Wesleyan University

Affine and Projective Planes
(and the Skew Fields Underneath Them)

by

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Abstract

In this paper, we study affine and projective planes, and their algebraic foundations. Following the early sections of Emil Artin’s book *Geometric Algebra* [1], we develop a theory of affine and projective planes. His text develops a correspondence between affine planes and skew fields. The notation and terminology used in Artin’s text is somewhat dated, and so we use more modern algebraic constructs and ideas to make his ideas more legible to a modern reader.

By establishing a correspondence between skew fields and affine or projective planes, we can move from a synthetic setting of axiomatic models of planes, to a constructive model of geometry. Basing geometry on algebraic structures allows us to appeal to ideas from algebra to discuss properties of geometry.
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Chapter 1

Introduction

Affine space is often seen as a simple model for the basics of planar geometry. It is meant to capture the basic relations that one can imagine when considering points and lines lying on a plane: points are incident to lines, some lines intersect, some lines do not, every pair of points are collinear, and so forth. In fact, one might think the following rules are sufficient to describe such a structure.

1. For any two points there is a line between them.

2. For any two lines, they are parallel or intersect in one point.

3. For every point and every line, there is a line containing that point parallel to the first line.

4. There are three points such that no line contains the three of them.

However, we will soon see that this is not enough. We would like for structures which behave nicely enough (that is, those which follow the above four rules) to be easily visualized. For our purposes, to be “easily visualized” means there is a coordinate system which characterizes it. As it turns out, it is necessary to add in Desargues’ Theorem, which are statements pertaining to the relationship of parallel lines in a particular configuration. In our initial treatment of affine
planes, we use axioms which are equivalent to Desargues’ Theorem, as they are more convenient for the purposes of Chapter 1.

We also will be discussing projective planes, which are another type of incidence structure. These are incidence structures with no notion of parallelism. Every pair of lines intersect. The meaning behind this is more aesthetic. A motivating example comes from my high school math teacher, though I have heard it from many other sources as well. When one is standing on a long railroad track, it appears as though the lines meet towards the horizon. As is the case with affine planes, most of the projective planes one might be interested in studying are those which have Desargues’ Theorem.

The unifying matter this paper discusses is a movement from a synthetic approach to geometry to a constructive one. The axioms of a projective plane and of an affine plane discuss objects called “points” and “lines.” While our intuition guides our understanding of what these objects ought to represent, they may behave quite differently than we expect. Enter algebra. Linear algebra is a powerful tool that we will use to make ideas more concrete. Lines and points are viewed as in a coordinate plane of a field, for example \( \mathbb{R}^2 \). The axioms for the synthetic constructs of projective planes and affine planes become theorems in this new setting. A strong understanding of the algebraic foundation of geometry allows for a richer understanding of the geometry itself, and vice versa.

Before we begin, here are some brief notes to the reader. First, there are several sources which helped me develop an understanding of the subject, but are not cited directly. These texts are Samuel and Levy’s *Projective Geometry* [2], Buekenhout and Cohen’s *Diagram Geometry Related to Classical Groups and Buildings* [3], Kryftis’ *A constructive approach to affine and projective planes* [4], Coxeter’s *Projective Geometry* [5], and Wylie’s *Introduction to Projective Geometry* [6]. Second, I have included diagrams to assist in the reader’s understanding of the
proofs, where appropriate, using GeoGebra’s drawing tools [7]. However, there are still some proofs for which the reader’s understanding is enhanced by drawing the steps themselves. If something is unclear, I would encourage making a diagram.
Chapter 2

Affine Planes

We begin by introducing the notion of an abstract affine plane. Often we visualize such a system on a coordinate plane, and that is how we will approach the discussion of abstract affine planes. We will take a skew field and construct an affine plane based on it, and discuss the general properties of an abstract affine plane in conjunction with the affine plane of this field. The purpose of this chapter is determining whether this construction is reversible: is there a canonical way of constructing a skew field given an abstract affine plane, such that the affine plane of this skew field is the same as the abstract affine plane we began with?

This chapter follows the methods of Chapter 2 in Artin’s text, *Geometric Algebra* [1]. We modify much of the notation, and introduce some new proofs where more exposition is necessary.

**Definition 2.0.1.** Let \( P \) be a set which we will call the **point-set**, whose elements are **points**, and let \( L \) be a set which we will call the **line-set**, whose elements are **lines**, and let \( \varepsilon \subseteq P \times L \) be a binary relation. Then, the ordered triple \( G = (P, L, \varepsilon) \) is an **abstract affine plane** if the 6 axioms hold, which we will delineate shortly.

*If \( p \in l \), then we say \( p \) is incident to \( l \), \( p \) lies on \( l \), or \( p \) is on \( l \), \( l \) contains \( p \), and
so forth.

If \( p \) is the only point on two lines \( l \) and \( m \), then we say \( p \) is in the intersection of \( l \) and \( m \).

We also introduce the notion of parallelism, as it is fundamental to the idea of planar geometry. There will be other terminology used in the axioms, which we will develop later on in the chapter.

**Definition 2.0.2.** Let \( l \) and \( m \) be lines. We say \( l \) and \( m \) are parallel, written \( l \parallel m \), if \( l = m \) or if \( p \in l \) exactly when \( p \in m \). I reiterate: lines are parallel to themselves! If \( l \) and \( m \) are not parallel, we write \( l \not\parallel m \).

**Axiom 1.** For any two distinct points \( p,q \in P \), there is a unique line which contains both of them. We denote this line by \( \ell(p,q) \). Note that we distinguish \( \ell \) from \( l \). The former is used in the line constructors, whereas the latter is used as a variable to name lines.

**Axiom 2.** For any two lines, if they are not parallel, then there is a unique point in their intersection.

**Axiom 3.** For any line \( l \) and any point \( p \), there is a unique line \( m \) which is parallel to \( l \) and contains \( p \). We write this as \( m = \ell(p \parallel l) \).

**Axiom 4.** There exist three points such that no line contains all three of them. Points for which there is indeed such a line are called **collinear**. Otherwise, they are **non-collinear**. Axiom 1 tells us that any two points are collinear, but it may not be the case that three or more points are collinear.

For a structure satisfying Axioms 1 through 4, we may define a group called the group of translations, which is defined in Chapter 2, Section 2. For now, you may view a translation as a map from points to points which preserves collinearity and parallelism. More details will be given later.
Axiom 5. The group of translations, $\mathbb{T}$, acts regularly on $\mathcal{P}$. For distinct points $p$ and $q$, the translation which maps $p$ to $q$ will be denoted $\tau_{p,q}$.

Axiom 6. For any equivalence class of parallel lines $\pi$, $TPH(\mathcal{G})$ acts transitively on $\mathbb{T}_\pi \setminus \{id\}$, where $TPH(\mathcal{G})$ is the skew field of trace-preserving homomorphisms of $\mathbb{T}$ and $\mathbb{T}_\pi$ is the group of translations with direction $\pi$. We prove that parallelism is indeed an equivalence relation in Proposition 2.1.8.

Later, we will show that Axiom 6 is equivalent to the following axiom.

Axiom (6b). For any three collinear, pair-wise distinct points $p, q$, and $r$ there is a dilatation $\sigma$ such that $\sigma(p) = p$ and $\sigma(q) = r$.

2.1 The Affine Plane of a Skew Field

Definition 2.1.1. Let $K$ be a skew field. Define the following sets:

$$A_K := K^2, \text{ viewed as a left } K\text{-vector space.}$$

$$L_K := \{a + Kb : a, b \in A_K, b \neq 0\}, \text{ viewed as cosets}$$

The triple $A_K = (A_K, L_K, \in)$ is the affine plane of $K$. As we will show through the chapter, it is indeed an affine plane with point-set $A_K$, line-set $L_K$, and incidence relation $\in$.

Towards showing that the first three axioms hold in the affine plane of $K$, we need the following lemma which delineates how lines can interact with one another.

Lemma 2.1.2. Let $l_1 = a + Kb, l_2 = p + Kq$. Then:

1. If $b$ and $q$ are left linearly independent, then $l_1 \cap l_2$ is a singleton.
2. Suppose \( b \) and \( q \) are left linearly dependent. Then \( l_1 \parallel l_2 \). Moreover, if \( p - a \notin Kb \), then \( l_1 \cap l_2 = \emptyset \). Otherwise \( l_1 = l_2 \).

Proof. We will prove the first claim. Let \( l_1, l_2 \) be given as in the hypothesis, and suppose \( b \) and \( q \) are linearly independent. We prove there exist unique \( t, u \in K \) such that \( a + tb = p + uq \). In other words, we prove the two lines intersect at exactly one point. We know \( b \) and \( q \) are linearly independent in a vector space of dimension 2, so they form a basis. So there exist unique \( x, y \in K \) such that \( p - a = xb + yq \). By letting \( t = x \) and \( u = -y \), we conclude that \( a + tb = p + uq \), for unique \( t \) and \( u \), so the two lines intersect at exactly one point.

Now, we prove the second claim. Let \( x, y \in K \) such that they are not both zero, and \( xb + yq = 0 \). Without a loss of generality, we may assume \( y \) is non-zero, and from this we deduce that \( x \) must also be non-zero. This allows us to write \( q = -y^{-1}xb \). For simplicity, we will write \( z = -y^{-1}x \) and \( q = zb \). For \( l_2 \), we have

\[
l_2 = p + Kq = p + Kzb = p + Kb.
\]

Thus we may write \( q = b \) without a loss of generality. See that \( l_1 \) and \( l_2 \) have a non-trivial intersection exactly when there exist \( t, u \in K \) such that \( a + tb = p + uq \), that is, when \( tb - uq = p - a \). In other words, the lines intersect exactly when \( (t - u)b = p - a \).

Now, we break into cases as guided by the lemma.

Case 1 Suppose that \( p - a \) is not a left multiple of \( b \); that is \( p - a \notin Kb \). Then, the equation has no solution and \( l_1 \cap l_2 \) is empty, so \( l_1 \parallel l_2 \).

Case 2 On the other hand, if \( p - a \in Kb \), then \( p = a + \delta b \), for some \( \delta \in K \).
We conclude

\[ l_2 = p + Kb \]
\[ = a + \delta b + Kb \]
\[ = a + Kb \]

Thus, \( l_1 = l_2 \) in this case, as desired.

This lemma gives us a technical framework of the conditions under which lines intersect everywhere, nowhere, and at exactly one point. But the most relevant result is this corollary, which will be used frequently:

**Corollary 2.1.3.** For any \( l_1, l_2 \in L_K \), either \( l_1 \cap l_2 \) is a singleton, or \( l_1 \parallel l_2 \).

Now, we are able to prove the first four axioms hold in the affine plane of \( K \).

**Proposition 2.1.4** (Axiom 1). For any distinct points \( p, q \in A_K \), there is a unique line \( l \in L_K \) containing both \( p \) and \( q \).

**Proof.** Let \( p, q \in A_K \) be distinct points. Let \( l = p + K(q - p) \). Then, \( p = p + 0(q - p) \) and \( q = p + 1(q - p) \). If there is some line \( l' \) which contains \( p \) and \( q \), Corollary 2.1.3 allows us to conclude that \( l \parallel l' \) and therefore \( l = l' \). Thus, \( l \) is unique.

The proof for Axiom 2 is a direct result of Corollary 2.1.3:

**Proposition 2.1.5** (Axiom 2). For any distinct lines \( l, m \in L_K \), if they are not parallel, then there is a unique point in their intersection.

**Proof.** Let \( l \) and \( m \) be distinct, non-parallel lines in \( L_K \). By Corollary 2.1.3, their intersection is a singleton. That is, there is a unique point in their intersection.

**Proposition 2.1.6** (Axiom 3). For any line \( l \) and any point \( p \), there is a unique line \( m \) which is parallel to \( l \) and contains \( p \).
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Proof. Let \( p \) be any point, and let \( l = x + Ky \). Then, the line \( l' = p + Ky \) is parallel to \( l \) by our lemma, for \( y \) is certainly linearly dependent with itself. Clearly \( p \) is on the line, for \( p = p + 0 \cdot y \). \( \Box \)

Proposition 2.1.7 (Axiom 4). There are points \( p, q \) and \( r \) \( \in L_K \) for which no line contains all three points.

Proof. Let \( p = (0,0), q = (0,1), r = (1,0) \). Then, \( l = 0 + Kq \) is the unique line containing \( q \) and \( p \), and it does not contain \( r \). Since it is unique, no line can contain all three points, so they are non-collinear. \( \Box \)

Let us briefly move back to the abstract setting. While it is clear that points in \( A_K \) truly are elements of lines of \( A_K \), it seems the lines and points of an abstract affine plane can be any objects, so long as they satisfy the axioms. However, it is helpful to view lines as sets of points. We need only the axioms we've already established in order to prove the following for an abstract affine plane:

Proposition 2.1.8. Let \( l \) and \( m \) be lines. Define \( \hat{l} := \{ p \in P : p \in l \} \), and define \( \hat{m} \) similarly. Then, \( l = m \) if and only if \( \hat{l} = \hat{m} \).

We will prove this proposition shortly. In the meantime, you may already see that the proof of “\( l = m \) implies \( \hat{l} = \hat{m} \)” is trivial. The proof that \( \hat{l} = \hat{m} \) implies \( l = m \) seems trivial on the surface, but we are missing a component for this proof. The outline of the argument is that if you have two points \( p \) and \( q \) on \( l \), then you have these points \( p \) and \( q \) also on \( m \). Axiom 1 tells us that two points uniquely determine a line, and so \( l \) and \( m \) are in fact the same line. However, it is not easy to see that there necessarily exist two points on \( l \). It is worth noting, for we use the fact often in our proofs, that every line contains at least one point. This is guaranteed because for every line, there is a line which is not parallel to it (Axiom 3), and their intersection will yield exactly one point (Axiom 2). Now, we will proceed to count points on lines with a chain of propositions.
Proposition 2.1.9. Parallelism is an equivalence relation.

Proof. Parallelism is trivially reflexive and symmetric. To show transitivity, suppose \( l_1 \parallel l_2 \) and \( l_2 \parallel l_3 \). If \( \hat{l}_1 \cap \hat{l}_2 = \emptyset \), then we are done. So suppose that is not the case: say \( \hat{l}_1 = \hat{l}_2 \) and let \( p \in \hat{l}_1 \cap \hat{l}_2 \). By Axiom 3, there is a unique a line \( l'_2 \) which is parallel to \( l_2 \) and contains \( p \). But then \( l'_2 = l_1 \), as \( l_1 \) is parallel to \( l_2 \) and contains \( p \). Likewise, \( l_3 \) is parallel to \( l_2 \) and contains \( p \), so \( l_3 = l'_2 = l_1 \), so \( l_3 \parallel l_1 \), as desired. \( \square \)

Definition 2.1.10. An equivalence class of parallel lines is called a pencil of parallel lines.

Proposition 2.1.11. Suppose that there exist three distinct pencils, \( \pi_1, \pi_2 \) and, \( \pi_3 \). Then, for any pencils \( \pi, \pi' \) we have \( |\pi| = |\pi'| \). Further, for any line \( l \), \( |\pi| = |\hat{l}| \).

Proof. Let \( l \) be a line in \( \pi_1 \). Define a map \( f : \hat{l} \rightarrow \pi_2 \) by mapping a point \( p \) to the line in \( \pi_2 \) which contains it. This map is injective, by Axiom 3. It is also surjective, because if \( m \) is any line in \( \pi_2 \), it has \( \hat{l} \cap \hat{m} \) as its pre-image. Thus, \( |\hat{l}| = |\pi_2| \), since \( f \) is a bijection. A similar argument will show that \( |\hat{l}| = |\pi_3| \), by replacing instances of \( \pi_2 \) with \( \pi_3 \). Moreover, a similar argument will show that if \( m \in \pi_2 \), then \( |\hat{m}| = |\pi_1| = |\pi_2| \). Thus, \( |\pi_1| = |\pi_2| = |\pi_3| \).

Now, let \( \pi \) be any pencil. Without a loss of generality, let \( \pi \neq \pi_1, \pi \neq \pi_2 \). For any \( l \in \pi_1, |\hat{l}| = |\pi| = |\pi_2| \), by constructing an argument analogous to the one above. Now, we may conclude that for any pencils \( \pi, \pi' \) that \( |\pi| = |\pi'| \). \( \square \)

Now, since there are at least three pencils as guaranteed by Axiom 4, each line has at least 3 points. Thus, we may proceed as follows:

Proof of Proposition 2.1.8. Let \( \hat{l} = \hat{m} \). Let \( p, q \in \hat{l} \). Then, \( p, q \in \hat{m} \). By axiom 1, \( l = m \), as two points determine a line uniquely. If \( l = m \), then \( \hat{l} = \hat{m} \) immediately. \( \square \)
Thanks to this result, we may view lines as sets of points and that is what we will do for the remainder of the paper in the case of any abstract affine plane. In general, for an abstract affine plane we will suppress the incidence relation and simply write \( G = (\mathcal{P}, \mathcal{L}) \).

### 2.2 Transformations of the affine plane

Before we introduce the next axioms, we must introduce the notions of translations and dilatations for an abstract affine plane. A dilatation can be thought of as moving the plane around in a bijective way while preserving the parallel relationship of lines.

Before we give a general notion of what dilatations and translations are, we will discuss them for the affine plane of \( K \). Fix a non-zero element \( \alpha \in K \), and any point \( c \in A_K \). Consider the map \( p \mapsto \alpha p + c \), which we will denote by \( \sigma_{\alpha, c} \). It is clear that this map is injective. It is surjective, as any point \( q \) has as a pre-image \( \alpha^{-1}(q - c) \). Moreover, this map preserves parallelism: every line \( l = p + Kq \), is parallel to its image \( (\alpha p + c) + Kq \) by Lemma 2.1.2. We give a detailed proof of this in Proposition 2.2.6.

What will differentiate a dilatation from a translation is whether the map has fixed points. We split into cases:

**Case 1:** \( \alpha \neq 1 \). Then, if \( x \) is a fixed point, we have

\[
\begin{align*}
x &= \alpha x + c \\
x - \alpha x &= c \\
(1 - \alpha)x &= c \\
x &= (1 - \alpha)^{-1}c.
\end{align*}
\]
That is, there is exactly one fixed point, and it is \((1 - \alpha)^{-1}c\).

**Case 2:** \(\alpha = 1\). If \(c \neq 0\), then there are no fixed points. If \(c = 0\), then all points are fixed– \(\sigma_{1,0} = \text{id}_P\).

In the affine plane of \(K\), the translations are maps of the form \(\sigma_{1,c}\). Since \(\alpha\) is always 1 in this case, we denote these simply by \(\tau_c\). These are themselves dilatations, but have an additional restriction: they either fix all points or no points. Here are the formulations for translations and dilatations in an abstract affine plane:

**Definition 2.2.1.** Let \(G = (P, L)\) be an abstract affine plane. A map \(\sigma : P \rightarrow P\) is a **dilatation** if for distinct points \(p, q\) we have \(\sigma(q) \in \ell(\sigma(p) \parallel \ell(p,q))\). If \(\tau : P \rightarrow P\) is the identity function or is a dilatation which fixes no points then we say \(\tau\) is a **translation**.

The restriction on dilatations is a rather strict one, as is shown in the next proposition.

**Proposition 2.2.2.** A dilatation is uniquely determined by the image of two distinct points.

**Proof.** Let \(\sigma\) be a dilatation, let \(p, q\) be distinct points. Write \(p' = \sigma(p), q' = \sigma(q), l = \ell(p,q), l' = \ell(p', q')\). We will determine the image of a point \(r \notin l\). Let \(r' = \sigma(r)\), let \(m = \ell(p,r), n = \ell(q, r)\). Let \(m' = \ell(p' \parallel m)\) and let \(n' = \ell(q' \parallel n)\). By the definition of a dilatation, \(r'\) must lie on both \(m'\) and \(n'\). So \(r'\) is the point of intersection of these lines, which is unique by axiom 2.
Now, suppose $s$ is some point on $\ell(p, q)$. To determine the image of $s$ under $\sigma$, simply apply the above argument, with $p$ staying as it is, $r$ playing the role of $q$, and $s$ playing the role of $r$. 
To be explicit, suppose $\sigma'$ is some other dilatation with $p' = \sigma(p) = \sigma'(p)$ and $q' = \sigma(q) = \sigma'(q)$. Let $x$ be a point distinct from $p$ and $q$. By our above argument, $\sigma(x) = \sigma'(x)$. Thus, $\sigma = \sigma'$. \hfill \qed

The definition of a dilatation does not rule out constant functions. In fact, any dilatation which maps two points to the same point must be constant everywhere. We call such a dilatation degenerate.

**Proposition 2.2.3.** Let $\sigma$ be a dilatation. Let $p$ and $q$ be distinct points. If $\sigma(p) = \sigma(q)$ then $\sigma$ is degenerate. Otherwise, $\sigma$ is a bijection.

**Proof.** Let $\sigma$ be a dilatation, and let $p$ and $q$ be distinct points. Suppose that $\sigma(p) = \sigma(q)$. Let $\tau$ be the degenerate dilatation defined by $x \mapsto \sigma(p)$. Since $\sigma(p) = \tau(p)$ and $\sigma(q) = \tau(q)$, we conclude that $\sigma = \tau$ by Proposition 2.2.2. So $\sigma$ is degenerate, too.

On the other hand, let $p' = \sigma(p)$ and $q' = \sigma(q)$ and suppose $p' \neq q'$. If any pair of distinct points had equal images under $\sigma$, then by the previous argument, $p' = q'$. But $p' \neq q'$, so no pair of distinct points have equal images, so $\sigma$ is injective.

Now, let $r'$ be any point not on $\ell(p', q')$. We will construct a point $r$ such that $\sigma(r) = r'$. Let $l' = \ell(p', r')$, $m' = \ell(q', r')$. Let $l = \ell(p \parallel l')$ be the line through $p$ parallel to $l'$ and let $m = \ell(q \parallel m')$ be the line through $q$ parallel to $m'$. Then $l \parallel m$, because $r' \notin \ell(p', q')$ and so $l' \parallel m'$. Moreover, $r \notin \ell(p, q)$, for otherwise $\ell(p, q) = l = m$. Then, proceeding using the method in the proof of Proposition 2.2.2, we see that $\sigma(r) = r'$. 
If \( r' \) is a point on \( \ell(p', q') \), let \( s' \) be a point not on \( \ell(p', q') \). Repeat the previous argument using \( p' \) and \( s' \) rather than \( p' \) and \( q' \). Thus, \( \sigma \) is a bijection, as desired. 

Degenerate dilatations will be of little use to us, so when we say “dilatation,” it is assumed that we are referring to non-degenerate dilatations unless otherwise stated. This gives us a simpler definition for dilatations, which will be the definition we use throughout the remainder of the text:

**Definition 2.2.4.** A dilatation is a bijection \( \sigma : \mathcal{P} \to \mathcal{P} \) such that for all distinct points \( p \) and \( q \), \( \ell(p, q) \parallel \ell(\sigma(p), \sigma(q)) \).

Now that we are working primarily with non-degenerate dilatations, we have the following corollary of Propositions 2.2.2 and 2.2.3.

**Corollary 2.2.5.** If \( \sigma \) is a dilatation which fixes two points, it is the identity map.

Moving back to the dilatations of \( A_K \), we should verify that the map \( \sigma_{\alpha, \epsilon} \) introduced in the beginning of the section is a dilatation.
Proposition 2.2.6. Let $\alpha \in K$ be a non-zero element, and let $c \in K^2$. Then, the map $\sigma_{\alpha,c} : \mathcal{P} \to \mathcal{P}$ defined by $\sigma_{\alpha,c}(p) = \alpha p + c$ is a dilatation.

Proof. Let $\alpha$ and $c$ be given. It is clear that $\sigma_{\alpha,c}$ is an injection. If $q$ is any point in $A_K$, it has the point $\alpha^{-1}(q - c)$ as a pre-image. Thus, $\sigma_{\alpha,c}$ is a bijection.

To see that it is a dilatation, let $p$ and $q$ be distinct points. Then, we have

\[
\ell(p, q) = p + K(q - p)
\]
\[
\ell(\sigma_{\alpha,c}(p), \sigma_{\alpha,c}(q)) = \alpha p + c + K(\alpha q + c - \alpha p - c)
\]
\[
= \alpha p + c + K(\alpha(q - p))
\]
\[
= \alpha p + c + K(q - p).
\]

But then, the lines $\ell(p, q) \parallel \ell(\sigma_{\alpha,c}(p), \sigma_{\alpha,c}(q))$, by Lemma 2.1.2. Thus, $\sigma_{\alpha,c}$ is a dilatation. \qed

Moving on to translations in the abstract setting, the lack of fixed points in a non-identity translation is a restriction even tighter than the one on dilatations. The action of a translation is uniquely determined by the image of any one point! Before we prove this fact, we need some terminology and a few lemmas.

Definition 2.2.7. Let $\sigma$ be a dilatation and $p$ be a point. A $\sigma$-trace of $p$ is a line containing $p$ and $\sigma(p)$. If $\sigma(p) \neq p$, this trace is unique.

The set of all traces of a dilatation $\sigma$ is written $\text{Tr}(\sigma) := \{l \in \mathcal{L} : p, \sigma(p) \in l \text{ for some } p \in \mathcal{P}\}$.

The following lemma will be used frequently in the discussion of dilatations, as traces come up frequently.

Lemma 2.2.8. Let $\sigma$ be a dilatation, $p$ be a point, and $l$ be a $\sigma$-trace of $p$. If $q \in l$ then $\sigma(q) \in l$. 
2.2. TRANSFORMATIONS OF THE AFFINE PLANE

Proof. Let \( \sigma, p, \) and \( l \) be given by the hypothesis. Let \( q \neq p \) such that \( q \in l \). Then, \( l = \ell(p, q) \). Moreover, \( \sigma(p) \neq \sigma(q) \) and \( \ell(\sigma(p), \sigma(q)) \parallel l \), as \( \sigma \) is a dilatation. But \( \sigma(p) \) lies on both \( \ell(\sigma(p), \sigma(q)) \) and \( l \), so the two lines are equal. We conclude that \( \sigma(q) \in l \). \[ \square \]

Here, we classify the possible traces of a dilatation:

**Proposition 2.2.9.** Let \( \sigma \) be a dilatation. Then, the possibilities of \( \text{Tr}(\sigma) \) are as follows:

1. All lines, if and only if \( \sigma \) is the identity.
2. All lines through a point \( p \) if and only if \( \sigma \) is non-identity and \( p \) is fixed.
3. A pencil of parallel lines if and only if \( \sigma \) is a non-identity translation.

Proof. **First claim.** Let \( \sigma \) be the identity. If \( l \) is any line and \( p \) is any point on that line, \( \sigma(p) = p \in l \) so \( l \in \text{Tr}(\sigma) \). On the other hand, suppose every line is a \( \sigma \)-trace for some point. Let \( l \) be a line, say the \( \sigma \)-trace of \( p \). Let \( m \) be a line through \( p \) which is not parallel to \( l \). Then, \( m \) is a trace of \( \sigma \) as well. By Lemma 2.2.8, \( \sigma(p) \) is on \( m \). But \( \sigma(p) \) is on \( l \) by hypothesis. So \( \sigma(p) \) is in the intersection of \( m \) and \( l \), but \( p \) was defined to be the intersection, so \( p = \sigma(p) \). We may repeat this process for some other point \( q \) on \( l \), and see that \( \sigma(q) = q \). Since \( \sigma \) has two fixed points, it is the identity.

**Second claim.** Suppose \( p \) is fixed by \( \sigma \) where \( \sigma \) is a non-identity dilatation. Let \( l \) be any line through \( p \). Then \( l \) is a \( \sigma \)-trace of \( p \), for it contains both \( p \) and \( \sigma(p) \). On the other hand, fix a point \( p \) and suppose \( \text{Tr}(\sigma) = \{ \ell(p, q) : q \in P \setminus \{p\} \} \). See that by the first statement of this proposition, this dilatation \( \sigma \) is not the identity. So we need only prove that \( \sigma(p) = p \). Towards a contradiction, suppose \( \sigma(p) = r \neq p \). Since affine planes have three non-collinear points, let \( q \) be point not incident to \( l_1 = \ell(p, r) = \ell(p, \sigma(p)) \). Then, the point \( \sigma(q) \in l_2 = \ell(p, q) \),
by definition of a trace. But then \( l_1 = \ell(p, q) = \ell(p, \sigma(q)) \parallel \ell(\sigma(q), \sigma(p)) \), which contradicts the hypothesis that \( \sigma \) is a dilatation.

**Third claim.** Let \( \sigma \) be a non-identity translation. First, we prove that for any points \( p, q \) that \( \ell(p, \sigma(p)) \parallel \ell(q, \sigma(q)) \). If \( q \in \ell(p, \sigma(p)) \) then the lines are parallel by Lemma 2.2.8. So suppose \( q \notin \ell(p, \sigma(p)) \). For the sake of contradiction, suppose the lines are not parallel. Let \( r \) be in their intersection. Since both \( \ell(p, \sigma(p)) \) and \( \ell(q, \sigma(q)) \) are \( \sigma \)-traces of a point, \( \sigma(r) \) lies on both of them, by Lemma 2.2.8. That is, \( r = \sigma(r) \). But \( \sigma \) is a non-identity translation, so it has no fixed points— a contradiction. So the two lines are indeed parallel. Thus, the set \( \text{Tr}(\sigma) \) is contained in some pencil.

Now, let \( l = \ell(p, \sigma(p)) \) be a \( \sigma \)-trace of \( p \). We prove that if \( m \parallel l \), then \( m \in \text{Tr}(\sigma) \). Let \( m \) be parallel to \( l \), let \( q \in m \). For the sake of contradiction, suppose \( \sigma(q) \) is not on \( m \). Then \( \ell(q, \sigma(q)) \) is immediately not parallel to \( m \), and therefore not parallel to \( l \). But all we’ve already shown that all traces are parallel— a contradiction. Thus, we conclude that \( \text{Tr}(\sigma) \) is a pencil of parallel lines. We call this pencil the **direction** of \( \sigma \).

On the other hand, suppose \( \text{Tr}(\sigma) \) is a pencil of parallel lines. If \( \sigma \) fixed one point, then its trace would have a mutual intersection in a point, as shown above. Thus, \( \sigma \) does not fix any points.

Now, we may finally prove that translations are uniquely determined by the image of one point.

**Proposition 2.2.10.** Let \( \tau \) and \( \tau' \) be translations. If \( \tau(p) = \tau'(p) \) for some point \( p \), then \( \tau = \tau' \).

*Proof.* Let \( \tau, \tau' \) be translations, and suppose \( p' = \tau(p) = \tau'(p) \). Let \( l = \ell(p, \tau(p)) = \ell(p, \tau'(p)) \). Let \( q \) be any point not on \( l \). Write \( \tau(q) = q' \) and \( \tau'(q) = q'' \). We will prove \( q' = q'' \). Let \( l' = \ell(q \parallel l) \). Then \( l' \) is a \( \tau \)-trace of \( q \), by Lemma 2.2.8,
so \( q' \in l' \). Likewise, \( q'' \in l' \). Let \( m = \ell(p, q) \) and let \( m' = \ell(p' \parallel m) \). But then by definition of a dilatation, \( \tau(q) \) lies on \( m' \). Likewise, \( q'' \) lies on \( m' \) So \( q' \) and \( q'' \) both lie on the intersection of \( m' \) and \( l' \). So \( q' = q'' \). But \( \tau \) and \( \tau' \) are themselves dilatations, so by Proposition 2.2.2, \( \tau = \tau' \). 

\[ \text{(Figure 2.4: Computing the image of a point not on the } \tau \text{-trace of } p \text{)} \]

Towards constructing the skew field associated to an affine plane, we exhibit some algebraic structure associated with translations and dilatations. The following two propositions are rather important to the discussion for this reason.

**Proposition 2.2.11.** Let \( \mathbb{D} = \{ \sigma : \mathcal{P} \to \mathcal{P} : \sigma \text{ is a dilatation} \} \). Then \( \mathbb{D} \) is a group under function composition.

*Proof.* See that the identity map \( \text{id}_\mathcal{P} \), is trivially a dilatation. First, we prove the composition of dilatations is a dilatation. Let \( \sigma, \sigma' \) be dilatations. Note that their composition is a bijection. Since parallelism is an equivalence relation, we have that

\[
\ell(p, q) \parallel \ell(\sigma'(p), \sigma'(q)) \parallel \ell(\sigma(\sigma'(p)), \sigma(\sigma'(q))).
\]
Therefore, $\sigma \sigma'$ is a dilatation.

Let $\sigma \in D$. Let $\sigma^{-1}$ be its inverse as a function. We prove that $\sigma^{-1} \in D$. Inverses of bijections are themselves bijections, so $\sigma^{-1}$ is a bijection. Then, we have

$$\ell(\sigma^{-1}(p), \sigma^{-1}(q)) \parallel \ell(\sigma(\sigma^{-1}(p)), \sigma(\sigma^{-1}(q))) = \ell(p, q),$$

so $\sigma^{-1}$ is a dilatation.

Since function composition is associative, we conclude that $D$ is a group.

**Proposition 2.2.12.** Let $T = \{\tau : P \to P \mid \tau \text{ is a translation}\}$. Then $T$ is a normal subgroup of $D$.

**Proof.** Note that $T \subseteq D$ and $id_P \in T$ by definition.

Let $\tau$ be a non-identity translation. We prove, $\tau^{-1}$ is a translation. For the sake of contradiction, suppose that $\tau^{-1}$ has a fixed point, $p$. Then $\tau^{-1}(p) = (p)$, which is true if and only if $\tau(p) = p$. But non-identity translations do not have fixed points, so we’ve reached a contradiction. So $\tau^{-1} \in T$.

Let $\tau$ and $\tau'$ be translations. If $\tau \tau'$ has no fixed points, then it is a translation, so suppose $\tau(\tau'(p)) = p$. Then $\tau^{-1}(p) = \tau'(p)$. Since translations are completely determined by the image of one point, we have $\tau' = \tau^{-1}$. Thus $\tau \tau' = id_P$. So whether the composition of translations fixes a point or not, their composition is a translation.

Now, we prove $T$ is a normal subgroup of $D$. Let $\sigma \in D, \tau \in T$. We prove that $\sigma \tau \sigma^{-1} = \tau' \in T$. Once again, if this composition does not have any fixed points, we are done. So suppose $\tau'(p) = p$. Then $\sigma \tau \sigma^{-1}(p) = p$, which means that $\tau(\sigma^{-1}(p)) = \sigma^{-1}(p)$. Therefore, $\sigma^{-1}(p)$ is also a fixed point of $\tau$. Therefore, $\tau = id_P$ and so $\tau' = id_P$. So either $\sigma \tau \sigma^{-1}$ has no fixed points or $\tau = id_P$. Thus, $T$ is a normal subgroup of $D$. \qed

When it is understood that we are working with any abstract affine plane, we
use $\mathbb{T}$ and $\mathbb{D}$ for the groups of translations and dilatations, respectively. When we want to refer to the translations or dilatations of a particular affine plane $\mathcal{G}$, we write $\mathbb{T}(\mathcal{G})$ and $\mathbb{D}(\mathcal{G})$. These next two lemmas are helpful insofar as they assist in the proof that $\mathbb{T}$ is abelian.

**Lemma 2.2.13.** If $\sigma$ is a dilatation and $\tau$ is a non-identity translation, then $\tau$ and $\sigma\tau\sigma^{-1}$ have the same direction. That is, the conjugation action of $\mathbb{D}$ on $\mathbb{T}$ is direction-preserving.

**Proof.** Let $\sigma$ be a dilatation and $\tau$ be a non-identity translation. Let $\pi$ be the pencil which is the direction of $\tau$. Since $\sigma$ is a dilatation, we have

$$\pi \ni \ell(p, \sigma^{-1}(p)) \parallel \ell(\sigma\sigma^{-1}(p), \sigma\tau\sigma^{-1}(p)) = \ell(p, \sigma\tau\sigma^{-1}(p)).$$

Thus, the line $\ell(p, \sigma\tau\sigma^{-1}(p)) \in \pi$ and is a $\sigma\tau\sigma^{-1}$-trace of $p$. So the direction of $\sigma\tau\sigma^{-1}$ is also $\pi$. \hfill $\square$

**Lemma 2.2.14.** Fix a pencil $\pi$. The set $\mathbb{T}_\pi := \{ \tau \in \mathbb{T} \mid \text{Tr}(\tau) = \pi \} \cup \{ \text{id}_P \}$ is a normal subgroup of $\mathbb{D}$.

**Proof.** First, we prove that this set is closed under composition. Suppose $\tau, \tau'$ are translations which have direction $\pi$. Then, the line $\ell(p, \tau'(p)) \in \pi$ (in particular, it is a trace of $\tau$ and $\tau'$) and it contains $\tau'(p)$. Therefore, it contains $\tau(\tau'(p))$, by Lemma 2.2.8. So $\ell(p, \tau'(p)) \in \pi$ is a trace of $\tau\tau'$. Thus, $\tau\tau'$ has direction $\pi$.

We prove that $\text{Tr}(\tau) = \text{Tr}(\tau^{-1})$. We have

$$\ell(p, \tau(p)) = \ell(\tau(p), \tau^{-1}(\tau(p)))$$

is a $\tau$-trace of $p$ and a $\tau^{-1}$-trace of $\tau(p)$. Since the direction of translations are an equivalence class, and the direction of $\tau$ intersects the direction of $\tau^{-1}$ non-trivially,
they have the same direction. The normality of $T_\pi$ in $D$ follows immediately from Lemma 2.2.13.

**Proposition 2.2.15.** If there exist translations with different directions, then $T$ is an abelian group.

**Proof.** Suppose $\tau$ and $\sigma$ are translations that have different directions. Then, by Lemma 2.2.13, the translation $\tau\sigma\tau^{-1}$ has the same direction as $\sigma$, and therefore the same direction as $\sigma^{-1}$, because inverting translations preserves direction. For the sake of contradiction, suppose $\tau\sigma\tau^{-1}\sigma^{-1} \neq \text{id}_P$. Then it has the same direction as $\sigma$, because composing translations with the same direction yields a translation with the same direction. But also, $\tau$ and $\sigma\tau\sigma^{-1}$ have the same direction, so then $\tau\sigma\tau^{-1}\sigma^{-1}$ also has the same direction as $\tau$. But this contradicts our hypothesis. Therefore, it must be the case that $\tau\sigma\tau^{-1}\sigma^{-1} = \text{id}_P$, so $\tau\sigma = \sigma\tau$. Therefore, translations with different directions commute.

Now, suppose $\tau$ and $\sigma$ are translations which have the same direction. Let $\tau_0$ be a translation with a different direction. Then, by the previous argument we see that $\tau\tau_0 = \tau_0\tau$. We can say that $\sigma\tau_0$ has a direction different from that of $\tau$, for otherwise $\sigma\sigma^{-1}\tau_0 = \tau_0$ would have the same direction as $\sigma$. Again, using the previous argument we know that

$$\tau(\sigma\tau_0) = (\sigma\tau_0)\tau = \sigma(\tau_0\tau) = \sigma(\tau_0\tau_0).$$

In other words, $\tau_0\tau = \sigma\tau\tau_0$, which implies that $\tau\sigma = \sigma\tau$. Thus, we conclude that all translations commute as long as there exist translations with different directions. 

It is indeed the case that $T$ is abelian, for translations of different directions exist.

**Corollary 2.2.16.** Let $G$ be an abstract affine plane. Then, $T(G)$ is abelian.
Proof. Let \( p, q, \) and \( r \) be non-collinear points. Recall that for distinct points \( p \) and \( q \), we write the translation mapping \( p \) to \( q \) as \( \tau_{p,q} \). Note that there is a comma here, which is important in distinguishing this from a similar bit of notation which is coming shortly. See that \( \tau_{p,q} \) and \( \tau_{p,r} \) have different directions, as \( \ell(p,q) \nparallel \ell(p,r) \).

By Proposition 2.2.15, \( T(G) \) is abelian.

At last, we have developed the vocabulary to understand Axiom 5, as well as the tools to prove it for \( A_K \).

**Proposition 2.2.17** (Axiom 5). The group of translations of \( A_K \), denoted \( T(A_K) \), acts transitively on \( A_K \).

Proof. For a point \( c \in K^2 \), define \( \tau_c(p) = p + c \). This is not to be confused with \( \tau_{p,q} \), which is differentiated from \( \tau_c \) by the presence of a comma. We prove that this is a translation. If \( c = 0 \) then \( \tau_c = \text{id}_{K^2} \in T(A_K) \). Suppose \( c \neq 0 \). Let \( p \) and \( q \) be distinct points. Then, the line containing \( p \) and \( q \) is given by \( l = p + K(q - p) \).

The line containing \( \tau_c(p) \) and \( \tau_c(q) \) is given by \( l' = p + c + K(q + c - c - p) = p + c + K(q - p) \) and by Lemma 2.1.2, we have \( l \parallel l' \). That is, \( \tau_c \) is a dilatation. It is clear that this map does not fix any points, so \( \tau_c \) is a non-degenerate dilatation, and a translation.

Now, we prove that \( T(A_K) \) indeed acts transitively on \( A_K \). Let \( p, q \in K^2 \). Then,

\[
\tau_{q-p}(p) = p + q - p = q
\]

We may conclude that \( \tau_c \) classifies all translations: if \( \tau \) is a non-identity translation, then surely it maps some point \( p \) to a point \( q \). That is \( \tau = \tau_{q-p} \), and we’ve seen that translations are uniquely determined by the image of one point.
2.3 Trace-preserving homomorphisms

Again, there is a bit of legwork needed to be done before presenting the next axiom, as it requires constructing the skew field to which this paper has previously alluded. The skew field will be the set of homomorphisms of translations whose operations are composition of the functions and composition of the images of the homomorphism. Not only is this skew field important in establishing the axioms for an affine plane, but it is canonically isomorphic to $K!$

We will begin by defining the elements of the field. For trace preserving homomorphisms, we use super-scripts to denote function application. You will see shortly that this notation is somewhat suggestive.

**Definition 2.3.1.** A trace-preserving homomorphism is a function $\varphi : \mathbb{T} \to \mathbb{T}$ such that

1. For all $\tau_1, \tau_2 \in \mathbb{T}$, $(\tau_1 \tau_2)^\varphi = \tau_1^\varphi \tau_2^\varphi$.

2. For all $\tau \in \mathbb{T}$, either $\tau^\varphi = \text{id}_P$ or $\text{Tr}(\tau) = \text{Tr}(\tau^\varphi)$.

Here are some examples of trace-preserving homomorphisms, which we will make use of frequently.

1. The 0 map, which sends all translations to the identity map: $\tau^0 = \text{id}_P$. This is the additive identity of the skew field.

2. The identity map, which will be denoted by 1. That is, $(\tau)^1 = \tau$, for all $\tau$. This is the multiplicative identity of the skew field.

3. The inversion map, denoted by $-1$, which maps each $\tau$ to its inverse $\tau^{-1}$. This requires some verification to see that it is a trace-preserving homomorphism:

$$(\tau_1 \tau_2)^{-1} = \tau_2^{-1} \tau_1^{-1} = \tau_1^{-1} \tau_2^{-1},$$
since $T$ is abelian. The preservation of direction comes from Lemma 2.2.14.

4. Fix a dilatation $\sigma$. The map $\tau \mapsto \sigma \tau \sigma^{-1}$ is a trace preserving homomorphism. While it does not itself get a name, it comes up frequently. In fact, all non-zero trace-preserving homomorphisms are of this form, as we will see later. To see the first condition holds, notice that

$$\sigma(\tau_1 \tau_2)\sigma^{-1} = \sigma \tau_1 \sigma^{-1} \sigma \tau_2 \sigma^{-1}.$$ 

We have already shown that $\tau$ and $\sigma \tau \sigma^{-1}$ have the same direction in Lemma 2.2.13.

Before we fully determine the trace preserving homomorphisms of $A_K$, we ought to exhibit that it is a ring. We will then prove that it is a skew field, using a lemma which classifies all non-zero trace-preserving homomorphisms.

**Definition 2.3.2.** Let $\mathcal{G}$ be an abstract affine plane with translation group $T$. We call the set of trace-preserving homomorphisms

$$TPH(\mathcal{G}) := \{ \varphi : T \to T \mid \varphi \text{ is a trace-preserving homomorphism} \}.$$ 

Let $\varphi, \psi \in TPH(\mathcal{G})$. Define $\varphi + \psi$ by

$$\tau^{\varphi + \psi} = \tau^\varphi \tau^\psi$$

and $\varphi \psi$ by

$$\tau^{\varphi \psi} = (\tau^\psi)^\varphi.$$ 

Notice that $\tau^{\varphi + \psi}$ is still a translation, whose direction is the same as that of $\tau$, by Lemma 2.2.14.
Proposition 2.3.3. \( \text{TPH}(\mathcal{G}) \) is a ring under the addition and multiplication defined above.

Proof. 1. First, we prove that \( \text{TPH}(\mathcal{G}) \) is closed under addition. Let \( \tau_1, \tau_2 \in \mathbb{T} \) and let \( \varphi, \psi \in \text{TPH}(\mathcal{G}) \). See that \( \varphi + \psi \in \text{TPH}(\mathcal{G}) \).

\[
(\tau_1 \tau_2)^{\varphi + \psi} = (\tau_1 \tau_2)^{\varphi}(\tau_1 \tau_2)^{\psi} \\
= \tau_1^{\varphi} \tau_2^{\psi} \tau_1^{\varphi} \tau_2^{\psi} \\
= \tau_1^{\varphi} \tau_2^{\varphi} \tau_1^{\psi} \tau_2^{\psi} = \tau_1^{\varphi + \psi} \tau_2^{\varphi + \psi}.
\]

Now we must verify that \( \varphi + \psi \) preserves trace. For the identity, \( \text{id}^{\varphi}_P = \text{id}^{\psi}_P = 1 \), and so \( \text{id}^{\varphi + \psi}_P = 1 \). On the other hand, let \( \tau \) be a non-identity translation. We know that \( \tau^{\varphi + \psi} = \tau^{\varphi} \tau^{\psi} \). But then each of \( \tau, \tau^{\varphi}, \tau^{\psi} \) have the same direction, and since \( \mathbb{T}_\pi \) is a group, composition of translations with the same direction yields a translation with that same direction. Thus \( \text{Tr}(\tau) = \text{Tr}(\tau^{\varphi} \tau^{\psi}) \), and so \( \varphi + \psi \in \text{TPH}(\mathcal{G}) \).

2. Now, we prove it is closed under multiplication. Again, let \( \tau_1, \tau_2 \in \mathbb{T} \) and let \( \varphi, \psi \in \text{TPH}(\mathcal{G}) \). Observe:

\[
(\tau_1 \tau_2)^{\varphi \psi} = ((\tau_1 \tau_2)^{\psi})^{\varphi} \\
= (\tau_1^{\psi} \tau_2^{\psi})^{\varphi} \\
= (\tau_1^{\psi})^{\varphi}(\tau_2^{\psi})^{\varphi} = \tau_1^{\varphi \psi} \tau_2^{\varphi \psi}.
\]

Moreover, it preserves direction, for \( \tau \) has the same direction as \( \tau^{\psi} \) and since \( \varphi \) preserves direction, then \( \tau \) will also have the same direction as \( (\tau^{\psi})^{\varphi} = \tau^{\varphi \psi} \).

3. Now, we prove additive associativity. Let \( f, g, h \in \text{TPH}(\mathcal{G}) \) and \( \tau \in \mathbb{T} \).
Then,
\[ \tau(f+g+h) = \tau f + g \tau h = \tau f \tau g \tau h = \tau f g + h = \tau f (g+h). \]

4. For additive commutativity, let \( \varphi, \psi \in \text{TPH}(\mathcal{G}), \tau \in \mathbb{T} \). Then,
\[ \tau^{\varphi + \psi} = \tau^{\varphi} \tau^{\psi} = \tau^{\psi} \tau^{\varphi} = \tau^{\psi + \varphi}. \]

5. For the zero element, see that
\[ \tau^{0+\varphi} = \tau^{0} \tau^{\varphi} = id_{\mathbb{T}} \tau^{\varphi} = \tau^{\varphi}. \]

6. To see that every element has an additive inverse, observe
\[ \tau^{\varphi + (-1)\varphi} = \tau^{\varphi} \tau^{(-1)\varphi} = \tau^{\varphi} (\tau^{\varphi})^{-1} = id_{\mathbb{T}} = \tau^{0}. \]

7. Now, we may prove that multiplication is left and right distributive.
Let \( f, g, h \in \text{TPH}(\mathcal{G}) \). We prove that \((g + h)f = gf + hf\):
\[ \tau^{(g+h)f} = (\tau^{f})^{g+h} = (\tau^{f})^{g} (\tau^{f})^{h} = \tau^{gf} \tau^{hf} = \tau^{gf + hf}. \]

On the other hand, we prove that \( f(g + h) = fg + fh\):
\[ \tau^{f(g+h)} = (\tau^{g+h})^{f} = (\tau^{g})^{f} (\tau^{h})^{f} = \tau^{fg} \tau^{fh} = \tau^{fg + fh}. \]

8. Here, we prove multiplicative associativity:
\[ \tau^{(fg)h} = (\tau^{h})^{fg} = ((\tau^{h})^{g})^{f} = (\tau^{gh})^{f} = \tau^{f(gh)}. \]
9. Lastly, we prove the **multiplicative identity:**

\[ \tau^1 \phi = (\tau^1)^1 = \tau \phi, \quad \tau \phi^1 = (\tau^1 \phi) = \tau \phi. \]

So, we conclude that $\text{TPH}(\mathcal{G})$ is a ring. \(\square\)

Since $\mathbb{T}$ is an abelian group, we may view it as a $\mathbb{Z}$-module. Then, $\text{TPH}(\mathcal{G})$ is the endomorphism ring of $\mathbb{T}$, with addition defined pointwise and multiplication given by composition. One may be tempted to invoke Schur’s Lemma, which states that the endomorphism ring of an $R$-module is a skew field, if the $R$-module is simple— that is, if the only proper submodules are the trivial one and $\mathbb{T}$ itself.

However, by Lemma 2.2.14, $\mathbb{T}$ is not a simple $\mathbb{Z}$-module. Thus, we need geometric properties to verify that it is a skew field.

This next theorem is important for constructing multiplicative inverses in $\text{TPH}(\mathcal{G})$. Also, it will completely characterize all non-zero trace-preserving homomorphisms.

**Lemma 2.3.4.** Let $\phi \in \text{TPH}(\mathcal{G})$ be a non-zero element. Let $p \in \mathcal{P}$. Then, there exists a unique dilatation $\sigma$ such that $\sigma(p) = p$ and for all $\tau \in \mathbb{T}, \tau \phi = \sigma \tau \sigma^{-1}$.

**Proof.** First, let $\phi \in \text{TPH}(\mathcal{G})$ and $p \in \mathcal{P}$ be given. Now, we will suppose such a $\sigma$ exists in order to demonstrate its uniqueness, as well as to provide some intuition as to how $\sigma$ acts on a point $q \in \mathcal{P}$. Let $q$ be any point. Consider the translation $\tau_{p,q}$. Then we have $\tau_{p,q} \phi = \sigma \tau_{p,q} \sigma^{-1}$. Then applying this to the point $p$ gives us

\[ \tau_{p,q}^\phi(p) = \sigma \tau_{p,q} \sigma^{-1}(p) = \sigma \tau_{p,q}(p) = \sigma(q). \]

So if the theorem holds, the $\sigma$ we obtain from it must have the following property: to obtain the image of any point $q$, compute $\tau_{p,q}^\phi(p)$. Since $\sigma$ fixes $p$ and the image of a point $q$ is determined this way, the dilatation $\sigma$ must be unique as its image.
is now completely determined.

Now, we can actually prove the proposition, using this insight we’ve gained. Let \( \varphi \in \text{TPH}(\mathcal{G}) \) be non-zero. Let \( p \) be a point. Define \( \sigma : \mathcal{P} \to \mathcal{P} \) by \( \sigma(q) = \tau_{p,q}^\varphi(p). \) First, we verify that this map satisfies the parallel property of dilatations. Let \( q, r \in \mathcal{P} \) be distinct. Note that \( \tau_{q,r} \tau_{p,q} = \tau_{p,r}, \) as translations are completely determined by the image of any one point. Now, applying \( \varphi \) to both sides gives us \( \tau_{q,r}^\varphi \tau_{p,q}^\varphi = \tau_{p,r}^\varphi. \) By applying the function to both sides of this equality to the point, we get the following:

\[
\tau_{p,r}^\varphi(p) = \tau_{q,r}^\varphi(\tau_{p,q}^\varphi(p)) \\
\sigma(r) = \tau_{q,r}^\varphi(\sigma(q)).
\]

Now, we have some more information with which to prove \( \sigma \) satisfies \( \ell(q, r) \parallel \ell(\sigma(q), \sigma(r)). \) Let \( l = (\sigma(q) \parallel \ell(q, r)). \) Since \( l \parallel \ell(q, r), \) it is a trace of \( \tau_{q,r} \) and therefore a trace of \( (\tau_{q,r})^\varphi. \) By Lemma 2.2.8 \( (\tau_{q,r})^\varphi(\sigma(q)) \in l. \) But this point is equal to \( \sigma(r). \) So \( \ell(\sigma(q), \sigma(r)) = l \parallel \ell(q, r), \) as desired.

Now, we prove \( \sigma \) fixes \( p. \) We have \( \sigma(p) = \tau_{pp}^\varphi(p) = \text{id}_p^\varphi(p) = \text{id}_p(p) = \text{id}_p. \) So \( \sigma \) fixes \( p. \)

Now, we will prove that \( \varphi \neq 0 \) implies that \( \sigma \) is a bijection. For the contrapositive, suppose \( \sigma \) is degenerate. Then every point is mapped to \( p, \) and then we have \( p = \tau_{pq}^\varphi(p), \) by definition of \( \sigma. \) In other words, for any \( q, \) \( \tau_{pq}^\varphi = \text{id}_p. \) Any translation can be written as \( \tau_{p,s} \) for some point \( s, \) so we have characterized the image of every translation under \( \varphi. \) That is, for all \( \tau \in \mathbb{T}, \) \( \tau^\varphi = \text{id}_p. \) In other words \( \varphi = 0. \) That is, if \( \sigma \) is degenerate, \( \varphi = 0. \) The contrapositive of this allows us to conclude that if \( \varphi \neq 0, \) then \( \sigma \) is not degenerate. So \( \sigma \) is a non-degenerate dilatation, as desired.

Lastly, we prove the last statement of the theorem: for all \( \tau \in \mathbb{T}, \) \( \tau^\varphi = \sigma \tau \sigma^{-1}. \)
Since $\sigma$ fixes $p$, we can rewrite the definition of $\sigma$ as, for any point $q$

$$\sigma(q) = \tau_{p,q}^\varphi(\sigma(p)).$$

We can rewrite this as

$$q = \sigma^{-1}(\tau_{p,q}^\varphi(\sigma(p))).$$

So this is a translation which moves $p$ into $q$, which tells us that

$$\sigma^{-1}\tau_{p,q}^\varphi \sigma = \tau_{p,q}.$$

Again, rewriting, we have that

$$\tau_{p,q}^\varphi = \sigma\tau_{p,q}\sigma^{-1}.$$

However, as we previously stated, every translation can be characterized by $\tau_{p,x}$ for some point $x$. Therefore, this result we’ve proved for $\tau_{p,q}$ holds for all translations $\tau$. So we can conclude that for all $\tau \in \mathbb{T}$, $\tau^\varphi = \sigma\tau\sigma^{-1}$. This concludes the proof of the lemma.

This result allows us to construct multiplicative inverses in $\text{TPH}(G)$.

**Proposition 2.3.5.** $\text{TPH}(G)$ is a skew field.

**Proof.** Let $\varphi \in \text{TPH}(G)$ be a non-zero trace-preserving homomorphism. Then, by Lemma 2.3.4, there is a unique $\sigma$ such that for all translations $\tau$, $\tau^\varphi = \sigma\tau\sigma^{-1}$. Then, define a map, which we will call $\varphi^{-1}$, by $\tau \mapsto \sigma^{-1}\tau\sigma$. This is indeed our multiplicative inverse:

$$\tau^{\varphi^{-1}} = (\tau^{\varphi^{-1}})^\varphi = \sigma(\sigma^{-1}\tau\sigma)\sigma^{-1} = \tau = \tau^1.$$
2.3. TRACE-PRESERVING HOMOMORPHISMS

It is also a left inverse:

\[ \tau^{\varphi^{-1} \varphi} = (\tau^\varphi)^{\varphi^{-1}} = \sigma^{-1}(\sigma \tau \sigma^{-1})\sigma = \tau = \tau_1 \]

Now, we may also apply Lemma 2.3.4 to prove that Axiom 6 is equivalent to Axiom 6b. Depending on the situation, it may be more convenient to use Axiom 6 or Axiom 6b. This gives an easier way of showing that \( A_\mathcal{K} \) is an affine plane.

Recall the statements of Axioms 6 and 6b:

**Axiom.** (6) For any pencil \( \pi \), \( TPH(\mathcal{G}) \) acts transitively on \( \mathbb{T}_\pi \setminus \{\text{id}_P\} \).

**Axiom.** (6b) For any three collinear, pair-wise distinct points \( p, q, \) and \( r \) there is a dilatation \( \sigma \) such that \( \sigma(p) = p \) and \( \sigma(q) = r \).

**Theorem 2.3.6.** Axiom 6 is equivalent to Axiom 6b.

**Proof.** First, we prove that Axiom 6b implies Axiom 6. Let \( \tau_1, \tau_2 \) be distinct non-identity translations with the same direction. Let \( p \in \mathcal{P} \) be any point, and say \( \tau_1(p) = q \) and \( \tau_2(p) = r \). See that \( p, q, \) and \( r \) are collinear, for \( \tau_1 \) and \( \tau_2 \) have the same direction. Let \( \sigma \) be the dilatation fixing \( p \) and mapping \( q \) to \( r \). Define \( \varphi : \tau \mapsto \sigma \tau \sigma^{-1} \). We have already shown that such a map is a trace-preserving homomorphism. Now, see that \( \tau_1^{\varphi} = \tau_2^{\varphi} \):

\[ \sigma \tau_1 \sigma^{-1}(p) = \sigma \tau_1(p) = \sigma(q) = r. \]

Then, the map \( \varphi : \tau \mapsto \sigma \tau \sigma^{-1} \) is a trace-preserving homomorphism with \( \tau_2 = \tau_1^{\varphi} \).

Now, suppose Axiom 6 holds. Let \( p, q, \) and \( r \) be distinct collinear points. Then, \( \tau_{p,q} \) and \( \tau_{p,r} \) have the same direction, but are distinct and both non-identity. So there exists a \( \varphi \in TPH(\mathcal{G}) \) such that \( \tau_{p,q}^{\varphi} = \tau_{p,r} \). By Lemma 2.3.4, there is a
(unique) dilatation \( \sigma \) such that \( \sigma(p) = p \) and for all \( \tau, \tau' = \sigma \tau \sigma^{-1} \). Now, to see \( \sigma(q) = r \), observe
\[
\tau_{p,r} = \tau_{p,q} = \sigma \tau_{p,q} \sigma^{-1}.
\]
This gives us that \( \tau_{p,r} \sigma = \sigma \tau_{p,q} \). In other words,
\[
\tau_{p,r} \sigma(p) = \sigma \tau_{p,q}(p) = r = \sigma(q).
\]
So \( \sigma \) witnesses Axiom 6b. \( \square \)

Now, to show that \( A_K \) satisfies Axiom 6, we prove it satisfies Axiom 6b.

**Proposition 2.3.7** (Axiom 6b). For any three collinear, pairwise distinct points \( p, q, r \in A_K \), there is a unique dilatation which fixes \( p \) and maps \( q \) to \( r \).

**Proof.** Write the line containing \( p, q \) and \( r \) as \( l = p + K(q - p) \). Then, there is some \( \alpha \in K \) such that \( r = p + \alpha(q - p) \). This \( \alpha \) is non-zero, as \( p \) and \( r \) were chosen to be distinct. Then, the map
\[
x \mapsto \alpha x + p - \alpha p
\]
is a dilatation by Proposition 2.2.6, as this map is \( \sigma_{\alpha,(p-\alpha p)} \). It fixes \( p \) and maps \( q \) to \( r \) as desired. The uniqueness follows from the fact that dilatations are uniquely determined by the image of two points. \( \square \)

We may conclude that any dilatation which fixes a point is of the form \( \sigma_{\alpha,e} \), for if an arbitrary dilatation \( \sigma \) fixes a point \( p \), then surely it maps a point \( q \) which lies on a line through \( p \) to another point, which must lie on that line. Now, we have the following straightforward notation for our dilatations and translation groups:

\[
\mathbb{D}(A_K) = \{ \sigma_{\alpha,e} : \alpha \in K, \alpha \neq 0, e \in A_K \},
\]
\[ \mathbb{T}(A_K) = \{ \tau_c : c \in A_K \}. \]

To complete our understanding of \( A_K \), it is necessary to fully determine its trace-preserving homomorphisms. We expect it to be isomorphic to \( K \), as we are hoping that what we are doing is canonical. Let \( \varphi \) be any trace-preserving homomorphism of \( \mathbb{T}(A_K) \). We will apply Lemma 2.3.4 here. Consider the origin, \( o = (0, 0) \). Then, there is a unique dilatation \( \sigma \) which fixes the origin and such that for all \( \tau \in \mathbb{T}(A_K) \), \( \tau^\varphi = \sigma \tau \sigma^{-1} \). In other words, any non-zero trace-preserving homomorphism is of the form \( \tau \mapsto \sigma \tau \sigma^{-1} \). However, we can be more specific. Any dilatation which fixes the origin is of the form \( p \mapsto \alpha p, \) for some non-zero \( \alpha \in K \). Such a dilatation is uniquely determined by \( \varphi \), and therefore \( \alpha \) is uniquely determined by \( \varphi \). Note that \( \sigma^{-1}(p) = \alpha^{-1}p \). Thus, for any translation \( \tau_c \), we can compute its image:

\[
\begin{align*}
\sigma \tau_c \sigma^{-1}(p) &= \sigma \tau_c (\alpha^{-1}p) \\
&= \sigma (\alpha^{-1}p + c) \\
&= \alpha (\alpha^{-1}p + c) \\
&= p + \alpha c = \tau_{\alpha c}(p).
\end{align*}
\]

Therefore, \( \tau_c^\varphi = \tau_{\alpha c} \). The 0 trace-preserving homomorphism can be realized by setting \( \alpha = 0 \). To make notation more clear, for \( \alpha \in K \), define \( \varphi_\alpha \) by \( \tau_c \mapsto \tau_{\alpha c} \).

In summary, we have \( \text{TPH}(A_K) = \{ \varphi_\alpha : \alpha \in K \} \). This allows us to proceed with the following

**Theorem 2.3.8.** Let \( K \) be a skew field. Then, \( \text{TPH}(A_K) \cong K \).

**Proof.** For ease of notation, we switch to prefix notation for application of trace-preserving homomorphisms. Define \( \varphi_\alpha : K \to \text{TPH}(A_K) \) by \( \alpha \mapsto \varphi_\alpha \). First, see that this is a surjection, as justified above. This is an injection as well, for \( \varphi_\alpha = \varphi_\beta \)
if and only if $\alpha = \beta$, by definition of $\tau_p$ for a point $p$.

Now, we prove that $\varphi_\bullet$ respects addition and multiplication. For addition, see that $(\varphi_\alpha + \varphi_\beta)(\tau_c) = \tau_{\alpha \epsilon + \beta \epsilon}$.

Then, we have

$$
\begin{align*}
\tau_{\alpha \epsilon + \beta \epsilon}(p) &= \tau_{\alpha \epsilon}(p + \beta \epsilon) \\
&= p + \beta \epsilon + \alpha \epsilon \\
&= \tau_{\alpha + \beta \epsilon}(p).
\end{align*}
$$

For multiplication, we have $\varphi_\alpha \varphi_\beta(\tau_c) = \tau_{\alpha \beta \epsilon} = \varphi_{\alpha \beta}(\tau_c)$. Thus, $\varphi_\bullet$ is an isomorphism witnessing $\text{TPH}(A_K) \cong K$. 

2.4 Constructing a skew field from an affine plane

We have seen that for any skew field, there is a corresponding affine plane. We have also shown, by the construction of trace-preserving homomorphisms, that for any abstract affine plane, there is a corresponding skew field. A reasonable question to ask is whether this is unique or canonical in any way (up to isomorphism, of course). In other words, for an abstract affine plane $\mathcal{G}$, is it the case that $\mathcal{G} \cong A_{\text{TPH}(\mathcal{G})}$? Here, we will address that. For this section, fix an abstract affine plane $\mathcal{G} = (P, \mathcal{L})$. We begin by introducing a way in which to generate $T(\mathcal{G})$, the translations of $\mathcal{G}$. Before that, a brief lemma.

**Lemma 2.4.1.** Let $\tau$ be a translation and let $\alpha$ and $\beta$ be trace-preserving homomorphisms. If $\tau^\alpha = \text{id}_P$, then $\alpha = 0$ or $\tau = \text{id}_P$. If $\tau^\alpha = \tau^\beta$, then $\tau = \text{id}_P$ or $\alpha = \beta$.

**Proof.** Let $\tau, \alpha$, and $\beta$ be given. Suppose $\tau^\alpha = \text{id}_P$ and $\alpha \neq 0$. Then,

$$
\tau = (\tau^\alpha)^{-1} = \text{id}_P^{-1} = \text{id}.
$$
Now suppose $\tau^\alpha = \tau^\beta$. Then, $\tau^{\alpha-\beta} = \text{id}_P$ and so $\alpha = \beta$ or $\tau = \text{id}_P$, as desired.

**Proposition 2.4.2.** Let $\tau_1, \tau_2$ be non-identity translations with different directions. Then, for any translation $\tau$, there exist unique trace preserving homomorphisms $\varphi, \psi$ such that

$$\tau = \tau_1^\varphi \tau_2^\psi = \tau_2^\psi \tau_1^\varphi.$$

**Proof.** Let $\tau_1, \tau_2$ be given. Let $\tau$ be any translation. Say $\tau = \tau_{p,q}$ for some points $p$ and $q$. Consider the $\tau_1$-trace through $p$ which we call $l_1$, and the $\tau_2$-trace through $p$ which we call $l_2$. The translations $\tau_1$ and $\tau_2$ have different directions, so $l_1$ and $l_2$ intersect in a point, say $r$. Since $p$ and $r$ are both incident with $l_1$, the translation $\tau_{p,r}$ is either the identity, or has the same direction as $\tau_1$. Likewise, since $q$ and $r$ both lie on $l_2$, the translation $\tau_{r,q}$ is either the identity, or has the same direction as $\tau_2$. In any case, by Axiom 6 there exist trace preserving homomorphisms $\varphi$ and $\psi$ such that $\tau_{p,r} = \tau_1^\varphi$ and $\tau_{r,q} = \tau_2^\psi$. Then, we conclude that

$$\tau_{p,q} = \tau_{r,q} \tau_{p,r} = \tau_2^\psi \tau_1^\varphi = \tau_1^\varphi \tau_2^\psi.$$

Note that these translations commute, as $\mathbb{T}$ is abelian so long as translations of different directions exist.

Now, we prove that the trace-preserving homomorphisms $\varphi, \psi$ are unique. Suppose $\tau_1^\varphi \tau_2^\psi = \tau_1^\alpha \tau_2^\beta$. Then, we have $\tau_1^{\varphi-\alpha} = \tau_2^{\beta-\psi}$. If these translations are non-identity, then they have the same directions. But these are trace-preserving homomorphisms, so this cannot be the case. Thus, $\tau_1^{\varphi-\alpha} = \tau_2^{\beta-\psi} = 1$, and thus we have $\varphi = \alpha$ and $\psi = \beta$.

This notion of building translations using two fixed translations and trace preserving homomorphisms is the basis for coordinatizing our abstract affine plane. Fix some point $o$ which will denote the origin. Fix non-identity translations $\tau_1, \tau_2$ with different directions. View the $\tau_1$ and $\tau_2$-traces of $o$ as the coordinate axes.
To get the coordinates of some point \( p \), write \( \tau_{o,p} = \tau_1^\alpha \tau_2^\beta \). The coordinates of \( p \) are then \((\alpha, \beta)\).

By this method, we see that the origin has coordinates \((0, 0)\). The “unit points” on our axes are given by \( \tau_1(o) \) and \( \tau_2(o) \), and have coordinates \((1, 0)\) and \((0, 1)\) respectively.

Now, we may parametrize lines. Consider a line \( l \), and a point \( p \) with coordinates \((\alpha, \beta)\). Let \( \tau \) be a non-identity translation which has \( l \) as a trace; write \( \tau = \tau_1^\gamma \tau_2^\delta \). For any trace preserving homomorphism \( \nu \), there is a point \( q \in l \) such that \( \tau'^\nu = \tau_{p,q} \). To see this, notice that \( \tau \) and \( \tau'^\nu \) have the same direction. Therefore, \( l \) is a trace of \( \tau'^\nu \) and so \( \tau'^\nu(p) = q \), for some point \( q \in l \). On the other hand, for any point \( q \in l \), we know the translation \( \tau_{p,q} = 1 \) or has the same direction as \( \tau \). Then, there is some trace-preserving homomorphism \( \nu \) such that \( \tau_{p,q} = \tau'^\nu \).

Combining these facts, the coordinates of a point \( q \in l \) is given by computing \( \tau_{o,q} \). Let \( \nu \) be the trace-preserving homomorphism such that \( \tau_{p,q} = \tau'^\nu \). Then

\[
\tau_{o,q} = \tau_{p,q} \tau_{o,p} = (\tau_1^\gamma \tau_2^\delta \tau_1^\alpha \tau_2^\beta)^\nu = \tau_1^{\nu\gamma + \alpha} \tau_2^{\nu\delta + \beta}.
\]

Then \( l = \{(\nu\gamma + \alpha, \nu\delta + \beta) : \nu \in \text{TPH}(\mathcal{G})\} \). We can then conclude that any line is of this form, so long as \( \gamma \) and \( \delta \) are not both zero (in that case, the \( \tau \) we chose would be the identity).

If we now let \( a = (\gamma, \delta) \) and \( b = (\alpha, \beta) \), we have a more traditional view of lines, which should be very familiar: \( l = a + \text{TPH}(\mathcal{G})b \).

Now that we have a well-defined way of moving from an abstract affine plane to a skew field, and vice-versa, we should show that we do so in a unique way. First, we should have a meaningful notion of what it means for affine planes to be equivalent. Seeing as affine planes are merely points, lines, and a binary relation, our morphism need only preserve the incidence relation.
**Definition 2.4.3.** Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}), \mathcal{G}' = (\mathcal{P}', \mathcal{L}')$ be abstract affine planes. An isomorphism from $\mathcal{G}$ to $\mathcal{G}'$ is a pair of bijections $f_P : \mathcal{P} \to \mathcal{P}'$ and $f_\ell : \mathcal{L} \to \mathcal{L}'$ such that for all $x \in \mathcal{P}$ and all $l \in \mathcal{L}$, we have $x \in l$ if and only if $f_P(x) \in f_\ell(l)$.

If such an isomorphism exists, we say $\mathcal{G} \cong \mathcal{G}'$.

Finally, we may assert:

**Theorem 2.4.4.** Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be an abstract affine plane. Then, $\mathcal{G} \cong A_{\text{TPH}(\mathcal{G})}$.

**Proof.** Let $K$ denote the skew field of trace-preserving homomorphisms. Fix a point $o \in \mathcal{P}$, and let $\tau_1$ and $\tau_2$ be translations in $\mathcal{T}(\mathcal{G})$ with different directions. Then, define $f_P : \mathcal{P} \to A_K$ using the coordinatization described above. That is, for any point $p \in \mathcal{P}$, there exist unique $\varphi, \psi$ such that $\tau_{o,p} = \tau_1^\varphi \tau_2^\psi$. Then, $f_P(p) = (\varphi, \psi)$. This is well-defined by the uniqueness described in Proposition 2.4.2. It is also an injection: let $p$ and $q$ be distinct points in $\mathcal{P}$. If $f_P(p) = f_P(q)$, we would have $\tau_{o,p} = \tau_{o,q}$, which would imply that $p = q$ as translations are completely determined by their action on one point. To show surjectivity, let $(\alpha, \beta)$ be any point in $A_K$. Then, let $p = (\tau_1^\alpha \tau_2^\beta)(o)$, and see that $f_P(p) = (\alpha, \beta)$.

Now, we define our line map. Define $f_\ell : \mathcal{L} \to L_K$ as follows. Let $l \in \mathcal{L}$ be any line. Then, $l = \ell(p, q)$ for some points $p, q \in l$. Write $p = f_P(p)$ and $q = f_P(q)$. Define $f_\ell(l) = p + K(q - p)$. Before we prove that this is a bijection, we will prove that this is indeed a morphism. That is, we prove that $p \in l$ if and only if $f_P(p) \in f_\ell(l)$.

Let $p \in l$. We prove $f_P(p) \in f_\ell(l)$. Fix a point $q \in l$. Then, $f_\ell(l) = p + K(q - p)$, where $p = f_P(p)$ and $q = f_P(q)$. Then, $p$ is clearly an element of $f_\ell(l)$. On the other hand, suppose $f_P(p) \in f_\ell(l)$. We may parametrize $f_\ell(l) = p + K(q - p)$. Again, it is clear that $p \in l$.

Now, we proceed with the proof that $f_\ell$ is bijective. For the proof of injectivity, let $f_\ell(l) = f_\ell(m)$. Then, the points $p, q$ are on $f_\ell(m)$ if and only if they are on
$f_\ell(l)$. But $f_\ell$ is a morphism, so the pre-images of $p$ and $q$ are both on $l$ and $m$. Since two points determine a line, we conclude $l = m$. To see that $f_\ell$ is surjective, let $l = p + Kq$ be some line in $L_K$. Say $p = (\alpha, \beta)$ and $q = (\gamma, \delta)$. Then, the points $p = (\alpha, \beta)$ and $(p + q) = (\alpha + \gamma, \beta + \delta)$ both lie on $l$. Define points in $\mathcal{P}$ by $p = (\tau_1^\alpha, \tau_2^\beta)(o)$ and $r = (\tau_1^{\alpha+\gamma}, \tau_2^{\beta+\delta})(o)$. Then, we have $f_\ell(\ell(p, r)) = l$, as desired. Therefore, $\mathcal{G} \cong A_{\text{TPH}(\mathcal{G})}$.

This shows us that our constructs were the right ones. That is, we have shown that every affine plane can be viewed as an affine plane over some skew field. This is a bridge from an axiomatic, synthetic view to a constructive view of geometry. In lieu of constructing arguments about affine planes by pulling out drafting equipment and drawing points and lines, we can instead reason about affine planes using the vast pool of knowledge mathematicians have developed studying algebra.
Chapter 3

Projective Planes

The other geometric objects of study here are projective planes. They can be thought of as a “completion” of an affine plane, in that an affine plane is missing the points at which parallel lines intersect. That is to say, every pair of lines in a projective plane intersect non-trivially. There are no such things as parallel lines in a projective plane.

We will first give an abstract notion of a projective plane. Then, as in the previous section, we will give a concrete, common example of a projective plane—a projective plane based on a skew field. We will then see that the affine plane of a skew field, $K$ is the same as the affine plane resulting from removing a line from the projective plane based on a skew field. The chapter will conclude by discussing how a Desarguesian projective plane relates to an affine plane.

Unlike with the first chapter of the thesis, the sources of the results in this chapter are more eclectic. We will give sources for the results whose proofs are more involved than simple linear algebra exercises.
Definition 3.0.1. Let $\Pi$ and $\Lambda$ be sets, whose elements are called points and lines respectively. Let $\varepsilon$ be a binary relation on $\Pi \times \Lambda$. Then, the triple $G = (\Pi, \Lambda, \varepsilon)$ is an abstract projective plane if it satisfies the following three axioms:

1. For all distinct $x, y \in P$, there is a unique $l \in L$ such that $x \in l$ and $y \in l$.
   Like with affine planes, we denote this $\ell(x, y)$.

2. For all distinct $l, l' \in L$, there is a unique $x \in P$ such that $x \in l$ and $x \in l'$.

3. There exist four distinct points in $P$ such that no line contains any three of them.

There is a fourth property which some projective planes have. Traditionally, it is known as Desargues' Theorem. However, it is not true for every projective plane. Every projective plane based on an skew field is Desarguesian, which we will see in the next section. However, there exist projective planes which are not Desarguesian. We will construct such a projective plane once we develop a method of turning an affine plane into a projective plane.

Definition 3.0.2. Let $G = (\Pi, \Lambda, \varepsilon)$ be an abstract projective plane. Let $q, q', r, r', s,$ and $s'$ be pairwise distinct points such that $\ell(q, q'), \ell(r, r')$, and $\ell(s, s')$ are distinct lines which meet in a point $p$. Suppose further that $p$ is distinct from all of the other points. Let $a = \ell(q, r) \cap \ell(q', r')$, $b = \ell(q, s) \cap \ell(q', s')$ and $c = \ell(r, s) \cap \ell(r', s')$. If $a, b,$ and $c$ are collinear, then we say $G$ is a Desarguesian projective plane, and that it satisfies Desargues’ Theorem.
3.1. Constructing a Projective Plane from a Skew Field

Like in the case of affine planes, we can construct a projective plane from an arbitrary skew field. Let $K$ be a skew field. We define the triple $G_K = (\Pi_K, \Lambda_K, \varepsilon)$ as follows: Let $\Pi_K = \{Kx : x \in K^3\}$, the set of one-dimensional subspaces of $K^3$. Let $\Lambda_K = \{Kx + Ky : x, y \in K^3\}$, the set of all two dimensional subspaces of $K^3$. Lastly, the $\varepsilon$ relation is simply the subset relation, $\subset$. Not only is this construction a projective plane, but it is Desarguesian.
Proposition 3.1.1 (Wylie). Let $K$ be a skew field. Then, $G_K$ is a Desarguesian projective plane.

Proof. 1. Let $Kx$ and $Ky$ be distinct points. Then, $x$ and $y$ are linearly independent, so $Kx + Ky$ is a two-dimensional subspace containing $x$ and $y$.

2. Let $l = Ka + Kb$ and $l' = Kc + Kd$ be distinct lines. These are distinct subspaces of $K^3$ with dimension two, so their intersection is a subspace of dimension one—namely, it is an element of $\Pi_K$.

3. Consider the vectors $x_1 = (1, 0, 0), x_2 = (0, 1, 0), x_3 = (0, 0, 1)$, and $x_4 = (1, 0, 1)$. Notice that any three of these vectors are linearly independent. Thus, a subspace of $K^3$ containing any 3 of these vectors is a 3-dimensional space, and thus is not a line in $\Lambda_K$.

4. Let $Kq, Kq', Kr, Kr', ks, ks' \in \Pi_K$ be pairwise distinct points such that $l_1 = Kq + Kq', l_2 = Kr + Kr'$ and $l_3 = Ks + Ks'$ meet in a point $Kp$, which is distinct from the other points. Let $Ka = (Kq + Kr) \cap (Kq' + Kr')$, $Kb = (Kq + Ks) \cap (Kq' + Ks')$ and $Kc = (Kr + Ks) \cap (Kr' + Ks')$. To verify that $G_K$ is a Desarguesian projective plane, we prove that $Ka, Kb$, and $Kc$ are collinear. That is, we prove that $Kc \subset Ka + Kb$. To do so, we will verify that there exist $\alpha, \beta \in K$ such that $c = \alpha a + \beta b$. First, since $Kp$ lies on $l_1, l_2$, and $l_3$, there exists $\lambda_i \in K$ such that

$$p = \lambda_q q + \lambda_{q'} q' = \lambda_r r + \lambda_{r'} r' = \lambda_s s + \lambda_{s'} s'.$$

By rewriting, we have that

$$\lambda_r r - \lambda_s s = -\lambda_r r' + \lambda_{s'} s'.$$
3.2. DESARGUES’ THEOREM FOR AFFINE PLANES

See that $\lambda_r r - \lambda_s s$ is not the zero vector, for otherwise $K r = K s$, a contradiction. Thus, $K(\lambda_r r - \lambda_s s) \subset (Kr + Ks) \cap (Kr' + Ks')$. But the intersection of lines is unique, so $Ka = K(\lambda_r r - \lambda_s s)$. A similar demonstration shows that $Kb = K(\lambda_q q - \lambda_s s)$ and $Kc = K(\lambda_q q - \lambda_r r)$. See that $a = b - c$. In other words, $Ka \subset Kb + Kc$. So $Ka$, $Kb$, and $Kc$ are collinear.

\[ \square \]

3.2 Desargues’ Theorem for Affine Planes

There is also an affine interpretation of Desargues’ Theorem. Since there is a notion of parallelism, which does not exist in projective planes, there are two circumstances to consider when establishing the hypothesis of Desargues’ Theorem.

**Theorem 3.2.1** (Desargues’ Theorem for parallel lines in an affine plane). Let $\mathcal{G} = (\mathcal{P}, \mathcal{L})$ be an abstract affine plane. Let $l_1, l_2$, and $l_3$ be distinct parallel lines. Let $q, q' \in l_1, r, r' \in l_2$, and $s, s' \in l_3$ be pairwise distinct points. Suppose that $\ell(q, r) \parallel \ell(q', r')$ and $\ell(q, s) \parallel \ell(q', s')$. Then, $\ell(r, s) \parallel \ell(r', s')$. 

Theorem 3.2.2 (Desargues’ Theorem for lines which meet in a point). Let \( \mathcal{G} = (\mathcal{P}, \mathcal{L}) \) be an abstract affine plane. Let \( l_1, l_2, \) and \( l_3 \) be distinct lines which all intersect in a point \( p \). Let \( q, q' \in l_1, r, r' \in l_2, \) and \( s, s' \in l_3 \) be points pairwise distinct from each other and distinct from \( p \). Suppose that \( \ell(q, r) \parallel \ell(q', r') \) and \( \ell(q, s) \parallel (q', s') \). Then, \( \ell(r, s) \parallel \ell(r', s') \).
Assuming the first four axioms of an affine plane, these two theorems above are equivalent to Axioms 5 and 6b for affine planes, respectively. That is, we can look at them as geometric interpretations of Axioms 5 and 6, which themselves seem more algebraic than geometric in nature. Moreover, it provides us with a way of confirming that we can make a projective plane into an affine plane.

Proof of equivalence of Desargues’ theorem for parallel lines and Axiom 5, Artin.

Suppose Axiom 5 holds. That is, for every pair of points $p, q \in \mathcal{P}$, there is a translation $\tau_{p,q}$ such that $\tau_{p,q}(p) = q$. Suppose the hypothesis of Desargues’ Theorem for parallel lines. Let $\tau = \tau_{q,q'}$ be the translation which maps $q$ to $q'$. Then, $l_1$ is a trace of $\tau$. Since $l_2$ and $l_3$ are parallel to $l_1$, they are also traces of $\tau$. So by Lemma 2.2.8, $\tau(r) \in l_2$. By the properties of translations, we also have that

$$\ell(q, r) \parallel \ell(\tau(q), \tau(r)) = \ell(q', \tau(r)) = \ell(q' \parallel \ell(q, r)) = \ell(q', r').$$
That is, \( r' \) and \( \tau(r) \) are in the intersection of \( l_2 \) and \( \ell(q', r') \), and so \( r' = \tau(r) \). We also see that \( s' = \tau(s) \) by similar reasoning. Then, by the properties of translations we have
\[
\ell(r, s) \parallel \ell(\tau(r), \tau(s)) = \ell(r', s').
\]

Now, let us suppose Desargues’ Theorem for parallel lines holds. We must first handle an edge case – suppose every line contains only two points. Then, by Proposition 2.1.11, there are two lines in a given pencil. Thus, there are four points in all. Lines in an affine plane are uniquely determined by two points, so there is only one geometry satisfying the property that every line has exactly two points. So this geometry must be \( A_K \), where \( K \) is the field with two elements. We have already shown that this satisfies Axiom 5.

Suppose every line contains at least three points. Then, every pencil has at least 3 lines in it. Let \( q, q' \) be distinct points. Let \( l_1 = \ell(q, q') \). Define a map \( \tau^q : P \setminus l_1 \to P \setminus l_1 \) by \( \tau^q(x) = \ell(x \parallel l_1) \cap \ell(q' \parallel \ell(q, x)) \). Fix an arbitrary point \( r \notin l_1 \). Let \( r' = \tau^q(r) \), let \( l_2 = \ell(r, r') \) and define \( \tau^r : P \setminus l_2 \to P \setminus l_2 \) by \( \tau^r(x) = \ell(x \parallel l_2) \cap \ell(r' \parallel \ell(r, x)) \). Lastly, fix a point \( s \notin l_1 \cup l_2 \). Let \( s' = \tau^q(s) \) and \( l_3 = \ell(s, s') \). Define a map \( \tau^s : P \setminus l_3 \to P \setminus l_3 \) by \( \tau^s(x) = \ell(x \parallel l_3) \cap \ell(s' \parallel \ell(s, x)) \).

Now, we prove \( \tau^q = \tau^r \) on all points for which they are defined. Since \( s \) is simply an arbitrary point in the domain of both \( \tau^q \) and \( \tau^r \), we will prove \( s' = \tau^r(s) \). See that \( l_1 \parallel l_2 \parallel l_3 \), by definition of \( \tau^q \). They are also pairwise distinct by choice of \( r \) and \( s \). Moreover, \( \ell(q, r) \parallel \ell(q', r') \) and \( \ell(q, s) \parallel \ell(q', s') \). By Desargues’ Theorem for parallel lines, we have \( \ell(r, s) \parallel \ell(r', s') \). This fact, in conjunction with \( \ell(r, r') \parallel \ell(s, s') \) gives us that \( s' = \tau^r(s) = \tau^q(s) \). A similar proof can show that \( \tau^q \) agrees with \( \tau^s \) on all points for which they are defined, using \( r \) instead of \( s \). Likewise, we can show that \( \tau^r \) agrees with \( \tau^s \) at all points for which they are
both defined. Thus, we can define a map \( \tau : \mathcal{P} \rightarrow \mathcal{P} \) by

\[
\tau(x) = \begin{cases} 
\tau^q(x) & \text{if } x \notin l_1 \\
\tau^r(x) & \text{if } x \notin l_2 \\
\tau^s(x) & \text{if } x \notin l_3.
\end{cases}
\]

If we verify that \( \tau(q) = q' \), and that \( \tau \) is a dilatation, we can conclude that \( \tau \) is a translation, for we see that the traces are parallel lines and so no point is fixed.

To see that \( \tau(q) = q' \), observe

\[
\tau(q) = \tau^r(q) = \ell(q \parallel l_2) \cap \ell(r', q) = l_1 \cap \ell(r', q') = q'.
\]

Lastly, we prove \( \tau \) is a dilatation. Let \( u \) and \( v \) be distinct points. Write \( u' = \tau(u), v' = \tau(v) \). Without a loss of generality, suppose \( u \) and \( v \) are both not incident to \( l_1 \). If \( \ell(u, v) \parallel l_1 \), then \( u', v' \in \ell(u, v) \), so \( \ell(u, v) = \ell(u', v') \). Suppose \( \ell(u, v) \nparallel l_1 \). Then, the proof that \( \ell(u, v) \parallel \ell(u', v') \) is identical to the proof given above that \( \ell(r, s) \parallel \ell(r', s') \), as \( \tau(u) = \tau^q(u) \) and \( \tau(v) = \tau^q(v) \). Thus, \( \tau \) is a translation which maps \( q \) to \( q' \). \( \square \)

Proof that Axiom 6b is equivalent to Desargues’ Theorem for intersecting lines, Artin.

[1] Suppose Axiom 6b holds. Suppose the hypothesis of Desargues’ Theorem for lines which meet in a point. Let \( \sigma \) be the dilatation which fixes \( p \) and maps \( q \) to \( q' \). Then, \( l_1, l_2, \) and \( l_3 \) are traces of \( \sigma \), therefore \( \sigma(r) \in l_2 \) and \( \sigma(s) \in l_3 \). Since \( \sigma \) is a dilatation, we have that

\[
\ell(q, r) \parallel \ell(\tau(q), \tau(r)) = \ell(q', \tau(r)) = \ell(q' \parallel \ell(q, r)) = \ell(q', r').
\]

So \( r' \) and \( \tau(r) \) are both in the intersection of \( l_2 \) and \( \ell(q', r') \) so \( \tau(r) = r' \). A similar
argument shows that $\tau(s) = s'$. Thus, we conclude that

$$\ell(r, s) \parallel \ell(\tau(r), \tau(s)) = \ell(r', s').$$

Now, suppose Desargues’ Theorem for intersecting lines holds. Again, we must handle an edge case in which every line has exactly two points. Again, since two points determine a line, there is only one such affine plane: $A_K$, where $K$ is the field of two elements. We already know that Axiom 6b holds here. Suppose every line has at least three points. Let $p, q,$ and $q'$ be distinct collinear points. We will construct a dilatation $\sigma$ such that $\sigma(p) = p$ and $\sigma(q) = q'$. We will proceed in a method similar to the one employed for the previous proof. Let $l_1 = \ell(q, q')$. Define $\sigma^q : \mathcal{P} \setminus l_1 \to \mathcal{P} \setminus l_1$ by $\sigma^q(x) = \ell(p, x) \cap \ell(q' \parallel \ell(q, x))$. Fix a point $r \notin l_1$. Let $r' = \sigma^q(r)$, let $l_2 = \ell(r, r')$ and define $\sigma^r : \mathcal{P} \setminus l_2 \to \mathcal{P} \setminus l_2$ by $\sigma^r(x) = \ell(p, x) \cap \ell(r' \parallel \ell(r, x))$. Fix a point $s \notin l_1 \cup l_2$. Let $s' = \sigma^q(s)$ and $l_3 = \ell(s, s')$. Define a map $\sigma^s : \mathcal{P} \setminus l_3 \to \mathcal{P} \setminus l_3$ by $\sigma^s(x) = \ell(p, x) \cap \ell(s' \parallel \ell(s', x))$.

As before, we will prove that $\sigma^q$ agrees with $\sigma^r$ on all the points for which they are both defined. Since $s$ was chosen arbitrarily, we will prove it for $s$. Recall that $l_1, l_2,$ and $l_3$ all meet in $p$. By construction, we have $\ell(q, r) \parallel \ell(q', r')$ and $\ell(q, s) \parallel \ell(q', s')$. By Desargues’ Theorem for intersecting lines, we have that $\ell(r, s) \parallel (r', s')$. So then $\sigma^r(s) = s' = \sigma^q(s)$, as desired. A similar argument shows that $\sigma^q$ and $\sigma^s$ agree on all points for which they are defined, and also that $\sigma^r$ and $\sigma^s$ agree on all points for which they are defined. Now, we can define the desired
map \( \sigma : \mathcal{P} \to \mathcal{P} \) by

\[
\sigma(x) = \begin{cases} 
    p & \text{if } x = p \\
    \sigma^4(x) & \text{if } x \neq p \text{ and } x \notin l_1 \\
    \sigma^r(x) & \text{if } x \neq p \text{ and } x \notin l_2 \\
    \sigma^s(x) & \text{if } x \neq p \text{ and } x \notin l_3.
\end{cases}
\]

Now, we prove that this is a dilatation. Let \( u \) and \( v \) be distinct points. If one of them is \( p \), we immediately have that \( \ell(u, v) = \ell(u, p) = \ell(\sigma(u), \sigma(p)) = \ell(\sigma(u), \sigma(v)) \). So suppose neither \( u \) nor \( v \) is equal to \( p \). Without a loss of generality, suppose neither of them lie on \( l_1 \). Again, the proof that \( \ell(u, v) \parallel \ell(\sigma(u), \sigma(p)) \) is identical to the proof that \( \ell(r, s) \parallel \ell(r', s') \). Therefore, \( \sigma \) is the desired dilatation.

\[\square\]

### 3.3 From the Projective to the Affine

Now, we may extract an affine plane from any Desarguesian projective plane. The intuition should be that in a projective plane, lines that are “parallel” actually meet at “infinity.” To be more clear, there is a line \( \rho \) which we adjoin to the affine plane, such that if \( \pi \) is a pencil of parallel lines in the affine plane, they have a mutual intersection at a point \( x \in \rho \). Each point of \( \rho \) is such an intersection. Based on our four axioms of a projective plane, there is no such line specifically designated. Thus, to turn a projective into an affine plane, we may remove any one line and the points on it.

**Proposition 3.3.1.** Let \( G = (\Pi, \Lambda, \varepsilon) \) be an abstract projective plane. Fix \( \rho \in \Lambda \), and define

\[
A_\rho G = (P(A_\rho G), L(A_\rho G), \varepsilon_\rho)
\]
by

\[ P(A_\rho G) = \{x \in \Pi : x \notin \rho\} \]

\[ L(A_\rho G) = \Lambda \setminus \{\rho\} \]

\[ \varepsilon_\rho = \varepsilon \cap (P(A_\rho G) \times L(A_\rho G)). \]

If \( G \) is Desarguesian, \( A_\rho G \) is an affine plane.

**Proof.** We verify the axioms of an affine plane, one by one.

Axiom 1 of affine planes is immediate from the second axiom of projective planes.

For axiom 2 of affine planes, any two lines which are not parallel are not equal by definition, and so this is immediate from the third axiom of projective planes.

For axiom 3, let \( l \in L(A_\rho G) \) and \( p \in P(A_\rho G) \). If \( p \notin l \), then the axiom holds, so suppose it does not lie on \( l \). Then, consider the point \( x = l \cap \rho \). Let \( y \) be a point on \( \ell(p, x) \) which is not equal to \( p \) and not equal to \( x \). Then, \( m = \ell(p, y) \) is a line parallel to \( l \) containing \( p \).

Axiom 4 is immediate from the fourth axiom of projective planes.

In lieu of proving Axiom 5, we will prove Desargues’ Theorem for parallel lines. Let \( l_1, l_2, l_3 \) be pairwise distinct parallel lines. Let \( q, q' \in l_1, r, r' \in l_2, \) and \( s, s' \in l_3 \) be pairwise distinct points, such that \( \ell(q, r) \parallel \ell(q', r') \) and \( \ell(q, s) \parallel \ell(q', s') \). We prove that \( \ell(r, s) \parallel \ell(q', s') \). Since \( l_1, l_2, \) and \( l_3 \) are distinct and parallel, they are disjoint. Every line intersects in a projective plane, so it must be that \( p = l_1 \cap l_2 \in \rho \). Likewise, \( l_2 \cap l_3 \in \rho \), and \( l_1 \cap l_3 \in \rho \). But then \( p = l_2 \cap l_3 \) and \( p = l_1 \cap l_3 \).

Viewing \( q, q', r, r', s, \) and \( s' \) as points in \( G \) and \( l_1, l_2, l_3 \) as lines in \( G \), we conclude that \( a = \ell(q, r) \cap \ell(q', r'), b = \ell(q, s) \cap \ell(q', s'), \) and \( c = \ell(r, s) \cap (r', s') \) are collinear. But \( \ell(q, r) \parallel \ell(q', r') \) by hypothesis. So \( a \in \rho \). Likewise, \( b \in \rho \). Thus, \( c \in \rho \). So \( \ell(r, s) \) does not intersect \( \ell(r', s') \). That is, they are parallel.
3.3. FROM THE PROJECTIVE TO THE AFFINE

Like before, to prove Axiom 6b, we prove Desargues’ Theorem for intersecting lines. Suppose $l_1, l_2,$ and $l_3$ are pairwise distinct lines which intersect at a point $p$. Note that this time, $p \in P(A_{\rho} G)$. Let $q, q' \in l_1, r, r' \in l_2$ and $s, s' \in l_3$ be pairwise distinct, such that $\ell(q, r) \parallel \ell(q', r')$ and $\ell(q, s) \parallel \ell(q', s')$. We prove that $\ell(r, s) \parallel \ell(r', s')$. That is, we prove that $\ell(r, s) \cap \ell(r', s') \in \rho$. The intersections $\ell(q, r) \cap \ell(q', r')$ and $\ell(q, s) \cap \ell(q', s')$ both lie on $\rho$, for they are parallel in $A_{\rho}(G)$. Since $G$ is Desarguesian, we have that $\ell(r, s) \cap \ell(r', s') \in \rho$, for the three intersections are collinear. Since $\rho \notin L(A_{\rho} G)$, this means that $\ell(r, s) \parallel \ell(r', s')$ in $A_{\rho} G$. 

This has an immediate corollary, which is relevant to our discussion as we are looking primarily at geometries based on skew fields.

**Corollary 3.3.2.** For a skew field $K$, $A_{\rho} G_K$ is an affine plane.

It would seem as though $G$ being Desarguesian is necessary for $A_{\rho} G$ to be an affine plane. Indeed, if one examined $A_{\rho} G$, omitting the assumption that $G$ is a Desarguesian projective plane, one would run into trouble verifying that $A_{\rho} G$ is also Desarguesian. We will now prove this fact. That is, we wish to verify that if $G$ is a projective plane which is not Desarguesian, $A_{\rho} G$ is not an affine plane – it satisfies only the first four axioms of an affine plane. To do so, we will develop a general method for building a projective plane from an affine plane, and show that such a projective plane is Desarguesian. Our goal can be expressed more concisely by the theorem below:

**Theorem 3.3.3.** A projective plane $G$ is Desarguesian if and only if $A_{\rho} G$ is an affine plane.

In order to prove this, we need to develop a sensible way of turning an affine plane into a projective plane in a way that is compatible with our $A_{\rho} G$ construct. It also will be necessary to introduce projective plane isomorphisms. These iso-
morphisms are defined very similarly to isomorphisms of affine planes, in that they need only be bijections which preserve incidence

**Definition 3.3.4.** A *isomorphism* of projective planes $G = (\Pi, \Lambda, \varepsilon)$ to $G' = (\Pi', \Lambda', \varepsilon')$ is a pair of bijections $f_\Pi : \Pi \to \Pi'$ and $f_\Lambda : \Lambda \to \Lambda'$ such that $p \varepsilon l$ if and only if $f_\Pi(p) \varepsilon' f_\Lambda(l)$.

We will approach proving Theorem 3.3.3 by proving every Desarguesian projective plane is isomorphic to a projective plane over some skew field $K$. We will make rigorous the notion of “completing” an affine plane. Note that we switch back from $\in$ notation to $\varepsilon$ notation for the incidence relation of points on lines.

**Proposition 3.3.5.** Let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \varepsilon)$ be an abstract affine plane. Define $l_\infty$ to be the set of pencils of $\mathcal{L}$. Define $\mathcal{P}_\infty = \mathcal{P} \cup l_\infty$ and $\mathcal{L}_\infty = \mathcal{L} \cup \{l_\infty\}$. Define the incidence relation $\varepsilon_\infty$ by

$$
\varepsilon_\infty = \varepsilon \cup \{(\pi, l) \in l_\infty \times \mathcal{L}_\infty \mid l = l_\infty \lor \pi \subseteq l\}.
$$

Then, $\mathcal{G}_\infty = (\mathcal{P}_\infty, \mathcal{L}_\infty, \varepsilon_\infty)$ is a projective plane.

**Proof.** Let $\mathcal{G}$ and $\mathcal{G}_\infty$ be given. We will prove the projective plane axioms for $\mathcal{G}_\infty$, one by one.

1. Let $x, y \in \mathcal{P}_\infty$ be distinct points. If neither of them are a pencil, this property is true by Axiom 1 of affine planes. So suppose that $x = [l]$, where $[l]$ is some pencil. Then, by Axiom 3 of affine planes, there is a unique line $m \in [l]$ such that $y \in m$. Then, $y \in m$ and $[l] \varepsilon_\infty m$ is immediate. If both $x$ and $y$ are pencils, then $l_\infty$ is the unique line containing them. Every line is a member of exactly one pencil, so the only line with multiple pencils on it as points is $l_\infty$. 
2. Let \( l, m \in \mathcal{L}_\infty \) be distinct lines. First, suppose neither of them are equal to \( l_\infty \). If they are parallel as lines in \( \mathcal{G} \), then they intersect in \([l]\). If they are not parallel, their intersection is uniquely given by Axiom 2 of affine planes. Now, suppose \( l = l_\infty \). Then, \( m \) intersects \( l_\infty \) in \([m]\). This intersection is unique, for every line of an affine plane is in exactly one pencil.

3. We prove there are four distinct points of \( \mathcal{P}_\infty \) such that no three of them are collinear. Let \( p, q, r \) be the three non-collinear points given by Axiom 4 of affine planes. Define a point \( s = \ell(q \parallel \ell(p, r)) \cap \ell(p \parallel \ell(q, r)) \). Then, \( s \) is distinct from \( p, q, \) and \( r \), for otherwise \( p, q, r \) are collinear. Let \( \pi = \ell(s, r) \cap l_\infty \). Then, \( p, q, r, \) and \( \pi \) are four points, no three of which are collinear. Notice that \( \ell(p, s), \ell(q, s), \) and \( \ell(r, s) \) are pairwise non-parallel, for they all intersect in \( s \) and nowhere else. Thus, no two of \( p, q, \) or \( r \) are collinear with \( \pi \).

\[ \square \]

To paint a clearer picture of how to interpret \( \mathcal{G}_\infty \) relative to \( \mathcal{G} \), we have a lemma.

**Lemma 3.3.6.** Let \( \mathcal{G} = (\mathcal{P}, \mathcal{L}, \varepsilon) \) be an affine plane, and let \( l, m \in \mathcal{L} \) be distinct lines. Then \( l \parallel m \) if and only if \( (l \cap m) \varepsilon_\infty l_\infty \).

**Proof.** Suppose \( l \parallel m \). Since every pair of lines intersect in \( \mathcal{G}_\infty \) but \( l \) and \( m \) do not intersect in \( \mathcal{G} \), it must be that they intersect in \( l_\infty \). On the other hand if \( (l \cap m) \varepsilon_\infty l_\infty \), we have that their intersection in \( \mathcal{G} \) is empty, since the intersection of lines in a projective plane is unique. That is, their intersection does not exist in \( A_\rho \mathcal{G} \), so \( l \parallel m \). \[ \square \]

Now, we must verify that our completion of the affine plane is compatible with our method of extracting an affine plane from a projective plane.
Lemma 3.3.7. Let $G = (\Pi, \Lambda, \varepsilon)$ be an abstract projective plane. Let $\rho \in \Lambda$. Then, $(A_\rho G)_\infty \cong G$.

Proof. Define $f_\Lambda : L(A_\rho G)_\infty \to \Lambda$ by

$$l \mapsto \begin{cases} 
  l & \text{if } l \neq l_\infty \\
  \rho & \text{if } l = l_\infty.
\end{cases}$$

To define $f_\Pi : P(A_\rho G)_\infty \to \Pi$, consider a point $x \in l_\infty$. Then, $x$ is a pencil of parallel lines in $A_\rho G$, so write $x = [l]$, where $l$ is some line in $L(A_\rho G)$ and $[l]$ denotes the pencil of lines parallel to $l$. Now, we may define $f_\Pi$ as follows:

$$x \mapsto \begin{cases} 
  x & \text{if } x \in_\infty l_\infty \\
  l \cap \rho & \text{if } x \in_\infty l_\infty.
\end{cases}$$

We prove that this is a morphism. We prove $p \in_\infty l$ if and only if $f_\Pi(p) \in f_\Lambda(l)$. First suppose $l \neq l_\infty$. Then, if $p \in_\infty l_\infty$, the functions $f_\Pi$ and $f_\Lambda$ fix $p$ and $l$ respectively. If $p \in_\infty l_\infty$, then $p = [m]$ for some line $m \in A_\rho G$. By definition of $\in_\infty$, we know that $p \in_\infty l$ if and only if $l \in [m]$ or if $l = l_\infty$. So it must be that $l \in [m]$. But $f_\Pi([m]) = l \cap \rho$, and $f_\Lambda(l) = l$. So in the case where $l \neq l_\infty$ $f_\Pi(p) \in f_\Lambda(l)$ exactly when $p \in_\infty l$.

On the other hand, suppose $l = l_\infty$. Then $p \in_\infty l_\infty$ is impossible, so we will treat the case where $p \in_\infty l_\infty$. Write $p = [l]$, for some line $l \in A_\rho G$. Then, $[l] \in_\infty l_\infty$ if and only if $f_\Pi([l]) = (l \cap \rho) \in \rho = f_\Lambda(l_\infty)$. Thus, $f_\Pi$ and $f_\Lambda$ are a morphism of projective planes.

Now, we prove that these maps are bijections. It is clear that $f_\Lambda$ is injective. If $l \in \Lambda$ is any line not equal to $\rho$, then $f_\Lambda(l) = l$, and so $f_\Lambda$ is a bijection.

To prove $f_\Pi$ is a bijection, let $x \in G$ be any point. If $x \notin \rho$, then $f_\Pi(x) = x$. If $x \in \rho$, there is some line $l \in \Lambda$ such that $x \in l$ and such that $l \neq \rho$, by Axiom
3 of projective planes. That is, $x = l \cap \rho$. Consider $p = f_{\Lambda}^{-1}(l) \cap l_\infty = l \cap l_\infty$. Then, $p = [l]$. Since the maps are morphisms, we have that $f_\Pi(p) = x$. So $f_\Pi$ is a surjection.

Suppose $f_\Pi(p) = f_\Pi(q)$. In the case where $f_\Pi(p) = f_\Pi(q) \in \rho$, see that $p, q \in l_\infty$ as $f_\Pi$ and $f_\Lambda$ are morphisms. But then $f_\Pi$ acts as the identity map on $p$ and $q$, so $f_\Pi(p) = f_\Pi(q)$ is equivalent to $p = q$. Now, suppose $f_\Pi(p) = f_\Pi(q) \in \rho$. Then $p, q = [l], [m]$ respectively. That is, $l \cap \rho = m \cap \rho$. Thus, $l \cap m \in \mathcal{E}_\infty$. That is, $l \parallel m$ in $A_\rho G$ and so $[l] = [m]$. So $f_\Pi$ and $f_\Lambda$ are a morphism of projective planes, and we conclude that $(A_\rho G)_\infty \cong G$.

Our completion of an affine plane is compatible with our extraction of an affine plane from a projective plane.

Before we can finally prove that $A_\rho G$ being affine implies $G$ is Desarguesian, we need two more lemma relating affine planes over skew fields to projective planes over skew fields. Here is some notation to do so:

**Definition 3.3.8.** Let $K$ be a skew field. Define $e_1 = (1, 0, 0) \in K^3, e_2 = (0, 1, 0) \in K^3$ and $e_3 = (0, 0, 1)$. For a vector $a = (a_1, a_2) \in K^2$, define $a^0 = (a_1, a_2, 0)$ and $a^1 = (a_1, a_2, 1)$.

**Lemma 3.3.9.** Let $K$ be a skew field. Then, $(A_K)_\infty \cong G_K$.

*Proof.* We prove that $(A_K)_\infty \cong G_K$. The line at infinity for $A_K$ is the set of all pencils in $A_K$. By Lemma 2.1.2, two lines $l_1 = a + Kb$ and $l_2 = p + q$ are parallel if $b$ and $q$ are linearly dependent if $b \in Kq$. Thus, every pencil is of the form $Kx$, for some $x \in K^2$. Thus, $l_\infty = \{Kb : b \neq 0\}$. A point in $(A_K)_\infty$ is either $x$ or $Kx$, where $x \in K^2$. A line in $(A_K)_\infty$ is either $a + Kb$ where $a, b \in K^2$, or $l_\infty$ itself. A point $p \in (A_K)_\infty$ is incident to a line $l \in (L_K)_\infty$ if $p \in A_K$ and $l \in L_K$ or if $p = Kx$ and $l = l_\infty$. Now, we define our morphism. Define $f_\Pi : (A_K)_\infty \to \Pi_K$.
by
\[
\begin{align*}
    f_\Pi(x) &= \begin{cases} 
        Kx^1 & \text{if } x = x \in K^2 \\
        Kx^0 & \text{if } x = Kx \in \infty l_\infty.
    \end{cases}
\end{align*}
\]

We define the map of lines by \( f_\Lambda : (L_K)_\infty \to \Lambda_K \) by
\[
\begin{align*}
    f_\Lambda(l) &= \begin{cases} 
        Kb^0 + Ka^1 & \text{if } x = a + Kb \in L_K \\
        Ke_1 + Ke_2 & \text{if } x = l_\infty.
    \end{cases}
\end{align*}
\]

We prove that \( p \in \infty l \) if and only if \( f_\Pi(p) \subseteq f_\Lambda(l) \). Let \( p = Kp \), then \( f_\Pi(p) = Kp^0 \), and \( l = l_\infty \). Then, \( Kp^0 \subseteq Ke_1 + Ke_2 \). If \( p \in K^2 \), then we may write \( p = p \) and \( l = p + Kq \), for some \( q \in K^2 \). Clearly \( Kp^1 \subseteq Kp^1 + Kq^0 \).

On the other hand, suppose \( f_\Pi(p) \in \infty f_\Lambda(l) \). If \( f_\Pi(p) = Kp^0 \) for some \( p \in K^2 \), it must have been that \( p = Kp \). Moreover, it must also be the case that \( l = l_\infty \), and so \( p \in \infty l \). Now, suppose \( f_\Pi(p) = Kp^1 \) for some \( p \in K^2 \). Then, \( f_\Lambda(l) \neq Ke_1 + Ke_2 \), and so \( l = a + Kb \), for some \( a \) and \( b \in K^2 \). Since \( Kp^1 \subseteq Kb^0 + Ka^1 \), we have that \( p^1 = \alpha a^1 + \beta b^0 \), for some \( \alpha, \beta \in K \). Notice that \( \alpha \neq 0 \). Let \( \gamma = \alpha^{-1} \beta \). We may write \( p^1 = \gamma b^0 + a^1 \). This implies \( p \in a + Kb \), as desired. So \((f_\Pi, f_\Lambda)\) is a morphism of \((A_K)_\infty \) and \( G_K \).

Now, we prove that \( f_\Pi \) is a bijection. Let \( Kx \in \Pi_K \), where \( x \in K^3 \). Write \( x = (x_1, x_2, x_3) \). If \( x_3 = 0 \), then \( f_\Pi(K(x_1, x_2)) = Kx \). If \( x_3 \neq 0 \), then \( f_\Pi(x_1x_3^{-1}, x_2x_3^{-1}) = Kx \). Thus, \( f_\Pi \) is a surjection. To see that it is an injection, suppose \( K(x_1, x_2, 0) = K(y_1, y_2, 0) \). Then clearly \( K(x_1, x_2) = K(y_1, y_2) \). Now, suppose \( K(x_1, x_2, 1) = K(y_1, y_2, 1) \). Once again, it is clear that \( (x_1, x_2) = (y_1, y_2) \). So \( f_\Pi \) is a bijection.

We will now prove that \( f_\Lambda \) is a bijection. Let \( Kx + Ky \) be a line in \( \Lambda_K \). If \( Kx + Ky = Ke_1 + Ke_2 \), then \( f_\Lambda(l_\infty) = Kx + Ky \). So suppose this is not the case. Then, \( x \) and \( y \) cannot both have 0 in their third component, so without a loss of generality, write \( x = (x_1, x_2, 1) \) and \( y = (y_1, y_2, y_3) \). If \( y_3 = 0 \), then
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\[ f_\lambda((x_1, x_2) + K(y_1, y_2)) = Kx + y. \] If \( y_3 \neq 0 \), then \( f_\lambda((x_1, x_2) + K(y_1 - x_1, y_2 - x_2)) = Kx + Ky. \) To see that \( f_\lambda \) is injective, suppose \( Kb^0 + Ka^1 = Kd^0 + Kc^1 \). We will show that \( a + Kb = c + Kd \). Write \( a^1 = (a_1, a_2, 1) \) and \( b^0 = (b_1, b_2, 0) \). We have that \( Kb^0 \) and \( K(a^1 + b^0) \) both lie on \( Kb^0 + Ka^1 \) and on \( Kd^0 + Kc^1 \). Thus, the pre-images of \( Ka^1 \) and of \( K(a^1 + b^0) \) lie on both \( a + Kb \) and \( c + Kd \). That is, \( a + Kb \) and \( c + Kd \) intersect in two distinct points, so they are equal. We conclude that \((A_K)_\infty \cong G_K\). 

**Lemma 3.3.10.** If \( G = (\mathcal{P}, \mathcal{L}) \) and \( G' = (\mathcal{P}', \mathcal{L}') \) are isomorphic affine planes, then \( G_\infty \cong G'_\infty \).

*Proof.* Let \( f_p : \mathcal{P} \to \mathcal{P}' \) and \( f_\ell : \mathcal{L} \to \mathcal{L}' \) be an isomorphism of \( G \) and \( G' \). Define \( f_\Pi : \mathcal{P}_\infty \to \mathcal{P}'_\infty \) by

\[
p \mapsto \begin{cases} 
  f_p(p) & \text{if } p \not\in \infty, \\
  [f_\ell(l)] & \text{if } p = [l], \text{ for some line } l \in \mathcal{L}.
\end{cases}
\]

If \([l] = [m]\), then \( f_\ell(l) \parallel f_\ell(m)\), and so \([f_\ell(l)] = [f_\ell(m)]\) since \( f_\ell \) is a morphism. Thus \( f_\Pi \) is well-defined. Define \( f_\Lambda : \mathcal{L}_\infty \to \mathcal{L}'_\infty \) by

\[
l \mapsto \begin{cases} 
  f_\ell(l) & \text{if } l \neq l_\infty, \\
  l'_\infty & \text{if } l = l_\infty.
\end{cases}
\]

The map \( f_\Lambda \) is clearly a bijection. If \( p \in \mathcal{P}'_\infty \) is a point in \( \mathcal{P}' \), then it has a preimage under \( f_\Pi \) since \( f_p \) is a bijection. If \( p \) is not a point in \( \mathcal{P}' \), then write \( p = [l] \) where \( l \) is some line in \( \mathcal{L}' \). The pencil \([l]\) has \([f_\ell^{-1}(l)]\) as a pre-image, since \( f_\ell \) is a bijection. Thus, \( f_\Pi \) is a surjection.

To see that \( f_\Pi \) is injective, let \( f_\Pi(p) = f_\Pi(q) \). If these points do not lie on \( l'_\infty \), then \( f_\Pi \) fixes \( p \) and \( q \), and so \( p = q \). On the other hand, suppose \( f_\Pi(p) =
Let $l$ be a line distinct from $l'_{\infty}$ containing $f_{\Pi}(p)$. Such a line exists by Axiom 3 of projective planes. Let $m = f_{\lambda}^{-1}(l)$. Then, $p, q \in m$, since the maps are morphisms. But we also have that $p, q \in l_{\infty}$. But the intersection of $m$ and $l_{\infty}$ is unique by Axiom 2 of projective planes, and so $p = q$. Thus, $G_{\infty} \cong G'_{\infty}$.

Finally, we may prove Theorem 3.3.3 by combining the lemmas proven above.

Proof of Theorem 3.3.3. We have already proven that if $G$ is Desarguesian, then $A_{\rho}G$ is an affine plane in Proposition 3.3.1. For the other direction of implication, suppose $A_{\rho}G$ is an affine plane. Then, $A_{\rho}G \cong A_{K}$, for some skew field $K$ by Theorem 2.4.4. But $G \cong (A_{\rho}G)_{\infty}$ by Lemma 3.3.7, $(A_{\rho}G)_{\infty} \cong (A_{K})_{\infty}$ by Lemma 3.3.10, and $(A_{K})_{\infty} \cong G_{K}$ by Lemma 3.3.9. Therefore $G \cong G_{K}$, and since projective planes over skew fields are Desarguesian, we have that $G$ is Desarguesian.

Embedded in the proof is an immediate corollary, which is a pithier version of the result of Theorem 3.3.3.

Corollary 3.3.11. If $G$ is a Desarguesian projective plane, then there exists a skew field $K$ such that $G \cong G_{K}$.

3.4 A projective plane which is not Desarguesian

Indeed, many of the projective planes that one would like to study, such as $\mathbb{R}P^2$, are Desarguesian. There is a rather degenerate incidence structure that we define which makes a projective plane which is not Desarguesian. We define the Moulton Plane, which was first described by Forest Ray Moulton [8] and expanded upon in Beutelspacher and Rosenbaum’s textbook, Projective Geometry: From Foundations to Applications [9].
Definition 3.4.1. Let $\mathbb{P} = \mathbb{R}^2$. Let $a, b \in \mathbb{R}^2$, where $b \neq 0$. For real numbers $m$ and $b$, define the sets

$$A_{m,b} = \{(x, y) \in \mathbb{R}^2 : x \leq 0 \text{ and } y = mx + b\} \cup \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y = 2mx + b\}.$$ 

$$L = \{a + \mathbb{R}(b_1, b_2) : a, (b_1, b_2) \in \mathbb{R}^2 \text{ and } b_1b_2 \geq 0\} \cup \{A_{m,b} : m, b \in \mathbb{R}^2, m < 0\}.$$ 

The triple $\mathbb{M} = (\mathbb{P}, L, \in)$ is called the **Moulton Plane**.

Note that the Moulton Plane is not an affine plane. It satisfies the first four axioms of a projective plane, but not the 6th. The points are simply elements on $\mathbb{R}^2$, but the lines are different from the lines in $A_\mathbb{R}$. Here, vertical lines and lines with non-negative slope are the same as in $A_\mathbb{R}$, lines with negative slope double in slope when it passes the $y$-axis.

**Proposition 3.4.2** (Beutelspacher and Rosenbaum). The Moulton plane is a triple for which Axioms 1 through 4 hold, but is not an affine plane.
Proof. We prove that for every pair of points, there is a line to which they are both incident. Let \((x_0, y_0)\) and \((x_1, y_1)\) \(\in \mathbb{P}\). If \(x_0 = x_1\), then \((x_0, y_0)\) and \((x_1, y_1)\) are both incident to the line \((x_0, y_0) + \mathbb{R}(0, 1)\). So suppose \(x_0 \neq x_1\) and without a loss of generality, suppose \(x_0 < x_1\). Then, we split into cases.

Case 1 Suppose \(x_0, x_1 \leq 0\). Let \(m = (y_0 - y_1)/(x_0 - x_1) < 0\). Then \((x_0, y_0)\) and \((x_1, y_1)\) are both incident to the line \(A_{m,b} \) where \(b = (y_1x_0 - y_0x_1)/(x_0 - x_1)\). If \(m \geq 0\), then \((x_0, y_0)\) and \((x_1, y_1)\) are both incident to the line \((x_0, y_0) + \mathbb{R}(x_1 - x_0, y_1 - y_0)\).

Case 2 Suppose \(x_0, x_1 \geq 0\). Let \(m\) be defined as in case 1. If \(m < 0\), then \((x_0, y_0)\) and \((x_1, y_1)\) are both incident to the line \(A_{2, b} \) where \(b\) is defined as in case 1. If \(m \geq 0\), then \((x_0, y_0)\) and \((x_1, y_1)\) are both incident to the line \((x_0, y_0) + \mathbb{R}(x_1 - x_0, y_1 - y_0)\).

Case 3 Suppose \(x_0 < 0\) and \(x_1 > 0\). If \(y_0 \leq y_1\), then \((x_0, y_0)\) and \((x_1, y_1)\) are incident to \((x_0, y_0) + \mathbb{R}(x_1 - x_0, y_1 - y_0)\). On the other hand, suppose \(y_0 > y_1\). To find a line incident to both the points, we need \(m\) and \(b \in \mathbb{R}\) such that \(y_0 = mx_0 + b\) and \(y_1 = 2mx_1 + b\). This gives \(m = (y_0 - y_1)/(x_0 - 2x_1)\) and \(b = (y_1x_0 - 2y_0x_1)/(x_0 - 2x_1)\), and so \((x_0, y_0)\) and \((x_1, y_1)\) are incident to the line \(A_{m,b}\) with \(m\) and \(b\) as above.

To prove Axiom 2 of affine planes, let \(l_1, l_2 \in \mathbb{L}\). If \(l_1 = a + \mathbb{R}(b_1, b_2)\) and \(l_2 = x + \mathbb{R}(y_1, y_2)\) where \((b_1, b_2), (y_1, y_2) \neq 0\) and \(b_1 b_2, y_1 y_2 \geq 0\), then \(l_1 \parallel l_2\) if \((y_1, y_2) = c(b_1, b_2)\) for some \(c \in \mathbb{R}\). If this is not the case, then they intersect in a singleton, by Lemma 2.1.2. If \(l_1 = A_{m,b}\) and \(l_2 = A_{m',b'}\), then they are parallel if \(m = m'\). If \(m \neq m'\), then surely \(l_1\) intersects \(l_2\). We will prove that this intersection is unique. Suppose their intersection is in a point \((x, y)\) with \(x \leq 0\). Since \(l_1\) and \(l_2\) behave like ordinary affine lines when \(x\) is not positive, if there was another intersection of \(l_1\) and \(l_2\), it would be with positive \(x\). Suppose \((x_0, y_0)\) is
such an intersection. This gives us the system of equations

\[ mx + b = m'x_0 + b'2mx_0 + b = 2m'x_0 + b'. \]

By rewriting, we get

\[ 2x_0(m' - m) = b - b' = x(m' - m). \]

That is, \( 2x_0 = x \). But \( x \leq 0 \) and \( x_0 > 0 \), a contradiction. On the other hand, suppose \( l_1 \) and \( l_2 \) intersect in a point \((x, y)\) with \( x > 0 \). If \( l_1 \) and \( l_2 \) were to intersect, it would be in a point \((x_0, y_0)\) with \( x_0 \leq 0 \). This leads to a contradiction analogous to the case where \( x \leq 0 \). Thus, if \( l_1 = A_{m,b} \) and \( l_2 = A_{m',b'} \), they are parallel when \( m = m' \) and intersect in a unique point when \( m \neq m' \). Lastly, let \( l_1 = x + R_y \) and \( l_2 = A_{m,b} \). Clearly, these lines are never parallel. Suppose they intersect in a point \((u, v)\). Let \( u \leq 0 \). If there were another intersection, it would be in some point \((u', v')\) with \( u' > 0 \). Since \((u, v)\) and \((u', v')\) are both elements of \( l_1 \) and \( u < u' \), we have that \( v \leq v' \). But since \((u, v)\) and \((u', v')\) are both elements of \( l_2 \) and \( u < u' \), we have \( v > v' \), which is a contradiction. Thus, \( l_1 \) and \( l_2 \) have a unique intersection.

We prove Axiom 3 of affine planes. Let \( \mathbf{x} = (x_0, x_1) \in \mathbb{P} \). Let \( l = a + \mathbb{R}b \). The line parallel to \( l \) containing \( \mathbf{x} \) is given by \( l' = \mathbf{x} + \mathbb{R}b \). Now suppose \( l = A_{m,b} \) for \( m < 0, b \in \mathbb{R} \). If \( x_0 \leq 0 \), then \( l' = A_{m,x_1 - mx_0} \). If \( x_0 > 0 \), then \( l' = A_{m,x_1 - 2x_0} \). See that in either case, \( l \parallel l' \) and \( \mathbf{x} \in l' \).

For Axiom 4, consider the points \((1, 0), (0, 1)\), and \((0, 0)\). These are clearly non-collinear.

Axiom 6b does not hold in the Moulton Plane, for it does not satisfy Desargues’ Theorem for lines which meet in a point. Consider the following points and lines, which satisfy the hypothesis: \( l_1 = A_{-1,2}, l_2 = \mathbb{R}(1, 0), l_3 = (1, 0) + \mathbb{R}(1, 1) \).
Notice these lines all intersect in \((1, 0)\). Let \(a = (-2, 4), a' = (1/2, 1) \in l_1\). Let \(b = (-5, 0), b' = (-1/4, 0) \in l_2\). Let \(c = (-2, -3), c' = (1/2, -1/2)\). The desired lines are parallel:

\[
\ell(a, b) = (5, 0) + \mathbb{R}(3, 4) \| \left(\frac{1}{4}, 0\right) + \mathbb{R}\left(\frac{3}{4}, 1\right) = \ell(a', b').
\]

\[
\ell(a, c) = (-2, 0) + \mathbb{R}(1, 0) \| \left(\frac{1}{2}, 0\right) + \mathbb{R}(1, 0) = \ell(a', c').
\]

However, \(\ell(b, c) = A_{-1, -5}\) while \(\ell(b', c') = A_{-\frac{2}{5} - \frac{1}{4}}\). These lines are not parallel, so the Moulton Plane is not an affine plane.

\[\square\]

**Corollary 3.4.3.** The projective plane obtained from adding the line at infinity...
to the Moulton Plane is non-Desarguesian. Equivalently there is no skew field $K$ for which $M_\infty \cong G_K$.

Proof. By the contrapositive of Proposition 3.3.1 and by Lemma 3.3.7, we have that $M_\infty$ is non-Desarguesian. \qed
Chapter 4

Conclusion

Some further work can be done in looking at these geometric structures from the perspective of category theory. Can our results be simplified by viewing the collections of affine planes and of projective planes as categories? What new results arise from examining the functorial properties of $G \mapsto A_{\rho}G$? As mathematicians develop new structures, perhaps they can be used to derive even more properties of affine and projective planes, without treating properties specific to particular planes. For now, the algebraic approach is more than sufficient.

To put it tersely, the main ideas of this paper can be obtained by Theorem 2.3.8, Theorem 2.4.4, and Theorem 3.3.3. An examination of Figures 3.2 and 3.2 shows us why Axioms 5 and 6 are so important. When studying planar geometry, we hope that Desargues’ Theorems hold, because they align with our intuition—they feel sensible. The fact that these two theorems are equivalent to Axioms 5 and 6 respectively demonstrates that the framework enabling us to view every abstract affine plane as the affine plane over some skew field is the same framework that makes a planar geometry sensible. Such is the guiding force of this thesis; we can study a geometry by studying algebraic structures associated to them.
Bibliography


