Quantum Computing: Temperature Scaling of Decoherence

by

Angus Kan

Faculty Advisor: Dr. Reinhold Blümel

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Abstract

After years of effort by quantum scientists, decoherence due to thermal effects still remains a huge hindrance to the upscaling of quantum computers. This thesis presents the temperature scaling of decoherence in a trapped-ion qubit. Microscopic quantum noise simulations are carried out. The resulting numerical scaling laws of coherence times, $T_1$ and $T_2$, are presented. The results enable experimentalists to determine, as a function of temperature, an upper bound for the time over which a quantum computer can be operated without a disastrous loss of quantum coherence, and the duration over which a memory qubit retains its information. Given the tradeoff between the enormous resource overhead demanded by quantum error correction protocols and the adverse effects of decoherence, the results provide guidance for the implementation and optimization of practical error suppression methods in the current era of quantum computers.
Dedication

To my family.
I would like to thank my research advisor, Professor Reinhold Blümel, for being a great mentor and role model. I would not have been able to write this thesis without your knowledge and insight. In addition, you have demonstrated to me how to be a professional research physicist. I am honored to be your student and will not forget my time under your tutelage.

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Chapter 1

Introduction

1.1 The Quantum Revolution

Quantum mechanics has left a huge imprint on modern technology. Scientists and engineers have leveraged quantum effects to construct devices, such as lasers, LEDs, GPS, MRI, solar cells, and perhaps most importantly, transistors. The invention of transistors has led to more compact, powerful, and faster computers, which underpin the Information Age and marked the climax of the first quantum revolution.

Now, we are in the initial stages of the second quantum revolution. Physicists and engineers nowadays are developing technologies, which, unlike the ones from the first quantum revolution, heavily rely on quantum control of individual particles, making use of quantum superposition, quantum entanglement, and quantum coherence. A few emerging quantum technologies include (1) quantum key distribution systems, which enable provably secure communication [1, 2], (2) quantum satellites, which provide secure intercontinental communication [3], and (3) quan-
The ultimate goal of the second quantum revolution is to construct a scalable, fault-tolerant, universal quantum computer. While we still have some ground to cover concerning large-scale quantum computers, we have already built small-scale quantum computers with up to about 20 qubits, and are very close to building intermediate-scale quantum computers with 50–100 qubits, which can perform tasks that are beyond the capabilities of current classical digital computers. While an intermediate-scale quantum computer is certainly not powerful enough to break RSA encryption, it is capable of realizing Feynman’s proposal—simulating quantum many-body systems that are intractable using a classical computer. In fact, proof-of-principle quantum simulations have already been executed on quantum computers to solve many-body problems from quantum chemistry [6], condensed matter [7, 8], nuclear physics [9] and high-energy physics [10]. In addition, quantum computers have shown flashes of potential in machine-learning applications [11, 12], thereby gaining attention from Silicon Valley companies [13].

However, the effect of noise seriously limits the computational power of intermediate-scale quantum computers. In particular, noise imposes a ceiling on the number of gate operations that can be performed before it overwhelms the signal and corrupts the computational task. There are multifarious sources of errors, which are specific to the chosen architecture [14]. For trapped-ion quantum computers, error sources include laser fluctuations, and heating of trapped ions due to trap imperfections [15, 16]. In theory, this is where quantum error correction (QEC) comes to the rescue and suppresses small errors in quantum circuits. However, the physical implementation of QEC protocols involves a significant overhead cost. Therefore, as far as experimentalists are concerned, QEC does not solve the immi-
nent scaling issues. On the bright side, there are alternative methods that extend the operation time, i.e., the coherence time, of a near-term quantum computer, for example, sympathetic cooling of an ion-qubit [17].

1.2 Outline

In this thesis, we investigate how fast a trapped-ion qubit becomes corrupt at different temperatures. In particular, we show that lowering the temperature will suppress errors in a trapped-ion quantum computer. We focus on the trapped-ion architecture because it is a highly promising platform for fault-tolerant quantum computers [18]. Moreover, trapped ions are widely used in other quantum technologies, such as quantum sensing [4] and quantum networks [19]. The results will be useful to experimentalists and engineers who are realizing these quantum technologies.
Chapter 2

Review of Coherence

In this chapter, we review the notion of coherence in optical fields and quantum systems. This will pave the way for future chapters where we investigate the interplay between optical and quantum coherence.

2.1 Coherence in Classical Optics

The aim of this section is to develop an understanding of coherence in classical optics. We first introduce temporal and spatial coherence of light through interferometry. Then, we discuss how to quantify coherence through correlation functions.

2.1.1 Temporal and Spatial coherence

Interferometers are used to study interference effects. Interference describes the phenomena where a wave pattern, called interference fringes, is formed by su-
perposed waves. Depending on the relative difference in the phases of a pair of waves, they will interfere constructively or destructively, thereby creating bright and dark fringes respectively. If the phase relationship between them is constant, they are called *coherent* waves.

Suppose we have a monochromatic, plane light-wave source. To create interference fringes, we may use a *Michelson interferometer* (see fig. 2.1). A wave from the source is split into two waves, and after a path delay $\Delta l = c\Delta t$, they are brought back together. The constant path delay introduces a constant phase difference $\Delta\phi = \omega\Delta t$ between the two waves when they superpose. Hence, they are coherent. By adjusting the delay, the interference pattern observed on the screen will differ.

Now, let us consider a source with an effective bandwidth of $\Delta\nu$. Within the bandwidth, different frequencies may interfere destructively, and destroy the interference pattern. In general, the interference fringes will only appear on the screen if $\Delta t\Delta\nu \lesssim 1$ \[20\]. Therefore, the appearance of the fringes is a manifesta-

Figure 2.1: Schematic diagram of a Michelson interferometer.
Figure 2.2: *Left*: An ideal plane wave has infinite temporal ($\tau$) and spatial coherence length. Hence, it has infinite area of coherence ($A$). *Center*: This wave has a finite temporal coherence length, but an infinite spatial coherence length. Hence, it has an infinite area of coherence. *Right*: This wave has finite temporal and spatial coherence lengths. Hence, it has a finite area of coherence.

A measure of *temporal coherence* between the beams. The time delay $\Delta t \sim \frac{1}{\Delta \nu}$ is called the *coherence time* or *temporal coherence length*. The superposed waves are expected to remain partially coherent within $\Delta t$. The path delay $\Delta l = c \Delta t$ is called the *spatial coherence length*. The meanings of temporal and spatial coherence are best conveyed with the help of a visualization (see fig. 2.2). Temporal coherence is a measure for the correlation between the phases of a light wave at different points in time. It also tells us how long a coherent wave train is in the direction of propagation. Spatial coherence is a correlation measure between the phases at different points in space. It tells us the length of a coherent wave train perpendicular to the propagation direction. The area of coherence is defined to be the area within which a wave remains both temporally and spatially coherent.
2.1.2 Quantifying Coherence

In general, the coherence of an electric field is characterized by the first-order correlation functions:

$$\langle E_i(r_1, t_1) E_j^* (r_2, t_2) \rangle,$$  \hspace{1cm} (2.1)

where $i$ and $j$ can be any combinations of the field components orthogonal to the propagation direction, and the angle brackets denote time-averaging. For instance, if we want to extract the temporal coherence of two real, one-component fields, we may use normalized correlation function

$$g^{(1)}(\delta) = \frac{\langle E(t_1) E(t_2) \rangle}{\langle E(t_1) E(t_1) \rangle} = \frac{\langle E(t_1) E(t_1 + \delta) \rangle}{\langle E^2(t_1) \rangle},$$ \hspace{1cm} (2.2)

where $\delta = t_1 - t_2$ is the time delay between two measurements at the same spatial position. This function is called the first-order degree of temporal coherence.

Similarly, we may define the second-order degree of temporal coherence:

$$g^{(2)}(\delta) = \frac{\langle E^2(t_1) E^2(t_1 + \delta) \rangle}{\langle E^2(t_1) \rangle^2} = \frac{\langle I(t_1) I(t_1 + \delta) \rangle}{\langle I(t_1) \rangle^2},$$ \hspace{1cm} (2.3)

to quantify the correlation of the intensity of electric fields.

2.2 Coherence in Quantum Systems

Let us consider a quantum eigenstate of a Hamiltonian. Since it is a stationary state by definition, while its modulus may show a dependence on position, it does not change in time. A quantum state governed by the Hamiltonian might be in a
superposition of the Hamiltonian’s basis states with well-defined amplitudes and phase relationships. In general, quantum coherence refers to the amplitude and phase relationships between the basis states, and decoherence describes the loss of quantum coherence. For instance, it may refer to amplitude decay, or the decay of phase relations between the superposed states.

In principle, a quantum eigenstate will not decohere if its Hamiltonian is fully specified, since its evolution will then be completely described by the Schrödinger equation. Indeed, decoherence in a quantum system can only come from the degrees of freedom and interactions that are neglected in the original Hamiltonian. A realistic quantum system is always physically situated in, and coupled to, an environment or bath. Naturally, the system-bath coupling then induces decoherence because the quantum eigenstate eventually acquires components in the bath. It is paramount to note that while the system and bath are parts of a bigger system, when one defines the system and bath, they must be well-isolated and that the interaction between them is weak, but still significant enough to affect the system’s dynamics.

### 2.2.1 Decoherence in Quantum Computers

Decoherence is closely related to quantum computing because quantum information is encoded in the amplitudes and phases of qubits [21]. The decoherence of qubits will corrupt a computational task and eventually render a quantum computer powerless. When constructing a quantum computer, it is imperative that one must isolate the qubits from the environment at least to such an extent that the coherence time of a qubit is much larger than the average gate operation time. Currently, there are various quantum computer architectures, and
decoherence arises via processes unique to each architecture [14]. In publications, terminologies like dephasing, delocalization, relaxation, scattering, etc. are used rather liberally and confusingly to describe such processes. Fortunately, regardless of the architecture, one can phenomenologically describe the decoherence of a qubit with two timescales, $T_1$ and $T_2$. $T_1$ is the population relaxation time, and $T_2$ is the phase relaxation time.

While the original definition of $T_1$ and $T_2$ applies to nuclear magnetic resonance, the two timescales are widely used in other fields, such as electron spin resonance [22], quantum optics [23], and laser physics [24]. However, it is important to note that $T_1$ and $T_2$ are only average measures of decoherence, and usually, more system-specific parameters will be needed to describe the mechanisms that lead to decoherence. Yet, in practice, whenever one can define a two-level system, $T_1$ and $T_2$ are often sufficient to characterize decoherence.

### 2.2.2 Decoherence in a Qubit

We shall introduce the notion of decoherence in the case of a laser-driven qubit. This example closely demonstrates the dynamics of a trapped atomic-ion qubit, which is the subject of this thesis. The Hamiltonian of such a system has two parts, the qubit Hamiltonian, and the interaction Hamiltonian. The qubit Hamiltonian, $H_0$, is defined by:

$$H_0 = \frac{\hbar \omega_0}{2} (|0\rangle\langle 0| + |1\rangle\langle 1|) ,$$  \hspace{1cm} (2.4)

where $\hbar \omega_0$ is the energy separation between the ground state, $|0\rangle$, and excited state, $|1\rangle$, of the qubit. Assuming the transition is dipole-allowed, the interaction Hamiltonian, $H_{int}$, is as follows:

$$H_{int} = -\mathbf{d} \cdot \mathbf{E}(t) ,$$  \hspace{1cm} (2.5)
where $d$ is the dipole operator and $E(t)$ is the laser field. Now, $d$ can be written as

$$
\begin{align*}
    d &= \langle 0|d|1\rangle \langle 1| + \langle 1|d|0\rangle \langle 0| \\
    &= \langle 0|d|1\rangle \sigma + \langle 1|d|0\rangle \sigma^\dagger \\
    &= \langle 0|d|1\rangle (\sigma + \sigma^\dagger) \\
    &= d^{(+)} + d^{(-)},
\end{align*}
$$

(2.6)

where $\sigma$ and $\sigma^\dagger$ are the lowering and raising operators, and the last step only holds if we choose a real phase for the dipole matrix element $\langle 0|d|1\rangle$. Let us turn to separate the laser field, $E(t)$, into its positively rotating part, $E^{(+)}$, and negatively rotating part, $E^{(-)}$.

$$
\begin{align*}
    E(t) &= \hat{\epsilon} E_0 \cos(\omega t) \\
    &= \frac{\hat{\epsilon} E_0}{2} (e^{-i\omega t} + e^{i\omega t}) \\
    &= \frac{E_0}{2} (e^{-i\omega t} + e^{i\omega t}) \\
    &= E^{(+)}(t) + E^{(-)}(t),
\end{align*}
$$

(2.7)

where $\omega$ is the laser frequency, and $\hat{\epsilon}$ is the polarization vector.

The interaction Hamiltonian now becomes

$$
\begin{align*}
    H_{int} &= - (d^{(+)} + d^{(-)}) \cdot (E^{(+)} + E^{(-)}) \\
    &= -d^{(+)} \cdot E^{(+)} - d^{(+)} \cdot E^{(-)} - d^{(-)} \cdot E^{(+)} - d^{(-)} \cdot E^{(-)}. \\
\end{align*}
$$

(2.8)

Note that since $\sigma = |0\rangle \langle 1|$ evolves under $H_0$, $\sigma$ has time dependence $e^{-i\omega_0 t}$. Moreover, the fact that $d^{(+)} \sim \sigma$ implies $d^{(+)} \sim e^{-i\omega_0 t}$. Similarly, $d^{(-)} \sim e^{+i\omega_0 t}$. Recall the time dependence of the laser field, $E^{(\pm)} \sim e^{\mp i\omega_0 t}$. Therefore, by assuming $|\omega - \omega_0| \ll \omega + \omega_0$, we may apply the rotating-wave approximation (RWA). We
are then left with the slowly oscillating cross terms and get that
\begin{align}
H_{\text{int}} &= -\mathbf{d}^{(+)} \cdot \mathbf{E}^{(-)} - \mathbf{d}^{(-)} \cdot \mathbf{E}^{(+)} \\
&= -\langle 0 | \hat{\mathbf{e}} \cdot \mathbf{d} | 1 \rangle \frac{E_0}{2} (\sigma e^{i\omega t} + \sigma^\dagger e^{-i\omega t}) \\
&= \frac{\hbar \Omega}{2} (\sigma e^{i\omega t} + \sigma^\dagger e^{-i\omega t}),
\end{align}
(2.9)
where \( \Omega \equiv -\langle 0 | \hat{\mathbf{e}} \cdot \mathbf{d} | 1 \rangle \frac{E_0}{\hbar} \). Here, we assume that the laser is linearly polarized in the \( x \)-direction. Then, we may write the full Hamiltonian, \( H \), in matrix form:
\begin{equation}
H = \frac{\hbar}{2} \begin{pmatrix}
\omega_0 & \Omega e^{-i\omega t} \\
\Omega e^{i\omega t} & -\omega_0
\end{pmatrix}.
\end{equation}
(2.10)
Using the density-matrix formalism, the dynamics of the qubit is dictated by the von Neumann equation,
\begin{equation}
\dot{\rho} = [H, \rho].
\end{equation}
(2.11)
and
\[ i\hbar \dot{\rho}_{01} = -\hbar \delta \rho_{01} + \frac{\hbar \Omega}{2} (\rho_{11} - \rho_{00}). \] (2.15)

These equations describe a phenomenon known as Rabi flopping [23], where \( \Omega \) is the frequency at which the qubit rotates on the Bloch sphere. This process is completely coherent, meaning the qubit never decoheres. However, \( H \), as written in eq. 2.10, is an idealization because a qubit is never fully isolated. For instance, a trapped-ion qubit may be influenced by noise in the applied control fields [16].

In order to quantitatively determine how the external or bath degrees of freedom affect the coherence of a qubit, one must include them in the Hamiltonian, and then use techniques such as master equations or Fokker-Planck equations [23–25] to investigate the qubit’s dynamics. Even if the decoherence mechanisms are specified in the Hamiltonian, this problem is still quite complicated. However, it is phenomenologically observed that an idle qubit, starting from \( \rho_{11} = 1 \) and \( \rho_{00} = 0 \), under the influence of noise, approaches the equilibrium state according to an exponential law, i.e., \( \rho_{11} \sim \frac{1}{2} (e^{-t/T_1} + 1) \) and \( \rho_{00} \sim \frac{1}{2} (1 - e^{-t/T_1}) \). For the off-diagonal elements, e.g., starting in the state \((|0\rangle + |1\rangle)/\sqrt{2}\), a similar exponential decay with decay constant \( T_2 \) is observed, i.e., \( |\rho_{01}|^2 = |\rho_{10}|^2 = \frac{1}{2} e^{-t/T_2} \). We emphasize that although some of the matrix elements of \( \rho \) decay exponentially, the time evolution of the qubit is unitary, i.e., electrons are never lost from the computation and two-level system.

The two time constants, \( T_1 \) and \( T_2 \), are also observed in our simulation to be discussed in ch. 4. The two time constants can be obtained from experiments or numerical simulations, or calculated when sufficient knowledge about the bath is available. This simple approach successfully describes various phenomena, from NMR [26] to quantum optics [23].

We interpret \( T_1 \) as the timescale at which the qubit is brought to a global thermal
equilibrium, and $T_2$ as the timescale at which the qubit’s phase is damped. In general, inelastic processes contribute to both population-, i.e. $T_1$, and phase-relaxation, i.e. $T_2$, but elastic processes can cause the phases to decay separately from the population $[24]$. Therefore, $T_1$ is often longer than $T_2$ $[14, 24]$, which means $T_2$ is more important for setting limits to the coherence time in quantum computation.
Chapter 3

Idle Qubit in a Thermal Field

In this chapter, we investigate the effects of blackbody radiation on an idle atomic-ion qubit. We only consider an idle, instead of a driven, qubit because a qubit’s idle time is generally much longer than the average gate operation time in a quantum computational task. In particular, we are interested in how the coherence time of a qubit scales with temperature. We believe this investigation shows that temperature control is a valid means to suppress decoherence.

In sec. 3.1 we introduce the two common types of trapped-ion qubits, optical qubits and hyperfine qubits, and discuss their differences and similarities. Then, in sec. 3.2 we examine a relevant paper, which investigates delocalization of atoms in a cavity due to thermal noise, and its implications on our investigation.
3.1 Optical and Hyperfine Ion Qubits

There are mainly two types of trapped-ion qubits: (1) an optical qubit encoded in a ground state and an excited metastable state, separated by an optical frequency, and (2) a hyperfine qubit encoded in the ground-state and an excited hyperfine level, separated by a microwave frequency.

In the case of the $^{40}\text{Ca}^+$ ion, a commonly used optical qubit \cite{27, 28}, the computational levels, $|0\rangle$ and $|1\rangle$, are stored in atomic levels, $|D_{5/2}\rangle$ and $|S_{1/2}\rangle$, respectively \cite{27, 28}. The transition between the two levels is dipole forbidden; it is an electric quadrupole transition at $729\text{nm}$. The Rabi frequency, which represents the laser-ion coupling, is well-defined and can be derived similarly to the dipole case in sec. 2.2.2 by substituting the dipole transition elements with the quadrupole transition elements. For further details on quadrupole transitions, refer to ref. \cite{29}.

The $^{171}\text{Yb}^+$ ion is a popular hyperfine qubit \cite{17, 30}. Its computational levels, $|0\rangle$ and $|1\rangle$, are realized as the $|F = 0, m_F = 0\rangle$ and $|F = 1, m_F = 0\rangle$ hyperfine levels of the $S_{1/2}$ ground state. Like the $^{40}\text{Ca}^+$ qubit, the transition is dipole forbidden. It is a magnetic dipole transition at about $12.6\text{GHz}$. One way to drive this transition is to use microwaves. However, the wavelength of a $12.6\text{GHz}$ microwave is about $2cm$, which is much larger than the average qubit-qubit separation at $5\mu m$. Hence, it is impossible to address individual qubits, and perform single- or two-qubit operations with microwaves. To circumvent this problem, the coherent manipulation of qubits is performed with multiple lasers via stimulated Raman transitions.

A general stimulated Raman transition involves three levels, including the two computational levels, $|0\rangle$ and $|1\rangle$, each of which is dipole-coupled to a third high-
Figure 3.1: $\omega_p$ and $\omega_s$ are the frequencies of the pump and Stokes lasers, respectively. In order to ensure fast population transfer from $|2\rangle$ to $|0\rangle$ or $|1\rangle$, the detunings must be chosen such that $\Delta \gg \delta$.

lying auxiliary level, $|2\rangle$. The level diagram of a Raman system is depicted in fig. 3.1. For instance, in order to transfer population from $|0\rangle$ to $|1\rangle$, we use a pump laser to couple $|0\rangle$ to $|2\rangle$, and a Stokes laser to couple $|2\rangle$ to $|1\rangle$. Note that there are two detunings, $\delta$ and $\Delta$, which are determined by various factors, including suppressing gate errors due to spontaneous scattering [31], controlling the light shifts caused by the Raman lasers, and optimizing the Raman transition rate [32].

Now we show that the dynamics of this three-level system is effectively that of a two-level system involving just the computational levels. Similar to the example in 2.2.2, we can split the Hamiltonian into the atomic part and the interaction part. The atomic Hamiltonian is

$$H_0 = \epsilon_0 |0\rangle \langle 0| + \epsilon_1 |1\rangle \langle 1| + \epsilon_2 |2\rangle \langle 2|,$$

(3.1)

where $\epsilon_i$ is the energy of $|i\rangle$. In the RWA, the interaction Hamiltonian becomes

$$H_{int} = \frac{\hbar \Omega_p}{2} (e^{-i\omega_p t} |2\rangle \langle 0| + h.c.) + \frac{\hbar \Omega_s}{2} (e^{-i\omega_s t} |2\rangle \langle 1| + h.c.),$$

(3.2)
where $\Omega_p = \frac{\langle 0|d E_p^2|2 \rangle}{\hbar}$ and $\Omega_s = \frac{\langle 1|d E_s^2|2 \rangle}{\hbar}$ are the Rabi frequencies for the transitions induced by the pump-laser and Stokes-laser, respectively. Now, we enter the lasers’ rotating frame, which is defined by:

$$|\tilde{\Psi}\rangle = U^\dagger |\Psi\rangle,$$

and

$$\tilde{H} = U^\dagger H U + i\hbar \frac{\partial U^\dagger}{\partial t} U,$$

where $U = e^{i\omega_p t}|0\rangle\langle 0| + e^{i\omega_s t}|1\rangle\langle 1|$. After performing the transformations, the Raman system’s Hamiltonian becomes

$$\tilde{H} = (\epsilon_0 + \hbar \omega_p)|0\rangle\langle 0| + (\epsilon_1 + \hbar \omega_s)|1\rangle\langle 1| + \epsilon_2|2\rangle\langle 2| + \frac{\hbar \Omega_p}{2}(|2\rangle\langle 0| + h.c.) + \frac{\hbar \Omega_s}{2}(|2\rangle\langle 1| + h.c.)$$

$$= [(\epsilon_1 - \epsilon_0 + \hbar (\omega_p - \omega_s))|1\rangle\langle 1| + (\epsilon_2 - \epsilon_0 - \hbar \omega_p)|2\rangle\langle 2|$$

$$+ \frac{\hbar \Omega_p}{2}(|2\rangle\langle 0| + h.c.) + \frac{\hbar \Omega_s}{2}(|2\rangle\langle 1| + h.c.)$$

$$= -\hbar \delta|1\rangle\langle 1| - \hbar \Delta|2\rangle\langle 2| + \frac{\hbar \Omega_p}{2}(|2\rangle\langle 0| + h.c.) + \frac{\hbar \Omega_s}{2}(|2\rangle\langle 1| + h.c.).$$

The second step is justified by setting the energy of $|0\rangle$ to be zero. In general, we may write the system’s time-dependent quantum state as $|\tilde{\Psi}(t)\rangle = c_0(t)|0\rangle + c_1(t)|1\rangle + c_2(t)|2\rangle$. After applying $\tilde{H}$ to $|\tilde{\Psi}(t)\rangle$, we get the following equations of motion:

$$i\dot{c}_0(t) = \frac{\Omega_p}{2} c_2(t),$$

$$i\dot{c}_1(t) = -\delta c_1(t) + \frac{\Omega_s}{2} c_2(t),$$

$$i\dot{c}_2(t) = \frac{\Omega_p}{2} c_0(t) + \frac{\Omega_s}{2} c_1(t) - \Delta c_2(t).$$

By assuming $\Delta \gg \Omega_p, \Omega_s$, and $\delta$, the population in $|2\rangle$ oscillates much faster than those in $|0\rangle$ and $|1\rangle$. Therefore, we may assume that over a large number of cycles, when the system reaches steady state, $\dot{c}_2(t) = 0$. Then, we can effectively reduce the dynamics of the three-level system to that of a two-level system, where the
effective Hamiltonian becomes

\[ H_{\text{eff}} = \frac{\hbar}{4} \left( \frac{\Omega_p^2}{\Delta} \frac{\Omega_s^2}{\Delta} - \frac{\Omega_p^2}{4\Delta} - 4\delta \right). \] (3.4)

The off-diagonal element is the Rabi frequency, and it represents the coupling between the computational levels. The diagonal terms are the light-shift or AC-Stark-shift terms, where the differential Stark shift of the transition frequency, \( \omega_0 \) in fig. 3.1, is \( \frac{\hbar(\Omega_p^2 - \Omega_s^2)}{4\Delta} \).

It is important to note that in an experiment, a Raman system often involves more than two lasers because the pump laser may couple \( |0\rangle \) to more than one high-lying state, and hence, more than one Stokes laser are needed to transfer the population back to \( |1\rangle \) [30]. Compared to the single-laser manipulation of an optical qubit, the Raman manipulation of a hyperfine qubit is much more complicated. Nonetheless, one of the reasons why hyperfine qubits are attractive to many experimentalists is that they have much longer radiative lifetimes [27, 33].

### 3.2 The 1/T law

Blümel et al. showed that under the influence of thermal noise, and in the high-temperature limit, the delocalization time of Rydberg atoms in a waveguide scales like \( 1/T \), where \( T \) represents temperature [25]. We conjecture that the coherence time of a qubit has the same scaling law.

They derived and solved a Markovian master equation for microwave-driven Rydberg atoms in a noisy waveguide. In particular, they introduced thermal noise into the waveguide by leaving two ends in one direction open. The master equation’s analytical solutions for the atoms’ density matrix elements indicates that
the decay times of the diagonal and off-diagonal elements, which represent the population and phases, scales inversely proportional to temperature or noise intensity. They further corroborated their theoretical predictions with numerical simulations, and experiments using Rubidium atoms.

One might point out that since they included more than two atomic levels in their simulations, the results might not hold true for qubits. However, their simulations were only for a specific case of the analytical solutions. The solutions themselves are general enough to be applied to a non-driven, idle qubit. Another difference is that in their theory, the atomic levels were coupled to the thermal field via dipole-interactions, while as shown in sec. 3.1 the computational levels of a qubit are not. However, decoherence is not restricted just to the two computational levels. One must keep in mind that an idle qubit is an atomic ion, which consists of many levels that are dipole-coupled to the computational levels. Therefore, as it decoheres, the populations in the computational levels can leak into other levels via dipole-interactions, regardless of the type of ion-qubit. Lastly, since Rubidium shares similar atomic structure to ion qubits, such as Calcium, Barium, and Ytterbium ions, they have similar decoherence mechanisms due to thermal effects. Therefore, we believe an idle ion-qubit’s coherence time should also scale inversely proportional to temperature.
Chapter 4

Noise Simulation

In this chapter, we develop a numerical model for an idle qubit in a blackbody field, and present the scaling laws of the qubit’s coherence time versus temperature. Then, we compare our results with the $1/T$ law discussed in sec. 3.2. We adopt a different approach here from that used in the $1/T$ law paper [25]. We employ the wavefunction formalism, instead of the density-matrix formalism.

In particular, we create a microscopic, stochastic model, which mimics the effects of blackbody radiation on a qubit. Then, we perform a Monte Carlo simulation of the quantum motion of a qubit, according to the model. After ensemble-averaging, we glean the expected dynamics of the qubit.
4.1 Microscopic Noise Model

The spectral energy density of blackbody radiation follows Planck’s distribution:

$$U(\omega) = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{e^{\frac{kT}{\hbar \omega}} - 1},$$  \hspace{1cm} (4.1)

where \(\omega\) is the frequency of radiation, \(c\) is the speed of light, and \(T\) is the temperature. As shown by Einstein, the Planck distribution arises from the spontaneous emission, stimulated emission, and absorption of ensembles of atoms, each of which has discrete energy levels with distinct frequencies [31]. Moreover, the population is distributed over the energy levels in accordance with the Maxwell-Boltzmann distribution.

Let us now consider a qubit placed in a blackbody field. We know from the example of a driven dipole qubit in sec. 2.2.2 that off-resonant radiation can induce transitions between two quantum levels. Therefore, in our simulations, we obtain accurate quantum dynamics by imposing a bandwidth of radiation around the resonant frequency of the qubit.

4.1.1 Simulated Equation of Motion

The equation of motion that describes a qubit interacting with a bandwidth of radiation is:

$$i\hbar \frac{d}{dt} |\Psi(t)\rangle = \left[ \frac{\hbar \omega_0}{2} \sigma_z - \sum_n d \cdot E_n \cos(\omega_n t + \phi_n) \sigma_x \right] |\Psi(t)\rangle,$$  \hspace{1cm} (4.2)

where \(d\) is the dipole matrix element of the qubit, \(E_n\) is the field strength of the radiation at frequency \(\omega_n\), \(\phi_n\) is the phase, and \(\sigma_i\) represents the Pauli matrix in the \(i\)th direction. We non-dimensionalize the equation by letting \(\tau = \omega_0 t\) and
obtain
\[
\dot{i}|\Psi(\tau)\rangle = \left[ \frac{1}{2}\sigma_z - \sum_n \frac{d_n}{\hbar\omega_0} \cos\left(\frac{\omega_n}{\omega_0}\tau + \phi_n\right)\right] |\Psi(t)\rangle
\]
\[
= \left[ \frac{1}{2}\sigma_z - \sum_n \Gamma_n \cos(r_n\tau + \phi_n)\right] |\Psi(t)\rangle,
\] (4.3)

where \(\Gamma_n\) is the ratio between the dipole-interaction energy and transition energy, and \(r_n\) is the ratio between the radiation frequency and the transition frequency. According to this equation, the drives at all frequencies are always in phase. However, this is not the case for blackbody radiation. According to Einstein, blackbody radiation is due to emissions from atoms. Moreover, the times at which different atoms radiate are random. Therefore, the phase relations between different modes of radiation must be time-dependent and stochastic. Accordingly, we modify eq. 4.3 according to:

\[
\dot{i}|\Psi(\tau)\rangle = \left[ \frac{1}{2}\sigma_z - \sum_n \Gamma_n \cos(r_n\tau + \phi_n)\right] |\Psi(\tau)\rangle,
\] (4.4)

where each \(\phi_n\) independently jumps from one value to another value, in the interval between 0 and 2\(\pi\). Therefore, the waiting time between any two consecutive jumps follows the Poisson distribution.

Eq. 4.4 does not describe decoherence because the dynamics described by it remains within the Hilbert space defined by the qubit’s levels. Moreover, the dynamics is completely coherent because between any two consecutive phase jumps, the qubit is just driven by a complicated, but coherent multi-frequency field. The effect of decoherence is revealed only after ensemble-averaging over different realizations of eq. 4.4 with different the random phases. This is equivalent to the ensemble-averaging carried out in the mixed-state density matrix.
4.2 Methodology and Results

There are three independent parameters in the simulation: $\Gamma_n$, $r_n$, and the mean waiting time between two consecutive phase jumps. The $\Gamma_n$'s are assumed to be constant for a narrow band of drive frequencies, which ranges from $0.95\omega_0$ to $1.05\omega_0$. Therefore, hereafter, we drop the subscript of $\Gamma_n$, and write $\Gamma$. We coarse-grain the frequency-band and include 50 equally spaced frequencies about the transition frequency. The phases, denoted by the $\phi_n$'s, are generated by a uniform random number generator between 0 and $2\pi$. In order to generate a Poissonian-distributed waiting time about its mean, we use an exponentially distributed random number generator. We denote the mean waiting time by $\tau_j$, and it is also the temporal coherence length of the driving field.

4.2.1 $T_2$ time

From the discussion in sec. 2.2.2, we know that $T_2$ is the decay time for the off-diagonal elements of the qubit’s density matrix. Suppose we initialize the qubit in a pure state, $|\Psi\rangle$. After subjecting it to a random process, it evolves into a mixed state, which is represented by $\sum_j P_j |\Psi_j\rangle\langle\Psi_j|$, where $P_j$ is the probability of $|\Psi_j\rangle$. We may obtain the probability distribution by collecting statistics over multiple simulations, in which case $P_j = 1/N$, where $N$ is the number of simulations. If $|\Psi_j\rangle = \alpha_j |0\rangle + \beta_j |1\rangle$, then we are interested in $\langle \alpha_j \beta_j^* \rangle$, where the angle brackets denote ensemble-averaging, as a function of time.

In our $T_2$-simulations, we set $N = 200$, and $\alpha = 1/\sqrt{2}$ and $\beta = 1/\sqrt{2}$ at $\tau = 0$. For each set of 200 simulations, we fix the values of $\Gamma$ and $\tau_j$. To demonstrate how we extract the corresponding $T_2$ time, we look at the data generated by the...
Figure 4.1: The off-diagonal element (coherence) as a function of time $\tau$ for $(\Gamma, \tau_j) = (0.02, 1)$. *Left:* Imaginary part of $\alpha_j\beta_j^\ast$ as a function of $\tau$. *Right:* Real part of $\alpha_j\beta_j^\ast$ as a function of $\tau$. Notice the revivals in the qubit’s coherence; decoherence does not occur for this single simulation.

Over one realization of the random phases $\phi_n$, we see that the qubit’s relative phase experiences revivals after it first reaches zero (see fig. 4.1).

After ensemble-averaging, we obtain an exponential dephasing of the qubit. The average squared magnitude of the qubit’s relative phase, $\langle|\alpha_j\beta_j^\ast|^2\rangle$, is plotted against time in fig. 4.2. We then fit an exponential function to the decay curve, and extract $T_2$.

The goal is to relate $T_2$ to temperature. We do this by recognizing that $|E_n|^2$ and hence, $|\Gamma|^2$ is proportional to the spectral energy density in eq. 4.1. In the high-temperature limit, we may expand the exponential term in eq. 4.1 up to first order, and obtain a linear relation between the spectral energy density and temperature $T$. Therefore, by assuming that $\tau_j$ is independent of temperature, we extract the temperature scaling of $T_2$ by running simulations at various values of $\Gamma$ and a fixed value of $\tau_j$. Then, we repeat the same procedures for various
Figure 4.2: The coherence of the qubit as a function of $\tau$ for $(\Gamma, \tau_j) = (0.02, 1)$, averaged over 200 simulations. **Left:** $\langle |\alpha_j\beta_j^*|^2 \rangle$ as a function of $\tau$; we observe an exponential decay of this quantity. **Right:** Same as the figure on the left, but plotted on a semi-log scale to prove the decay is indeed exponential. We fit the curve on the left with an exponential function $0.25e^{-0.022\tau}$, where $T_2 = 1/0.022\tau^{-1} \approx 45.45\tau^{-1}$.

$\tau_j$'s: 0.1, 0.3, 0.5, 1, $\pi$, 2$\pi$, 10, 20, and 30. We find that for $\tau_j = 0.1, 0.3, 0.5, \text{and } 1$, $T_2 \propto \Gamma^{-2}$ and hence, $T_2 \propto 1/T$. And then, as we increase $\tau_j$, the scaling becomes sublinear. In particular, $T_2 \propto T^{-0.9}$ for $\tau_j = \pi$, and $2\pi$, while $T_2 \propto T^{-0.8}$ for $\tau_j = 10, 20 \text{ and } 30$ (see fig. 4.4). Therefore, we find that the $1/T$ law only holds for small $\tau_j$-values.

4.2.2 $T_1$ time

The initial qubit state in our $T_1$-simulations is $|\Psi\rangle = |0\rangle$. Therefore, in this case, $T_1$ is the decay time for the population in $|0\rangle$. We chose $N = 10$, which is much smaller than that in our $T_2$-simulations, because $T_1$ is much longer than $T_2$ and we are limited by our computational power. Since we are averaging over
Figure 4.3: The $T_2$ times as a function of $\Gamma^2$ on a log-log scale. The solid lines are the fits, and the dots are the data. **Left:** The curves here depict a linear relation between $T_1$ and temperature. From top to bottom, $\tau_j = 0.1, 0.3, 0.5,$ and $1$. **Right:** The curves here depict sublinear relations between $T_1$ and temperature. From top to bottom, $\tau_j = \pi, 2\pi, 10, 20,$ and $30$. Note that the data for $\tau_j = 20$ and 30 closely overlap and can be fitted to the same function.

Figure 4.4: The exponents, denoted by $\gamma$, of the $T_2$-power laws are plotted against $\tau_j$. 
Section 4.3. Discussion

Figure 4.5: Here, \((\Gamma, \tau_j) = (0.005, 0.5)\). \(<|\alpha_j|^2|\) is plotted against \(\tau\). While we observe a decay, it is impossible to tell whether it is exponential due to a small ensemble size. According to our definition, \(T_1 \approx 2194\tau^{-1}\).

In a small ensemble, the exponential behavior of the decay is not revealed (see fig. 4.5). Therefore, we resort to a different definition: \(T_1\) is the time at which the population in \(|0\rangle\) first reaches 0.5.

We then extract the temperature scaling of \(T_1\) using the same method we employ to obtain that of \(T_2\). Similarly, we find that \(T_1\) is related to temperature via power laws, meaning \(T_1 \propto T^{-\gamma}\), as depicted in fig. 4.6. However, contrary to the \(T_2\) results, the exponents \(\gamma\), fluctuate between 0.8 and 1, and do not correlate with \(\tau_j\) (see fig. 4.7). The random behavior of the exponents may be due to the small ensemble size.

4.3 Discussion

The deviations from the \(1/T\) law may be due to various assumptions and simplifications made in our model. First, in our coarse-graining model for the drive frequencies, the frequencies are equally spaced. The regularity of drive frequencies
Figure 4.6: $T_1$ times as a function of $\Gamma^2$ represented in the form of a log-log plot. The solid lines are the fits, and the dots are the data. *Left:* The curves here depict a linear relation between $T_1$ and temperature. From top to bottom, $\tau_j = 0, 0.5, 1, \pi,$ and $2\pi$. *Right:* The curves here depict a sublinear relation between $T_1$ and temperature. From top to bottom, $\tau_j = 0.05, 0.3, 10, 20,$ and $30$.

Figure 4.7: The exponents, denoted by $\gamma$, of the $T_1$-power laws are plotted against $\tau_j$. 
may lead to more regular dynamics, and for instance, extend the coherence time of the qubit. However, we have tried (1) selecting 10, and 20 frequencies, instead of 50, (2) 50 random frequencies within the bandwidth, and (3) one slightly detuned frequency with 50 different $\phi_n$’s. The results are similar to that obtained in the original model in the following sense: for (1) and (2), the exponential decays remain, and for (3), both the exponential decays and power laws remain.

Moreover, our temperature measure may be an oversimplification. We assumed that (1) $\Gamma^2 \propto U(\omega)$, and that (2) the bandwidth of the drive, $\tau_j$, and $\Gamma^2$ are decoupled. Here, we discuss both (1) and (2) briefly. More detailed investigations are beyond the scope of this thesis. Let us consider (1), which implies that $\left( \int U(\omega) d\omega \right)^{1/2} \propto \int \Gamma^2(\omega) d\omega$. We can write $\int \Gamma^2(\omega) d\omega$ as a Riemann sum $\sum_n \Gamma_n^2 \Delta\omega$. We notice that $\Delta\omega$ is missing in the simulated equation of motion (see eq. 4.4). This may lead to issues with dimension and normalization. Assumption (2) can be violated in reality. In our simulations, the bandwidth of the drive is acting as the linewidth of the transition, and the linewidth can be coupled to $\Gamma^2$ via power broadening [35]. Moreover, the linewidth can be used as a band-pass filter, and the temporal coherence length of the filtered light is inversely proportional to the linewidth [20]. Therefore, whether our model captures the essence of blackbody radiation’s effect on a qubit awaits further investigation.

Nonetheless, we have simulated Poisson noise, or electric shot noise. Poisson noise originates from current fluctuations due to the discreteness of the electrical charge, and is independent of temperature and frequency. There is a temperature-dependent electric noise known as Johnson-Nyquist noise. These two kinds of electric noise lead to heating and breaking of Coulomb crystals in ion traps [15]. At high temperature, Johnson-Nyquist noise dominates over shot noise. At high frequency and low temperature, shot noise becomes dominant. In order to im-
prove trapping, experimentalists are moving away from room temperature to low temperature settings, which means the effect of shot noise needs to be taken into consideration. In the context of shot noise, $\Gamma^2$ represents noise intensity, instead of temperature, which means the power laws depicted in fig. 4.6 and 4.3 relate a qubit’s coherence time to noise intensity.

In addition to the noise-intensity power laws, we discover an unexpected power-law relationship between $T_2$ and $\tau_j$ (see fig. 4.8), where $T_2 \propto \tau_j^{-0.8}$ for $\Gamma = 0.007, 0.01$. The fact that $T_2$ is inversely related to $\tau_j$ can be explained by the power spectra, shown in fig. 4.8. The power spectrum becomes sharper about the transition frequency as $\tau_j$ increases, which means the drive field becomes more efficient at inducing the transition. Note that $\tau_j$ represents the temporal coherence length of the noise field. Since both field intensity and field correlation are experimentally measurable quantities, all the power laws presented in this thesis could be used to estimate the coherence time of a qubit under the influence of shot noise.
Chapter 5

Outlook

In this thesis, we have presented scaling laws for the coherence times, $T_1$ and $T_2$, of a qubit, under the influence of blackbody noise and shot noise. The scaling laws relate $T_1$ and $T_2$ to the intensity of noise via power laws. In the short term, the shot-noise scaling is relevant since ion-trap experiments are moving to low temperature settings. The blackbody noise is not dominant in the current era of quantum computers [15] because a qubit’s coherence time under blackbody noise is much longer than any computation executable in the current era of quantum computers. However, in the long term, useful quantum computation will easily involve millions of gate operations, which will take a long enough time for blackbody-induced decoherence to occur [25]. Therefore, this thesis acts as a guide for experiments, both in the near term and the more distant future.

Aside from noise-scaling, we are also interested in seeing how the coherence time of coupled qubits scales with the number of qubits, $N$. It is commonly believed that by assuming that the probability of decoherence for each qubit is independent of the coupling, the coherence time of qubits scales like $1/N$. By including more
qubits in our simulation, we may test the validity of this argument.
Appendix A

The Density Matrix

Here, we will briefly discuss the motivations behind using the density matrix, and introduce the theory of density matrix. For a detailed report, Dirk Ter Haar’s review article [36] serves as a great reference.

The density matrix was introduced by John von Neumann in 1927 [37] to describe statistical concepts in quantum mechanics. Just like in classical systems, quantum systems often consist of many particles, or involve many degrees of freedom. It is impossible to fully describe the state of these systems. However, statistical physics enables us to closely predict the general behaviors of these systems via statistical ensembles. The density matrix, which is an ensemble of wave functions, is the quantum analogue to the classical ensemble.
Appendix A. The Density Matrix

A.1 The Pure-State Density Matrix

A quantum system, for which the wave function is known, is called a pure state. We shall show the connection between a wave function and the density matrix via a generic two-level system. We may write the state as:

\[ |\Psi\rangle = \alpha |0\rangle + \beta |1\rangle, \]  

(A.1)

where \( \alpha \) and \( \beta \) are the complex amplitudes. The density matrix is the outer product \( |\Psi\rangle \langle \Psi| \), which in matrix form is

\[
\rho = \begin{pmatrix}
0 & 1 \\
0 & \alpha^2 + \beta^2
\end{pmatrix}
\begin{pmatrix}
\alpha^* \\
\beta^*
\end{pmatrix}
= \begin{pmatrix}
\rho_{00} & \rho_{01} \\
\rho_{10} & \rho_{11}
\end{pmatrix}.
\]  

(A.2)

The expectation value of an observable, \( O \), is then

\[
\langle O \rangle = \langle \Psi | O | \Psi \rangle = \langle \alpha^* | O | \beta \rangle = \rho_{00}O_{00} + \rho_{01}O_{01} + \rho_{10}O_{10} + \rho_{11}O_{11}
\]  

(A.3)

Thus far, we see that when we can fully specify the wave function, there is no real advantage of using the density matrix over the wave function approach.
A.2 The Mixed-State Density Matrix

The main reason for using the density matrix is that, for systems with many
degrees of freedom, we often do not know their exact wave function. We only know
certain statistical properties of the system. These properties are incorporated into
the definition of the density matrix:

$$\rho = \sum_k P_k |\Psi_k\rangle \langle \Psi_k|,$$

(A.4)

where $P_k$ is the probability with which the system assumes the state $|\Psi_k\rangle$. It
can be easily shown that $\langle O \rangle = Tr [\rho O]$ as well. Perhaps an example will help
demonstrate a simple application of the density matrix. Suppose we have prepared
$N$ copies of the exact same two-level system such that the density matrix of this
ensemble is eq. A.2. We subject each two-level system to a random, unknown
process, and now the ensemble’s density matrix is given by:

$$\rho = \sum_k P_k \left( |\alpha_k|^2 \begin{pmatrix} \alpha_k \beta_k^* \\ \beta_k^* \alpha_k \end{pmatrix} |\beta_k|^2 \right).$$

(A.5)

Suppose each outcoming state obtained a phase $\phi_k$ between the otherwise un-
changed amplitudes, and the phases are uniformly distributed such that

$$|\Psi_k\rangle = \alpha |0\rangle + e^{i\phi_k} \beta |1\rangle.$$

(A.6)

The off-diagonal element or coherence $\rho_{01}$ becomes

$$\rho_{01} = \sum_k P_k \alpha_k \beta_k^*$$

$$= \frac{\alpha \beta^*}{N} \sum_k e^{-i\phi_k}$$

$$\approx \frac{\alpha \beta^*}{2\pi} \int_0^\infty e^{-i\phi} d\phi$$

$$= 0.$$

(A.7)
This example shows that a pure state can evolve into a mixed state, and that the coherence may decay to zero, i.e. decoherence. Note that (1) each $|\Psi_k\rangle$ may retain its coherence, and that (2) only after averaging do we see decoherence.
Bibliography


