2k + 1 \), any proper \((r + 2)\)-coloring of \( P \) can be extended to a proper \((r + 2)\)-coloring of \( G \). We also extend this result for cliques of order at most \( k \).
Chapter 2

Background

2.1 General Definitions

To start, we provide some general graph theory definitions that will be used throughout the paper. Many of the definitions we use are from J.A. Bondy and U.S.R. Murty [4]. A graph $G$ is an ordered pair $(V(G), E(G))$ which consists of a nonempty set $V(G)$ of vertices and a set $E(G)$ of 2-element subsets of $V(G)$ called edges. The number of vertices in a graph $G$ is the order of $G$. The number of edges in $G$ is the size of $G$. A graph is finite if both its vertex set and edge set are finite. We will only be considering finite graphs in this paper. If two vertices share an edge, they are adjacent. Two distinct adjacent vertices are called neighbors. A graph is simple if it has no loops or parallel edges. A loop is an edge that has identical ends. Parallel edges occur when there are two or more edges with the same pair of distinct ends. We will only be considering simple graphs in this paper. A $u - v$ path is a sequence of vertices beginning with $u$ and ending with $v$ such that consecutive vertices in the sequence are adjacent and each vertex appears only once in the sequence. $G$ is connected if $G$ contains a $u - v$ path for every pair of vertices $u, v$ of $G$. The distance between two vertices is the length of the shortest path from one vertex to the other, where the length of a path is the number of edges in the path.
A graph $H$ is a subgraph of a graph $G$, written $H \subset G$, if $V(H)$ is a subset of $V(G)$ and $E(H)$ is a subset of $E(G)$. A subgraph $F$ of a graph $G$ is an induced subgraph of $G$ if whenever $u$ and $v$ are vertices of $F$ and $uv$ is an edge of $G$, then $uv$ is an edge of $F$ as well. Given a set $X \subseteq V(G)$ we denote the subgraph induced by $X$ as $G[X]$. Thus, $G[X]$ is a subgraph of $G$ whose vertex set is $X$ and whose edge set consists of all edges of $G$ with both ends in $X$. A graph $G$ is complete if every pair of distinct vertices are adjacent in $G$. A clique of $G$ is a complete subgraph of $G$.

A $k$-vertex coloring of $G$, or just a $k$-coloring of $G$, is a function $f : V(G) \to S$, where $S$ is a set of $k$ distinct colors, that assigns to each vertex one of the $k$ colors in $S$. A $k$-coloring is proper if no two adjacent vertices are assigned the same color. We say a graph is $k$-colorable if it has a proper $k$-coloring. The minimum number $k$ for which a graph $G$ is $k$-colorable is called its chromatic number, which we denoted by $\chi(G)$. A planar graph is a graph that can drawn in the plane in such a way that edges meet only at points corresponding to their common ends. A very famous result is that all planar graphs are 4-colorable. That is, if $G$ is planar, then $\chi(G) = 4$. In this paper, we consider both planar graphs and nonplanar graphs. If we do not specifically indicate that we are talking about planar graphs, then what we are referring to applies to both planar or nonplanar graphs.

## 2.2 Precoloring Extensions

The precoloring extension problem was formulated by Biró, Hujter, and Tuza in 1991 [3]. The problem is set up as follows:

**Instance:** An integer $k \geq 2$, a graph $G = (V, E)$ with $|V| \geq k$, a vertex subset $W \subseteq V$, and a proper $k$-coloring of $G[W]$, where $G[W]$ denotes the subgraph induced by $W$.

**Question:** Can this proper $k$-coloring of $G[W]$ be extended to a proper $k$-coloring of the whole graph $G$?
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There are now many different variations of this problem. Later in this paper, the
majority of what we focus on is the instance when we have a graph $G$ with $\chi(G) = r$,
$W \subseteq V$ such that $G[W]$ is the union of cliques, and a proper $(r + 1)$-coloring of $G[W]$.
In most of the problems we consider in the paper, our precolored subgraph is colored
with an additional color, $r + 1$, where $\chi(G) = r$. The question in this case is: Can any
proper $(r + 1)$-coloring of $G[W]$ extend to a proper $(r + 1)$-coloring of the whole graph
$G$. The additional color is needed to guarantee that any proper coloring of the subgraph
is able to extend to the whole graph. This is because we can construct instances where
for a graph $G$ with $\chi(G) = r$, a proper $r$-coloring of the subgraph does not extend to
a proper $r$-coloring of the whole graph. For example, consider the graph $G$ that is the
path of $n$ vertices where $n$ is odd. We have that $\chi(G) = 2$. Now, consider the subgraph
$P$ that consists of the two vertices at the ends of the path. If we choose the precoloring
with 2 colors such that one vertex in $P$ is assigned the color 1 and the other is assigned
the color 2, then this proper 2-coloring of $P$ does not extend to a proper 2-coloring of
$G$. We can see that adding an additional color in this example would make us able
to extend any proper 3-coloring of $P$ to a proper 3-coloring of $G$. In additional to the
extra color, we must include a condition on the minimum distance between the cliques
in $G[W]$ and a constraint on the number of vertices in the cliques to be able to extend
the $(r + 1)$-coloring of $G[W]$ to the whole graph.

According to Michael Albertson [1], the complexity of extending a proper coloring
from a subgraph to the whole graph without using extra colors is NP-complete. This is
the case unless the graph is particularly simple to color.

Precoloring extensions have a wide range of applications. One that is widely refer-
enced is in scheduling problems. Biró, Hutjter, and Tuza [3] describe in detail a real life
scheduling problem that arose at Malév Hungarian Airlines. This scheduling problem
can be interpreted as a precoloring extension problem on an interval graph. An interval
graph is a graph in which each vertex is represented by an open real interval such that
two vertices are adjacent if and only if their intervals intersect each other. This problem
is a flight-maintenance scheduling problem where there are \( n \) jobs in total \( J_1, J_2, \ldots, J_n \), and there are \( k \) planes, \( M_1, M_2, \ldots, M_k \). The vertices are the jobs, the colors are the planes, and to color a vertex is to assign a plane to a job’s corresponding time interval. \( m \) of the jobs, \( J_1, \ldots, J_m \) correspond to time intervals of the required maintenance of the planes and \( n - m \) of the jobs, \( J_{m+1}, \ldots, J_n \), correspond to the time intervals of flights. The time that each plane is getting maintenance done is fixed and cannot be changed. Thus, the jobs corresponding to maintenance time intervals are the precolored vertices. After the planes are already assigned their time intervals for maintenance, the planes will be assigned to time intervals of flights. So, the jobs corresponding to flight time intervals are the vertices that have not been colored yet. Since we are considering an interval graph, vertices are adjacent if and only if their corresponding time intervals overlap and therefore, vertices are not adjacent if their corresponding time intervals are disjoint. The same plane can obviously only be assigned to jobs whose corresponding time intervals do not overlap because, for example, a plane cannot be receiving maintenance and flying at the same time. Thus, this is a precoloring extension problem on an interval graph where we have \( k \) colors, \( m \) precolored vertices, and \( n - m \) vertices that are not colored yet.

We will now look at simple example of a precoloring extension problem. Let \( G \) be the graph in Figures 2.1 and 2.2. Consider Figure 2.1 and let the precolored vertices, \( v_6, v_7, v_{10}, v_{11} \), induce the subgraph \( H \). We have a proper 2-coloring of \( H \). We ask the question: Can this proper 2-coloring of \( H \) extend to be a proper 2-coloring of the whole graph \( G \)? Looking at the next figure, Figure 2.2, we can see that the answer is yes. We have colored the vertices in \( G - H \), while keeping the vertices in \( H \) colored their same color, so that we have a proper 2-coloring of \( G \). We must color the vertices \( v_2, v_5, v_{12} \), and \( v_{15} \) the color red since they are all adjacent to vertices in \( H \) colored blue. We must color the vertices \( v_3, v_8, v_9 \), and \( v_{14} \) the color blue since they are all adjacent to vertices in \( H \) colored red. Based on the colors we just assigned some of vertices in \( G - H \), we must color \( v_1 \) and \( v_{16} \) the color blue and \( v_4 \) and \( v_{13} \) the color red. Since no two adjacent
vertices are assigned the same color, this is a proper 2-coloring on $G$.

### 2.3 List-Colorings

To define list-colorings, we follow the definitions in [4]. Let $G$ be a graph and let $L$ be a function that assigns to each vertex $v$ of $G$ a set $L(v)$ of positive integers. $L(v)$ is called the list of vertex $v$. A coloring $c : V(G) \rightarrow \mathbb{N}$ such that $c(v) \in L(v)$ for all $v \in V(G)$ is called a list coloring of $G$ with respect to $L$, and we say that $G$ is $L$-colorable. We note that list-colorings are a generalization of $k$-colorings since if $L(v) = \{1, 2, \ldots, k\}$ for all $v \in V$, then the $L$-coloring is a $k$-coloring. This is because each of the vertices in $V$ has the option of being colored any of the $k$ colors since all $k$ colors are on the lists of every vertex. We also observe that assigning a vertex a list of length one is precoloring that vertex, as we have no choice of its color.

In the following example, we consider the same graph from Figures 2.1 and 2.2. Instead of thinking of that example as a precoloring extension problem, we will turn it into a list-coloring problem. To do this, in Figure 2.3, instead of precoloring the vertices in $H$, $v_6$, $v_7$, $v_{10}$, $v_{11}$, we will instead assign each of them a list of just one color, either $r = \text{red}$ or $b = \text{blue}$. Let $L(v_6) = \{b\}$, $L(v_7) = \{r\}$, $L(v_{10}) = \{r\}$, and $L(v_{11}) = \{b\}$. Then, we will let the lists for the vertices in $G - H$ consist of both colors, red and blue. So, $L(v) = \{r, b\}$ for all $v \in G - H$. Then, we ask the question: Are we able to color all the vertices in $G$ from colors on their assigned lists such that no two adjacent vertices are assigned the same color?

Since the vertices in $H$ only have one color on their lists, we must color these vertices that color. Then, we consider the vertices in $G - H$ that are adjacent to the vertices in $H$. These vertices in $G - H$ that are adjacent to vertices in $H$ are left with only one available color on their lists that they can be assigned. For example, since both $v_2$ and $v_5$ are adjacent to $v_6$, and $v_6$ must be colored blue, $v_2$ and $v_5$ must both be colored red. After coloring these 6 vertices in $G - H$ that are adjacent to vertices in $H$, we move
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Figure 2.1: Graph, $G$, has the precolored vertices, $v_6, v_7, v_{10}, v_{11}$. These four vertices induce the subgraph $H$. This is a proper 2-coloring of $H$.

Figure 2.2: We were able to color $G - H$ while keeping the precolored vertices assigned their same color. Thus, we extended the proper 2-coloring of $H$ to a proper 2-coloring of $G$. 
on to coloring the 4 remaining vertices in the corners of the graph. After coloring the 6 other vertices in $G - H$, we are left with only one available color on each of the 4 remaining vertices’ lists. For example, since we colored both $v_2$ and $v_5$ the color red, $v_1$ cannot be colored red since it is adjacent to both $v_2$ and $v_5$. Thus, $v_1$ must be colored blue. We can see in Figure 2.4, that by assigning each vertex a color from its list, we are able to color the graph so that no two adjacent vertices are assigned the same color. Therefore, $G$ is $L$-colorable.

To give a proper explanation of list-colorings, we will give the following additional definitions. A graph, $G$, is $k$-list-colorable, or $k$-choosable, if it has a list coloring whenever all lists have length $k$. We note that for $G$ to be $k$-list-colorable, $G$ must be list colorable whenever all lists have length $k$, no matter what colors are in each list. Then, the minimum value of $k$ for which $G$ is $k$-list-colorable is the list chromatic number of $G$, which we denote by $\chi_L(G)$.

A very important result about list-coloring is due to Carsten Thomassen [7]. In 1994, Thomassen proved that every planar graph is 5-choosable. We will use this result in Chapter 3.

### 2.4 Kempe Chains

One of the main arguments we use in the proofs in this paper involve paths of vertices of two alternating colors, called Kempe chains. These chains are named after Alfred Kempe, who used these types of paths and the idea of interchanging the colors on them, in his attempt to prove the Four Color Conjecture. His proof, however, had a mistake, which was pointed out by Percy John Heawood. Heawood ended up using Kempe’s idea of interchanging the colors on these chains in his proof of the Five Color Theorem, which states that every planar graph is 5-colorable. More information on the error that Kempe made in his proof, general background, and Heawood’s proof of the Five Color Theorem can be found in [4] and [6].
Figure 2.3: In this graph $G$ the vertices in $H$ are each assigned lists of only one color. The vertices in $G - H$ each have lists consisting of two colors.

Figure 2.4: Since each vertex in $H$ has only one color on its list, we must color each vertex its respective color on its list. Then, based on the color of the vertices in $H$, we assign to each vertex in $G - H$ one of the two colors from its list.
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We will now give a precise definition of what a Kempe chain is. Suppose $G$ has proper coloring and suppose $G$ has at least one vertex colored $a$ or $b$. Let this vertex be $x$. An $(a,b)$-Kempe chain containing $x$ is the maximal connected subgraph $K$ of $G$ containing $x$ and vertices colored only $a$ and $b$. In other words, the vertices in the $(a,b)$-Kempe chain containing $x$ are colored either $a$ or $b$ and are connected to $x$ by a path of vertices colored $a$ and $b$. [5] We refer to the procedure of interchanging the two colors on a Kempe chain, as a Kempe change. Now, suppose we have a proper coloring on a graph. A natural question that may arise after introducing the procedure of a Kempe change, is if interchanging two colors on a Kempe chain leaves us with a proper coloring of the graph. The following lemma addresses this question.

**Lemma 1.** Let $G$ be a graph, and let $\gamma$ be an $r$-coloring of $G$, where two of the colors used in the $r$-coloring are $a$ and $b$. Let vertex $x$ be colored $a$ by $\gamma$. If we interchange the colors on the $(a,b)$-Kempe chain containing $x$, then we still have a proper $r$-coloring of $G$.

**Proof.** Let $\gamma$ be an $r$-coloring of $G$, where two of the colors used in the $r$-coloring are $a$ and $b$. Let $M$ denote the vertices on the $(a,b)$-Kempe chain containing $x$. After we interchange the colors $a$ and $b$ on the vertices in $M$, vertices in $M$ will not be adjacent to vertices not in $M$ assigned either color, $a$ or $b$. This is because the subgraph induced by $M$ is a maximum connected subgraph containing vertices colored only $a$ or $b$. Suppose there were a vertex $v$ such that either $\gamma(v) = a$ or $\gamma(v) = b$ and such that $v$ is adjacent to a vertex in $M$. Then, $v$ is joined to $x$ by a path of vertices colored $a$ and $b$, and thus, must be in $M$ as well. So, this vertex will have its color interchanged as well. Also, since $\gamma$ is a proper $r$-coloring, before we interchange the colors $a$ and $b$, vertices colored $a$ are not adjacent to other vertices colored $a$, and vertices colored $b$ are not adjacent to other vertices colored $b$. So, when we interchange the colors on the $(a,b)$-Kempe chain containing $x$, this is still the case. Therefore, we still have a proper $r$-coloring on $G$. □

We provide an example of the process of Kempe changes in the following two figures.
In Figure 2.5, for the graph $G$, we have a proper 4-coloring of $G$. We will consider the $(\text{blue, red})$-Kempe chain containing the vertex, $v_1$. Let $M$ be the set of vertices included on this Kempe chain. Then $M = \{v_1, v_2, v_3, v_7, v_9, v_{12}, v_{13}, v_{14}, v_{15}, v_{17}, v_{18}\}$. Note that there are blue and red vertices not on the $(\text{blue, red})$-Kempe Chain containing the vertex, $v_1$. These vertices are $v_5$, $v_{21}$, $v_{24}$. We will interchange the colors along the $(\text{blue, red})$-Kempe chain containing $v_1$. The result is the coloring of the graph $G$ shown in Figure 2.6. The vertices on the Kempe chain that were colored blue in the previous figure, Figure 2.5, are now colored red. Vertices on the Kempe chain that were colored red are now colored blue. We can see that we still have a proper 4-coloring of $G$ since no two adjacent vertices are assigned the same color.
Figure 2.5: This is a 4-coloring of $G$.

Figure 2.6: After interchanging the colors of the vertices on the \((\text{blue, red})\) Kempe chain containing $v_1$, we still have a proper 4-coloring of $G$. 
Chapter 3

‘You Can’t Paint Yourself into a Corner’

According to Michael Albertson [1], Thomassen posed the following precoloring extension problem:

Suppose $G$ is a planar graph and $W \subset V(G)$ such that the distance between any two vertices in $W$ is at least 100. Can a 5-coloring of $W$ be extended to a 5-coloring of $G$?

Albertson responded by lowering the distant constraint between the vertices in the subgraph from 100 to 4 in the following result. One recoloring technique that is used in this result is insulating certain color changes on subsets of vertices so that those color changes do not affect other parts of the graph. This technique is similar to the one we use later in the paper in our revision of Albertson’s proof concerning precoloring a subgraph consisting of cliques. For the proof of the following result, we provide Albertson’s proof, with a few details added and notation changed so that it is similar to the proof we provide in Chapter 4. For example, we define a new coloring of $G$ each time we recolor a vertex in the subgraph. This is to ensure that at each step in the modification of our original coloring on the whole graph, we have a proper coloring of $G$. I also add details concerning why the new colorings we define are actually proper colorings.
Theorem 2 ([1]). Suppose $G$ is any planar graph and $W \subseteq V(G)$ such that the distance between any two vertices in $W$ is at least 4. Any 5-coloring of the vertices of $W$ can be extended to a 5-coloring of $G$.

Proof. Let $\gamma$ be a 4-coloring of the graph, $G$. Since $G$ is planar, $\chi(G) = 4$, so we know there is a 4-coloring of $G$. Let $c$ be a proper 5-coloring of $G[W]$. We will modify $\gamma$ so it agrees with the precoloring, $c$, of $W$. Let $|W| = t$ and let the vertices in $W$ be $x_1, \ldots, x_t$.

Assume that $c$ uses the colors $\{1, 2, 3, 4, 5\}$ and $\gamma$ uses only the colors $\{1, 2, 3, 4\}$. We will modify $\gamma$ by considering one vertex in $W$ at a time. After recoloring each vertex in $W$ its precoloring color, we will make sure that we have a proper 5-coloring of $G$.

First, we consider $x_1 \in W$. The new coloring we will define at this step is $f_1$. For $x_1 \in W$, it is either the case that $c(x_1) = \gamma(x_1)$ or $c(x_1) \neq \gamma(x_1)$. If $c(x_1) = \gamma(x_1)$, then we let $f_1(x_1) = c(x_1) = \gamma(x_1)$ and we move on to the next vertex in $W$ as this one is assigned its precolored color under $f_1$. In this case, $f_1$ is a proper coloring because we are keeping all vertices in $G$ colored the same color as their $\gamma$ color, and $\gamma$ is a proper coloring by assumption.

If $c(x_1) \neq \gamma(x_1)$, then let $f_1(x_1) = c(x_1)$. We must make color changes to other vertices in $G$ so that we still have a proper coloring of $G$. If $v \in G$ is adjacent to $x_1$ and $\gamma(v) = c(x_1)$, then recolor $v$ the color 5 under $f_1$. Note that if $c(x_1) = 5$, then no surrounding vertices in $G - W$ will need to be recolored, as $\gamma$ does not use the color 5 on $G$. Once all the neighbors, $v$, of $x_1$ that were assigned the color $c(x_1)$ by $\gamma$ are recolored 5 by $f_1$, $x_1$ has no neighbors colored $c(x_1)$ under $f_1$. Also, no two vertices colored 5 will be adjacent because for two vertices to be both colored 5 under $f_1$, they must be colored the same color by $\gamma$. Since $\gamma$ is a proper coloring, two vertices assigned the same color by $\gamma$ are not adjacent. Thus, $f_1$ is a proper 5-coloring of $G$.

Now assume the vertices $x_1, x_2, \ldots, x_{i-1}$ are colored their precolored colors. We will also assume that $f_{i-1}$ is a proper 5-coloring of $G$. Now, consider the $i^{th}$ vertex in $W$, $x_i$. For $x_i \in W$, it is either the case that $c(x_i) = f_{i-1}(x_i)$ or $c(x_i) \neq f_{i-1}(x_i)$. If $c(x_i) = f_{i-1}(x_i)$, then we let $f_i(x_i) = c(x_i) = f_{i-1}(x_i)$ and we move on to the next
vertex in $W$ as this one is assigned its precolored color under $f_i$. In this case, $f_i$ is a proper coloring because we are keeping all vertices in $G$ colored the same color as their color assigned under the previous coloring, $f_{i-1}$.

If $c(x_i) \neq f_{i-1}(x_i)$, let $f_i(x_i) = c(x_i)$. Then, we must make color changes to other vertices in $G$. If $v \in G$ is adjacent to $x_i$ and $f_{i-1}(v) = c(x_i)$, then we recolor $v$ the color 5 under $f_i$. By the same argument for the first vertex $x_1 \in W_1$, at this stage, $f_i$ is a proper 5-coloring of $G$.

We continue recoloring the vertices of $W$ until we recolor the last vertex, $x_t$. The coloring we have after recoloring all vertices in $W$ is $f_t$. The only vertices in $G - W$ that are recolored 5 are vertices that are adjacent to vertices in $W$. Vertices in $W$ must be at least distance 4 away from each other, so vertices recolored 5 under $f_t$ are at least distance 2 away from each other if they are adjacent to different vertices in $W$. If two vertices are adjacent to the same vertex $x \in W$ and are recolored 5 by $f_t$, then they were both originally colored the same color under $\gamma$, and therefore are not adjacent. Thus, in any case, vertices colored 5 under the last coloring, $f_t$, must be at least distance 2 apart. Thus, $f_t$ is a proper 5-coloring, and we have extended the 5-coloring of $G[W]$ to a 5-coloring of $G$. \hfill $\square$

Since we only use the planarity hypothesis in the previous theorem to create a 4-coloring, Albertson’s following result is immediate. The following theorem applies to planar graphs, as well as nonplanar graphs.

**Theorem 3.** [1] Suppose $\gamma(G) = r$ and $W \subseteq V(G)$ such that the distance between any two vertices in $W$ is at least 4. Any $(r + 1)$-coloring of the vertices of $W$ can be extended to a $(r + 1)$-coloring of $G$.

To change the proof of the Theorem 2 to work for Theorem 3, instead of assuming we have a 4-coloring of $G$ since $\chi(G) = 4$, we assume that we have an $r$-coloring of $G$ since $\chi(G) = r$. Also, instead of coloring a vertex adjacent to a precolored vertex the color 5, we recolor it $r + 1$. The remainder of the proof will follow in the same way that
the proof of Theorem 2 does.

Albertson also claims that there is an analogous argument that works if each component of \( W \) is a clique, not just a single vertex, like for the previous two results. As we mentioned previously, we will provide this result in Chapter 4. In this result, the cliques that make up the subgraph have to be sufficiently far apart, just like the single vertices in the subgraph have to be at least distance 4 apart. Although it seems like the argument in the clique case would be similar to argument for Theorem 2 and 3, there are many more situations and subtleties we need to consider.

Before we talk about the result of extending a precoloring on cliques in a graph to the whole graph, we will explore another one of Albertson’s results from [1]. This one is interesting because of the method used. Although the theorem is about proper \( k \)-colorings, we use hypotheses about list-colorings in the proof. We provide Albertson’s result below.

**Theorem 4.** [1] Suppose \( G \) is any planar graph and \( W \subseteq V(G) \) such that the distance between any two vertices in \( W \) is at least 3. Any 6-coloring of \( W \) can be extended to a 6-coloring of \( G \).

**Proof.** Fix a coloring of \( W \) that we are hoping to extend. Set \( G' = G - W \). Create a list for each vertex \( x \) in \( G' \) consisting of some five colors from \( \{1, 2, 3, 4, 5, 6\} \) not including any color used on a vertex from \( W \) adjacent to \( x \). Since each vertex in \( G' \) can be adjacent to at most one vertex in \( W \), there are five colors available for \( x \)'s list. Now since \( G' \) is planar, it can be 5-list colored [7]. This coloring of \( G' \) can be combined with the original coloring of \( W \) to be a 6-coloring of \( G \).

Here, the planarity hypothesis is only used so that we know \( G \) is 5-list colorable, or 5-choosable. Thus, Albertson’s following result is immediate and concerns both planar and nonplanar graphs.

**Theorem 5.** [1] Suppose \( G \) is \( r \)-list colorable and \( W \subseteq V(G) \) such that the distance
between any two vertices in $W$ is at least 3. Any $(r + 1)$-coloring of $W$ can be extended to an $(r + 1)$-coloring of $G$. 
Chapter 4

Revision of Albertson’s Proof

We will now provide Albertson’s result [1] on precoloring the vertices in a set of cliques using $r + 1$ colors. As previously mentioned, we revise his proof by adding in additional details and arguments so that it is more clear and addresses each of the possible situations we can have.

We use Albertson’s main arguments in our revised proof, such as performing Kempe changes to modify the colors of vertices in a clique and creating barriers so the Kempe changes do not get too far from a clique. However, in this proof we must split our argument up into two cases, one in which $r + 1$ is a color used by the precoloring on a clique and one in which $r + 1$ is not a color used by the precoloring on a clique. We then describe the procedure in which we must recolor the vertices on the clique. Other subtleties that are important in this proof are the order in which we need to recolor each vertex in a clique, which is based on the placement of barriers. Also, after each time we recolor a vertex in the precolored subgraph, we define a new coloring on the whole graph and show that it is, in fact, proper. By doing this, when we recolor the last vertex in the last clique, we know that the coloring we have on the whole graph is a proper $(r + 1)$-coloring.

Theorem 6. [1] Suppose $\chi(G) = r$ and $W \subseteq V(G)$ induces a subgraph of $G$ such that $G[W] = W_1 \cup W_2 \cup \cdots \cup W_m$, where each $W_i$ is a clique. If $|W_i| \leq k \, \forall \, i$ and
\[ \text{dist}(x, y) \geq 6k - 2 \] whenever \( x \in W_i, y \in W_j, \) and \( i \neq j, \) then any proper \((r + 1)\)-coloring of \(G[W]\) can be extended to a proper \((r + 1)\)-coloring of \(G.\)

**Proof.** Suppose \(c\) is the \((r + 1)\)-coloring of \(G[W]\) that we wish to extend to an \((r + 1)\)-coloring of \(G,\) where \(G[W] = W_1 \cup W_2 \cup \cdots \cup W_m,\) such that each \(W_i\) is a clique. We will \(r\)-color the graph \(G\) and then modify this coloring so it agrees with the \((r + 1)\)-precoloring of \(W.\) Let \(\gamma\) be the \(r\)-coloring of \(G.\) Let \(c\) use the colors from \(\{1, \ldots, (r + 1)\}\) and \(\gamma\) use the colors from \(\{1, \ldots, r\}\). We will perform these modifications to one clique at a time, starting with \(W_1.\)

For each vertex in a clique we will recolor the vertex its color assigned by the precoloring, \(c.\) We need to perform these modifications in a specific order, which we will explain in more detail later in the proof. Once we recolor a vertex in a clique its color assigned by \(c,\) we may have to recolor surrounding vertices so the graph remains properly colored. To recolor the surrounding vertices, we perform Kempe changes, which interchange colors on a path of vertices with two alternating colors. To make it so the interchanging of colors does not get too far away from the specific clique we are modifying, we will create barriers surrounding the clique. We do not want the color changes made on the vertices surrounding one clique to affect the colors of vertices surrounding other cliques since we are only considering one clique at a time. This is so we guarantee that the current coloring at every stage is a proper \((r + 1)\)-coloring of \(G.\)

We now define a way of talking about vertices at a certain distance away from a clique. For the clique, \(W_i,\) of \(G,\) let \(N_t(W_i)\) denote the vertices in \(G\) whose distance from \(W_i\) is exactly \(t.\) When it is clear which clique we are referring to, we will just say \(N_t.\) A vertex \(v \in G\) is distance \(t\) away from a clique \(W_i\) if the length of the shortest path from \(v\) to any vertex in \(W_i\) is exactly \(t.\) So, \(N_1\) is the set of vertices whose distance from \(W_i\) is exactly 1, \(N_2\) is the set of vertices whose distance from \(W_i\) is exactly 2, and so on.

We provide an example in Figure 4.1. The clique is composed of the 4 grey vertices and the 6 edges between them. The blue vertices are all distance 1 away from clique, so
Figure 4.1: The clique is composed of the grey vertices. The set of blue vertices make up $N_1$, the set of green vertices make up $N_2$, and the set of red vertices make up $N_3$.

the set of blue vertices makes up $N_1$. The green vertices are all distance 2 away from the clique, so the set of green vertices makes up $N_2$, and the red vertices are all distance 3 away from the clique, so the set of red vertices makes up $N_3$.

Let $W_1$ be the first clique in $G$ that we recolor. Since $W_1$ is a clique and $|W_1| \leq k$, there must be exactly $|W_1| \leq k$ colors used on $W_1$ by $c$ and by $\gamma$. Also, since $G$ is $r$-colorable, $k \leq r$. Let $|W_1| = q_1 \leq k \leq r$.

We will consider two cases: In the first case, $r + 1$ is a color used by $c$ on $W_1$. In the second case, $r + 1$ is not a color used by $c$ on $W_1$.

**Case 1:**

Assume there is a vertex $x_i$ in $W_1$ such that $c(x_i) = r + 1$. So, $\{c(x)|x \in W_1\} \subset \{1, 2, \ldots, r + 1\}$. Let the colors $c$ uses on $W_1$ be $s_1, s_2, \ldots, s_{q_1} \in \{1, 2, \ldots, r + 1\}$. Let $s_1 = r + 1$. Then let $s_2, \ldots, s_{q_1}$ go in numerical order from least to greatest, where $s_2$ is the minimum color $c$ uses on $W_1$ and \{s_2, \ldots, s_{q_1}\} \subset \{1, \ldots, r\}$. Let $\{a_1, a_2, \ldots, a_{q_1}\}$ be the set of colors $\gamma$ uses on $W_1$, where $a_p$ is the color $\gamma$ assigns to the vertex that is colored $s_p$ by $c$. We will also index the vertices in $W_1$ so that $x_p$ is the vertex that is colored $s_p$ by $c$ and $a_p$ by $\gamma$.

We will define a new coloring $f_0^1$ on $G$ to make the barriers surrounding $W_1$. Let the first barrier for $W_1$ be $N_3 \cup N_4$. If $v \in N_3 \cup N_4$ and $\gamma(v) = s_2$, then $f_0^1$ recolors $v$. 
the color $r + 1$. If $v \in N_3 \cup N_4$ and $\gamma(v) \neq s_2$, then $v$ remains colored its color assigned by $\gamma$ under $f_0^1$. Let the second barrier be $N_6 \cup N_7$. If $v \in N_6 \cup N_7$ and $\gamma(v) = s_3$, then $f_0^1$ recolors $v$ the color $r + 1$. If $v \in N_6 \cup N_7$ and $\gamma(v) \neq s_3$, then $v$ remains colored its color assigned by $\gamma$ under $f_0^1$. In general, for $2 \leq t \leq q_1$, let the $(t - 1)^{th}$ barrier be $N_{3t-3} \cup N_{3t-2}$. If $v \in N_{3t-3} \cup N_{3t-2}$ and $\gamma(v) = s_t$, then $f_0^1$ recolors $v$ the color $r + 1$. If $v \in N_{3t-3} \cup N_{3t-2}$ and $\gamma(v) \neq s_t$, then $v$ remains colored its color assigned by $\gamma$ under $f_0^1$. Also, if $v /\in N_{3t-3} \cup N_{3t-2}$ for any $t \in \{2, \ldots, q_1\}$, $v$ remains colored its color assigned by $\gamma$ under $f_0^1$.

In general, for $v \in G$,

$$f_0^1(v) = \begin{cases} 
  r + 1 & \text{if } v \in N_{3t-3}(W_1) \cup N_{3t-2}(W_1) \text{ for } 2 \leq t \leq q_1 \text{ and } \gamma(v) = s_t \\
  \gamma(v) & \text{otherwise.} 
\end{cases}$$

Within the $j^{th}$ barrier, two vertices are only recolored $r + 1$ if they were originally assigned the same color, $s_{j-1}$, under $\gamma$. Since $\gamma$ is a proper coloring, two vertices within the same barrier that are both colored $r + 1$ by $f_0^1$ cannot be adjacent. In addition, each barrier is distance 2 away from any other barrier, so vertices in different barriers assigned the color $r + 1$ by $f_0^1$ will not be adjacent. Also, recoloring a vertex $r + 1$ does not create an issue with vertices that remain colored their color assigned by $\gamma$ under $f_0^1$ since $\gamma$ does not use the color $r + 1$ on $G$. Thus, $f_0^1$ is a proper $(r + 1)$-coloring of $G$.

Now, we start recoloring the vertices in $W_1$ the color they’re assigned under the precoloring, $c$. As we mentioned before, to modify the colors on the clique, we may have to make modifications on vertices outside the clique. To do this, we perform Kempe changes. We must modify the colors on the vertices in the clique in a specific order, so a Kempe change does not alter barriers that we need to stop subsequent Kempe changes from getting too far from $W_1$. By stopping a Kempe change from getting too far from $W_1$ we mean, stopping the interchanging of colors on vertices so the recoloring of vertices does not occur greater than distance $3k - 2$ away from $W_1$.

We start by considering the vertex $x_1 \in W_1$ such that $c(x_1) = s_1 = r + 1$. It must be
the case that \( c(x_1) \neq f^1_0(x_1) = \gamma(x_1) = a_1 \) since \( \gamma \) does not use the color \( r + 1 \) on \( G \). To recolor \( x_1 \), the color \( r + 1 \), we define a new coloring on \( G \), \( f^1_1 \). Under this new coloring, \( f^1_1 \), let \( f^1_1(x_1) = s_1 = r + 1 \). For all other vertices, \( v \in (G - \{x_1\}) \), let \( f^1_1(v) = f^0_0(v) \).

In general, for \( v \in G \),

\[
f^1_1(v) = \begin{cases} 
  r + 1 & \text{if } v = x_1, \text{ where } c(x_1) = r + 1 \\
  f^0_0(v) & \text{otherwise.}
\end{cases}
\]

\( x_1 \) will not be adjacent to any vertices in \( G \) also colored \( r + 1 \) under \( f^1_1 \) since all vertices in \( G - \{x_1\} \) will be assigned the same color as they were assigned under \( f^0_0 \). Other vertices colored \( r + 1 \) under \( f^1_1 \) are at least distance 3 away from \( x_1 \), since the closest vertices to \( x_1 \) colored \( r + 1 \) under \( f^1_1 \) are in the first barrier, \( N_3 \cup N_4 \). In addition, all vertices in \( G - \{x_1\} \) will be assigned the same color as they were assigned under \( f^1_0 \) and \( f^0_0 \) is a proper \((r + 1)\)-coloring on \( G \). Thus, \( f^1_1 \) is a proper \((r + 1)\)-coloring on \( G \) as well.

Now assume that the vertices \( x_1, x_2, \ldots, x_{j-1} \) have all been assigned their color from the precoloring and that \( f^1_{j-1} \) is a proper \((r + 1)\)-coloring of \( G \). Consider the vertex \( x_j \in W_1 \) such that \( c(x_j) = s_j \) and the current color of \( x_j \) is \( f^1_{j-1}(x_j) \). Since each \( s_i \) is a unique color from \( \{1, 2, \ldots, r + 1\} \), we know that \( s_j \neq r + 1 \) because \( s_1 = r + 1 \). To recolor \( x_j \), the color \( s_j \), we define a new coloring on \( G \), \( f^1_j \). It is either the case that \( s_j = f^1_{j-1}(x_j) \) or that \( s_j \neq f^1_{j-1}(x_j) \). To simplify the notation, let \( f^1_{j-1}(x_j) = e_j \).

If \( c(x_j) = s_j = e_j \), then we keep \( x_j \) colored \( s_j = e_j \) under \( f^1_j \). So, let \( f^1_j(x_j) = f^1_{j-1}(x_j) \). Also, for all other vertices, \( v \in (G - x_j) \), let \( f^1_j(v) = f^1_{j-1}(v) \). Since all vertices in \( G \) are assigned the same color they were assigned by \( f^1_{j-1} \) under the new coloring, \( f^1_j \), \( f^1_j \) is a proper \((r + 1)\)-coloring of \( G \).

If \( c(x_j) = s_j \neq e_j \), we will interchange the colors on the \((s_j, e_j)\)-Kempe chain containing \( x_j \). The vertex \( x_j \) will be recolored from the color \( e_j \) to the color \( s_j \) under \( f^1_j \). All other vertices on the \((s_j, e_j)\)-Kempe chain containing \( x_j \) will be colored \( s_j \) under \( f^1_j \) if they were colored \( e_j \) under the previous coloring, \( f^1_{j-1} \). They will be colored \( e_j \)
under $f_j^1$ if they were colored $s_j$ under $f_{j-1}^1$.

The $(s_j, e_j)$-Kempe chain containing $x_j$ will end at most distance $3j - 3$ away from $W_1$ because of the barriers we made by the coloring $f_0^1$. Under $f_{j-1}^1$, vertices in the $j - 1^{th}$ barrier, $N_{3j-3} \cup N_{3j-2}$, are assigned the same color they were assigned by $f_0^1$. So, if a vertex in $N_{3j-3} \cup N_{3j-2}$ was assigned $r + 1$ under $f_0^1$, then under $f_{j-1}^1$, it is still assigned the color $r + 1$. Also, if a vertex in $N_{3j-3} \cup N_{3j-2}$ was assigned its $\gamma$ color under $f_0^1$, then under $f_{j-1}^1$, it is still assigned its $\gamma$ color. This is because no previous Kempe change is able to change the colors of any of the vertices in the $(j - 1)$th barrier because the $(j - 1)$th barrier is further away from $W_1$ than all previous barriers. This ensures that there are no vertices colored $s_j$ in $N_{3j-3} \cup N_{3j-2}$. This means that the $(s_j, e_j)$-Kempe chain containing $x_j$ cannot go past $N_{3j-3}$. Thus, the color changes made by this Kempe change will be at most distance $3j - 3$ away from $W_1$.

For the case when $s_j \neq e_j$, for $v \in G$, we have the following:

$$f_j^1(v) = \begin{cases} 
  s_j & \text{if } v = x_j, \text{ where } c(x_j) = s_j \\
  s_j & \text{if } f_{j-1}^1(v) = e_j, \text{ and } v \text{ is on the } (s_j, e_j) - \text{Kempe chain containing } x_j \\
  e_j & \text{if } f_{j-1}^1(v) = s_j, \text{ and } v \text{ is on the } (s_j, e_j) - \text{Kempe chain containing } x_j \\
  f_{j-1}^1(v) & \text{otherwise.}
\end{cases}$$

By Lemma 1, $f_j^1$ is a proper $(r + 1)$-coloring of $G$.

Eventually, we get to the last vertex, $x_{q_1}$, in $W_1$. By defining a new coloring on $G$, $f_{q_1}^1$, we recolor this vertex its color assigned by the precoloring, $s_{q_1}$ and then perform a Kempe change on any vertices that the recoloring of $x_{q_1}$ necessitates. This Kempe change will end at most distance $3q_1 - 3$ away from $W_1$ because there are no vertices in the last barrier, $N_{3q_1-3} \cup N_{3q_1-2}$, colored $s_{q_1}$ under $f_{q_1-1}^1$. This is because all vertices in this barrier that were assigned $s_{q_1}$ by $\gamma$ were recolored $r + 1$, and no previous color changes could have occurred at this distance from $W_1$. After doing this, we have that $f_{q_1}^1$ is a proper $(r + 1)$-coloring of $G$ by the same arguments that we made for the $j^{th}$ vertex of $W_1$, using Lemma 1. All vertices of $W_1$ are now assigned the color given to
them by the precoloring. Therefore, we have finished recoloring $W_1$ and the vertices within distance $3q_1 - 2 \leq 3k - 2$ of $W_1$. The farthest color changes that occurred during this process are on vertices at most distance $3q_1 - 2 \leq 3k - 2$ from $W_1$.

We now provide an example of making modifications to the $r$-coloring of a graph so that it agrees with colors on the precolored vertices in the cliques by following the procedure described above in Case 1. In our example, let $\gamma$ be a 3-coloring of $G$ that uses the colors from \{\textit{blue, orange, green}\}. Let $c$ be a 4-coloring of $G[W]$ that uses the colors from \{\textit{blue, orange, green, red}\}. We will consider the clique $W_1$, where $|W_1| = 3$. Let the colors $\gamma$ uses on $W_1$ be from \{\textit{blue, orange, green}\}, and let the colors $c$ uses on $W_1$ be from \{\textit{blue, orange, red}\}.

In Figure 4.2, we have the clique, $W_1$, that consists of three vertices, $x_1$, $x_2$, and $x_3$, and the vertices within distance 8 of $W_1$, which are the vertices in $\{v_1, v_2, \ldots, v_{24}\}$. What is shown in this figure is a subgraph of $G$. However, when we define new proper colorings on the graph, the colorings are on all of $G$, not just the subgraph shown. All the vertices in the graph are currently colored by $\gamma$. For the vertices in the clique, for reference, we provide their precolored color as the outer color of the nodes. Their inner color of each node is what it is currently colored by $\gamma$. Since red is used by $c$ on $W$ and not by $\gamma$ on $G$, red is acting as our $r + 1$ color. We will order the colors $c$ uses on the clique. Since red is our $r + 1$ color, let $s_1 = \text{red}$. Then we will go in alphabetical order, and let $s_2 = \text{blue}$ and $s_3 = \text{orange}$. Thus, $s_1 = c(x_1) = \text{red}$, $s_2 = c(x_2) = \text{blue}$, and $s_3 = c(x_3) = \text{orange}$. We will now label the colors that $\gamma$ uses on $W_1$. Since $s_1 = c(x_1) = \text{red}$, let $a_1 = \gamma(x_1) = \text{blue}$. Since $s_2 = c(x_2) = \text{blue}$, let $a_2 = \gamma(x_2) = \text{orange}$. Since $s_3 = c(x_3) = \text{orange}$, let $a_3 = \gamma(x_3) = \text{green}$. We will follow this example through the process of recoloring the entire clique, $W_1$.

In Figure 4.3, we show the creation of the barriers. By the barrier coloring procedure, under $f_0^1$, vertices in the first barrier, $N_3 \cup N_4$, that are colored $s_2 = \text{blue}$ and vertices in the second barrier, $N_6 \cup N_7$, that are colored $s_3 = \text{orange}$, get recolored red. All other vertices in this graph receive their same color from $\gamma$, shown in the previous figure,
Figure 4.2, under $f_0^1$. We can see that $f_0^1$ is a proper 4-coloring of the subgraph shown, and it will also be a proper 4-coloring of $G$.

In Figure 4.4, we show the recoloring of the first vertex in $W_1$, $x_1$. The vertex, $x_1$, is the first one we recolor since $c(x_1) = s_1 = \text{red}$. We define a new coloring on $G$, $f_1^1$, and we recolor $x_1$ from $f_0^1(x_1) = a_1 = \text{blue}$ to $c(x_1) = s_1 = \text{red}$ under the new coloring. All other vertices in the graph receive their same color from $f_0^1$, shown in Figure 4.3, under our new coloring, $f_1^1$. We can see that $f_1^1$ is a proper 4-coloring of the subgraph shown, and it will also be a proper 4-coloring of $G$.

In Figure 4.5, we recolor the vertex, $x_2$, since $c(x_2) = s_2 = \text{blue}$. We define a new coloring, $f_2^1$, and recolor $x_2$ from $f_1^1(x_2) = \text{orange}$ to $s_2 = \text{blue}$. There is a ($\text{blue}, \text{orange}$)-Kempe chain containing $x_2$ that we must recolor under $f_2^1$ as well. Other than $x_1$, the vertices on this chain are $v_1, v_2, v_4, v_5, v_8$. Thus, we perform a Kempe change and the vertices on this chain that were colored blue under $f_1^1$, $v_1$ and $v_4$, are recolored orange under $f_2^1$. The vertices on this chain that were colored orange under $f_1^1$, $v_2, v_5$, and $v_8$, are recolored blue under $f_2^1$. This Kempe chain stops at $N_3$ since in the first barrier, $N_3 \cup N_4$, all $s_2 = \text{blue}$ vertices were recolored red under $f_0^1$. All other vertices in the graph receive their same color from $f_1^1$, shown in Figure 4.4, under our new coloring, $f_2^1$. We can see that $f_2^1$ is a proper 4-coloring of the subgraph shown, and it will also be a proper 4-coloring of $G$.

In Figure 4.6, we recolor the vertex, $x_3$, since $c(x_3) = s_3 = \text{orange}$. We define a new coloring, $f_3^1$, and recolor $x_3$ from $f_2^1(x_3) = \text{green}$ to $s_3 = \text{orange}$. There is an ($\text{orange}, \text{green}$)-Kempe chain containing $x_3$ that we must recolor under $f_3^1$ as well. Other than $x_3$, the vertices on this chain are $v_1, v_3, v_4, v_6, v_9, v_{11}, v_{12}, v_{14}, v_{15}, v_{18}$. We perform a Kempe change and the vertices on this chain that were colored orange under $f_2^1$, $v_1, v_4, v_{11}$, and $v_{14}$, are recolored green under $f_3^1$. The vertices on this chain that were colored green under $f_2^1$, $v_3, v_6, v_9, v_{12}, v_{15}$, and $v_{18}$, are recolored orange under $f_3^1$. This Kempe chain stops at $N_6$ since in the second barrier, $N_6 \cup N_7$, all $s_3 = \text{orange}$ vertices were recolored red under $f_0^1$. All other vertices in the graph
Figure 4.2: The whole graph is currently colored by \( \gamma \). \( \gamma \) is a proper 3-coloring of the subgraph shown and of the whole graph \( G \). Each vertex in \( W_1 \), has its color assigned by \( \gamma \) shown in the center of the node and its color assigned by \( c \) shown on the outside of the node.

receive their same color from \( f^1_2 \), shown in Figure 4.5, under our new coloring, \( f^1_3 \). We can see that \( f^1_3 \) is a proper 4-coloring of the subgraph shown, and it will also be a proper 4-coloring of \( G \). We are done recoloring \( W_1 \) and its surrounding vertices, and the vertices in \( W_1 \) are all assigned their precoloring colors. In this example, the furthest color change from \( W_1 \) that occurred is in \( N_7 \), by the barrier color change, \( f^1_0 \). We can see that no vertex in \( N_8 \) had its color altered. We would now move on to recoloring the next clique, \( W_2 \).
Figure 4.3: Under $f^1_0$, we created the barriers for $W_1$. $f^1_0$ is a proper 4-coloring of the subgraph shown and of $G$.

Figure 4.4: Under $f^1_1$, we assigned $x_1$ its precoloring color, red. All other vertices in the graph are assigned their same color from $f^1_0$ under $f^1_1$. $f^1_1$ is a proper 4-coloring of the subgraph shown and of $G$. 
Figure 4.5: Under \( f^1_2 \), we assigned \( x_2 \) its precoloring color, \textit{blue}, and performed a Kempe change along the \((\text{blue, orange})\)-Kempe Chain containing \( x_2 \). All other vertices in the graph are assigned their same color from \( f^1_1 \) under \( f^1_2 \). \( f^1_2 \) is a proper 4-coloring of the subgraph shown and of \( G \).

Figure 4.6: Under \( f^1_3 \), we assigned \( x_3 \) its precoloring color, \textit{orange}, and we performed a Kempe change along the \((\text{orange, green})\)-Kempe Chain containing \( x_3 \). All other vertices in the graph are assigned their same color from \( f^1_2 \) under \( f^1_3 \). \( f^1_3 \) is a proper 4-coloring of the subgraph shown and of \( G \). We have finished modifying \( \gamma \) on this clique and its surrounding vertices, so each vertex in \( W_1 \) is now assigned its precoloring color.
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Case 2:

Assume there is no vertex \( x_i \) in \( W_1 \) such that \( c(x_i) = r + 1 \). So, \( \{c(x) \mid x \in W_1\} \subseteq \{1, 2, \ldots, r\} \). Recall that \( |W_1| = q_1 \). Case 2 will have 2 subcases. In Case 2a, we will assume there is a vertex \( x_i \in W_1 \) in which either \( c(x_i) \notin \{\gamma(x) \mid x \in W_1\} \) or \( c(x_i) = \gamma(x_i) \).

In Case 2b, we will assume the set of colors used on \( W_1 \) by \( \gamma \) is equal to the set of colors used on \( W_1 \) by \( c \), so \( \{c(x) \mid x \in W_1\} = \{\gamma(x) \mid x \in W_1\} \), and for all \( x_i \in W_1 \), \( c(x_i) \neq \gamma(x_i) \).

Case 2a:

Let there be an \( x_i \in W_1 \) in which one of the two following conditions hold:

1. \( c(x_i) \notin \{\gamma(x) \mid x \in W_1\} \).

2. \( c(x_i) = \gamma(x_i) \).

If there are multiple vertices that satisfy either of these conditions, choose the vertex with minimum \( c \) color. Label this vertex \( x_1 \) and let \( s_1 = c(x_1) \) where \( c(x_1) \) is the minimum \( c \) color such that either \( c(x_1) \notin \{\gamma(x) \mid x \in W_1\} \) or \( c(x_1) = \gamma(x_1) \). Then let \( s_2, \ldots, s_{q_1} \) go in numerical order from least to greatest, where \( s_2 \) is the minimum color \( c \) uses on \( (W_1 - \{x_1\}) \). We will also label the vertices in \( W_1 \) and colors \( \gamma \) uses on \( W_1 \) so that their subscripts match up with the subscripts of the colors used by \( c \) on \( W_1 \). Let \( \{x_1, x_2, \ldots, x_{q_1}\} \) be the set vertices in \( W_1 \) where \( x_p \) is the vertex that is colored \( s_p \) by \( c \). Let \( \{a_1, a_2, \ldots, a_{q_1}\} \) be the set of colors \( \gamma \) uses on \( W_1 \) where \( a_p \) is the color \( \gamma \) uses on the vertex that is colored \( s_p \) by \( c \). We have that either \( s_1 \notin \{a_1, a_2, \ldots, a_{q_1}\} \) or \( s_1 = a_1 \).

Similar to Case 1, we will define a new coloring, \( f^1_0 \), on \( G \) to create barriers surrounding \( W_1 \). Let the first barrier for \( W_1 \) be \( N_1 \). If \( v \in N_1 \) and \( \gamma(v) = s_1 \), then \( f^1_0 \) recolors \( v \) the color \( r + 1 \). If \( v \in N_1 \), but \( \gamma(v) \neq s_1 \), then \( v \) remains colored its color assigned by \( \gamma \) under \( f^1_0 \). Let the second barrier be \( N_3 \cup N_4 \). If \( v \in N_3 \cup N_4 \) and \( \gamma(v) = s_2 \), then \( f^1_0 \) recolors \( v \) the color \( r + 1 \). If \( v \in N_3 \cup N_4 \), but \( \gamma(v) \neq s_2 \), then \( v \) remains colored its color assigned by \( \gamma \) under \( f^1_0 \). In general, for \( 2 \leq t \leq q_1 \), let the \( t^{th} \) barrier be \( N_{3t-3} \cup N_{3t-2} \). If
v \in N_{3t-3} \cup N_{3t-2} and \gamma(v) = s_t, then \(f_1^1\) recolors \(v\) the color \(r + 1\). If \(v \in N_{3t-3} \cup N_{3t-2}\), but \(\gamma(v) \neq s_t\), then \(v\) remains colored its color assigned by \(\gamma\) under \(f_0^1\). Also, if \(v \notin N_1\) or if \(v \notin N_{3t-3} \cup N_{3t-2}\) for any \(t \in \{2, \ldots, q_1\}\), \(v\) also remains colored its color assigned by \(\gamma\) under \(f_0^1\).

In general, for \(v \in G\),

\[
f_0^1(v) = \begin{cases} 
  r + 1 & \text{if } v \in N_1(W_1) \text{ and } \gamma(v) = s_1 \\
  r + 1 & \text{if } v \in N_{3t-3}(W_1) \cup N_{3t-2}(W_1) \text{ for } 2 \leq t \leq q_1 \text{ and } \gamma(v) = s_t \\
  \gamma(v) & \text{otherwise.}
\end{cases}
\]

Within the \(j\)th barrier, two vertices are only recolored \(r + 1\) if they were originally assigned the same color, \(s_j\), under \(\gamma\). Since \(\gamma\) is a proper coloring, two vertices within the same barrier that are both colored \(r + 1\) by \(f_0^1\) cannot be adjacent. In addition, each barrier is distance 2 away from any other barrier, so vertices in different barriers assigned the color \(r + 1\) by \(f_0^1\) will not be adjacent. Also, recoloring a vertex \(r + 1\) does not create an issue with vertices that remain colored their color assigned by \(\gamma\) under \(f_0^1\) since \(\gamma\) does not use the color \(r + 1\) on \(G\). Thus, \(f_0^1\) is a proper \((r + 1)\)-coloring of \(G\).

Once the barriers for \(W_1\) are made, we start recoloring the vertices of \(W_1\) the color they are assigned by \(c\). Again, we must do this in a specific order just like we did for Case 1. We start by considering the vertex \(x_1 \in W_1\) such that \(c(x_1) = s_1\). It’s either the case that \(s_1 = a_1\) or that \(s_1 \notin \{a_1, \ldots, a_{q_1}\}\).

Let us first suppose we have the case where \(s_1 = f_0^1(x_1) = \gamma(x_1) = a_1\). To recolor \(x_1\) its color assigned by \(c\), we define a new coloring on \(G\), \(f_1^1\). Under our new coloring, \(f_1^1\), \(x_1\) keeps its color \(s_1 = f_0^1(x_1)\). Under \(f_1^1\), all other vertices in \(G\) also keep their color assigned by the previous coloring, \(f_0^1\). Thus, for all \(v \in G\), \(f_1^1(v) = f_0^1(v)\). \(f_1^1\) is clearly a proper \((r + 1)\)-coloring of \(G\) since it is the same coloring as \(f_0^1\), which we previously showed is a proper \((r + 1)\)-coloring of \(G\).

Now, let us suppose we have the case that \(s_1 \notin \{a_1, \ldots, a_{q_1}\}\). Define a new coloring on \(G\), \(f_1^1\). Under the new coloring, \(f_1^1\), we recolor \(x_1\) from \(f_0^1(x_1) = \gamma(x_1) = a_1\) to the
color $s_1$. For all other vertices, $v \in (G - \{x_1\})$, let $f_1^1(v) = f_0^1(v)$. So, in general, for $v \in G$, we have the following:

$$f_1^1(v) = \begin{cases} s_1 & \text{if } v = x_1, \text{ where } c(x_1) = s_1 \neq a_1 \\ f_0^1(v) & \text{otherwise.} \end{cases}$$

There are no vertices in $(W_1 - \{x_1\})$ colored $s_1$ under $f_1^1$ because all vertices in $(W_1 - \{x_1\})$ are colored their color given by $\gamma$ under $f_1^1$, which are the colors in $\{a_2, \ldots, a_{q_1}\}$. By assumption, $s_1 \notin \{a_1, \ldots, a_{q_1}\}$. Also, because of the barrier coloring, $f_0^1$, all vertices in $N_1$ that were colored $s_1$ by $\gamma$ have been recolored $r + 1$ under $f_0^1$ and remain colored $r + 1$ under $f_1^1$. Therefore, there are no vertices adjacent to $x_1$ that are colored $s_1$ under $f_1^1$ since the only vertices that are adjacent to $x_1$ are in $W_1$ and $N_1$. Also, since all other vertices in $G$ receive their $f_0^1$ color under $f_1^1$, and since $f_0^1$ is a proper $(r + 1)$-coloring of $G$, we have that $f_1^1$ is also a proper $(r + 1)$-coloring of $G$.

Therefore, in either case, $s_1 = a_1$ or $s_1 \notin \{a_1, \ldots, a_{q_1}\}$, we are able to recolor $x_1$ its color assigned by $c$ and we still have a proper $(r + 1)$-coloring on all of $G$.

Now, assume that the vertices $x_1$, $x_2, \ldots, x_j$ have all been assigned their color from the precoloring, $c$, and that $f_{j-1}^1$ is a proper $r + 1$-coloring of $G$. Consider the vertex $x_j \in W_1$ such that $c(x_j) = s_j$ and the current color of $x_j$ is $f_{j-1}^1(x_j)$. To recolor $x_j$ the color $s_j$, we define a new coloring, $f_j^1$, on $G$. There are two cases for $x_j$. Either $s_j = f_{j-1}^1(x_j)$ or $s_j \neq f_{j-1}^1(x_j)$. To simplify the notation, let $f_{j-1}^1(x_j) = e_j$. If $s_j = e_j$, then we keep $x_j$ colored $s_j = e_j$ under $f_j^1$. So, let $f_j^1(x_j) = f_{j-1}^1(x_j)$. Also, for all other vertices, $v \in (G - x_j)$, let $f_j^1(v) = f_{j-1}^1(v)$. Since all vertices in $G$ are assigned the same color they were assigned by $f_{j-1}^1$ under the new coloring, $f_j^1$, and since $f_{j-1}^1$ is a proper $(r + 1)$-coloring of $G$, $f_j^1$ is a proper $(r + 1)$-coloring of $G$ as well.

If $c(x_j) = s_j \neq e_j$, we interchange the colors on the $(s_j, e_j)$- Kempe chain containing $x_j$. Under $f_j^1$, $x_j$ is recolored the color $s_j$. All other vertices on the $(s_j, e_j)$- Kempe chain containing $x_j$ will be recolored $s_j$ under $f_j^1$ if they were colored $e_j$ under the
previous coloring, \( f_{j-1}^1 \). They will be recolored \( e_j \) under \( f_j^1 \) if they were colored \( s_j \) under \( f_{j-1}^1 \). The \((s_j, e_j)\)- Kempe chain containing \( x_j \) will end at most distance \( 3j - 3 \) away from \( W_1 \) because of the barriers we made by the coloring \( f_0^1 \). Under \( f_{j-1}^1 \), vertices in the \( j^{th} \) barrier, \( N_{3j-3} \cup N_{3j-2} \), are assigned the same color they were assigned by \( f_0^1 \). So, if a vertex in \( N_{3j-3} \cup N_{3j-2} \) was assigned \( r+1 \) under \( f_0^1 \), then under \( f_{j-1}^1 \), it is still assigned the color \( r+1 \). Also, if a vertex in \( N_{3j-3} \cup N_{3j-2} \) was assigned its color given by \( \gamma \) under \( f_0^1 \), then under \( f_{j-1}^1 \), it is still assigned its color given by \( \gamma \). This is because no previous Kempe change is able to change the colors of any of the vertices in the \( j^{th} \) barrier because the \( j^{th} \) barrier is further away from \( W_1 \) than all previous barriers. This makes it so there are no vertices colored \( s_j \) in \( N_{3j-3} \cup N_{3j-2} \) under \( f_{j-1}^1 \). This means that the \((s_j, e_j)\)- Kempe chain containing \( x_j \) cannot go past \( N_{3j-3} \). Thus, the color changes made by this Kempe change will be at most distance \( 3j - 3 \) away from \( W_1 \).

For the case when \( s_j \neq e_j \), for \( v \in G \), we have the following:

\[
f_j^1(v) = \begin{cases} 
  s_j & \text{if } v = x_j, \text{ where } c(x_j) = s_j \\
  s_j & \text{if } f_{j-1}^1(v) = e_j, \text{ and } v \text{ is on the } (s_j, e_j) - \text{Kempe chain containing } x_j \\
  e_j & \text{if } f_{j-1}^1(v) = s_j, \text{ and } v \text{ is on the } (s_j, e_j) - \text{Kempe chain containing } x_j \\
  f_{j-1}^1(v) & \text{otherwise.}
\end{cases}
\]

By Lemma 1, \( f_j^1 \) is a proper \((r+1)\)-coloring of \( G \).

Eventually, we get to the last vertex, \( x_{q_1} \), in \( W_1 \). By defining a new coloring \( f_{q_1}^1 \), we recolor this vertex its color assigned by the precoloring, \( s_{q_1} \), and then perform a Kempe change on any vertices that the recoloring of \( x_{q_1} \) necessitates. This Kempe change will end at most distance \( 3q_1 - 3 \) away from \( W_1 \) because there are no vertices in the last barrier, \( N_{3q_1-3} \cup N_{3q_1-2} \), colored \( s_{q_1} \) under \( f_{q_1-1}^1 \). After doing this, we have that \( f_{q_1}^1 \) is a proper \((r+1)\)-coloring of \( G \) by the same arguments that we made for the \( j^{th} \) vertex of \( W_1 \), using Lemma 1. All vertices of \( W_1 \) are now assigned the color given to them by the precoloring, \( c \). Therefore, we have finished recoloring \( W_1 \) and the vertices within distance \( 3q_1 - 2 \leq 3k - 2 \) of \( W_1 \). The farthest color changes that occurred during this
process were on vertices at most distance $3q_1 - 2 \leq 3k - 2$ from $W_1$.

We now provide an example of making modifications to the $r$-coloring of a graph so that it agrees with colors on the precolored vertices in the cliques by following the procedure described above in Case 2a. In our example, let $\gamma$ be a 4-coloring of $G$ that uses the colors from $\{\text{blue, green, orange, yellow}\}$. Let $c$ be a 5-coloring of $G[W]$ that uses the colors from $\{\text{blue, green, orange, yellow, red}\}$. We will consider the clique $W_1$, where $|W_1| = 3$. Let the colors $\gamma$ uses on $W_1$, be from $\{\text{blue, green, orange}\}$. Let the colors $c$ uses on $W_1$, be from $\{\text{blue, green, yellow}\}$.

In Figure 4.7, we have the clique, $W_1$, that consists of three vertices, $x_1$, $x_2$, and $x_3$, and the vertices within distance 8 of $W_1$, which are the vertices in $\{v_1, v_2, \ldots, v_{32}\}$. In this figure we have a subgraph of $G$. However, when we define a new coloring on the graph, the coloring is on all of $G$, not just the subgraph shown. All the vertices in the graph are currently colored by $\gamma$. For each vertex in the clique, for reference, we provide its precolored color as the outer color of the node. The node’s inner color is what it is currently colored by $\gamma$. Since $c(x_1) = \text{yellow} \notin \{\gamma(x) \mid x \in W_1\} = \{\text{blue, green, orange}\}$, we let $s_1 = c(x_1) = \text{yellow}$. Then we will go in alphabetical order, and let $s_2 = c(x_2) = \text{blue}$ and $s_3 = c(x_3) = \text{green}$. We will now label the colors that $\gamma$ uses on $W_1$. Since $s_1 = c(x_1) = \text{yellow}$, let $a_1 = \gamma(x_1) = \text{blue}$. Since $s_2 = c(x_2) = \text{blue}$, let $a_2 = \gamma(x_2) = \text{green}$. Since $s_3 = c(x_3) = \text{green}$, let $a_3 = \gamma(x_3) = \text{orange}$. We will follow this example through the process of recoloring the entire clique, $W_1$.

In Figure 4.8, we show the creation of the barriers. By the barrier coloring procedure, under $f_0^1$, vertices in the first barrier, $N_1$, that are colored $s_1 = \text{yellow}$, get recolored red. Vertices in the second barrier, $N_3 \cup N_4$, that are colored $s_2 = \text{blue}$, get recolored red. Vertices in the third barrier, $N_6 \cup N_7$, that are colored $s_3 = \text{green}$, get recolored red. All other vertices in this graph receive their $\gamma$ color, shown in the previous figure, Figure 4.7, under $f_0^1$. We can see that $f_0^1$ is a proper 5-coloring of the subgraph shown, and it will also be a proper 5-coloring of $G$.

In Figure 4.9, we show the recoloring of the first vertex in $W_1$, $x_1$. The vertex, $x_1$,
is the first one we recolor since \( c(x_1) = s_1 = \text{yellow} \). We define a new coloring on \( G \), \( f_1^1 \), and we recolor \( x_1 \) from \( f_0^1(x_1) = a_1 = \text{blue} \) to \( s_1 = c(x_1) = \text{yellow} \) under the new coloring. All other colors in the graph receive their same color from \( f_0^1 \), shown in Figure 4.8, under the new coloring, \( f_1^1 \). We can see that \( f_1^1 \) is a proper 5-coloring of the subgraph shown, and it will also be a proper 5-coloring of \( G \).

In Figure 4.10, we now recolor the vertex, \( x_2 \), since \( c(x_2) = s_2 = \text{blue} \). We define a new coloring, \( f_1^2 \), and recolor \( x_2 \) from \( f_1^2(x_2) = \text{green} \) to \( s_2 = \text{blue} \). There is \((\text{blue, green})\)-Kempe chain containing \( x_2 \) that we must interchange the colors on under \( f_2^1 \). Other than \( x_1 \), the vertices on this chain are \( v_1, v_2, v_5, v_6, v_{10} \). We perform a Kempe change and the vertices on this chain that were colored \( \text{blue} \) under \( f_1^1, v_1, v_5, \) are recolored \( \text{green} \) under \( f_2^1 \). The vertices on this chain that were colored \( \text{green} \) under \( f_1^1, v_2, v_6, v_{10}, \) are recolored \( \text{blue} \) under \( f_2^1 \). This Kempe chain stops at \( N_3 \) since in the 2\(^{nd} \) barrier, \( N_3 \cup N_4 \), all \( s_2 = \text{blue} \) vertices were recolored \( \text{red} \) under \( f_0^1 \). All other vertices in the graph receive their same color from \( f_1^1 \), shown Figure 4.9, under the new coloring, \( f_2^1 \). We can see that \( f_2^1 \) is a proper 5-coloring of the subgraph shown, and it will also be a proper 5-coloring of \( G \).

In Figure 4.11, we recolor the vertex, \( x_3 \), since \( c(x_3) = s_3 = \text{green} \). We define a new coloring, \( f_3^1 \), and recolor \( x_3 \) from \( f_3^1(x_3) = \text{orange} \) to \( s_3 = \text{green} \). There is \((\text{green, orange})\)-Kempe chain containing \( x_3 \) that we must interchange the colors on under \( f_3^1 \) as well. Other than \( x_3 \), the vertices on this chain are \( v_1, v_3, v_5, v_7, v_{11}, v_{14}, v_{15}, v_{18}, v_{19}, v_{23} \). Thus, we perform a Kempe change and the vertices on this chain that were colored \( \text{green} \) under \( f_2^1, v_1, v_5, v_{14}, v_{18}, \) are recolored \( \text{orange} \) under \( f_3^1 \). The vertices on this chain that were colored \( \text{orange} \) under \( f_2^1, v_3, v_7, v_{11}, v_{15}, v_{19}, v_{23}, \) are recolored \( \text{green} \) under \( f_3^1 \). Now, we have finished modifying \( \gamma \) on this clique and its surrounding vertices so that vertices in the clique are assigned their precoloring color. Our new coloring \( f_3^1 \) is a proper 5-coloring on our graph. This Kempe chain stops at \( N_6 \) since in the third barrier, \( N_6 \cup N_7 \), all \( s_3 = \text{green} \) vertices were recolored \( \text{red} \) under \( f_0^1 \). All other vertices in the graph receive their same color from \( f_2^1 \), shown in Figure 4.10,
under our new coloring, $f_3^1$. We can see that $f_3^1$ is a proper 5-coloring of the subgraph shown, and it will also be a proper 5-coloring of $G$. We are done recoloring $W_1$ and its surrounding vertices, and the vertices in the $W_1$ are assigned their precoloring color. In this example, the furthest color change from $W_1$ that occurred is in $N_7$, by the barrier color change, $f_0^1$. We can see that no vertices in $N_8$ had their color altered. We would now move on to recoloring the next clique, $W_2$. 
Figure 4.7: The whole graph is currently colored by $\gamma$. $\gamma$ is a proper 4-coloring of the subgraph shown and of the whole graph $G$. Each vertex in $W_1$, has its color assigned by $\gamma$ shown in the center of the node and its color assigned by $c$ shown on the outside of the node.

Figure 4.8: Under $f^1_0$, we created the barriers for $W_1$. $f^1_0$ is a proper 5-coloring of the subgraph shown and of $G$. 

Figure 4.9: Under $f_1^1$, we assigned $x_1$ its precoloring color, yellow. All other vertices in the graph are assigned their same color from $f_0^1$ under $f_1^1$. $f_1^1$ is a proper 5-coloring of the subgraph shown and of $G$.

Figure 4.10: Under $f_2^1$, we assigned $x_2$ is assigned its precoloring color, blue, and performed a Kempe change along the (blue, green)-Kempe Chain containing $x_2$. All other vertices in the graph are assigned their same color from $f_1^1$ under $f_2^1$. $f_2^1$ is a proper 5-coloring of the subgraph shown and of $G$. 
Figure 4.11: Under $f_3^1$, we assigned $x_3$ its precoloring color, green, and we performed a Kempe change along the (green, orange)-Kempe Chain containing $x_3$. All other vertices in the graph are assigned their same color from $f_2^1$ under $f_3^1$. $f_3^1$ is a proper 5-coloring of the subgraph shown and of $G$. We have finished modifying $\gamma$ on this clique and its surrounding vertices, so each vertex in $W_1$ is now assigned its precoloring color.
CHAPTER 4. REVISION OF ALBERTSON’S PROOF

Case 2b:

Assume that there is no vertex $x_i \in W_1$ in which either of following conditions hold:

1. $c(x_i) \notin \{\gamma(x)|x \in W_1\}$.
2. $c(x_i) = \gamma(x_i)$.

Since there is no vertex in $W_1$ such that either of the above conditions holds, we have that $\{c(x)|x \in W_1\} = \{\gamma(x)|x \in W_1\}$, and for all $x_i \in W_1$, $c(x_i) \neq \gamma(x_i)$. Assume that the colors $c$ uses on $W_1$ are from $\{s_1, s_2, \ldots, s_{q_1}\} \subseteq \{1, 2, \ldots, r\}$. Let $s_1, s_2, \ldots, s_{q_1}$ go in numerical order from least to greatest, where $s_1$ is the minimum color $c$ uses on $W_1$. Let $\{a_1, a_2, \ldots, a_{q_1}\}$ be the set of colors $\gamma$ uses on $W_1$, where $a_p$ is the color $\gamma$ uses on the vertex that is colored $s_p$ by $c$. We will also index the vertices in $W_1$ so that $x_p$ is the vertex that is colored $s_p$ by $c$. After labeling the colors, we have that $\{s_1, \ldots, s_{q_1}\} = \{a_1, \ldots, a_{q_1}\}$ and $a_i \neq s_i$ for all $i \in \{1, \ldots, q_1\}$.

We now construct the barriers by defining the coloring, $f_0^1$, on $G$. The procedure to create the barriers is the same exact procedure as in Case 2a to create the barriers.

In general, for $v \in G$,

$$f_0^1(v) = \begin{cases} 
  r + 1 & \text{if } v \in N_1(W_1) \text{ and } \gamma(v) = s_1 \\
  r + 1 & \text{if } v \in N_{3t-3}(W_1) \cup N_{3t-2}(W_1) \text{ for } 2 \leq t \leq q_1 \text{ and } \gamma(v) = s_t \\
  \gamma(v) & \text{otherwise.}
\end{cases}$$

By the same argument in Case 2a for why the barrier coloring is a proper coloring, the coloring we have just defined, $f_0^1$, is a proper $(r + 1)$-coloring on $G$.

After making the barriers, we want to first recolor $x_1$ the color $s_1$. To do this, we define a new coloring on $G$, $f_1^1$. By our assumptions for this case, we know that $c(x_1) = s_1 \neq a_1$. We also know that there must be a vertex in $(W_1 - \{x_1\})$ colored $s_1$ under $f_0^1$ because $\{s_1, \ldots, s_{q_1}\} = \{a_1, \ldots, a_{q_1}\}$. We will interchange the colors on the $(s_1, a_1)$-Kempe chain containing $x_1$. The vertex $x_1$ will be recolored the color $s_1$ under $f_1^1$. All other vertices on the $(s_1, a_1)$-Kempe chain containing $x_1$ will be recolored $s_1$. 
under $f_1$ if they were colored $a_1$ under the previous coloring, $f_0$, and will be recolored
$a_1$ under $f_1$ if they were colored $s_1$ under $f_1$.

The $(s_1, a_1)$- Kempe chain containing $x_1$ will eventually end because of the barriers
we made under the coloring $f_0$. However, the barrier for $s_1$, $N_1$, is only composed of one
layer of vertices. Since this barrier is only one layer, instead of two, the $(s_1, a_1)$- Kempe
chain is able to pass by it. As we mentioned before, in this case, there must be a vertex
in $(W_1 - \{x_1\})$ colored $s_1$ under $f_0$. Let $x_i$ be the vertex in $(W_1 - \{x_1\})$ such that
$f_0(x_i) = a_i = s_1$. If there is a vertex in $N_1$ colored $a_1$ that is adjacent
to $x_i \in W_1$, the $(s_1, a_1)$- Kempe chain containing $x_1$ can continue past $N_1$. However
because $\{s_1, \ldots, s_{q_1}\} = \{a_1, \ldots, a_{q_1}\}$, $a_1 = s_j$ for some $j \in \{2, \ldots, q_1\}$. Thus, there is a
barrier for $a_1 = s_j$ that is made up of two layers of vertices. This barrier, $N_{3j-2} \cup N_{3j-3}$,
guarantees that the $(s_1, a_1)$- Kempe chain will be stopped at most distance $3j - 3$ away
from $W_1$.

For $v \in G$, we have the following:

$$f_1(v) = \begin{cases} 
  s_1 & \text{if } v = x_1, \text{ where } c(x_1) = s_1 \\
  s_1 & \text{if } f_0(v) = a_1, \text{ and } v \text{ is on the } (s_1, a_1) - \text{ Kempe chain containing } x_1 \\
  a_1 & \text{if } f_0(v) = s_1, \text{ and } v \text{ is on the } (s_1, a_1) - \text{ Kempe chain containing } x_1 \\
  f_0(v) & \text{otherwise.}
\end{cases}$$

By Lemma 1, $f_1$ is a proper $(r + 1)$-coloring of $G$.

We note that interchanging the colors on the $(s_1, a_1)$- Kempe chain can recolor a
vertex in the first layer of the $s_j = a_1$ barrier, $N_{3j-3}$, the color $a_1 = s_j$. Consider
interchanging the colors on a future Kempe chain involving $s_j$ and some other color,
b. Let $v_1$ be a vertex in $N_{3j-3}$ that was colored $s_j$ by the interchanging of colors
on the $(s_1, a_1)$- Kempe chain containing $x_1$, where $a_1 = s_j$. Let $v_2$ be a vertex in
$N_{3j-2}$ currently colored $b$ by the most recently coloring of $G$, and let $v_3$ be a vertex in
$N_{3j-1}$, outside of the $s_j$ barrier, currently colored $s_j$ by the most recently coloring of
$G$. Let $v_1$ be adjacent to $v_2$, let $v_2$ be adjacent to $v_3$, and let all three vertices be on
the \((s_j, b)\)-Kempe chain containing \(x_j\). Then when we interchange the colors on the \((s_j, b)\)-Kempe Chain containing \(x_j\), \(v_1 \in N_{3j-3}\) is recolored \(b\), \(v_2 \in N_{3j-2}\) is recolored \(s_j\), and \(v_3 \in N_{3j-1}\) is recolored \(b\). This means that the \((s_j, b)\)-Kempe change is not stopped by the \(s_j\) barrier. Thus, once a barrier is used to stop a Kempe change involving that barrier’s associated color, it can no longer be used to stop future Kempe changes involving that barrier’s color.

The process of possibly ‘destroying’ a barrier after we use it to stop a Kempe change is what happens in Case 1 and Case 2a, as well. However, in these two previous cases, we have not needed to use a barrier more than once, so it has not been an issue. The following is what occurs in both Case 1 and 2a. Say we are recoloring a vertex, \(x_i\) the color \(s_i\). Then at this step, we may have to do a Kempe change with \(x_i\)’s precoloring color, \(s_i\), and \(x_i\)’s current color. The barrier for \(s_i\) will stop this Kempe change, but a vertex in \(s_i\)’s barrier may be recolored \(s_i\) as a consequence. The color \(s_i\) is the vertex’s color assigned by \(c\), and once we color a vertex it’s precoloring color, it does not appear in any subsequent Kempe change. We will explain why that is the case below. Thus, the barrier for \(s_i\) does not need to be intact to stop any future Kempe changes involving \(s_i\) because there will not be any.

The difference in this case, Case 2b, however, is that the barrier we use to stop the \((s_1, a_1)\)-Kempe change is the barrier for \(x_1\)’s current color, \(a_1 = s_j\), not the barrier for \(x_1\)’s precoloring color, \(s_1\). Since \(a_1 = s_j\) for some \(j \in \{2, \ldots, q_1\}\), \(s_j\) can appear in subsequent Kempe changes. We will address this issue and how to resolve it below.

Assume that the vertices \(x_1, x_2, \ldots, x_{j-1}\) have all been assigned their precoloring color, \(s_1, s_2, \ldots, s_{j-1}\), and that \(f^1_{j-1}\) is a proper \((r + 1)\)-coloring of \(G\). The Kempe changes necessitated by recoloring each vertex in \(\{x_1, x_2, \ldots, x_{j-1}\}\) its color assigned under \(c\) will be able to be stopped by each vertex’s respective \(c\) color barrier. Before recoloring a vertex in \(W_1\) its color assigned by \(c\), the barrier for the vertex’s current color is closer to \(W_1\) than the barrier for the vertex’s current color. This is because of the order we chose to recolor the vertices in and because of our specific placement of the barriers.
Also, before recoloring a vertex in \( \{x_1, x_2, \ldots, x_{j-1}\} \), its \( c \) color barriers has not been used previously to stop any other Kempe Changes by assumption. Thus, each vertex’s respective \( c \) color barrier will stop the possible Kempe Change involving the vertex’s current color and the vertex’s precoloring color.

This is the case until we get to the \( j \)th vertex, \( x_j \in W_1 \), such that \( c(x_j) = s_j \) where \( s_j = a_1 \). When we recolored the first vertex in the clique, \( x_1 \), the color \( c(x_1) = s_1 \), the barrier for \( s_j = a_1 \) was needed to ensure that the \((s_1, a_1)\)-Kempe chain containing \( x_1 \) did not get too far from \( W_1 \). Currently, \( x_j \) is colored \( f^1_{j-1}(x_j) \) under \( f^1_{j-1} \). If \( f^1_{j-1}(x_j) = s_j \), then no color change is needed at this step and consequently, no interchanging of colors on the \((s_j, f^1_{j-1}(x_j))\)-Kempe chain containing \( x_j \) needs to be performed. So, let us assume that it is the case that \( f^1_{j-1}(x_j) \neq s_j \). Then, we need a barrier to make sure the \((s_j, f^1_{j-1}(x_j))\)-Kempe chain containing \( x_j \) does not get too far from \( W_1 \). Since the \( a_1 = s_j \) barrier has been used previously in the recoloring of \( x_1 \), the barrier for \( f^1_{j-1}(x_j) \) must be used in this step in the recoloring of \( x_j \). Before we can use the barrier for \( f^1_{j-1}(x_j) \), we must show that there is a barrier for \( f^1_{j-1}(x_j) \) and that it has not been used to stop previous Kempe changes. This is so that we know that it will actually be able to stop the \((s_j, f^1_{j-1}(x_j))\)-Kempe Change.

The color \( f^1_{j-1}(x_j) \) is either \( x_j \)'s original color assigned by \( \gamma \), or \( x_j \) has been recolored in one or more than one Kempe change. Either way, since the colors involved in Kempe changes must be in \( \{s_1, \ldots, s_{q_1}\} = \{a_1, \ldots, a_{q_1}\} \), \( f^1_{j-1}(x_j) = s_p \) where \( s_p = c(x_p) \in \{s_1, \ldots, s_{q_1}\} \). We also know that \( p > j \) because once a vertex \( x \) is assigned its \( c \) color at its respective step, this \( c \) color cannot appear in any subsequent Kempe changes. This is because this \( c \) color can be neither the \( c \) color or the current color of a vertex in the clique we have not yet colored. It cannot be the \( c \) color of a subsequent vertex in the clique because \( c \) assigns each vertex in the clique a unique color since \( c \) is a proper coloring. It cannot be the current color of a subsequent vertex in the clique either, because at each step we have a proper coloring and vertex \( x \) in the clique is already assigned this color. Thus, \( f^1_{j-1}(x_j) = s_p \), where \( s_p \in \{s_1, \ldots, s_{q_1}\} \) and \( p > j \). Since we
created a barrier for all the colors in \{s_1, \ldots, s_p\}, there is a barrier for the color \(f^1_{j-1}(x_j) = s_p\), namely \(N_{3p-3} \cup N_{3p-2}\).

Since \(f^1_{j-1}(x_j) = s_p\) where \(p > j\), we know that the barrier for \(f^1_{j-1}(x_j) = s_p\) has not been used to stop any previous Kempe Change. \(f^1_{j-1}(x_j) = s_p\) is the color assigned to \(x_p\) by the precoloring, \(c\), and thus, is not the color assigned to any previous vertices in \(W_1\) by the precoloring. So far, by assumption, the barriers used to stop Kempe changes that have occurred have been the \(c\) color barriers, except for when we recolored \(x_1\) and used the \(a_1 = s_j\) barrier instead of the \(s_1\) barrier. Since \(f^1_{j-1}(x_j) = s_p \neq s_1\) and \(f^1_{j-1}(x_j) = s_p \neq a_1 = s_j\), the barrier for \(f^1_{j-1}(x_j) = s_p\) for \(p > j\) has not been used previously before the recoloring of \(x_j \in W_1\). Thus, the barrier for \(f^1_{j-1}(x_i) = s_p\) will be able to stop the \((s_j, f^1_{j-1}(x_j))\)-Kempe Change.

We now define a new coloring, \(f^1_j\), on \(G\), and interchange the colors on the \((s_j, f^1_{j-1}(x_j))\)-Kempe chain containing \(x_1 = j\). \(x_j\) will be recolored the color \(s_j\) under \(f^1_j\). All other vertices on the \((s_j, f^1_{j-1}(x_j))\)-Kempe chain containing \(x_j\) will be recolored \(s_j\) under \(f^1_j\) if they were colored the color \(f^1_{j-1}(x_j) = s_p\) under the previous coloring, \(f^1_{j-1}\), and they will be recolored \(f^1_{j-1}(x_j) = s_p\) under \(f^1_j\) if they were colored \(s_j\) under \(f^1_{j-1}\). The barrier for \(f^1_{j-1}(x_j) = s_p\), \(N_{3p-3} \cup N_{3p-2}\), will guarantees that the \((s_j, f^1_{j-1}(x_j))\)-Kempe chain will be stopped at most distance \(3p - 3\) away from \(W_1\).

For \(v \in G\), we have the following:

\[
f^1_j(v) = \begin{cases} 
  s_j & \text{if } v = x_j, \text{ where } c(x_j) = s_j \\
  s_j & \text{if } f^1_{j-1}(v) = f^1_{j-1}(x_j) = s_p, \text{ and } v \text{ is on the } (s_j, f^1_{j-1}(x_j))\text{-Kempe chain containing } x_j \\
  s_p & \text{if } f^1_{j-1}(v) = s_j, \text{ and } v \text{ is on the } (s_j, f^1_{j-1}(x_j))\text{-Kempe chain containing } x_j \\
  f^1_{j-1}(v) & \text{otherwise.}
\end{cases}
\]

By Lemma 1, \(f^1_j\) is a proper \((r + 1)\)-coloring of \(G\).

Recoloring the \(p^{th}\) vertex in \(W_1\), \(x_p\), where \(c(x_p) = s_p = f^1_{j-1}(x_j)\), follows the same
process as the the recoloring of the $j^{th}$ vertex, $x_j$ in $W_1$. For the $j^{th}$ recoloring, we needed to use the barrier for $f_{j-1}^1(x_j)$ since the barrier for $c(x_j) = s_j = f_0^1(x_1) = a_1$ was used in the recoloring process of the first vertex in the clique, $x_1$. Similarly, for the recoloring process of the $p^{th}$ vertex, we must use the barrier for $f_{p-1}^1(x_p)$ since the barrier for $c(x_p) = s_p = f_{j-1}^1(x_j)$ was used in the recoloring of the $j^{th}$ vertex.

Eventually, however, this process of needing to use a vertex’s current color barrier, instead of its $c$ color barrier will end. This is because as previously explained, once a vertex in the clique is assigned its color given under $c$, that color does not appear in subsequent Kempe changes. So, once we get to the last vertex in the clique, $x_{q_1}$, where $c(x_{q_1}) = s_{q_1}$, it must be the case that $f_{q_1-1}^1(x_{q_1}) = s_{q_1}$. This is because $f_{q_1-1}^1(x_{q_1})$ can’t be equal to any other $c$ color, $s_1, ..., s_{q_1-1}$, as they have already been assigned to the vertices $x_1, ..., x_{q_1-1}$. Thus, for $x_{q_1}$, we have that $f_{q_1-1}^1(x_{q_1}) = c(x_{q_1})$. Consequently, no color change is needed to assign $x_{q_1}$ its color assigned by $c$, since its current color $f_{q_1-1}^1(x_{q_1})$ is equal to its color assigned under $c$, $s_{q_1}$. This implies that no Kempe change takes place, and thus, no barrier for $s_{q_1}$ needs to be available to stop a Kempe change involving $s_{q_1}$. Under the new coloring we define, $f_{q_1}^1$, all vertices in $G$ keep their color assigned by $f_{q_1}^1$, which is a proper $(r + 1)$-coloring of $G$. All vertices of $W_1$ are now assigned the color given to them by the precoloring, $c$, and $f_{q_1}^1$ is a proper $(r + 1)$-coloring of $G$. Therefore, we have finished recoloring $W_1$ and the vertices within distance $3q_1 - 2 \leq 3k - 2$ of $W_1$. The farthest color changes that occurred during this process were on vertices at most distance $3q_1 - 2 \leq 3k - 2$ from $W_1$.

We now provide an example of making modifications to the $r$-coloring of a graph so that it agrees with colors on the precolored vertices in the cliques by following the procedure described above in Case 2b. In our example, let $\gamma$ be a 3-coloring of $G$ that uses the colors from \{blue, green, orange\}. Let $c$ be a 4-coloring of $G[W]$ that uses the colors from \{blue, green, orange, red\}. We will consider the clique $W_1$ where $|W_1| = 3$. Let the colors $\gamma$ and $c$ both use on $W_1$, be from \{blue, green, orange\}.

In Figure 4.12, we have the clique, $W_1$, that consists of three vertices, $x_1$, $x_2$, and
x_3, and the vertices within distance 8 of W_1, which are the vertices in \{v_1, v_2, \ldots, v_{24}\}. What is shown in this figure is a subgraph of G. However, when we define new colorings on the graph, the colorings are on all of G, not just the subgraph shown. All the vertices in the graph are currently colored by \( \gamma \). For the vertices in the clique, for reference, we provide their precolored color as the outer color of the nodes. A node’s inner color is what it is currently colored by \( \gamma \). We will order the colors \( c \) uses on the clique. We will order them alphabetically, and let \( s_1 = c(x_1) = blue, s_2 = c(x_2) = green, \) and \( s_3 = c(x_3) = orange \). We will now label the colors that \( \gamma \) uses on \( W_1 \). Since \( s_1 = c(x_1) = blue \), let \( a_1 = \gamma(x_1) = orange \). Since \( s_2 = c(x_2) = green \), let \( a_2 = \gamma(x_2) = blue \). Since \( s_3 = c(x_3) = orange \), let \( a_3 = \gamma(x_3) = green \). We will follow this example through the process of recoloring the entire clique, \( W_1 \).

In Figure 4.13, we show the creation of the barriers. By the barrier coloring procedure, under a new coloring of \( G, f_1^0 \), vertices in the first barrier, \( N_1 \), that are colored \( s_1 = blue \), get recolored \( red \). Vertices in the second barrier \( N_3 \cup N_4 \), that are colored \( s_2 = green \), get recolored \( red \). Also, vertices in the third barrier, \( N_6 \cup N_7 \), that are colored \( s_3 = orange \), get recolored \( red \). All other vertices in this graph receive their same color from \( \gamma \), shown in the previous figure, Figure 4.2, under \( f_0^1 \). We can see that \( f_0^1 \) is a proper 4-coloring of the subgraph shown, and it will also be a proper 4-coloring of \( G \).

In Figure 4.14, we show the recoloring of the first vertex in \( W_1, x_1 \). We recolor \( x_1 \) first since \( c(x_1) = s_1 = blue \). We define a new coloring on \( G, f_1^1 \), and we recolor \( x_1 \) from \( f_0^1(x_1) = a_1 = orange \) to \( c(x_1) = s_1 = blue \) under the new coloring. There is \((blue, orange)\)-Kempe chain containing \( x_1 \) that we must recolor under \( f_1^1 \) as well. Other than \( x_1 \), the vertices on this chain are \( x_2, v_1, v_4, v_5, v_7, v_8, v_{10}, v_{11}, v_{13}, v_{14}, v_{17} \). Thus, we perform a Kempe change and the vertices on this chain that were colored \( blue \) under \( f_0^1, x_2, v_5, v_8, v_{11}, v_{14}, \) and \( v_{17} \), are recolored \( orange \) under \( f_1^1 \). The vertices on this chain that were colored \( orange \) under \( f_0^1, v_1, v_4, v_7, v_{10}, \) and \( v_{13} \), are recolored \( blue \) under \( f_1^1 \). This Kempe change was able to pass the barrier for \( s_1 = blue \), but the
barrier for $a_1 = \text{orange}$ is able to stop it. So this Kempe chain stops at $N_6$ since in the the third barrier, $N_6 \cup N_7$, all $a_1 = s_3 = \text{orange}$ vertices were recolored red under $f_0^1$. We can see that in $N_6$ there is now a vertex colored orange, which makes this barrier, unable to be used again. All other vertices in the graph receive their same color from $f_1^1$, shown in the previous figure, Figure 4.14, under our new coloring, $f_1^1$. We can see that $f_1^1$ is a proper 4-coloring of the subgraph shown, and it will also be a proper 4-coloring of $G$.

In Figure 4.15, we recolor the vertex, $x_2$, since $c(x_2) = s_2 = \text{blue}$. We define a new coloring, $f_2^1$, on $G$, and recolor $x_2$ from $f_1^1(x_2) = \text{orange}$ to $s_2 = c(x_2) = \text{green}$. There is (green, orange)- Kempe chain containing $x_2$ that we must recolor under $f_2^1$ as well. Other than $x_2$, the vertices on this chain are $x_3$, $v_3$, $v_5$, $v_6$, $v_8$. Thus, we perform a Kempe change and the vertices on this chain that were colored green under $f_2^1$, $x_3$, $v_3$, and $v_6$, are recolored orange under $f_2^1$. The vertices on this chain that were colored orange under $f_1^1$, $v_5$ and $v_8$, are recolored green under $f_2^1$. This Kempe chain stops at $N_3$ since in the first barrier, $N_3 \cup N_4$, all $s_2 = \text{green}$ vertices were recolored red under $f_0^1$. All other vertices in the graph receive their same color from $f_1^1$, shown in Figure 4.14, under our new coloring, $f_1^1$. We can see that $f_2^1$ is a proper 4- coloring of the subgraph shown, and it will also be a proper 4- coloring of $G$. When we recolored $x_3$ in the Kempe change under $f_2^1$, we recolored it the color orange, which is the precoloring color for $x_3$. So, we define a new coloring, $f_3^1$ on $G$, and all vertices receive their same coloring from $f_2^1$. Since $f_2^1$ is a proper 4-coloring of $G$, $f_3^1$ is as well. We are now done recoloring $W_1$ and its surrounding vertices, and the vertices in $W_1$ are assigned their precoloring colors. In this example, the furthest color change from $W_1$ that occurred is in $N_7$, by the barrier color change, $f_0^1$. We can see that no vertex $N_8$ had its color altered. We would now move on to recoloring $W_2$. 
Figure 4.12: The whole graph is currently colored by $\gamma$. $\gamma$ is a proper 3-coloring of the subgraph shown and of the whole graph, $G$. Each vertex in $W_1$, has its color assigned by $\gamma$ shown in the center of the node and its color assigned by $c$ shown on the outside of the node.

Figure 4.13: Under $f^1_0$, we created the barriers for $W_1$. $f^1_0$ is a proper 4-coloring of the subgraph shown and of the whole graph, $G$. 
CHAPTER 4. REVISION OF ALBERTSON’S PROOF

Figure 4.14: Under $f_1^1$, we assigned $x_1$ its precoloring color, blue, and performed a Kempe change along the (blue, orange)-Kempe Chain containing $x_1$. All other vertices in the graph are assigned their same color from $f_0^1$ under $f_1^1$. $f_1^1$ is a proper 4-coloring of the subgraph shown and of the whole graph, $G$.

Figure 4.15: Under $f_2^1$, we assigned $x_2$ its precoloring color, blue, and performed a Kempe change along the (green, blue)-Kempe Chain containing $x_2$. All other vertices in the graph are assigned their same color from $f_1^1$ under $f_2^1$. $f_2^1$ is a proper 4-coloring of the subgraph shown and of the whole graph, $G$. We note that by the Kempe change performed, $x_3$ is assigned its precoloring color, $s_3 = orange$. So, we define a new coloring of $G$, $f_3^1$, and all vertices receive their same color from $f_2^1$. $f_3^1$ is a proper 4-coloring of the subgraph shown and of the whole graph, $G$. We have finished modifying $\gamma$ on this clique and its surrounding vertices, so each vertex in $W_1$ is now assigned its precoloring color.
Now that the first clique, $W_1$, is assigned its colors from the precoloring of $W$, we move on to $W_2$ and repeat this process by following the procedure described in the case $W_2$ satisfies. We then continue coloring all of the cliques in numerical order until we finish with the last clique, $W_m$. We note that when we recolor any clique besides the first clique, $W_1$, the vertices in the clique and its surrounding vertices will be assigned their $\gamma$ color, but they are colored under the last proper coloring from the previous clique. So when we first look at the $i^{th}$ clique, $W_i$, all vertices, $v$, in $W_i$ and within $3q_i - 2$ of $W_i$ are currently colored by $f_{q_{i-1}}^{i-1}$, but for each $v \in W_i$, $f_{q_{i-1}}^{i-1}(v) = \gamma(v)$. After all the cliques have been assigned their precoloring color, the last $(r+1)$-coloring of $G$ is $f_{q_m}^m$.

Since the cliques are at least distance $6k - 2$ apart from each other, the vertices recolored by doing this process to one clique will not be adjacent to the vertices recolored by doing this process to any other clique. For each clique, the farthest away a color change can occur is at most distance $3k - 2$, so no color changes can occur at distance $3k - 1$. Since $(3k - 2) + (3k - 2) = 6k - 4$ and $(3k - 1) + (3k - 1) = 6k - 2$, vertices recolored surrounding one clique will be at least distance $(6k - 4) - (6k - 2) = 2$ away from vertices recolored surrounding any other clique. Thus, the result from all of this is an extension of a proper $(r+1)$-coloring of $G[W]$ to a proper $(r+1)$-coloring of $G$.

We note that each clique in $W$ does not necessarily have $k$ vertices- each clique has at most $k$ vertices. Thus, for two cliques, $W_i$ and $W_j$, where $|W_i| = q_i \leq k$ and $|W_j| = q_j \leq k$, the furthest color changes associated to $W_i$ can occur from $W_i$ is $3q_i - 2$, and the furthest color changes associated to $W_j$ can occur from $W_j$ is $3q_j - 2$. For vertices recolored by performing this process on $W_i$ and vertices recolored by performing this process on $W_j$ to not be adjacent, $W_i$ and $W_j$ must be at least distance $(3q_i - 2) + (3q_j - 2) + 2 = 3q_1 + 3q_2 - 2 \leq 6k - 2$. 

\[\square\]
Chapter 5

Improved Results

After writing the proof above, we discovered that Alexandr Kostochka improved Albertson’s result by decreasing the distance needed between cliques from \(6k - 2\) to \(4k\) [2]. We note that Kostochka’s result is found in [2] by Albertson and Moore. The result is provided below:

**Theorem 7.** [2] Suppose \(\chi(G) = r\) and \(P \subseteq V(G)\) induces a subgraph of \(G\) that consists of \(k\)-cliques. If the distance between any two cliques of the subgraph induced by \(P\) is at least \(4k\), then any \((r + 1)\)-coloring of \(P\) can be extended to an \((r + 1)\)-coloring of \(G\).

**Proof.** The strategy will be to \(r\)-color the graph \(G\), and then modify this coloring so it agrees with the precoloring of \(P\). The modification will be done one clique at a time. Let \(K\) denote a clique in the subgraph induced by \(P\), and label its vertices \(v_1, v_2, \ldots, v_k\). Suppose \(c_1, c_2, \ldots, c_k\) are the colors assigned to these vertices in the \(r\)-coloring of \(G\) and \(d_1, d_2, \ldots, d_k\) are the colors assigned to these vertices in the precoloring of \(P\). If the precoloring of \(K\) uses the color \(r + 1\) we choose the labeling so that this occurs on vertex \(v_k\). Starting with \(v_1\), if \(c_1 \neq d_1\) we switch the color on \(v_1\) from \(c_1\) to \(r + 1\). This can be done without altering the color on any neighbor of \(v_1\). Then we interchange colors on the \((d_1, r + 1)\)-Kempe chain originating at \(v_1\). At this moment \(v_1\) is assigned its proper color and at worst a vertex at distance one from \(K\) has been colored \(r + 1\). At any time in the process, \(D\) represents the distance from the clique to the farthest vertex.
colored $r+1$ by this process. Now assume vertices $v_1, v_2, \ldots, v_{i-1}$ have the correct colors $d_1, d_2, \ldots, d_{i-1}$, and $D \leq 2i - 3$. Let $c_i$ be the current color of vertex $v_i$. If $c_i \neq d_i$ first switch the color of $v_i$ to $r + 1$ by switching the colors on the $(c_i, r + 1)$-Kempe chain containing $v_i$. Then change its color to the correct $d_i$ by switching the colors of the $(d_i, r + 1)$-Kempe chain. Since each Kempe change involves color $r + 1$, distance $D$ can grow by no more than two- one for each Kempe change. That is, $D \leq 2i - 1$. When we finish with the clique, $D \leq 2k - 1$. Since $k$-cliques of $P$ are at least $4k$ from each other, the vertices changed by applying this algorithm to two cliques cannot be adjacent to each other. The result is an extension of the $(r + 1)$-coloring of $P$ to all of $G$. \qed

We find it useful to follow this proof by making a table with the vertices in the clique. At each step, we record each vertex’s current color and keep track of when and how many Kempe Changes are needed throughout recoloring the entire clique.

We provide an example of making modifications to the $r$-coloring of a graph so that it agrees with the colors assigned to the vertices in the cliques by the $(r + 1)$-precoloring. We follow the procedure described in the Kostochka’s proof above. In our example, $P \subseteq V(G)$ induces a subgraph of $G$ that consists of 3-cliques. Let there be a 3-coloring of $G$ that uses the colors from \{blue, green, orange\}, and let there be a 4-coloring of $P$ that uses the colors from \{blue, green orange, red\}. We will consider the clique $K_1$. The colors the 3-coloring uses on $K_1$ are from \{blue, orange, green\}, and let the colors the precoloring uses on $K_1$ be from \{blue, green, red\}. In this example, since red are used on $P$ by the precoloring, but not on $G$ by the 3-coloring, red represents the $r + 1$ color, which in this case is 4.

In Figure 5.1, we have the clique, $K_1$ that consists of three vertices, $v_1, v_2,$ and $v_3$, and the vertices within distance 6 of $K_1$, which are the vertices in \{x_1, x_2, \ldots, x_{18}\}. What is shown in this figure is a subgraph of $G$. For reference, for each vertex in $K_1$, we provide its precolored color as the outer color of the node. The inner color on each node in $K_1$ is what the vertex is currently colored. Since the precoloring uses the color red, we choose the labeling of the vertices so this occurs on the last vertex in $K_1$, vertex $v_3$. 
We now label the other 2 vertices arbitrary. We will let the vertex that the precoloring colors blue be $v_1$ and the vertex the the precoloring colors green to be $v_2$.

The first vertex in the clique we will recolor is $v_1$. To recolor $v_1$ its correct color, blue, we first recolor $v_1$ the color red. Since red is not used on $G$ by the 3-coloring, there are no other vertices currently colored red. We show this step in Figure 5.2. After making this change, the coloring we have on the graph is a proper 4-coloring.

Now, to color $v_1$ the color blue, we interchange the colors on the (blue, red)-Kempe chain originating at $v_1$. The vertex $v_1$ gets recolored from red to its correct color, blue. Other than $v_1$, the vertices on this chain are $v_3$ and $x_3$. Since both $v_3$ and $x_3$ are currently colored blue, they get recolored red. This is shown in Figure 5.3. The current coloring we have on the graph is now a proper 4-coloring.

The next vertex in the clique we will recolor is $v_2$. To recolor $v_2$ its correct color, green, we first recolor $v_2$ the color red. To do this, we interchange the colors on the (orange, red)-Kempe chain originating at $v_2$. The vertex $v_2$ gets recolored from its current color, orange, to red. Other than $v_2$, the vertices on this chain are $v_3$, $x_2$, $x_3$, and $x_5$. Since both $v_3$ and $x_3$ are currently colored red, they get recolored orange. Since both $x_2$ and $x_5$ are currently colored orange, they get recolored red. This is shown in Figure 5.4. The current coloring we have on the graph is a proper 4-coloring.

Now, to color $v_2$ the color green, we interchange the colors on the (green, red)-Kempe chain originating at $v_2$. The vertex $v_2$ gets recolored from red to its correct color, green. Other than $v_2$, the vertices on this chain are $x_1$, $x_2$, $x_4$, $x_5$, and $x_7$. Since $x_1$, $x_4$, and $x_7$ are currently colored green, they get recolored red. Since both $x_2$ and $x_5$ are currently colored red, they get recolored green. This is shown in Figure 5.5. The current coloring we have on the graph is a proper 4-coloring.

The next vertex in the clique we recolor is $v_3$. To color $v_3$ its correct color, red, we interchange the colors on the (orange, red)-Kempe chain originating at $v_3$. The vertex $v_3$ gets recolored from its current color, orange, to red. Other than $v_2$, the vertices on this chain are $x_1$, $x_3$, $x_7$, $x_8$, and $x_{11}$. Since $x_1$, $x_4$, and $x_7$ are currently colored
Figure 5.1: This is a proper 3-coloring of the subgraph shown and of the whole graph, \( G \). The inner color of the vertices in \( K_1 \) is their current color. The outer color of the vertices in \( K_1 \) are their precolored color.

Figure 5.2: To recolor \( v_1 \) its correct color, blue, we first recolored \( v_1 \) from green to red. red, they get recolored orange. Since \( x_3 \), \( x_8 \), and \( x_{11} \) are currently colored orange, they get recolored red. This is shown in Figure 5.6. The current coloring we have on the graph is still a proper 4-coloring. Now, all vertices in \( K_1 \) have been assigned their correct color. In this example, the color changes only occurred at distance \( 4 < 2(3) - 1 = 5 \) from \( K_1 \). We would now move on to recoloring \( K_2 \).

We note the similarities in the thinking behind the approaches used in the proof of Theorem 7 and in Albertson’s proof of Theorem 6. For Albertson’s proof of Theorem 6, we creates barriers to make sure Kempe changes originating at the clique we are recoloring do not get too far away. For Theorem 7, we accomplish this by making it so
Figure 5.3: We recolored \( v_1 \) its correct color, blue, by interchanging the colors on the \((\text{blue}, \text{red})\)-Kempe Chain originating at \( v_1 \). This Kempe chain also includes the vertices \( v_3 \) and \( x_3 \).

Figure 5.4: To recolor \( v_2 \) its correct color, we first had to recolor \( v_2 \) from orange to red. To do this, we interchanged the colors on the \((\text{orange}, \text{red})\)-Kempe Chain originating at \( v_2 \). This Kempe chain also includes the vertices \( v_3 \), \( x_2 \), \( x_3 \) and \( x_5 \).

Figure 5.5: We recolored \( v_2 \) its correct color, green, by interchanging the colors on the \((\text{green}, \text{red})\)-Kempe Chain originating at \( v_2 \). This Kempe chain also includes the vertices \( x_1 \), \( x_2 \), \( x_4 \), \( x_5 \), and \( x_7 \).
Figure 5.6: We recolored $v_3$ its correct color, red, by interchanging the colors on the $(orange, red)$- Kempe Chain originating at $v_3$. This Kempe chain also includes the vertices $x_1, x_3, x_4, x_7, x_8,$ and $x_{11}$. All the vertices in $K_1$ are recolored and we have a proper 4-coloring of this graph.

each Kempe change we perform involves the color $r + 1$ and therefore can not get too far from the clique because $r + 1$ is not a color used outside the clique by the original coloring on the whole graph. Both approaches are used so that changes made to one clique and its surrounding vertices cannot affect another clique and its surrounding vertices. However, by performing the modifications of the coloring by having $r + 1$ involved in ever Kempe change, Theorem 7 shows that the modifications to vertices outside the clique only alter the colors of vertices at most distance $2k - 1$ from the clique, instead of $3k - 2$.

Albertson and Moore [2] state that a consequence of this proof is the more general result:

**Theorem 8.** [2] Suppose $\chi(G) = r$ and $P \subseteq V(G)$ induces a graph consisting of cliques, if for any pair of cliques of sizes $k_i$ and $k_j$ the distance between them is at least $2k_i + 2k_j$, then any $(r + 1)$-coloring of $P$ can be extended to an $(r + 1)$-coloring of $G$.

It was proved that for a clique of order $k$, the farthest from the clique that color changes associated to that clique can occur is distance $2k - 1$. Thus, if the clique is of order $k_i \leq k$, the farthest from the clique that color changes can occur is distance $2k_i - 1 \leq 2k - 1$. Also, since the color changes associated to two different cliques of order $k$ need to be distance 2 apart, the distance between any two $k$-cliques need to be
(2k − 1) + (2k − 1) + 2 = 4k. Thus, for two cliques of order $k_i \leq k$ and $k_j \leq k$, where the farthest color changes from the cliques are at most distance $2k_i - 1$ and $2k_j - 1$, respectively, the distance needed between this pair of cliques is $(2k_i - 1) + (2k_j - 1) + 2 = 2k_i + 2k_j \leq 4k$.

For Theorem 7, we noticed that if we order the vertices in the clique so that the vertices that are assigned their correct color at the start, are at the beginning, then we will not need to perform as many Kempe changes. By being assigned their correct color at the start, we mean that for a vertex $x$ in a clique, the color assigned by the precoloring to $x$ is the same as the color assigned by the $r$-coloring to $x$. For each vertex in a clique of order $k$, except for the first vertex, there are two possible Kempe changes that need to be performed to color the vertex its correct color. So, if there are $n$ vertices in which it is the case that they are already assigned their correct color from the start, then we are able to avoid performing $2n$ possible Kempe changes. Since each Kempe change makes a color change at a distance one more than after the previous Kempe change, by avoiding performing $2n$ Kempe changes, the color changes made by recoloring this clique with occur at most distance $(2k - 1) - 2n \leq 2k - 1$ from the clique.

Albertson and Moore [2] point out that in the proof of Theorem 7, we have a set, arbitrary order in which we recolor the vertices in a clique. But, if we change the order as we are recoloring the clique depending on the colors of the vertices in the clique, the distance needed between the cliques could be reduced. When we recolor a vertex, $v_j$, such that its correct color is $s_j$, we perform at most two Kempe changes. One involves $v_j$’s current color and the color $r + 1$, and the other involves $v_j$’s correct color, $s_j$, and $r + 1$. When we perform the second Kempe change, the one involving $s_j$ and $r + 1$, there is the possibility of recoloring a vertex in the clique that is not assigned its correct color yet, the color $r + 1$. This would be the case if for a vertex in the clique, its current color at the time is equal to $s_j$. Then, if we switched the order that we recolor the vertices in the clique and recolor this vertex that was just recolored $r + 1$ next, then we only have to perform at most one Kempe change in the next step. Since this vertex will already
be colored $r + 1$, we do not need to perform our preliminary Kempe change to color this vertex $r + 1$ before being able to color it its correct color. The following result by Albertson and Moore takes advantage of this observation and decreases the distance needed between cliques in certain situations.

**Theorem 9.** [2] Suppose $\chi(G) = r$ and $P \subseteq V(G)$ induces a subgraph of $G$ that consists of $k$-cliques, and $k \leq r \leq 2k$. If the distance between any two cliques is at least $2k + 2 \left\lceil \frac{r}{2} \right\rceil$, then any $(r + 1)$-coloring of $P$ can be extended to an $(r + 1)$-coloring of $G$. 
Chapter 6

(r+2)- Precoloring Extension

A natural extension to Theorem 7, was to explore what the distance needs to be between the cliques if we have the same set up of Theorem 7, but instead of the induced subgraph being colored with \( r + 1 \) colors, it is colored with \( r + 2 \) colors.

Building off of the same methods used in Kostochka’s result in 7, we were able to prove that when the distance between \( k \)-cliques is \( 2k + 2 \), we are able to extend any proper \( (r+2) \)-coloring of \( P \) to a proper \( (r+2) \)-coloring of \( G \). We will also be changing the notation from that proof and we will be using notation similar to that used in the proof of Theorem 6. We will be showing this result for \( k \)-cliques, but we will discuss how it can be generalized for cliques of different orders after.

**Theorem 10.** Suppose \( \chi(G) = r \) and \( P \subseteq V(G) \) induces a subgraph of \( G \) that consists of \( k \)-cliques. If the distance between any two cliques of the subgraph induced by \( P \) is at least \( 2k + 2 \), then any \( (r+2) \)-coloring of \( P \) can be extended to an \( (r+2) \)-coloring of \( G \).

**Proof.** Let there be an \( (r+2) \)-precoloring of \( P \). We will start with an \( r \)-coloring of \( G \), and we will alter the coloring so it agrees with the vertices in the subgraph that are precolored. We will modify the coloring on one clique at a time. To do this, we need to make sure these modifications to a clique and consequently, to its surrounding vertices, do not affect the colors of another clique and its surrounding vertices.
Let $K_1$ be the clique in the subgraph induced by $P$ that we will consider first. We are assuming every clique in the subgraph is of order $k$, so we label the vertices of $K_1$, $v_1, v_2, \ldots, v_k$. Let $a_1, a_2, \ldots, a_k$ denote the colors assigned to the vertices of $K_1$ by the $r$-coloring and let $s_1, s_2, \ldots, s_k$ denote the colors assigned to the vertices of $K_1$ by the $(r+2)$-coloring.

If $k$ is even and the $(r+2)$-precoloring uses both colors $r+1$ and $r+2$ on $K_1$, we choose the labeling so that $s_{k-1} = r+1$ occurs on vertex $v_{k-1}$ and that $s_k = r+2$ occurs on vertex $v_k$. If $k$ is odd and the $(r+2)$-precoloring uses both colors $r+1$ and $r+2$ on $K_1$, we choose the labeling so that $s_{k-1} = r+2$ occurs on vertex $v_{k-1}$ and that $s_k = r+1$ occurs on vertex $v_k$. No matter the parity of $k$, if the precoloring only uses one of $r+1$ and $r+2$ on $K_1$, then we choose the labeling so that $s_k = r+1$ or $s_k = r+2$ occurs on the vertex $v_k$.

Starting with vertex $v_1$ in $K_1$, if $a_1 \neq s_1$, we switch the color on $v_1$ to the color $r+1$. There is no $a_i \in \{a_1, a_2, \ldots, a_k\}$ such that $a_i = r+1$ since the vertices of $K$ are still assigned their color from the $r$-coloring of $G$. Also, there is no vertex colored $r+1$ outside of the $k$-cliques of $P$, so we do not have to alter the color on any neighbor of $v_1$ for this to still be a proper coloring of $G$. To color $v_1$ its correct color, $s_1$, we interchange the colors on the $(s_1, r+1)$-Kempe chain originating at $v_1$. If there is a vertex $v_j \in K_1$ currently colored $a_j = s_1$, the Kempe change will recolor $v_j$ the color $r+1$. If there is a vertex $x$ outside of $K_1$ that is adjacent to $v_1$ and currently colored $s_1$, then $x$ will be recolored $r+1$. The furthest away from $K_1$ that a vertex can be recolored $r+1$ is distance one. For a vertex farther than distance one away from $K_1$ to be recolored, there would have to have been a vertex either distance one or two away from $K_1$ originally colored $r+1$. There are no vertices colored $r+1$ distance one or two from $K_1$ before this step, so the farthest vertex that can be changed in this step is a vertex distance one away from $K_1$ originally colored $s_1$. Now, $v_1$ is assigned its correct color, $s_1$, and at worst a vertex at distance one from $K_1$ has been assigned the color $r+1$. By Lemma 1, we have a proper $(r+1)$-coloring of $G$. 
We now consider $v_2 \in K_1$. Let the current color of $v_2$ be $a_2$. If $a_2 \neq s_2$, we switch the color on $v_2$ to the color $r + 2$. There is no $a_i \in \{a_1, a_2, \ldots, a_k\}$ such that $a_i = r + 2$, and there is no vertex colored $r + 2$ outside of $P$, so we do not have to alter the color on any neighbor of $v_2$ for this to still be a proper coloring of $G$. To color $v_2$ its correct color, $s_2$, we interchange the colors on the $(s_2, r + 2)$-Kempe Chain originating at $v_2$. If there is a vertex, $v_j \in K_1$ currently colored $a_j = s_2$, the Kempe change will recolor it $r + 2$. If there is a vertex $x$ outside of $K_1$ that is adjacent to $v_1$ and currently colored $s_2$, then $x$ will be recolored $r + 2$. After interchanging the colors $s_2$ and $r + 2$ on the $(s_2, r + 2)$-Kempe Chain originating at $v_2$, $v_2$ is now assigned its correct color, $s_2$, and the farthest away from $K$ a vertex will be recolored $r + 2$ is distance one. By Lemma 1, we now have a proper $(r + 2)$-coloring of $G$.

Let $D_1$ denote the distance from $K_1$ to the farthest vertex colored $r + 1$ by this process and let $D_2$ denote the distance from the $K_1$ to the farthest vertex colored $r + 2$ by this process. After recoloring vertex $v_1$ and $v_2$ their correct colors, $s_1$ and $s_2$, respectively, $D_1 \leq 1$ and $D_2 \leq 1$.

Now assume the vertices $v_1, v_2, \ldots, v_i$ have their correct colors, $s_1, s_2, \ldots, s_{i-1}$, both $D_1 \leq i - 2$ and $D_2 \leq i - 2$, where $i$ is odd, and we currently have a proper $(r + 2)$-coloring of $G$.

Let $b_i$ be the current color of vertex $v_i$. If $b_i \neq s_i$, since $i$ is odd, we will first switch the color of $v_i$ to $r + 1$ by switching the colors on the $(b_i, r + 1)$-Kempe chain containing $v_i$. Note that if $b_i = r + 1$, no color change is needed, and thus, no color alterations to other vertices are needed either. So, we assume $b_i \neq r + 1$. Before interchanging the colors $b_i$ and $r + 1$, $D_1 \leq i - 2$. Interchanging the colors on the $(b_i, r + 1)$-Kempe chain containing $v_i$, may recolor a vertex the color $r + 1$ that is at most distance $i - 1$ from $K_1$. So, now $D_1 \leq i - 1$. Now, to assign $v_i$ its correct color, $s_i$, we will switch the color of $v_i$ from $r + 1$ to $s_i$ by switching the colors on the $(s_i, r + 1)$-Kempe chain containing $v_i$. The interchanging of the colors on the $(s_i, r + 1)$-Kempe chain containing $v_i$, may recolor a vertex the color $r + 1$ that is at most distance $i$ from $K_1$. Since each Kempe
change on the vertices in $K_1$ with odd subscripts involves the color $r + 1$, correcting the color on each one of these vertices makes $D_1$ increase by at most 2, one for each Kempe change performed. Thus, we now have that $D_1 \leq i$. Note that to recolor $v_1$ the correct color, we only needed to perform one Kempe change.

Now, we consider $v_{i+1}$. Let $b_{i+1}$ be the current color of vertex $v_{i+1}$. If $b_{i+1} \neq s_{i+1}$, since $i+1$ is even, we will first switch the color of $v_{i+1}$ to $r + 2$ by switching the colors on the $(b_{i+1}, r + 2)$-Kempe chain containing $v_{i+1}$. Note that if $b_{i+1} = r + 2$, no color change is needed, and thus, no color alterations to other vertices are needed either. So, assume $b_{i+1} \neq r + 2$. Before interchanging the colors $b_{i+1}$ and $r + 2$, $D_2 \leq i - 2$. Interchanging of the colors on the $(b_{i+1}, r + 2)$-Kempe chain containing $v_{i+1}$, may recolor a vertex the color $r + 2$ that is at most distance $i - 1$ from $K_1$. So, now $D_2 \leq i - 1$. To assign $v_{i+1}$ its correct color, $s_{i+1}$, we will switch the color of $v_{i+1}$ from $r + 2$ to $s_{i+1}$ by switching the colors on the $(s_{i+1}, r + 2)$-Kempe chain containing $v_{i+1}$. The interchanging of the colors on the $(s_{i+1}, r + 2)$-Kempe chain containing $v_{i+1}$ may recolor a vertex the color $r + 2$ that is at most distance $i$ from $K_1$. Since each Kempe change on vertices in $K_1$ with even subscripts involves the color $r + 2$, correcting the color on each one of these vertices makes $D_2$ increase by at most 2, one for each Kempe change performed. Thus, we now have that $D_2 \leq i$. Note that to recolor $v_2$ the correct color, we only needed to perform one Kempe change.

Note that subsequent Kempe changes associated with $K_1$ will not affect the vertices in $K_1$ assigned their correct color already. Let a vertex, $v_j$, be colored its correct color $s_j$. Then a Kempe change associated to the vertex $v_{j+1}$ will not involve the color, $s_j$, and thus we do not need to recolor $v_j$. Let $v_{j+1}$ currently be colored $b_{j+1}$. The possible Kempe changes needed to assign $v_{j+1}$ its correct color, $s_{j+1}$ will be ones involving $b_{j+1}$, $r + 1$ or $r + 2$, and $s_{j+1}$. $s_j \neq r + 1$ and $s_j \neq r + 2$ because if it were, then it would need to be the second to last vertex in $K_1$, $s_j \neq a_{j+1}$ because if it were, then in the previous step when we interchanged the colors on the $(s_j, r + 1)$-Kempe Chain or the $(s_j, r + 2)$-Kempe chain, $v_{j+1}$ would have been recolored $r + 1$ or $r + 2$, respectively.
Also, \( s_j \neq s_{j+1} \) since each \( s_i \) is a unique color.

To finish recoloring the clique \( K_1 \) we have to alter our process depending on if both \( r+1 \) and \( r+2 \) are used by the precoloring of \( K_1 \), if only one of \( r+1 \) and \( r+2 \) is used by the precoloring of \( K_1 \), or if neither \( r+1 \) nor \( r+2 \) is used by the precoloring of \( K_1 \). So, we have three cases for these three situations, but within each case, we have subcases depending on if \( k \) is even or odd.

**Case 1:**

First, we will assume both \( r+1 \) and \( r+2 \) are used by the precoloring of \( K_1 \).

**Case 1a:**

First, assume \( k \) is odd. Since \( k \) is odd, we chose the labeling so that \( s_{k-1} = r+2 \) occurs on vertex \( v_{k-1} \) and that \( s_k = r+1 \) occurs on vertex \( v_k \). We are assuming that \( v_1, v_2, \ldots, v_{k-2} \) have been assigned their correct colors, \( s_1, s_2, \ldots, s_{k-2} \). Since \( k \) is odd, and thus, \( k-2 \) is odd, the recoloring of the most previous vertex, \( v_{k-2} \), involved at most two Kempe changes with the color \( r+1 \). So, after recoloring \( v_{k-2} \) its correct color, \( D_1 \leq k-2 \), and \( D_2 \leq k-4 \). \( D_1 \leq k-2 \) since \( (k-1)^2 - 1 \) vertices have needed at most two Kempe changes performed involving the color \( r+1 \), and so far, one vertex has needed at most one Kempe change performed involving the color \( r+1 \). Thus, the total Kempe changes needed so far involving \( r+1 \) is \( 2(k-1)^2 - 1 + 1 = (k-3) + 1 = k-2 \). \( D_2 \leq k-4 \) since \( (k-3)^2 - 1 \) vertices have needed at most two Kempe changes performed involving the color \( r+2 \), and so far, one vertex has needed at most one Kempe change performed involving the color \( r+2 \). Thus, the total Kempe changes needed so far involving \( r+2 \) is \( 2(k-3)^2 - 1 + 1 = (k-5) + 1 = k-4 \).

At the next vertex, \( v_{k-1} \), where \( s_{k-1} = r+2 \), we only have to perform at most one Kempe change. Let \( b_{k-1} \) be the current color of \( v_{k-1} \). If \( b_{k-1} = r+2 \) no Kempe change is needed since \( v_{k-1} \) is already colored it correct color. If \( b_{k-1} \neq r+2 \), then we switch the color of \( v_{k-1} \) to \( r+2 \) by switching the colors on the \( (b_{k-1}, r+2) \)- Kempe chain containing \( v_{k-1} \). No further Kempe changes are needed because \( v_{k-1} \) is now colored its
correct color $s_{k-1} = r + 2$. Thus, $D_2 \leq k - 3$ since we performed at most one additional Kempe change with $r + 2$, and at the previous step $D_2 \leq k - 4$.

Considering the last vertex, $v_k$, where $s_k = r + 1$, we only have to perform at most one Kempe change here as well. Let $b_k$ be the current color of $v_k$. If $b_k = r + 1$ no Kempe change is needed since $v_k$ is already colored it correct color. If $b_k \neq r + 1$, then we switch the color of $v_k$ to $r + 1$ by switching the colors on the $(b_k, r + 1)$- Kempe chain containing $v_k$. No further Kempe changes are needed because $v_k$ is now colored its correct color $s_k = r + 1$. Thus, $D_1 \leq k - 1$ since we performed one additional Kempe change with $r + 1$, and at the previous step $D_1 \leq k - 2$. Now all vertices are assigned their correct color. The farthest color changes that can be made by this process are distance $k - 1$ from $K_1$.

**Case 1b:**

Now, assume $k$ is even. Since $k$ is even, we chose the labeling so that $s_{k-1} = r + 1$ occurs on vertex $v_{k-1}$ and that $s_k = r + 2$ occurs on vertex $v_k$. We are assuming that $v_1, v_2, \ldots, v_{k-2}$ have their correct colors, $s_1, s_2, \ldots, s_{k-2}$. Since $k$ is even, and thus, $k - 2$ is even, the recoloring of the most previous vertex, $v_{k-2}$, involved at most two Kempe changes with the color $r + 2$. After recoloring $v_{k-2}$ its correct color, $D_2 \leq k - 3$. We have that $D_1 \leq k - 3$ as well.

At our next vertex $v_{k-1}$, where $s_{k-1} = r + 1$, we only have to perform at most one Kempe change involving the color $r + 1$ so that $v_k$ is recolored $r + 1$. Thus, $D_1 \leq k - 2$ since at the previous step $D_1 \leq k - 3$.

Now consider the last vertex, $v_k$, where $s_k = r + 2$. We only have to perform at most one Kempe change here as well, this time involving the color $r + 2$ to color $v_k$ its correct color, $r + 1$. Thus, $D_2 \leq k - 2$ since at the previous step $D_2 \leq k - 3$. Now all vertices are assigned their correct color. The farthest color changes made by this process are at most distance $k - 2$ from $K_1$.

**Case 2:**
Now, assume only one of \( r + 1 \) or \( r + 2 \) is used by the precoloring of the vertices on \( K_1 \).

Case 2a:

Assume that the precoloring uses the color \( r + 1 \) on the vertices of \( K_1 \), but does not use the color \( r + 2 \).

Case 2a1:

Assume that \( k \) is odd. We are assuming that \( v_1, v_2, \ldots, v_{k-1} \) are assigned their correct colors, \( s_1, s_2, \ldots, s_{k-1} \). Since \( k \) is odd, and thus, \( k - 1 \) is even, the recoloring of our most previous vertex, \( v_{k-1} \), involved at most two Kempe changes with the color \( r + 2 \). After performing these two possible Kempe changes, \( D_2 \leq k - 2 \). We have that \( D_1 \leq k - 2 \) as well. Moving to our next and last vertex, \( v_k \), where \( s_k = r + 1 \), we only have to perform at most one Kempe change involving \( r + 1 \) to recolor \( v_k \) the color \( r + 1 \). Thus, \( D_1 \leq k - 1 \) since at the previous step, \( D_1 \leq k - 2 \). Now all vertices are assigned their correct color. The farthest color changes made are at most distance \( k - 1 \) from \( K_1 \).

Case 2a2:

Now, assume \( k \) is even. We are assuming that \( v_1, v_2, \ldots, v_{k-2} \) have the correct colors, \( s_1, s_2, \ldots, s_{k-2} \). Since \( k \) is even, and thus, \( k - 2 \) is even, the recoloring of our most previous vertex, \( v_{k-2} \), involved at most two Kempe changes with the color \( r + 2 \). After performing these two possible Kempe changes, \( D_2 \leq k - 3 \). We also have that \( D_1 \leq k - 3 \).

Now, we move to the second to last vertex, \( v_{k-1} \). Since \( k \) is even, and thus, \( k - 1 \) is odd, at this step, we would normally do Kempe changes involving \( r + 1 \) since we alternate between \( r + 1 \) and \( r + 2 \) based on the parity of the subscript of the vertices in \( K_1 \). However, in this specific case, we will instead perform the at most two Kempe changes with \( r + 2 \). This is to decrease the distance from \( K_1 \) that color changes are
made. Then we have that, $D_2 \leq k - 1$ since we performed at most two additional Kempe changes with $r + 2$ and at the previous step, $D_2 \leq k - 3$.

Now, consider the last vertex, $v_k$, where $s_k = r + 1$. We only have to perform at most one Kempe change involving $r+1$ so that $v_k$ is colored its correct color, $r + 1$. Thus, $D_1 \leq k - 2$ since at the previous step $D_1 \leq k - 3$. Now all vertices are assigned their correct color. The farthest color changes made are at most distance $k - 1$ from $K_1$.

Note that if for vertex, $v_{k-1}$, we performed the Kempe changes with $r + 1$ instead of $r + 2$, like we did, then at that step we would have had $D_1 \leq k - 1$. Then after recoloring the last vertex $v_k$, we would have had $D_1 \leq k$. By switching the Kempe changes associated with $v_{k-1}$ to involve $r + 2$ instead of $r + 1$, we are able to guarantee that the farthest color changes made were at most distance $k - 1$ from $K_1$, instead of at most distance $k$ from $K_1$.

**Case 2b:**

Assume that the precoloring uses the color $r + 2$ on the vertices of $K$, but does not use the color $r + 1$.

**Case 2b1:**

Assume that $k$ is odd. We are also assuming that $v_1$, $v_2$, $\ldots$, $v_{k-1}$ are assigned their correct colors, $s_1$, $s_2$, $\ldots$, $s_{k-1}$. Since $k$ is odd, and thus, $k - 1$ is even, the recoloring of the most previous vertex, $v_{k-1}$, involved at most two Kempe changes with the color $r + 2$. After performing these two possible Kempe changes, $D_2 \leq k - 2$. We have that $D_1 \leq k - 2$ as well.

Now, consider the last vertex, $v_k$, where $s_k = r + 2$, we only have to perform at most one Kempe change involving $r + 2$ to color $v_k$ its correct color, $r + 2$. Thus, $D_2 \leq k - 1$ since at the previous step $D_2 \leq k - 2$. Now all vertices are assigned their correct color. The farthest color changes made are at most distance $k - 1$ from $K_1$.

**Case 2b2:**
Now, assume \( k \) is even. We are also assuming that \( v_1, v_2, \ldots, v_{k-1} \) are assigned their correct colors, \( s_1, s_2, \ldots, s_{k-1} \). Since \( k \) is even, and thus, \( k - 1 \) is odd, the recoloring of the most previous vertex, \( v_{k-1} \), involved at most two Kempe changes with the color \( r + 1 \). After performing these two possible Kempe changes, \( D_1 \leq k - 1 \). We have that \( D_2 \leq k - 3 \). This is because we started the first vertex with \( r + 1 \), so there may have been more possibilities for Kempe changes with \( r + 1 \) than \( r + 2 \).

Now, consider the last vertex, \( v_k \), where \( s_k = r + 2 \). We only have to perform at most one Kempe change involving \( r + 2 \) to color \( v_k \) its correct color, \( r + 2 \). Thus, \( D_2 \leq k - 2 \) since at the previous step \( D_2 \leq k - 3 \). Now all vertices are assigned their correct color. The farthest color changes made are at most distance \( k - 1 \) from \( K_1 \).

**Case 3:**

Assume that neither \( r + 1 \) nor \( r + 2 \) is used by the precoloring on \( K \).

**Case 3a:**

Assume \( k \) is odd. We are also assuming that \( v_1, v_2, \ldots, v_{k-1} \) are assigned their correct colors, \( s_1, s_2, \ldots, s_{k-1} \). Since \( k \) is odd, and thus, \( k - 1 \) is even, the recoloring of the most previous vertex, \( v_{k-1} \), involved at most two Kempe changes with the color \( r + 2 \). After performing these two possible Kempe changes, \( D_2 \leq k - 2 \). We have that \( D_1 \leq k - 2 \) as well.

Moving to our last vertex, \( v_k \), where \( s_k \neq r + 2 \). Since \( s_k \neq r + 2 \), we have to perform at most two Kempe changes to color \( v_k \) its correct color, \( s_k \). Let the current color of \( v_k \) be \( b_k \). If \( b_k \notin \{s_k, r + 1, r + 2\} \), we will first switch \( v_k \) to \( r + 1 \) by interchanging the colors on the \((b_k, r + 1)\)-Kempe chain containing \( v_k \). Then we must change \( v_k \) from \( r + 1 \) to \( s_k \) by interchanging the colors on the \((s_k, r + 1)\)-Kempe chain containing \( v_k \). Thus, in this situation, \( D_1 \leq k \) since we performed two additional Kempe changes here and at the previous step \( D_1 \leq k - 2 \). Now all vertices are assigned their correct color. The farthest color changes made were at most distance \( k \) from \( K_1 \).
We note that we could have performed both of these Kempe changes to recolor \( v_k \) its correct color with the color \( r+2 \) instead of \( r+1 \). However, we would get that \( D_2 \leq k - 2 \), which means that color changes would still occur at most distance \( k \) from \( K_1 \).

**Case 3b:**

Now, assume \( k \) is even. We are also assuming that \( v_1, v_2, \ldots, v_{k-1} \) are assigned their correct colors, \( s_1 = s_2, \ldots, s_{k-1} \). Since \( k \) is even, and thus, \( k - 1 \) is odd, the recoloring of the most previous vertex, \( v_{k-1} \), involved at most two Kempe changes with the color \( r+1 \). After performing these two possible Kempe changes, \( D_1 \leq k - 1 \) and \( D_2 \leq k - 3 \). Moving to our last vertex, \( v_k \), where \( s_k \neq r+2 \), we have to perform at most two Kempe changes, like usual, to color \( v_k \) its correct color, \( s_k \). Thus, \( D_2 \leq k - 1 \) since at the previous step \( D_2 \leq k - 3 \). Now all vertices are assigned their correct color.

The farthest color changes made are at most distance \( k - 1 \) from \( K_1 \).

No matter which case we have, when we finish with recoloring \( K_1 \) so that its vertices are assigned their correct color and the coloring we have of \( G \) is a proper \((r+2)\)-coloring, \( D_1 \leq k \) and \( D_2 \leq k - 1 \). Thus, the farthest color changes made are at most distance \( k \) from \( K_1 \). Since the \( k \)-cliques of \( P \) are at least distance \( 2k + 2 \) from each other, the vertices changed by this process for one clique will be at least distance \( (2k+2) - 2k = 2 \) from the vertices changed by this process for another clique. Thus, we have an extension of the \((r+2)\)-coloring of \( P \) to an \((r+2)\)-coloring of the whole graph \( G \).

\[\square\]

We showed that for clique of order \( k \), the farthest from the clique that color changes associated to that clique can occur is distance \( k \). So, if a clique is of order \( k_i \leq k \), the farthest from the clique that color changes can occur is distance \( k_i \leq k \). Also, since the color changes associated to two different cliques of order \( k \) need to be distance 2 apart, the distance between any two \( k \)-cliques needs to be \( k + k + 2 = 2k + 2 \). Thus, for
two cliques of order \(k_i\) and \(k_j\), where the farthest color changes from the cliques are at most distance \(k_i \leq k\) and \(k_j \leq k\), respectively, the distance needed between this pair of cliques is \(k_i + k_j + 2 \leq 2k + 2\).

We provide an example of making modifications to the \(r\)-coloring of a graph so that it agrees with the colors assigned to the vertices in the cliques by the \((r + 2)\)-precoloring. We follow the procedure described in the proof above. In our example, \(P \subseteq V(G)\) induces a subgraph of \(G\) that consists of 3-cliques. Let \(\gamma\) be a 3-coloring of \(G\) that uses the colors from \{blue, green, orange\}, and let \(c\) be a 5-coloring of \(P\) that uses the colors from \{blue, green orange, yellow, red\}. We will consider the clique \(K_1\). The colors \(\gamma\) uses on \(K_1\) are from \{blue, orange, green\}, and let the colors \(c\) uses on \(K_1\) be from \{blue, green, red\}. In this example, since yellow and red are used on \(P\) by \(c\), but not on \(G\) by \(\gamma\), they represent to colors \(r + 1\) and \(r + 2\), which in this case are 4 and 5. Let yellow be 4 and let red be 5.

In Figure 6.1, we have the clique, \(K_1\), that consists of three vertices, \(v_1, v_2, \text{and } v_3\), and the vertices within distance 4 of \(K_1\), which are the vertices in \(\{x_1, x_2, \ldots, x_{12}\}\). What is shown in this figure is a subgraph of \(G\). All the vertices in the graph are currently colored by \(\gamma\). For reference, for each vertex in \(K_1\), we provide its precolored color as the outer color of the node. The inner color on each node in \(K_1\) is what the vertex is currently colored by \(\gamma\). red is acting as our \(r + 2\) color, so let \(s_3 = \text{red}\) be the precoloring color of \(v_3\). Then, we will let \(s_1 = \text{blue}\) be the precoloring color of \(v_1\), and \(s_2 = \text{green}\) be the precoloring color of \(v_2\). We will recolor the clique in the order based off of the indices of the correct colors. Then, we have that \(a_1 = \gamma(v_1) = \text{green}\), \(a_2 = \gamma(v_2) = \text{blue}\), and \(a_3 = \gamma(x_3) = \text{orange}\). We will follow this example through the process of recoloring the entire clique, \(K_1\).

The first vertex in the clique we will recolor is \(v_1\). To recolor \(v_1\) its correct color, \(s_1 = \text{blue}\), we first recolor \(v_1\) the color yellow. Since yellow is not used by \(\gamma\) on \(G\), there are no other vertices currently colored yellow. We show this step in Figure 6.2. The current coloring we have on the graph is a proper 4-coloring.
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Now, to color \( v_1 \) the color \( s_1 = \text{blue} \), we interchange the colors on the (blue, yellow)-Kempe chain originating at \( v_1 \). The vertex \( v_1 \) gets recolored from yellow to its correct color, blue. Other than \( v_1 \), the vertices on this chain are \( v_2 \) and \( x_2 \). Since both \( v_2 \) and \( x_2 \) are currently colored blue, they get recolored yellow. This is shown in Figure 6.3. The current coloring we have on the graph is a proper 4-coloring.

The next vertex in the clique we will recolor is \( v_2 \). To recolor \( v_2 \) its correct color, \( s_2 = \text{green} \), we first recolor \( v_2 \) the color red. Since red is not used by the current coloring on \( G \), there are no other vertices currently colored red. We show this step in Figure 6.4. The current coloring we have on the graph is now a proper 5-coloring.

Now, to color \( v_2 \) the color \( s_2 = \text{green} \), we interchange the colors on the (green, red)-Kempe chain originating at \( v_2 \). The vertex \( v_1 \), gets recolored from red to its correct color, green. The only other vertex on this Kempe chain is \( x_1 \). We recolor \( x_1 \) from green to red. This is shown in Figure 6.5. The current coloring we have on the graph is still a proper 5-coloring.

The next vertex in the clique we recolor is \( v_3 \). We follow the procedure in Case 2b_1 of the proof since we have that \( k = 3 \) is odd, and red = 5 is used on \( K_1 \), but yellow = 4 is not used on \( K_1 \) by the precoloring of \( P \). To recolor \( v_3 \) its correct color, \( s_3 = \text{red} \), we interchange the colors on the (orange, red)-Kempe chain originating at \( v_3 \). We recolor \( v_3 \) from orange to its correct color, red. Other than \( v_1 \), the vertices on this chain are \( x_1 \) and \( x_6 \). We recolor \( x_1 \) the color orange since it was colored red previously, and we recolor \( x_6 \) the color red since it was colored orange previously. We show this step in Figure 6.6. The current coloring we have on the graph is still a proper 5-coloring. Now, all vertices in \( K_1 \) have been assigned their correct color.

In this example, the color changes only occurred at distance 2 from \( K_1 \). No matter what example we have, the first two vertices we recolor in the clique only need one Kempe change performed to be recolored their correct color. Then, the next vertex we had to recolor in the clique was the last vertex and we only needed to perform one Kempe change to recolor this vertex as well.
Figure 6.1: The whole graph is currently colored by $\gamma$. $\gamma$ is a proper 3-coloring of the subgraph shown and of the whole graph, $G$.

Figure 6.2: We recolored $v_1$ from green to yellow.

Figure 6.3: We recolored $v_1$ its correct color, blue, by interchanging the colors on the (blue, yellow)-Kempe Chain originating at $v_1$. This Kempe chain also includes the vertices $v_2$ and $x_1$. 

\begin{align*}
K_1 & \quad 1 \quad 2 \quad 3 \quad 4 \\
\begin{array}{cccc}
\text{v1} & \text{x1} & \text{x4} & \text{x7} & \text{x10} \\
\text{v2} & \text{x2} & \text{x5} & \text{x8} & \text{x11} \\
\text{v3} & \text{x3} & \text{x6} & \text{x9} & \text{x12} \\
\end{array}
\end{align*}
Figure 6.4: We recolored $v_2$ from yellow to red.

Figure 6.5: We recolored $v_2$ its correct color, green, by interchanging the colors on the $(green, red)$- Kempe Chain originating at $v_2$. This Kempe chain also includes the vertex, $x_1$.

Figure 6.6: We recolored $v_3$ its correct color, red, by interchanging the colors on the $(orange, red)$- Kempe Chain originating at $v_3$. This Kempe chain also includes the vertices $x_1$ and $x_6$. All the vertices in $K_1$ are recolored and we have a proper 5-coloring of this graph.
Chapter 7

Conclusion

In this paper, we primarily consider precoloring extension problems where the subgraph that is precolored is composed of cliques. We consider this problem when the precoloring of the subgraph is an \((r + 1)\)-coloring and an \((r + 2)\)-coloring. There are many other results like the ones presented in this paper, but with different conditions on the graph and the subgraph. Variations on the graph can include its chromatic number of the graph, whether its planar or not, and the type of graph it is. Variations on the subgraph that is precolored can include, the type of subgraph, the coloring of the subgraph, and constraints on the distance between vertices in the subgraph.

Even though Theorem 6 has been improved, the methods used in this proof, such as the creation of barriers to stop Kempe changes, can possibly be extended to help with other precoloring extension problems. Also, further work can be done in trying to lower the distant constraint even more between the cliques in a precolored subgraph. In addition, for the \((r + 2)\) result, the distance constraint between the cliques could possibly be lowered if we consider the number of colors the \(r\)-coloring and the precoloring have in common on a clique. This is how Albertson and Moore made Kostochka’s result more precise in Theorem 9, which we discuss in Chapter 5.
Bibliography


