A Generalization of Tree Decomposition to Amalgamation Classes of Finite Structures

by

Miriam Parnes

Advisor: Cameron Donnay Hill

Assistant Professor of Mathematics

Wesleyan University

Middletown, CT

May, 2019

A Dissertation in Mathematics

submitted in partial fulfillment of the

requirements for the degree of Doctor of Philosophy

For my mother, Lois J. Radisch. For everything.

Abstract

A tree decomposition of a graph G is a tree such that each vertex of the tree is a subset of the vertices of G satisfying particular properties. The tree width of a graph comes from its tree decompositions and is a measure of how tree-like the graph is. Many questions which are hard to answer about graphs in general become easy to answer about classes of graphs when their tree width is bounded. It was shown by Seymour and Thomas in 1993 that for a particular game of Cops and Robbers played on graphs, k cops have a winning strategy for the game on the graph G if and only if G has tree width less than k. We generalize tree decomposition, tree width, and the Cops and Robbers game to certain amalgamation classes of finite structures.

Amalgamation classes are classes of finite or countable structures which satisfy specific properties guaranteeing the existence of a "generic model" – a unique countable structure $\mathcal{M}$ such that its substructures are precisely the elements of the class and any isomorphism between two finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. The class of all finite graphs is an example of a type of amalgamation class called a Fraïssé class. (It is an algebraically trivial Fraïssé class because its generic model, the random (Rado) graph, has no "special" points or finite sets of points.) We show that having a particular kind of independence relation, which we call an abstract free amalgamation relation, on the finite substructures of the generic model of an algebraically trivial Fraïssé class allows a unique decomposition of each finite substructure into components. Using this decomposition, we are able to define tree decomposition, tree width, and the Cops and Robbers game on elements of an algebraically trivial Fraïssé class with an abstract free amalgamation relation and show that the analogue of Seymour and Thomas's result holds in this setting.

Acknowledgements

First of all, I must thank my advisor, Cameron D. Hill, for his continuing patience, guidance, and support – thank you for finding me a problem that suited my dual interests so well and for not becoming too frustrated with me when things took longer than they should (even if I was frustrated with myself). I also thank my committee members, Philip Scowcroft and Dan Licata, for their patience (yes, this is a recurring theme) and willingness to carefully read what became a much longer document than I expected, in a much shorter span of time than any of us would have liked.

More generally, I need to thank the entire Department of Mathematics and Computer Science, students, faculty, and staff, for making Wesleyan the right place for me. In particular, thank you to my year-mates, Cameron Bishop and Lisa Kaylor – I'm so glad that we're completing this journey together, and my initial academic advisor and friend, David Pollack, for being there for me from the beginning.

I also want to thank my Wesleyan support network beyond the department: Cheryl Hagner and Barb Schukoske from the Office of Graduate Student Services, Dr. Jennifer D'Andrea from CAPS, and Meredith Hughes, the faculty advisor for Wesleyan Women in Science. Thanks to each of you for being there for me in times of both trouble and triumph.

It has been a very long journey to get to this point and there are any number of people who have played important roles in my life along the way. I would like to be able to thank each one of you individually, but I know that if I try, the Acknowledgements would probably end up being longer than the rest of the thesis and I would still, inevitably, miss someone. But please know that I do appreciate each of you and that I am thinking of you as I write this.

Finally, I have to thank my family. Thank you to my boyfriend, Vince – my life has been so much better since you came into it, I'm so glad you found me. To my sister, Anna – thank you for cheering me on throughout my journey and talking me off more than one proverbial cliff. And to my mother, Lois Radisch – any words I try to use to express my gratitude for your enduring love and support would fall hopelessly short, but thank you for everything; I love you.

Contents

Chapter 1

Introduction

Model theory is a branch of mathematical logic which investigates properties which hold across different kinds of mathematical structures. One of the original goals of modern model theory was to identify families of structures whose members can be classified by relatively simple combinatorial invariants. This kind of thinking has allowed model theorists to extend results known for one kind of structure to classes of structures of very different sorts that have just a few similar combinatorial properties. The goal of my thesis project is of this type; the aim is to generalize the concepts of tree decomposition, tree width, and a particular Cops and Robbers game from graphs to other kinds of structures.

The notions of tree decomposition and tree width came to prominence in Seymour and Robertson's graph minors project in the 1980s. The tree width of a graph is an important measure of how connected the graph is. Many questions which are hard to answer about graphs in general become easy to answer about classes of graphs when their tree width is bounded. The tree width of a graph G comes from its tree decompositions, certain trees associated with the graph. We will define tree decomposition and tree width more precisely in Section 2.2. In 1993, Seymour and Thomas showed that for a particular game of cops and robbers played on graphs, k cops have a winning strategy for the game on the graph G if and only if G has tree width less than k [6]. We will introduce an equivalent version of this game in Section 2.3.

Our goal is to generalize these concepts to certain amalgamation classes of finite structures, specifically algebraically trivial Fraïssé classes. This is a natural goal because the class of all finite graphs is itself an algebraically trivial Fraïssé class. In Section 3.2, we'll define amalgamation classes and Fraïssé classes. In Section 3.3, we'll see an existing generalization of tree decomposition and tree width to this setting, but also why we might want to find a different one. Since the ideas of paths and edges are central to the definitions of tree decomposition and the Cops and Robbers game, we will need to find a way to replace these ideas in classes where they may not be present. In Section 3.4, we will introduce an independence relation, which we will call an *abstract free amalgamation relation*, that will allows us to do this.

Using this independence relation and the idea of components, we will be able to generalize the Cops and Robbers game in a relatively natural way in Chapter

4. The generalization of tree decomposition which we introduce in Section 5.2 is less natural, but nevertheless seems to be the "right" one, since, as we show in Chapter 6, using these definitions allows us to prove the analogue of Seymour and Thomas's result. Along the way, we will see that elements of algebraically trivial Fraïssé classes with an abstract free amalgamation relation are, in some ways, very graph-like. This will allow us to prove an analogue of Menger's Theorem, a classic result of graph theory, in Chapter 7.

Chapter 2

Graph Theory Background

This thesis is concerned with generalizing certain concepts from graphs to other types of structures (which we will define more precisely in the next chapter). The first section of this chapter will quickly review some basic ideas from graph theory and define some of our notation. (For more thorough coverage, see the first chapter of *Introduction to Graph Theory*by Douglas West [7].) The second and third sections give more in-depth introductions to the primary concepts we are trying to generalize, namely tree decomposition and a particular game of Cops and Robbers played on graphs. In section four, we will see the graph-theoretic result connecting these ideas, the analogue of which in a broader context is the main goal of this thesis.

2.1 Basics

We are interested in generalizing certain concepts from graphs to other types of structures. While some of these concepts are defined for graphs with multiple edges or loops, or for directed graphs, we restrict our attention to simple, undirected graphs. Thus, if $V(G)$ is the vertex set of a graph G, we may consider the edge set, $E(G)$, to consist of two-element subsets of $V(G)$.

Since we will often be interested in the subsets of a particular size of another set, we define the following notation:

Convention. For a set A and integer k, we denote by $\binom{A}{k}$ the set of all subsets of A of size exactly k. $\binom{A}{\leq k}$ will be used to denote the set of all subsets of A of size less than or equal to k.

Using this shorthand, we have that for a graph G, $E(G) \subseteq \binom{V(G)}{2}$. If $\{u, v\} \in E(G)$, then we say that u and v are *adjacent* in G and call u and v *endpoints* of the edge. If $u \in V(G)$, then each $v \in V(G)$ such that $\{u, v\} \in E(G)$ is a *neighbor* of u.

In addition to being simple, graphs we consider will be finite (that is, they will have a finite set of vertices) unless otherwise noted.

A few specific kinds of graphs that are of interest to us are paths, cycles, and complete graphs. A *path* is a graph whose vertices can be listed (with no repeats) so that they are adjacent if and only if they appear consecutively in the list. So if $|V(G)| = n+1$, we can label the vertices $v_0, v_1, ..., v_n$ such that $E(G) = \{\{v_i, v_{i+1}\} | 0 \leq i \leq n-1\}$. In this case, v_0 and v_n are the *endpoints* of the path. We denote the path with n vertices by P_n. A *cycle* is a graph C such that if we remove any single edge $\{x, y\}$ from $E(C)$, the resulting graph is a path with endpoints x and y. So, equivalently, a cycle is a graph whose vertices can be listed (with no repeats) so they are adjacent if and only if they either appear consecutively in the list or are the first and last elements of the list. We denote the cycle with n vertices by C_n. We say that a graph is *complete* if $E(G) = \binom{V(G)}{2}$, that is, if it has all possible edges between its vertices. We denote the complete graph on n vertices by K_n.

We are particularly interested in the above types of graphs when they occur (or, sometimes, when they do not occur) as subgraphs of a given graph. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G) \cap \binom{V(H)}{2}$, that is, we keep some set of vertices from G and some of the edges which have both endpoints in that set of vertices. If $A \subseteq V(G)$, the *induced subgraph* of G on A is the subgraph H with $V(H) = A$ and $E(H) = E(G) \cap \binom{A}{2}$, where we keep *all* of the edges with both endpoints in A. We will most often be interested in the induced subgraph on the remaining vertices of G when some subset of vertices is taken away. If $C \subseteq V(G)$, we denote the induced subgraph on $V(G) \setminus C$ by $G - C$.

When a complete graph occurs as a subgraph of a graph G, we call the subset of $V(G)$ forming the vertices of that complete graph a *clique* in G. When a path P occurs as a subgraph of G, if u and v are the endpoints of the path, then we say that P is a u, v-path in G. When there is a u, v-path in G for each pair of vertices $u, v \in V(G)$, we say that G is *connected*. We will be particularly interested in the *components* of a graph, which are its maximal connected subgraphs. Most often, we will consider the components of $G - C$ for some $C \in V(G)$. For reasons that will become clear later, we may also refer to these as the components of G over C. Seymour and Thomas call the vertex set of a component H of $G - C$ a C-*flap* [6].

Since we are working only with simple graphs, we can specify a path or a walk in a graph G just by giving a sequence of vertices, without explicitly mentioning the edges between them. For vertices $u, v \in V(G)$, a u, v-*walk* in G is a sequence of vertices of G $x_0, x_1, ...x_n$ such that $x_0 = u$, $x_n = v$, and for each $0 \leq i \leq n-1$, $\{x_i, x_{i+1}\} \in E(G)$. We can understand a u, v-path in G to be either a subgraph of G as previously defined or a u, v-walk in G such that $x_i \neq x_j$ whenever $i \neq j$. The connection between these is that the subgraph version is the subgraph P of G such that $V(P)$ is the set of vertices in the list and $E(P)$ is the set of pairs of vertices that appear consecutively in the list.

We will make repeated use of the following fact about walks and paths:

Fact 2.1.1. If G is a graph, $u, v \in V(G)$, then every u, v-walk in G must contain a u, v-path.

We find this path by removing segments of the walk between the first and last occurrence of repeated vertices. In applying this fact, it is important to remember that, although we are not explicitly mentioning the edges, they are understood to be part of the walk or path under consideration. So the u, v-path contained in a u, v-walk cannot use any edges not used in the walk. Hence, two vertices can appear consecutively in the u, v-path only if they also appear consecutively in the u, v-walk.

The fact that we consider edges to be part of a walk or path even though they are not explicitly mentioned is also important in defining the length of a path or walk: the *length* of a path or a walk is the number of edges in the path or walk (with repeated edges counted as many times as they appear). Of course, the length can also be described as being one less than the number of vertices in the path or walk (again, counting repeated vertices as many times as they appear). For $u, v \in V(G)$, the distance between u and v, denoted $d(u, v)$, is the length of the shortest u, v-path in G if such a path exists, if no such path exists, we say $d(u, v) = \infty$.

One last type of graph which is of particular importance in this work is a tree. We follow the graph theorist's definition of a tree, rather than the logician's: a *tree* T is a connected graph with no cycles as subgraphs. Since T is connected but has no cycles, between any pair of vertices in T there is exactly one path.

(And a path is a kind of tree.) An equivalent characterization of a tree is that it is a connected graph with one fewer edge than vertices.

We observe that if $s, s', t \in V(T)$ for a tree T and $s, s' \neq t$, then if the neighbor of t on the s,t-path is not the same as the neighbor of t on the t,s'-path, then combining the s,t-path and the t,s'-path gives an s,s'-path, not just an s,s'-walk. (If there were any vertex other than t appearing on both the s,t-path and the t,s'-path, we would make a cycle by taking the vertex closest to t on the t,s'-path, call it v, and combining the t,v-path contained in the t,s'-path with the v,t-path contained in the s,t-path.)

Any vertex of a tree T which has only one neighbor in T is called a *leaf.* We may designate one vertex of a tree to be its *root.* We may choose a particular vertex to be the root of our tree for properties it has that are useful, but being the root of a tree does not in and of itself imply any particular properties.

2.2 Tree Decomposition

Definition 2.2.1. Let G be a graph. A *tree decomposition* of G is a tree T such that each $t \in V(T)$ is a subset of $V(G)$ and

- $\bigcup V(T) = V(G)$,
- for each $e \in E(G)$, $e \subseteq t$ for some $t \in V(T)$, and

- for all $t_1, t_2 \in V(T)$, if $t \in V(T)$ is on the unique path between t_1 and t_2, then $t_1 \cap t_2 \subseteq t$.

We should acknowledge here that this is not quite the standard definition of tree decomposition. Usually, rather than having each vertex of T be a subset of $V(G)$, we would instead associate with each $t \in T$ some $B_t \subseteq V(G)$. The first bullet point would become $\bigcup_{t \in V(T)} B_t = V(G)$, while the second and third would replace t, t_1, t_2 with the associated sets. The distinction is that if t and t' are distinct vertices of T in our definition, then they must be distinct subsets of $V(G)$, whereas the standard definition would allow t and t' to be associated with the same subset of $V(G)$, i.e., we could have $B_t = B_{t'}$.

For our work, this distinction will not matter, since if we have a standard tree decomposition T, we can find a tree decomposition that fits our definition, T', by the following lemma and its corollary. So we choose the definition given for simplicity of notation.

Lemma 2.2.2. *Let G be a graph and T a tree with B_t an associated subset of $V(G)$ for each $t \in T$ giving a standard tree decomposition of G. Suppose there exists $\{t_1, t_2\}$ in $E(T)$ such that $B_{t_1} \subseteq B_{t_2}$. Let T' be defined by*

$$V(T') = (V(T) \setminus \{t_1, t_2\}) \cup \{t^*\}$$

and

$$E(T') = \{\{t,t'\} | t,t' \in V(T') \text{ and } \{t,t'\} \in E(T)\}$$
$$\cup \{\{t,t^*\} | t \in V(T') \text{ and at least one of } \{t,t_1\} \text{ or } \{t,t_2\} \text{ is in } E(T)\}$$

(i.e., contract the edge $\{t_1, t_2\}$). If $B'_{t^*} = B_{t_2}$ and $B'_t = B_t$ for all other $t \in V(T')$, then T' with the assocated B'_t for each $t \in V(T')$ is a standard tree decomposition.

Proof. First, we observe that T' is still a tree since it must still be connected and, since we have decreased the number of edges and the number of vertices by one each, will have one fewer edge than vertex. Clearly, since $B_{t_1} \subseteq B_{t_2} = B'_{t^*}$, we will have $\bigcup_{t \in V(T')} B'_t = \bigcup_{t \in V(T)} B'_t$. It is also clear that if $e \in E(G)$ then, since $e \subseteq B_t$ for some $t \in V(T)$, $e \in B'_{t'}$ for some $t' \in V(T')$ (where $t' = t$ for $t \in V(T)$ and $t' = t^*$ if $t = t_1, t_2$).

Finally, suppose $t'_1, t'_2 \in V(T')$ and let t be any vertex on the unique path between them. We consider three cases:

- If $t'_1, t'_2, t \neq t^*$, then t must be on the unique path between t'_1 and t'_2 in T as well, so

  $$B'_{t'_1} \cap B'_{t'_2} = B_{t'_1} \cap B_{t'_2} \subseteq B_t = B'_t$$

  by T a standard tree decomposition.

- If $t'_1, t'_2 \neq t^*$, but $t = t^*$, then either t_1 or t_2 (or both) must lie on the path from t'_1 to t'_2 in T, so either $B_{t'_1} \cap B_{t'_2} \subseteq B_{t_1}$ or $B_{t'_1} \cap B_{t'_2} \subseteq B_{t_2}$ and in either case, $B'_{t'_1} \cap B'_{t'_2} = B_{t'_1} \cap B_{t'_2} \subseteq B_{t^*}$.

- The third possibility is that at least one of t'_1, t'_2 is equal to t^*; without loss of generality, assume that $t'_1 = t^*$. Regardless of whether t'_2 is also equal to t^* or not, if $t = t^*$, then we have $B'_{t'_1} \cap B'_{t'_2} \subseteq B'_{t'_1} = B'_{t^*} = B'_t$.

  On the other hand, if $t \neq t^*$ and, hence, $t'_2 \neq t^*$, then $t, t'_2 \in V(T)$ and t lies on the unique path from either t_1 or t_2 to t'_2. But since $t \neq t_1$, if t is on the unique path from t_1 to t'_2, it must also be on the unique path from t_2 to t'_2 – either t_2 is the unique neighbor of t_1 on the t_1, t'_2-path or, by an observation in Section 2.1, combining the t_2, t_1-path with the t_1, t'_2 path gives the unique path from t_2 to t'_2. Thus by T a standard tree decomposition we have

$$B'_{t'_1} \cap B'_{t'_2} = B'_{t^*} \cap B_{t'_2} = B_{t_2} \cap B_{t'_2} \subseteq B_t = B'_t.$$

Since in all possible cases we have $B'_{t'_1} \cap B'_{t'_2} = B'_t$, the third bullet point in the definition of a standard tree decomposition is satisfied. Having already shown the other necessary conditions, T' with the associated B'_t for each $t \in V(T')$ is a standard tree decomposition. □

Corollary 2.2.3. *Let G be a graph and T a tree with B_t an associated subset of $V(G)$ for each $t \in T$ giving a standard tree decomposition of G. Suppose there exist distinct $s_1, s_2 \in V(T)$ such that $B_{s_1} = B_{s_2}$. Let s be the unique neighbor of*

s_1 on the path from s_1 to s_2 and define T' by

$$V(T') = (V(T) \setminus \{s_1, s\}) \cup \{s^*\}$$

and

$$\begin{aligned} E(T') =& \{\{t,t'\} | t,t' \in V(T') \text{ and } \{t,t'\} \in E(T)\} \\ & \cup \{\{t,s^*\} | t \in V(T') \text{ and at least one of } \{t,s_1\} \text{ or } \{t,s\} \text{ is in } E(T)\} \end{aligned}$$

(i.e., contract the edge $\{s_1, s\}$). If $B'_{s^*} = B_s$ and $B'_t = B_t$ for all other $t \in V(T')$, then T' with the assocated B'_t for each $t \in V(T')$ is a standard tree decomposition.

Proof. Since T with the associated B_t for each $t \in V(T)$ is a standard tree decomposition of G and s is on the unique path from s_1 to s_2, $B_{s_1} = B_{s_1} \cap B_{s_2} \subseteq B_s$. Since T' and the associated B'_t are defined as in the previous lemma (with s_1 in the role of t_1 and s in the role of t_2), the fact that they give a standard tree decomposition follows immediately from that result. □

For any graph G and any standard tree decomposition of G, applying this corollary repeatedly will let us ultimately produce a standard tree decomposition T with associated $B_t \subseteq V(G)$ that form a standard tree decomposition of G in which $B_{t_1} = B_{t_2}$ implies $t_1 = t_2$[1]. We can then create a tree decomposition as defined in this chapter by taking the B_t such that $t \in V(T)$ as our vertices and putting edges between B_t and $B_{t'}$ if and only if there is an edge between t and t' in T.

(We can also create a standard tree decomposition from a tree decomposition as defined in this thesis in the obvious way, by making the associated vertex set B_t equal to t. This allows us to apply Lemma 2.2.2 to our version of tree decompositions as well. So we may assume that no vertex in a tree decomposition is a subset of any of its neighbors.)

In the process above, while we decrease the number of vertices in our standard tree decomposition, we never change the maximum size of the associated B_t. That is, if we start with a standard tree decomposition T with B_t the associated subset of $V(G)$ for each $t \in V(T)$ and finish with a tree decomposition T', then the maximum size of a vertex $t' \in V(T')$ will be equal to the maximum size of a set B_t such that $t \in V(T)$. This is essential because, as we will see shortly, the maximum size of a vertex in a tree decomposition (or of an associated set B_t in

[1] Since we assume that G itself is finite, we can deal with the case of infinite trees if necessary. We note that if $t_1, t_2, t_3, \ldots$ is an infinite path in T then for some $n \in \mathbb{N}$, by the third bullet point of the definition of standard tree decomposition, we must have $B_{t_n} \supseteq B_{t_{n+1}} \supseteq B_{t_{n+2}} \supseteq \ldots$ and a similar argument to the one in Lemma 2.2.2 will allow us to cut off everything above t_n.

Since $V(G)$ is finite, there can only be a finite number of distinct finite sequences of subsets of $V(G)$. So if we have an infinite number of branches from one root, after truncating all of them as discussed above to minimal finite length, there can be only be a finite number of branches from the root whose vertices give distinct sequences of associated subsets of $V(G)$. Removing all but one of each sequence will still leave a standard tree decomposition which is now finite.

a standard tree decomposition) will play an important role in our work.

For any graph G, one possible tree decomposition is the tree consisting of a single vertex, t, with $t = V(G)$. For most graphs there will also be tree decompositions which don't put all of the vertices of the graph into a single vertex of the tree. However, when G is complete, if T is a tree decomposition of G, then there must be some $t \in V(T)$ such that $t = V(G)$, as shown by the following useful observation.

Fact 2.2.4. Let G be a graph and T a tree decomposition of G. For every clique in G, there is some vertex in T containing all of the vertices of the clique.

Proof. We proceed by induction on c, the size of a clique.

Base cases: If $c = 1$, then the clique consists of a single vertex. Since T is a tree decomposition of G, each vertex of G is an element of some $t \in T$, so the vertices of the clique are contained in t. If $c = 2$, then the vertices of the clique are the endpoints of an edge in G and, again, since T is a tree decomposition of G, we must have both vertices of the clique contained in t for some $t \in V(T)$.

Inductive step: Assume that when $c \leq m$ for each clique in G with c vertices there must exist a $t \in V(T)$ containing the vertices of the clique. We must show that the same is true for each clique of size $2 < c = m + 1$. Suppose

C is the set of vertices of a clique of size $m+1$ in G. We need to show that there exists $t \in V(T)$ such that $C \subseteq t$.

Since $|C| = c \geq 3$, we may choose distinct vertices $v_0, v_1, v_2 \in C$. Since the vertices of C form a clique in G, so do the vertices of $C_i = C \setminus \{v_i\}$ for each $i \in 3$. Thus, by inductive assumption, for each $i \in 3$, there are $t_i \in V(T)$ such that $C_i \subseteq t_i$.

Adding the unique t_1, t_2-path to the unique t_0, t_1-path gives us a t_0, t_2-walk. This walk must contain a t_0, t_2-path, so in fact it contains the unique t_0, t_2-path. Let t be the vertex of maximal distance from t_0 which appears in both the t_0, t_1-path and the t_0, t_2-path. (Such a vertex must exist since t_0 itself is on both paths and, due to the uniqueness of paths in trees, no two distinct vertices on the t_0, t_1-path can have the same distance from t_0.) Since t is on both of these paths, it must contain both $C_0 \cap C_1 = C \setminus \{v_0, v_1\}$ and $C_0 \cap C_2 = C \setminus \{v_0, v_2\}$, so we have $C \setminus \{v_0\} \subseteq t$. If t is also on the unique path from t_1 to t_2, then it will contain v_0 as well and, hence, we will have $C \subseteq t$.

Consider the t_1, t_2-walk obtained by combining the unique t_1, t-path and the unique t, t_2-path. This walk must contain the unique t_1, t_2-path. Let t' be the vertex of maximal distance from t_1 which appears in both the t_1, t_2-path and the t_0, t_1-path. We want to show that $t' = t$.

Note that since t is the vertex of maximum distance from t_0 that is contained in both the unique t_0, t_1-path and the unique t_0, t_2-path, the unique t, t_2-path (which must be contained in the unique t_0, t_2-path) cannot contain any vertex of the t_0, t_1 path other than t. In particular, it does not contain t' unless $t' = t$ and cannot contain an edge from t to t'. Similarly, the unique t', t_2-path cannot contain any vertex of the t_0, t_1-path other than t' (and cannot contain an edge between t and t'). Putting together the unique t, t_2-path and the unique t_2, t'-path gives a t, t'-walk which must contain the unique t, t'-path. Therefore, the unique t, t'-path cannot contain any vertex of the t_0, t_1-path other than t and t'.

But since t and t' both appear in the unique t_0, t_1-path, the unique t, t'-path must also be contained in the t_0, t_1-path. This implies that the *only* vertices on the unique t, t'-path must be t and t' themselves. Since the t, t'-path is contained in a t, t'-walk which does not contain an edge from t to t', there cannot be any edges in the t, t'-path. So, in fact, we must have $t = t'$.

Therefore, t is a vertex on the uniqe path from t_1 to t_2 and $v_0 \in C \setminus \{v_1, v_2\} = C_1 \cap C_2 \subseteq t$. Since we already have $C \setminus \{v_0\} \subseteq t$, this shows that $C \subseteq t$, as desired.

$\square$

One consequence of this fact is that if we replace the second bullet point in the definition of tree decomposition with

- if $C \subseteq V(G)$ such that the induced subgraph of G on C is complete (i.e., C is the set of vertices of a clique in G), then there is some $t \in V(T)$ such that $C \subseteq t$

the resulting definition will be equivalent to the original defintion. We will return to this idea in Chapter 5 when we attempt to generalize tree decomposition.

So for any tree decomposition T of K_n, we must have a vertex $t \in T$ of size n. For any graph G, if $E(G)$ is not empty, then if T is a tree decomposition of G, T must have some vertex of size at least 2, since we must have some $e \in E(G)$ and that e must be contained in some vertex of T. For a path, 2 is enough: if our vertices are listed as $x_1, x_2, ..., x_n$, then we can create a tree decomposition T by setting $t_i = \{x_i, x_{i+1}\}$, $V(T) = \{t_i | i \in n\}$, and $E(T) = \{\{t_i, t_{i+1}\} | i \in n - 1\}$. In other words, we create a vertex in T for each edge in G and make two distinct vertices adjacent if and only if their intersection is nonempty. We can use a similar approach to show that for any tree T_0, we can find a tree decomposition T such that the maximum size of a vertex in T is 2.

The same approach won't work for cycles, because the resulting graph would again be a cycle. But for a cycle C, consider choosing one $v \in V(C)$ and removing it from the graph (i.e., look at the induced subgraph on $V(C) \setminus \{v\}$). The resulting graph is a path, so we can find a tree decomposition T of $C - \{v\}$ with maximum vertex size 2 as above. If we create T' by defining $t' = t \cup \{v\}$ for each $t \in V(T)$, then setting $V(T') = \{t' | t \in V(T)\}$ and $E(T') = \{\{t'_1, t'_2\} | \{t_1, t_2\} \in E(T)\}$, the result will be a tree decomposition of C with maximum vertex size 3. We have already seen that we cannot do "better" than this if $C = C_3 = K_3$. In fact, for any $n \geq 3$, there is no tree decomposition of C_n ($n \geq 3$) with maximum vertex size less than 3:

For the sake of contradiction, suppose that C_n with $n \geq 3$ is a cycle with tree decomposition T such that the maximum size of a vertex in T is 2. Note that since $|E(C_n)| = n \geq 3$ and each edge of C_n must be a subset of some vertex of T, if the maximum size of a vertex in T is 2, we must have at least n vertices in T. Since each vertex of C_n is contained in some edge of C_n (in fact, in two of them), we may assume by Lemma 2.2.2 that *each* vertex of T has size 2 (since any single point or empty vertex could be absorbed into a neighbor)[2]. Let $t_1, t_2 \in V(T)$ be adjacent. Since $t_1 \neq t_2$, $|t_1 \cap t_2| \leq 1$ and there must be some $v_1 \in t_1$ such that $v_1 \notin t_2$ and some $v_2 \in t_2$ such that $v_2 \notin t_1$. As we will later show in Lemma 5.2.4, for any graph G and any tree decomposition S of G, if s_1 and s_2 are adjacent in S, then for any $s, s' \in V(T)$ such that the edge $\{s_1, s_2\}$ lies on the unique path from s to s', any $x \in s \setminus (s_1 \cap s_2)$ and $y \in s' \setminus (s_1 \cap s_2)$ must be in different components of $G - (s_1 \cap s_2)$. Using this fact now, we reach a contradiction – since C_n is a cycle and $|t_1 \cap t_2| \leq 1$, $G - (t_1 \cap t_2)$ must be connected, so v_1 cannot be in a different component from v_2.

As we go from paths and trees to cycles to complete graphs, the graphs under consideration become more and more connected, that is, we would have to remove more and more vertices to break them into multiple components. We also see that the maximum vertex size any tree decomposition must have increases.

Definition 2.2.5. Let G be a graph. If T is a tree decomposition of G, then the *width* of T is

$$w(T) = \max\{|t| - 1 | t \in T\}.$$

The tree width of G is

$$tw(G) = \min\{w(T) | T \text{ is a tree decomposition of } G\}.$$

Tree width, as we have seen hints of, is an important measure of how connected a graph is. As we have seen, in any tree decomposition T of the complete graph on n vertices, there must be some vertex $t \in V(T)$ such that $|t| = n$, so the tree width of K_n is $n - 1$. The tree width of any tree (including paths) is 1 (this is the reason for the "-1" in the definition of width), while the tree width of any cycle is 2.

[2]For any vertex $t \in V(T)$, if $t = \{v\}$ for some $v \in V(C_n)$, let $e = \{v, v'\}$ be one of the edges in C_n containing v. $e \in t'$ for some $t' \in V(T)$. Since $t \neq t'$, some neighbor of t must lie on the unique path from t to t'. Because T is a tree decomposition, we must then have v an element of that neighbor and, thus, t is a subset of that neighbor.

Tree width is of particular interest because having a bound on the tree width for a class of graphs makes it easier to answer questions about those graphs. More precisely, Courcelle's Theorem says that for classes of graphs with bounded tree width, monadic second order statements can be decided in linear time.

2.3 Cops and Robbers

Consider the following game, for two players, played on a graph:

Definition 2.3.1. The Cops and Robbers game with k cops is played on a graph G in discrete rounds as follows:

Round 0**:** The cops choose a set $C_0 \in \binom{V(G)}{\leq k}$, then the robber chooses an element $z_0 \in V(G) \setminus C_0$.

Round $n+1$**:** The cops choose a set $C_{n+1} \in \binom{V(G)}{\leq k}$, then the robber chooses an element $z_{n+1} \in V(G) \setminus C_{n+1}$ such that there is a path from z_n to z_{n+1} in G which does not intersect $C_n \cap C_{n+1}$.

The cops win if the robber is ever unable to choose an appropriate element. The robber wins if he can force the game to continue forever.

The idea behind this game is that the cops can move wherever they want to without having to follow the paths in the graph, perhaps by helicopter or teleporter. The robber is stuck running through the paths of the graph (the streets). He can run really fast, so distance is no object. But since he doesn't want to get caught, he can't go somewhere if a cop is present and his path can never cross a vertex where there has been a cop present the whole time between his previous move and this one (hence, his path can't cross the intersection of the cops' previous position and their new one).

Let us analyze how many cops it takes to win the game on certain graphs. On any graph G, the cops can always win if there are at least as many cops as there are vertices, since they can just choose $C_0 = V(G)$. For a complete graph, K_n, they will *only* be able to win if there are at least n cops, since if they leave any vertex unoccupied, that vertex will always be connected to the robber's previous vertex by an edge and the robber will be able to move there.

For the same reason, if there are *any* edges in the graph, it will take at least two cops to catch the robber, since he could always pick one endpoint of the edge if there was only one cop. But two cops are enough to catch the robber on a path: if the path is $x_1, ..., x_n$, the cops can choose $C_0 = \{x_1, x_2\}$ as their initial position and then, regardless of the robber's choices, they can push him down the path by choosing $C_{i+1} = \{x_{i+2}, x_{i+3}\}$. Since $C_i \cap C_{i+1} = \{x_{i+2}\}$, the robber will never be able to move to any x_j with $j < i + 3$ in Round $i + 1$ and will eventually run out of room.

The same idea will work for trees in general. The cops start at a leaf and its unique neighbor, then move down the tree one vertex at a time. Each time the tree branches out, the robber will be forced to choose a branch once the lead cop occupies the vertex from which they branch out. Then the cops will push him down that branch and, since he can never jump over them, he will be unable to get to any other branch.

We can win with three cops on a cycle by having one cop stay put the whole time. This turns the remaining vertices into a path and the remaining two cops can proceed as above. But there is no way two cops can win on a cycle. In order to win on any graph, one cop must occupy the robber's previous position and each neighbor of that vertex must also be occupied by a cop (i.e., for the cops to win in Round $n+1$, we must have $C_{n+1} = \{z_n\} \cup \{z \in V(G) | z$ is a neighbor of $z_n\})$. On a cycle, this means we need at least three cops since z_n will have two neighbors.

If this pattern seems familiar, that is not coincidental. We will see the connection between the Cops and Robbers game and tree decompositions in the next section. In the meantime, to set up that result and also our later work, we will develop some more details related to the game and show that it is equivalent to two other games of Cops and Robbers played on graphs.

We start by being a bit more formal about the structure of the game and how we know whether one player or the other wins.

Definition 2.3.2. For the Cops and Robbers game with k cops on a graph G, a *robber-winning run* of length $m+1$ is a pair of sequences $(C_0, C_1, ..., C_m)$ and $(z_0, z_1, ..., z_m)$ where for each i, $C_i \in \binom{V(G)}{\leq k}$, $z_i \in V(G) \setminus C_i$, and if $i > 0$, then there is a path from z_i to z_{i-1} which doesn't intersect $C_{i-1} \cap C_i$. We will generally write $(C_i, z_i)_{i \leq m}$ as shorthand for the pair of sequences $(C_0, C_1, ..., C_m)$ and $(z_0, z_1, ..., z_m)$. We consider the "empty run," ε, to be the unique robber-winning run of length 0.

We denote the set of all robber-winning runs of length n by $RW_k(G, n)$. The set of all robber-winning runs of any finite length is $RW_k(G) = \bigcup_{n \in \mathbb{N}} RW_k(G, n)$.

Definition 2.3.3. A *robber strategy* for the Cops and Robbers game with k cops on G is a function

$$\varrho : RW_k(G) \times \binom{V(G)}{\leq k} \longrightarrow V(G).$$

A run $(C_i, z_i)_{i \leq m}$ *respects* ϱ if for each $i \leq m$,

$$z_i = \varrho\left((C_j, z_j)_{j<i}, C_i\right).$$

A robber strategy ϱ is *winning* if for every $(C_i, z_i)_{i \leq m}$ in $RW_k(G)$ that respects ϱ and every $C_{m+1} \in \binom{V(G)}{\leq k}$, if $z_{m+1} = \varrho\left((C_i, z_i)_{i \leq m}, C_{m+1}\right)$, then $(C_i, z_i)_{i \leq m+1}$ is a robber-winning run.

Definition 2.3.4. A *cop strategy* for the Cops and Robbers game with k cops on G is a function

$$\kappa : RW_k(G) \longrightarrow \binom{V(G)}{\leq k}.$$

A run $(C_i, z_i)_{i \leq m}$ *respects* κ if for each $i \leq m$,

$$C_i = \kappa\left((C_j, z_j)_{j<i}\right).$$

A cop strategy κ is *winning* if there is a number N such that for each robber winning run $(C_i, z_i)_{i \leq m}$ that respects κ, $m \leq N$.

We can now show that for any finite graph G, for any $k \in \mathbb{N}$, either the cops have a winning strategy in the Cops and Robbers game for k cops on G or the robber has a winning strategy in that game.

Lemma 2.3.5. *Let G be a graph and $k \in \mathbb{N}$. For the Cops and Robbers game with k cops on G, either the cops have a winning strategy or the robber has a winning strategy.*

Proof. Assume that k cops do not have a winning strategy on G. We will show that we can construct a winning robber strategy.

Define a tree T as follows. Let the vertex set of T consist of the empty sequence $\langle\rangle$, the single-element sequences $\langle C_0 \rangle$, and, for each $n \in N$, all sequences of the form

$$\langle C_0, z_0, ..., C_n, z_n \rangle$$

and all sequences of the form

$$\langle C_0, z_0, ..., C_n, z_n, C_{n+1} \rangle$$

such that

- for all i, $C_i \in \binom{V(G)}{\leq k}$,
- for all i, $z_i \in V$, and

- $(C_i, ..., C_n), (z_0, ..., z_n)$ is a robber-winning run.

Put an edge between two vertices of T if and only if one is an immediate subsequence of the other.

If we take $\langle\rangle$ to be the root of the tree, we can see that each path from the root gives one possible play of the Cops and Robbers game for k cops on G, with each vertex along the path giving the sequence of the cops' and robber's positions to that point.

Let us say that a vertex of T of the form $\langle C_0, z_0, ..., C_n, z_n\rangle$ has *depth* $\geq N$ for some $N \in \mathbb{N}$ if

$$\forall C_1' \exists z_1' \forall C_2' ... \exists z_N' (\langle C_0, z_0, ..., C_n, z_n, C_1', z_1', ..., z_N'\rangle \in V(T)).$$

Let

$$\text{depth}(w) = \max\{N \in \mathbb{N} : w \text{ has depth} \geq N\}$$

if the maximum exists; let $\text{depth}(w) = \infty$ if w has depth $\geq N$ for all $N \in \mathbb{N}$. $\text{depth}(w) = 0$ if $w = \langle C_0, z_0, ..., C_n, z_n\rangle$ does not have depth ≥ 1, so exactly when

$$\exists C_1' \forall z_1' (\langle C_0, z_0, ..., C_n, z_n, C_1', z_1'\rangle \notin V(T)).$$

That is, a vertex in T corresponding to the robber winning run $(C_i, z_i)_{i \leq n}$ will have depth 0 precisely when there is some cop position C_{n+1} which the cops may choose in Round $n+1$ such that there will be no legal robber choices in Round $n+1$. Note that we cannot have $\text{depth}(w) < 0$ for any $w \in V(T)$.

Clearly, given $w = \langle C_0, z_0, ..., C_n, z_n\rangle \in V(T)$,

$$\text{depth}(w) = 1 + \min_{C \in \binom{V(G)}{\leq k}} \left(\max_{z:\langle C_0, z_0, ..., C_n, z_n, C, z\rangle \in V(T)} \text{depth}(\langle C_0, z_0, ..., C_n, z_n, C, z\rangle) \right).$$

We claim that $\text{depth}(\langle\rangle) = \infty$. For the sake of a contradiction, suppose not. Then $\text{depth}(\langle\rangle) = N$ for some $N \in \mathbb{N}$. Since ε does not have depth $\geq N+1$, we must have

$$\exists C'_1 \forall z'_1 \exists C'_2 ... \forall z'_{N+1}(\langle C'_1, z'_1, ..., z'_{N+1}\rangle \notin V(T)).$$

In other words, $(C'_1, ..., C'_{N+1}), (z'_1, ..., z'_{N+1})$ is not a robber-winning run. Consider the following recursively defined cop strategy, κ.

- Since $\text{depth}(\langle\rangle) = N$, there must be some C'_1 such that for all $z'_1 \in V(G - C'_1)$ (i.e., all vertices such that (C'_1, z'_1) is a robber-winning run), the sequence $\langle C'_1, z'_1\rangle$ has depth at most $N - 1$. Define $\kappa(\varepsilon) = C'_1$.

- Assume that κ has been defined for all robber-winning runs of length less than or equal to m for some $m \geq 0$ in such a way that each robber-winning run $(C_i, z_i)_{i \leq l}$ of length $l + 1 \leq m + 1$ which respects κ, $\langle C_0, z_0, ..., C_l, z_l\rangle$ has depth at most $N - (l + 1)$.

  Then if $(C_i, z_i)_{i \leq m}$ is a robber-winning run of length $m + 1$ which respects κ, by assumption, $\text{depth}(\langle C_0, z_0, ..., C_m, z_m\rangle) \leq N - (m+1)$. So there must be some C'_1 such that for each z'_i making $\langle C_0, z_0, ..., C_m, z_m, C'_1, z'_1\rangle$ a vertex of T,

$$\text{depth}(\langle C_0, z_0, ..., C_m, z_m, C'_1, z'_1 \rangle) \leq N - (m+2).$$

We set $\kappa((C_i, z_i)_{i \leq m}) = C'_1$.

If $(C_i, z_i)_{i \leq m}$ is a robber-winning run of length $m+1$ which doesn't respect κ, define $\kappa\left((C_i, z_i)_{i \leq m}\right) = C_m$.

Note that if $(C_i, z_i)_{i \leq m+1}$ is a robber-winning run which respects κ, we must have $(C_i, z_i)_{i \leq m}$ respecting κ, so $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$ will imply that $\langle C_0, z_0, ..., C_m, z_m, C_{m+1}, z_{m+1} \rangle$ has depth at most $N - (m+2)$.

It follows immediately from the definition of κ that for any robber-winning run $(C_i, z_i)_{i \leq m}$ of length $m+1$ which respects κ, the corresponding vertext of T, $\langle C_0, z_0, ..., C_m, z_m \rangle$ must have depth less than or equal to $N - (m+1)$. Since the depth of every vertex of T must be at least 0, this means that the length of any robber winning run which respects κ is at most N, so κ is a winning strategy for k cops on G. But we have assumed that k cops do not have a winning strategy, a contradiction. Thus we must have $\text{depth}(\langle\rangle) = \infty$.

We now recursively define a robber strategy ϱ.

- Since $\text{depth}(\langle\rangle) = \infty$, for each $C \in V(G)$, there must exist some $z \in V(G)$ such that (C, z) is a robber-winning run and $\langle C, z \rangle$ has depth ∞. For each pair (ε, C) such that $C \in \binom{V(G)}{\leq k}$, define $\varrho\left((\varepsilon, C)\right)$ to be this z.

- If (C_0, z_0) is a robber-winning run such that $\text{depth}(\langle C_0, z_0\rangle) = \infty$ and $C_1 \in \binom{V(G)}{\leq k}$, then there must be some $z_1 \in V(G)$ such that $(C_i, z_i)_{i\leq 1}$ is a robber-winning run and $\langle C_0, z_0, C_1, z_1\rangle$ has depth ∞. Define $\varrho\left((C_0, z_0), C_1\right) = z_1$.

  If (C_0, z_0) is a robber winning run which does not satisfy $\text{depth}(\langle C_0, z_0\rangle) = \infty$ (note that this means (C_0, z_0) doesn't respect ϱ), define $\varrho\left((C_0, z_0), C_1\right) = z_0$.

- Assume ϱ has been defined for all ordered pairs in $RW_k \times \binom{V(G)}{\leq k}$ such that the robber winning run has length less than or equal to $m + 1$ and that for each ϱ-respecting robber winning run $(C_i, z_i)_{i\leq m}$ and each $C_{m+1} \in \binom{V(G)}{\leq k}$, if $z_{m+1} = \varrho\left((C_i, z_i)_{i\leq m}, C_{m+1}\right)$, then $\langle C_0, z_0, ..., C_m, z_m, C_{m+1}, z_{m+1}\rangle$ has depth ∞.

  If $(C_i, z_i)_{i\leq m+1}$ is any robber-winning run such that $\langle C_0, z_0, ..., C_{m+1}, z_{m+1}\rangle$ has depth ∞ (so, in particular, if $(C_i, z_i)_{i\leq m+1}$ respects ϱ), then for each $C_{m+2} \in \binom{V(G)}{\leq k}$ there exists some $z_{m+2} \in V(G)$ such that $(C_i, z_i)_{i\leq m+2}$ is a robber-winning run and

$$\text{depth}\left(\langle C_0, z_0, ..., C_{m+1}, z_{m+1}, C_{m+2}, z_{m+2}\rangle\right) = \infty.$$

  Define

$$\varrho\left((C_i, z_i)_{i\leq m+1}, C_{m+2}\right) = z_{m+2}.$$

  If $(C_i, z_i)_{i\leq m+1}$ is any robber-winning run such that $\langle C_0, z_0, ..., C_{m+1}, z_{m+1}\rangle$

has finite depth (in which case, $(C_i, z_i)_{i\leq m+1}$ must not respect ϱ), define

$$\varrho\left((C_i, z_i)_{i\leq m+1}, C_{m+2}\right) = z_{m+1}.$$

By definition, for any robber-winning run $(C_i, z_i)_{i\leq m}$ which respects ϱ and any $C_{m+1} \in \binom{V(G)}{\leq k}$, $\langle C_0, z_0, ..., C_m, z_m, C_{m+1}, z_{m+1}\rangle$ is a vertex of T (with depth ∞), so $(C_i, z_i)_{i\leq m+1}$ is a robber winning run. Thus ϱ is a winning robber strategy for the Cops and Robbers game with k cops on G.

$\square$

Here are two more games of Cops and Robbers discussed by Seymour and Thomas [6][3]:

Definition 2.3.6. The Cops and Robbers Search game with k cops is played on a graph G in discrete rounds as follows:

Round 0**:** The cops choose a set $C_0 \in \binom{V(G)}{\leq k}$, then the robber chooses a component X_0 of $G - C_0$.

Round $n+1$**:** The cops choose a set $C_{n+1} \in \binom{V(G)}{\leq k}$ such that either $C_{n+1} \subseteq C_n$ or $C_{n+1} \supseteq C_n$. Then the robber chooses a component X_{n+1} of $G - C_{n+1}$

[3] We do take two liberties in defining these games. First, we have the robbers choose components of $G - C$ for some $C \subseteq V(G)$, rather than the corresponding C-flaps. Second, we call it a win for the robber if he can force the game to continue forever (i.e., if the cops don't have a winning strategy). Neither of these changes has any significant effect, they are simply for convenience.

such that if $C_{n+1} \subseteq C_n$, X_n is a subgraph of X_{n+1}, and if $C_{n+1} \supseteq C_n$, X_{n+1} is a subgraph of X_n.

The cops win if the robber is ever unable to choose an appropriate component. The robber wins if he can force the game to continue forever.

Definition 2.3.7. The Cops and Robbers Jump Search game with k cops is played on a graph G in discrete rounds as follows:

Round 0**:** The cops choose a set $C_0 \in \binom{V(G)}{\leq k}$, then the robber chooses a component X_0 of $G - C_0$.

Round $n+1$**:** The cops choose a set $C_{n+1} \in \binom{V(G)}{\leq k}$. Then the robber chooses a component X_{n+1} of $G - C_{n+1}$ such that either $V(X_{n+1}) \cap V(X_n) \neq \emptyset$ or there exists some $x \in V(X_{n+1})$ which is adjacent in G to some $x' \in V(X_n)$.

The cops win if the robber is ever unable to choose an appropriate component. The robber wins if he can force the game to continue forever.

If k cops have a winning strategy in the first of these games on G, we say that k cops can *search* G. If k cops have a winning strategy in the second of these games on G, we say that k cops can *jump search* G.

In both of these games, rather than choosing individual points for his positions, the robber is choosing a component of $G - C$ for some $C \subseteq V(G)$. While this may seem like a big difference from our game, it actually isn't.

In our Cops and Robbers game on graphs, at each turn after the first, the robber is required to choose a vertex such that there is a path from his previous vertex to this new one which doesn't include any vertex continuously occupied by the cops between the robbers moves, i.e., if it is turn $n+1$, the path between the robbers vertices can't intersect $C_n \cap C_{n+1}$, the intersection of the cops' previous and new positions. Since what matters here is not so much any particular path, but just the existence of a path, this is really a statement about components. The definition could easily be rephrased to say that the robber must choose a vertex in the same component of $G - (C_n \cap C_{n+1})$ as his previous vertex.

But more than that, if z_n and z'_n are in the same component of $G - C_n$, then they will also be in the same component of $G - (C_n \cap C_{n+1})$, so the robber's choices at Round $n+1$ will be the same. So, effectively, for Round n the robber is not choosing a single vertex, but rather a whole component of $G - C_n$.

But if by choosing any vertex in a given component of $G - C_n$ the robber is effectively making the same choice, shouldn't the cops' responses also be the same? That is, if $(C_i, z_i)_{i \leq n}$ and $(C'_i, z'_i)_{i \leq n}$ are robber winning runs of length $n+1$ such that for each $0 \leq i \leq n-1$, $C_i = C'_i$ and $z_i = z'_i$, $C_n = C'_n$, and z_n and z'_n are in the same component of $G - C_n$, wouldn't it make sense for a cop strategy κ to have $\kappa\left((C_i, z_i)_{i \leq n}\right) = \kappa\left((C'_i, z'_i)_{i \leq n}\right)$? We'll call the cop strategies which do this "coarse."

Definition 2.3.8. Let G be a graph and $\kappa : RW_k(G) \longrightarrow \binom{V(G)}{\leq k}$ a cop strategy for the Cops and Robbers game with k cops on G. We say that κ is *coarse* if for any two robber winning runs of length $m+1$, $(C_i, z_i)_{i \leq m}$ and $(C'_i, z'_i)_{i \leq m}$, which respect κ and satisfy

- $C_i = C'_i$ for all $i \leq m$ and
- z_i, z'_i are in the same component of $G - C_i$ for all $i \leq m$,

we have $\kappa\left((C_i, z_i)_{i \leq m}\right) = \kappa\left((C'_i, z'_i)\right)$.

As one might guess from the discussion so far, the cops will have a coarse, winning strategy whenever they have any winning strategy.

Lemma 2.3.9. *Let G be a graph. If there exists a winning cop strategy κ for the Cops and Robbers game with k cops on G, then there is a coarse, winning cop strategy κ' for the Cops and Robbers game with k cops on G.*

Proof. Suppose κ is a winning cop strategy for the Cops and Robbers game with k cops on G. Define a cop strategy κ' inductively as follows:

- Let $\kappa'(\varepsilon) = \kappa(\varepsilon)$.
- For each component X of $G - \kappa'(\varepsilon) = G - \kappa(\varepsilon)$, choose some $z_X \in V(X)$. Note that $(\kappa(\varepsilon), z_X)$ is a robber winning run which respects κ. Then for each robber winning run (C_0, z_0), if $C_0 = \kappa'(\varepsilon) = \kappa(\varepsilon)$, define $\kappa'\left((C_0, z_0)\right)$ to be $\kappa\left((C_0, z_{X_0})\right)$ where X_0 is the component of $G - C_0$ containing z_0. If $C_0 \neq \kappa(epsilon)$ (i.e., if (C_0, z_0) does not respect κ), set $\kappa'\left((C_0, z_0)\right) = C_0$.

Note that this means that for each robber winning run $(C_i, z_i)_{i \leq 1}$ that respects κ', there will be some κ-respecting robber winning run $(C'_i, z'_i)_{i \leq 1}$ such that for $i \in \{0, 1\}$, $C'_i = C_i$ and z'_i is in the same component of $G - C_i = G - C'_i$ as z_i.

- Assume κ' has been defined for all robber winning runs of length at most m for some $m \geq 1$[4]so that for each robber winning $(C_i, z_i)_{i \leq m}$ that respects κ', there is some κ-respecting robber winning run (C'_i, z'_i) such that for $i \in \{0, 1, ..., m\}$, $C'_i = C_i$ and z'_i is in the same component of $G - C'_i = G - C_i$ as z_i.

  We can define an equivalence relation on the set of robber winning runs of length $m + 1$ which respect κ' by setting two robber winning runs to be equivalent if they both have the above relationship with the same κ-respecting robber winning run $(C'_i, z'_i)_{i \leq m}$. For each such equivalence class, choose one κ-respecting robber winning run $(C^*_i, z^*_i)_{i \leq m}$ such that for each $(C_i, z_i)_{i \leq m}$ in the equivalence class, for each $i \in 0, 1, ..., m$, $C^*_i = C_i$ and z^*_i is in the same component of $G - C^*_i = G - C_i$ as z_i.

  Let $(C_i, z_i)_{i \leq m}$ be a robber winning run.

[4]That is, assume we have defined κ' for all robber winning runs of the form $(C_i, z_i)_{i \leq n}$ for some $0 \leq n \leq m - 1$. So any robber winning run of length $m + 1$ which respects κ' must be of the form $(C_i, z_i)_{i \leq m}$ where $(C_i, z_i)_{i \leq m-1}$ respects κ' and $C_m = \kappa'((C_i, z_i)_{i \leq m-1})$.

– If $(C_i, z_i)_{i \leq m}$ respects κ', define $\kappa'\left((C_i, z_i)_{i \leq m}\right) = \kappa\left((C_i^*, z_i^*)_{i \leq m}\right)$ where $(C_i^*, z_i^*)_{i \leq m}$ is the chosen κ-respecting robber winning run corresponding to the equivalence class of $(C_i, z_i)_{i \leq m}$.

– Otherwise, define $\kappa\left((C_i, z_i)_{i \leq m}\right) = C_m$

Note that this implies that for each robber winning run $(C_i, z_i)_{i \leq m+1}$ which respects κ', there will be a κ-respecting robber winning run $(C_i', z_i')_{i \leq m+1}$ which respects κ such that for $i \in \{0, 1, ..., m+1\}$, $C_i' = C_i$ and z_i' is in the same component of $G - C_i' = G - C_i$ as z_i.

It follows immediately from the definition of κ' that for any robber winning run $(C_i, z_i)_{i \leq m}$ which respects κ', there must be some robber winning run $(C_i', z_i')_{i \leq m}$ with $C_i' = C_i$ and z_i' in the same component of $G - C_i' = G - C_i$ as z_i such that $\kappa'\left((C_i, z_i)_{i \leq m}\right) = \kappa\left((C_i', z_i')_{i \leq m}\right)$. Since κ is a winning cop strategy, there must be some $N \in \mathbb{N}$ such that each robber winning run which respects κ has length less than N. Thus, for the same N, each robber winning run which respects κ' must have length less than N and κ' is a winning cop strategy for the Cops and Robbers game with k cops on G.

Now, consider two κ'-respecting robber winning runs $(C_i, z_i)_{i \leq m}$ and $(D_i, y_i)_{i \leq m}$ such that for all $i \leq m$, $C_i = D_i$ and z_i, y_i are in the same component of $G - C_i = G - D_i$. If $m = 0$, then $\kappa'\left((C_i, z_i)_{i \leq m}\right) = \kappa'\left((D_i, y_i)_{i \leq m}\right)$ by the second bullet point in the definition of κ'. If $m \geq 1$, there must be a κ-respecting robber winning run $(C'_i, z'_i)_{i \leq m}$ with $C'_i = C_i$ and z'_i in the same component of $G - C'_i = G - C_i$ as z_i such that $\kappa'((C_i, z_i)_{i \leq m}) = \kappa((C'_i, z'_i)_{i \leq m})$. Clearly, we must also have $C'_i = D_i$ and z'_i in the same component of $G - C'_i = G - D_i$ as y_i. So $(C_i, z_i)_{i \leq m}$ and $(D_i, y_i)_{i \leq m}$ are in the same equivalence class and, therefore, $\kappa'((C_i, z_i)_{i \leq m}) = \kappa'((D_i, y_i)_{i \leq m})$. Thus we have shown that κ' is coarse.

Having produced a coarse winning strategy κ' from the winning strategy κ, we have shown that whenever k cops have a winning strategy for the Cops and Robbers game on G, they must also have a coarse winning strategy. □

We define one more cops and robbers game on a graph G as follows:

Definition 2.3.10. The Component Cops and Robbers game with k cops is played on a graph G in discrete rounds as follows:

Round 0**:** The cops choose a set $C_0 \in \binom{V(G)}{\leq k}$, then the robber chooses a component X_0 of $G - C_0$.

Round $n + 1$**:** The cops choose a set $C_{n+1} \in \binom{V(G)}{\leq k}$, then the robber chooses a component X_{n+1} of $G - C_{n+1}$ such that X_n and X_{n+1} are subgraphs of the same component of $G - (C_n \cap C_{n+1})$.

The cops win if the robber is ever unable to choose an appropriate component. The robber wins if he can force the game to continue forever.

If we define robber winning runs, robber strategies, cop strategies, etc., as one would expect for this new game, Lemma 2.3.10 will tell us that k cops will have a winning strategy for the Component Cops and Robbers game on G if and only if they have a winning strategy for the Cops and Robbers game on G. But we can also show equivalence with the two Seymour and Thomas games as well.

Proposition 2.3.11. *Let G be a graph. The following are equivalent:*

(1) k cops have a winning strategy in the Cops and Robbers game on G.

(2) k cops have a winning strategy in the Component Cops and Robbers game on G.

(3) k cops can jump search G.

(4) k cops can search G.

Proof.

(1) $\iff$ (2)

By Lemma 2.3.9, k cops have a winning strategy in the Cops and Robbers game on G if and only if they have a coarse, winning strategy in the Cops and Robbers game on G.

Suppose κ is a coarse winning strategy for k cops in the Cops and Robbers game on G. Define a cop strategy κ' for k cops in the Component Cops and Robbers game on G as follows:

- Let $\kappa'(\varepsilon) = \kappa(\varepsilon)$.
- Let $(C_i, X_i)_{i \leq m}$ be a robber winning run in the Component Cops and Robbers game for k cops on G. If there is a robber κ-respecting winning run $(C'_i, z_i)_{i \leq m}$ in the Cops and Robbers game for k cops on G such that for each $0 \leq i \leq m$, $C'_i = C_i$ and $z_i \in V(X_i)$, then set $\kappa'\left((C_i, X_i)_{i \leq m}\right) = \kappa\left((C'_i, z_i)_{i \leq m}\right)$. This is well defined since κ is coarse. Otherwise, let $\kappa'\left((C_i, X_i)_{i \leq m}\right) = C_m$.

For any robber winning run $(C_i, X_i)_{i \leq m}$ which respects κ', we must have a robber winning run $(C'_i, z_i)_{i \leq m}$ such that for each $0 \leq i \leq m$, $C'_i = C_i$ and $z_i \in V(X_i)$. (This follows immediately from the definition of κ' by induction on m and $X_i \neq \emptyset$.) Since κ is a winning cop strategy for the Cops and Robbers game with k cops on G, there is some $N \in \mathbb{N}$ such that for all robber winning runs $(C'_i, z_i)_{i \leq m}$ which respect κ, $m \leq N$. Thus each robber winning run $(C_i, X_i)_{i \leq m}$ which respects κ' must have $m \leq N$ for that same N. So κ' is a winning cop strategy for the Component Cops and Robbers game with k cops on G.

So we have shown that if k cops have a winning strategy in the Cops and

Robbers game on G, then they must also have a winning strategy in the Component Cops and Robbers game on G.

On the other hand, suppose that κ is a winning strategy for k cops in the Component Cops and Robbers game on G. Define a cop strategy κ' for k cops in the Cops and Robbers game on G as follows:

- Let $\kappa'(\varepsilon) = \kappa(\varepsilon)$.
- Let $(C_i, z_i)_{i \leq m}$ be a robber winning run in the Cops and Robbers game for k cops on G. For each $0 \leq i \leq m$, let X_i be the component of $G - C_i$ such that $z_i \in V(X_i)$. Then $(C_i, X_i)_{i \leq m}$ is a robber winning run for the Component Cops and Robbers game with k cops on G. (This is clear for $m = 0$. If $m \geq 1$, note that z_i must be in the same component of $G - (C_{i-1} \cap C_i)$ as z_{i-1}, so there is a path from z_i to z_{i-1} in G. Now for any $z \in X_i$ and $z' \in X_{i-1}$ we can find a z, z'-path not intersecting $C_{i-1} \cap C_i$ in the walk formed by combining a z, z_i-path in X_i, a z_i, z_{i-1}-path in $G - (C_{i-1} \cap C_i)$, and a z_{i-1}, z'-path in X_{i-1}. This shows that X_i is a subgraph of the same component of $G - (C_{i-1} \cap C_i)$ as X_{i-1}.) So we may define $\kappa'\left((C_i, z_i)_{i \leq m}\right)$ to be equal to $\kappa\left((C_i, X_i)_{i \leq m}\right)$.

From this definition, it is clear that if $(C_i, z_i)_{i \leq m}$ is a robber winning run in the Cops and Robbers game for k cops on G which respects κ', then

$(C_i, X_i)_{i \leq m}$ where X_i is the component of $G - C_i$ containing z_i will be a κ-respecting robber winning run for the Component Cops and Robbers game with k cops on G. Since κ is a winning cop strategy, there is some $N \in \mathbb{N}$ such that each robber winning run in the Component Cops and Robbers game for k cops that respects κ has length less than N. It follows that for this same N, each robber winning run $(C_i, z_i)_{i \leq m}$ in the Cops and Robbers game for k cops which respects κ' must have length less than N and, hence, κ' is a winning cop strategy.

(4) $\implies$ (2)

Let $C_n, C_{n+1} \subseteq V(G)$. Note that if $C_n \subseteq C_n + 1$, then $C_n \cap C_{n+1} = C_n$, so each component of $G - C_{n+1}$ is a subgraph of a unique component of $G - (C_n \cap C_{n+1}) = G - C_n$. Thus, for any component X_n of $G - C_n$, if the robber chooses a component of $G - C_{n+1}$ which is a subgraph of the same component of $G - (C_n \cap C_{n+1})$ as X_n, he must choose a component of $G - C_{n+1}$ which is contained in X_n as a subgraph.

On the other hand, if $C_n \supseteq C_{n+1}$, then $C_n \cap C_{n+1} = C_{n+1}$. In this case, each component of $G - C_n$ is contained in a unique component of $G - (C_n \cap C_{n+1}) = G - C_{n+1}$. It follows that if X_n is any component of $G - C_n$, if the robber chooses a component of $G - C_{n+1}$ which is a subgraph of the same component of $G - (C_n \cap C_{n+1})$ as X_n, he must choose the unique component which contains X_n as a subgraph.

This shows that the Cops and Robbers Search game is really the same game as the Component Cops and Robbers game except that there are more restrictions on the cops' positions. As a result, any winning strategy for k cops in the Cops and Robbers Search game on G will also be a winning strategy for k cops in the Component Cops and Robbers game on G.

(2) $\Longrightarrow$ (3)

While the Cops and Robbers Search game placed more constraints on the cops' choices than the Component Cops and Robbers game, the Cops and Robbers Jump Search game places more restrictions on the robber's choices.

Let $C_n, C_{n+1} \subseteq V(G)$, X_n a component of $G - C_n$ and X_{n+1} a component of $G - C_{n+1}$. Note that $V(X_n)$ and $V(X_{n+1})$ are both subsets of $V(G - (C_n \cap C_{n+1})$. So if there exists some $x \in V(X_n) \cap V(X_{n+1})$, then we can create a walk from any $x_1 \in X_{n+1}$ to any $x_0 \in X_n$ by combining a path from x_1 to x in X_{n+1} with a path from x to x_0. Since this walk contains only vertices of X_n and X_{n+1}, it doesn't intersect $C_n \cap C_{n+1}$. So there is a path (contained in the walk) from x_1 to x_0 in G which doesn't intersect $C_n \cap C_{n+1}$ and, hence, x_1 and x_0 must belong to the same component of $G - (C_n \cap C_{n+1})$. Thus X_n and X_{n+1} must be subgraphs of the same component of $G - (C_n \cap C_{n+1})$.

Similarly, if there is an edge from some $x \in X_n$ to some $x' \in X_{n+1}$, then we can create a path from any $x_0 \in X_n$ to any $x_1 \in X_{n+1}$ which doesn't intersect $C_n \cap C_{n+1}$. This time we combine a path from x_0 to x in X_n, the edge $\{x, x'\}$ and a path from x' to x_1 in X_{n+1} to get an x_0, x_1-walk in G not intersecting $C_n \cap C_{n+1}$ and take the x_0, x_1-path contained in that walk. Again, we see that X_n and X_{n+1} must be subgraphs of the same component of $G - (C_n \cap C_{n+1})$.

So in any particular round of the Cops and Robbers Jump Search Game, the cops are allowed to make the same choices as in the Component Cops and Robbers game, but the robber's choices in response are some subset of his choices in the Component game. Therefore, if k cops have a winning strategy in the Component Cops and Robbers game on G, then they must also have a winning strategy in the Cops and Robbers Jump Search game on G.

(3) $\Longrightarrow$ (4)

This comes directly from [6], where Seymour and Thomas show that k cops can search G if and only if k cops can jump search G.

$\square$

2.4 Seymour and Thomas

We have seen that it takes at least n cops to catch a robber on a complete graph on n vertices and that a complete graph on n vertices has tree width $n-1$. On a path (or any other tree with at least two vertices) it takes a minimum of two cops to catch a robber and the tree width of such a graph is 1. On a cycle, we need at least three cops and the tree width is 2. The pattern seems to be that the fewest cops that can catch a robber on a particular graph is exactly equal to the tree width of the graph plus one. This was proved for graphs in general by Seymour and Thomas in 1993 [6].

Theorem 2.4.1 (Seymour and Thomas)**.**
k cops have a winning strategy on a graph G if and only if G has tree width less than k.

Our goal is to generalize tree decomposition and the Cops and Robbers game in such a way that the analogue of this result will hold. The original statement of Seymour and Thomas's result included a number of other equivalent statements and, as a result, the proof is too long to reproduce here. We will, however, give an argument for one direction, since this will guide our later work. Specifically, we will show that if G has tree width less than k, then we can construct a winning cop strategy for k cops. To do this, we will make use of the following sequence of lemmas about tree decompositions:

Lemma 2.4.2. *Let G be a graph and T a tree decomposition of G. Let t_n, t_{n+1} be adjacent in T and let $x \in V(G) \setminus t_n$ such that t_{n+1} is on the unique path from t_n to s for some $s \in V(T)$ such that $x \in s$. For each x' in the same component of $G - t_n$ as x, there exists an $s' \in V(T)$ such that $x' \in s'$ and t_{n+1} is on the unique path from t_n to s'.*

Proof. We use induction on p, the length of the shortest path in $G - t_n$ from x to some x' in the same component over $G - t_n$. (p is the distance from x to x' in $G - t_n$, which may not be the same as the distance in G.)

Base case: If $p = 0$, then $x' = x$ and we have $s' = s$.

Inductive step: Assume the lemma holds for all x' in the same component of $G - t_n$ as x such that the shortest path from x to x' in $G - t_n$ has length m. Let x' be an element of the same component of $G - t_n$ as x such that the shortest path from x to x' in $G - t_n$ has length $m + 1$. Let $x = x_0, x_1, ..., x_m, x_{m+1} = x'$ be an x, x'-path in $G - t_n$ of length $m + 1$.

Since there is a path in $G - t_n$ of length m from x to x_m, by inductive assumption there must be some $s_m \in V(T)$, with $x_m \in s_m$, such that t_{n+1} is on the unique path from t_n to s_m. Also, because x_m is adjacent to x' and T is a tree decomposition of G, there must be some $s^* \in V(T)$ such that $x_m, x' \in s^*$. We claim that t_{n+1} must be on the unique path from t_n to s^*.

For the sake of contradiction, suppose not. Then combining the unique s_m, t_n-path and the t_n, s^*-path would give us an s_m, s^*-path (not just walk) in T. Since T is a tree decomposition, we would have $x_m \in t_n$. But x_m is a vertex on a path from x to x' in $G - t_n$ and, therefore, cannot be in t_n.

Thus we must have t_{n+1} on the unique path from t_n to s^*. Setting $s' = s^*$ gives the desired result.

□

Lemma 2.4.3. *Let G be a graph and T a tree decomposition of G. Let t_n, t_{n+1} be adjacent in T and let $x \in V(G) \setminus t_n$ such that t_{n+1} is on the unique path from t_n to s for some $s \in V(T)$ such that $x \in s$. Let H be the component of $G - (t_n \cap t_{n+1})$ containing x and let H' be the component of $G - t_n$ containing x. For each $y \in V(G) \setminus t_{n+1}$, if $y \in V(H)$, then $y \in V(H')$ (i.e., for each y not in t_{n+1}, if y is in the same component as x of G over $t_n \cap t_{n+1}$, then it is also in the same component over t_n).*

Proof. For the sake of a contradiction, suppose there exists $y \in (V(G) \setminus t_{n+1}) \cap V(H)$ such that $y \notin V(H')$. Since $y \in V(H)$, there is an x, y-path, P, in $G - (t_n \cap t_{n+1})$. If this path did not include any element of t_n, then it would also be a path in $G - t_n$ and y would be in H', so some vertex on the path must belong to t_n. Let y' be the element of t_n closest on the path to x. Note that $y' \neq x$, since $x \in V(G) \setminus t_n$. Let x' be the neighbor of y' on the x, y'-path contained in P.

Since no vertex of the x, x'-path contained in P can be an element of t_n, x' must be in H'. So, by the previous lemma, there exists an $s' \in V(T)$ such that $x' \in s'$ and t_{n+1} is on the unique path from t_n to s'. Because x' is adjacent to y' in G and T is a tree decomposition of G, there must also be some $s'' \in V(T)$ such that $x', y' \in s''$.

t_n adjacent to t_{n+1} in T implies that either t_{n+1} is on the unique path from t_n to s'' or t_n is on the unique path from s' to s''. In the former case, T a tree decomposition gives us $y' \in t_n \cap s'' \subseteq t_{n+1}$, in the latter, we have $x' \in s' \cap s'' \subseteq t_n$. Either way, we have reached a contradiction.

So we must indeed have that each $y \in (VG) \setminus t_{n+1}$ which is in $V(H)$ is also in $V(H')$. $\square$

Lemma 2.4.4. *Let G be a graph and T a tree decomposition of G. Let t_n, t_{n+1} be adjacent in T and let $x \in V(G) \setminus t_n$ such that t_{n+1} is on the unique path from t_n to s for some $s \in V(T)$ such that $x \in s$. For each $y \in V(G) \setminus t_{n+1}$ which is in the same component of $G - (t_n \cap t_{n+1})$ as x, there exists a unique neighbor $t_{n+2} \neq t_n$ of t_{n+1} such that t_{n+2} is on the unique path from t_{n+1} to s' for each $s' \in V(T)$ such that $y \in s'$.*

Proof. Let $y \in V(G) \backslash t_{n+1}$ such that y is in the same component of $G-(t_n \cap t_{n+1})$ as x. By the previous lemma, y is also in the same component of $G - t_n$ as x. So by the first lemma of this section, there exists an $s^* \in V(T)$ such that $y \in s^*$ and t_{n+1} is on the unique path from t_n to s^*. Let t_{n+2} be the neighbor of t_{n+1} which lies between t_{n+1} and s^* on the t_n, s^*-path in T. Clearly, $t_{n+2} \neq t_n$.

Let s' be any vertex of T which contains y. Since $y \notin t_{n+1}$, $s' \neq t_{n+1}$, so any path from t_{n+1} to s' must contain some neighbor of t. Also since $y \notin t_{n+1}$ and T is a tree decomposition of G, we cannot have t_{n+1} on the unique path from s^* to s' in T. This implies that t_{n+2} must be the neighbor of t_{n+1} on the unique path from t_{n+1} to s' (otherwise combining the s^*, t_{n+1}- and t_{n+1}, s'-paths would give us an s^*, s'-path containing t_{n+1}). Since s' was arbitrary, $t_{n+2} \neq t_n$ must be the neighbor of t_{n+1} on the unique t_{n+1}, s'-path for each $s' \in V(T)$ such that $y \in s'$. □

Lemma 2.4.2 shows that if $x, y \in V(G)$ are in the same component of $G - t$ for some $t \in V(T)$, then any $s_x, s_y \in V(T)$ such that $x \in s_x$, $y \in s_y$ must be in the same component of $T - \{t\}$. For the purposes of this chapter, Lemma 2.4.2, along with 2.4.3, primarily serves to allow us to prove Lemma 2.4.4. However, it will be useful to see these parts individually when we attempt to generalize tree decomposition.

Lemma 2.4.4 is key to our ability to build a winning strategy from a tree decomposition. It shows that by choosing their positions appropriately as a path among the vertices of the tree decomposition, the cops will be able to chase the robber down the tree, as we will see more precisely in the following proof.

Proof of the backwards direction of Seymour and Thomas Theorem. Assume that G has tree width less than k. Then there exists a tree decomposition of G having width less than k. Let T be such a tree. Note that for each $t \in V(T)$, $t \in \binom{V(G)}{\leq k}$. We will construct a winning strategy for the Cops and Robbers game with k cops on G using appropriately chosen vertices of T for the cops' positions.

We define a cop strategy κ as follows:

- Let t_0 be any vertex of T. Define $\kappa(\varepsilon)$ to be t_0.
- Let (C_0, z_0) be a robber winning run of length one. If $C_0 \neq t_0$, define $\kappa((C_0, z_0)) = t_0$. If $C_0 = t_0$, then we must have $z_0 \in V(G) \setminus t_0$. Since T is a tree decomposition of G, z_0 must be an element of some vertex of T. Since $z_0 \notin t_0$, some neighbor of t_0 must lie on the unique path from t_0 to a vertex of T containing z_0. That neighbor must be unique, since if t was a neighbor of t_0 on the unique path from t_0 to s for some s containing z_0 and $t' \neq t$ was a neighbor of t_0 on the unique path from t_0 to s' for some s' containing z_0, then t_0 would be on the unique path from s to s' and, by T a tree decomposition, we would have $z_0 \in s \cap s' \subseteq t_0$, a contradiction. Define $\kappa((C_0, z_0))$ to be this unique neighbor of t_0.

- Let $(C_i, z_i)_{i \leq m}$ be a robber winning run of length $m+1$. If $C_0, C_1, ..., C_m$ is a path in T and C_m is on the unique path from C_{m-1} to s for some $s \in V(T)$ such that $z_{m-1} \in s$, then by Lemma 2.4.4, since z_m must be in the same component of $G - (C_{m-1} \cap C_m)$ as z_{m-1} (because $(C_i, z_i)_{i \leq m}$ is a robber winning run), there exists a unique neighbor $t_{m+1} \neq C_{m-1}$ of C_m such that t_{m+1} is on the unique path from C_m to s' for each $s' \in V(T)$ such that $z_m \in s'$. In this case, define $\kappa\left((C_i, z_i)_{i \leq m}\right) = t_{m+1}$. Otherwise, define $\kappa\left((C_i, z_i)_{i \leq m}\right) = C_0$.

We need to show that κ is a winning cop strategy. That is, we must show that each robber winning run which respects κ must have length less than or equal to N for some $N \in \mathbb{N}$. We claim that if $(C_i, z_i)_{i \leq m}$ is a robber winning run of length at least two (i.e., $m \geq 1$) which respects κ, then C_m is on the unique path from C_{m-1} to s for some $s \in V(T)$ such that $z_{m-1} \in s$ and $C_0, C_1, ..., C_m$ must be a path in T.

By induction on m:

Base Case: Since $(C_i, z_i)_{i \leq m}$ respects κ, we must have $C_0 = \kappa(\varepsilon) = t_0$ for some vertex $t_0 \in V(T)$. If $m = 1$, then we must have $C_1 = \kappa\left((C_0, z_0)\right)$. Since $C_0 = t_0$, $\kappa\left((C_0, z_0)\right)$ must be the unique neighbor of t_0 which lies on a path from t_0 to some vertex $s \in V(T)$ containing z_0 and C_0, C_1 is a path in T.

Inductive Step: Suppose for some $l \in \mathbb{N}$, for each $1 \leq m \leq l$ and each κ

respecting robber winning run $(C_i, z_i)_{i\leq m}$, C_m is on the unique path from C_{m-1} to s for some $s \in V(T)$ such that $z_{m-1} \in s$ and $C_0, C_1, ..., C_m$ is a path in T. We need to show that the same holds for robber winning runs with $m = l + 1$.

Let $(C_i, z_i)_{i\leq l+1}$ be a robber winning run which respects κ. Then $(C_i, z_i)_{i\leq l}$ is a robber winning run which respects κ and, by assumption, $C_0, C_1, ..., C_l$ is a path in T with C_l on the unique path from C_{l-1} to s for some $s \in V(T)$ such that $z_{l-1} \in s$. We must have $C_{l+1} = \kappa\left((C_i, z_i)_{i<l+1}\right)$. By the third bullet point of the definition above, $\kappa\left((C_i, z_i)_{i<l+1}\right)$ must be the unique neighbor of C_l which lies on the path from C_l to s' for every $s' \in V(T)$ such that $z_l \in s'$. Therefore C_{l+1} does indeed lie on the unique path from C_l to s' for some $s' \in V(T)$ such that $z_l \in s'$ and $C_0, C - 1, ..., C_l, C_{l+1}$ is a path in T.

Since $C_0, C_1, ..., C_m$ is a path in T for any κ respecting robber winning run $(C_i, z_i)_{i\leq m}$ of length at least two , we cannot have a robber winning run of length greater than $N = \max\{1, |V(T)|\}$. Thus, κ is a winning cop strategy. $\square$

Chapter 3

Model Theory Background

In Chapter 2, we introduced the concepts from graph theory that we hope to generalize to a broader context. In this chapter, we will introduce that context.

The class of all finite graphs is an example of what model theorists call a Fraïssé class, one of a number of amalgamation classes of finite structures. We will discuss what these terms mean in Section 2 of this chapter, after a brief review of some basic concepts from model theory in Section 1. (More complete coverage of the material in Section 1 can be found in Chapter 1 of *Model Theory: An Introduction* by David Marker [5].) In the third section, we will describe a known generalization of tree decomposition to the structures we want to consider, but we will also see why that generalization may not be entirely satisfactory. Finally, in Section 4, we will introduce the kind of independence relation that will allow us to generalize tree decomposition in a new way.

3.1 Basics

Most mathematical structures can be described well in terms of certain constants, relations, and functions defined on the elements of that structure. For instance, we can say everything we need to about a graph by talking about its edges, which define a binary relation, E, on the vertices of the graph. Fields, on the other hand, can be described using constants 0 and 1, binary functions $+$ and $\cdot$, and a unary function $-$ (for the additive inverse).

In model theory, we formalize this. A *language*, $\mathcal{L}$, consists of a set of constant symbols, $\mathcal{C}$, a set of relation symbols, $\mathcal{R}$, and a set of function symbols, $\mathcal{F}$.[1] For each function symbol $f \in \mathcal{F}$, there is an accompanying "arity," $n_f \in \mathbb{N}$, giving the number of variables of the function. Similarly, for each $R \in \mathcal{R}$, there is an associated $n_R \in \mathbb{N}$ to indicate that R is an n_R-ary relation. We will usually write $\mathcal{L}$ as the union of the sets $\mathcal{C}$, $\mathcal{R}$, and $\mathcal{F}$ (rather than as the set $\{\mathcal{C}, \mathcal{R}, \mathcal{F}\}$), specifying which symbol type each element of $\mathcal{L}$ belongs to as needed (though we will often use notation that is suggestive enough to avoid explicitly identifying the kinds of symbols).

An *$\mathcal{L}$-structure* $\mathcal{M}$ is then made up of a set M, referred to as the *universe* of

[1] In defining a language this way, we are following Marker. Other authors, including Wilfrid Hodges whose *A Shorter Model Theory* [4] will be referenced in the next section, may refer to this as a *signature*.

$\mathcal{M}$, and interpretations of each constant, relation, and function symbol contained in a set in the signature as actual constants, relations, or functions on M. So, for instance, if $\mathcal{L} = \{R\}$ where the single element R of $\mathcal{L}$ is a binary relation symbol, any graph G will be an $\mathcal{L}$-structure where the universe is $V(G)$ and the interpretation of the single binary relation symbol R is the edge relation, $E(G)$. We write $R^G = \{(u,v)|\{u,v\} \in E(G)\}$, where the notation R^G indicates that this is the interpretation of R in G. (We would indicate the interpretations of any functions or constants similarly.)

Although we may choose other letters appropriate to the context (as above for graphs), we will generally use $\mathcal{M}$ and $\mathcal{N}$ for structures whose universes may be inifinite and the corresponding plain type letters M and N for their universes. For finite structures (that is, those with finite universes), we will usually use letters from the beginning of the alphabet and will often identify the structure and its universe, e.g., A may be used for both a structure with a finite universe and the universe that structure. If $\mathcal{M}$ is an $\mathcal{L}$-structure, we may also use the early letters of the alphabet to represent subsets of M, *even though these may not be substructures of* M (see the definition of substructure below). The usage should be clear from context.

If we are giving a language with the intent of using it to talk about a particular kind of structure, we may choose to have the constant, relation, and function symbols have the same appearance as their interpretations in that language.

So the language of graphs, instead of being $\{R\}$ for a binary relation symbol R, could be given as $\{E\}$ for a binary relation symbol E. We may write the language of fields as $\mathcal{L} = \{+, -, \cdot, 0, 1\}$ (with $-$ a unary function symbol) and then have any field F be an $\mathcal{L}$-structure by interpreting the symbols in the obvious way.

But not every $\mathcal{L}$-structure for a language $\mathcal{L}$ which we associate with a particular kind of structure is necessarily a structure of that type. The language of fields used above can also be used to describe rings, for instance, but it isn't true that every structure in that language is a ring either. Most mathematical structures are specified by a set of axioms that describe the behavior of their elements. In model theory, for a given language $\mathcal{L}$, we can create a set of $\mathcal{L}$-*formulas* made up from the symbols in $\mathcal{L}$, variables, and symbols from $\{=, \wedge, \vee, \neg, \forall, \exists, \rightarrow, \leftrightarrow\}$ to state axioms we might want an $\mathcal{L}$ structure to satisfy.

For instance, if $\mathcal{L}_G = \{E\}$ where E is a binary relation symbol, an $\mathcal{L}$ structure will be a (simple) graph if it satisfies two conditions

1. for each $x \in G$, there is no edge from x to x (i.e., G has no loops) and
2. for each $x, y \in G$, if there is an edge from x to y, then there is also an edge from y to x (i.e., G is not directed).

Written as $\mathcal{L}$-formulas, these axioms are, respectively,

1. $\forall x \; \neg E(x, x)$

   and

2. $\forall x \forall y \ (E(x,y) \to E(y,x))$.

$\mathcal{L}$-formulas, such as those above, which say something that must either be true or false about any particular $\mathcal{L}$-structure are called *sentences*. In such formulas, all of the variables are *bound* by quantifiers – we are saying something about all elements (at least all elements that do something, as in the second formula above) or we are saying that there exist elements that do something we care about.[2] A set of $\mathcal{L}$-sentences is called a *theory* and can be viewed as a set of axioms. If each sentence in a theory T is true for some $\mathcal{L}$-structure $\mathcal{M}$, we write $M \vDash T$ and say that $\mathcal{M}$ is a *model* of T or that $\mathcal{M}$ *satisfies* T.

$\mathcal{L}$-formulas which have variables that are not bound allow us to ask about whether properties hold for a particular element or set of elements in an $\mathcal{L}$-structure. For instance, in the language of graphs, we could write the formula

$$\varphi(x_1, x_2, x_3) = E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_1).$$

Now for any $\mathcal{L}_{\mathcal{G}}$-structure $\mathcal{M}$ and any $a_1, a_2, a_3 \in M$, we can ask whether $\varphi(a_1, a_2, a_3)$ is true in that structure. If it is, we write $\mathcal{M} \vDash \varphi(a_1, a_2, a_3)$ and say

[2] An example of an $\mathcal{L}$-sentence where the variables are bound by the quantifier $\exists$, instead of $\forall$, would be:

$$\exists x \exists y \exists z \ (E(x,y) \wedge E(y,z) \wedge E(z,x)).$$

If this were true in a $\mathcal{L}$-structure which also satisfies the graph axioms above, then the structure would be a graph which has a copy of C_3, a triangle, as a subgraph.

that $\mathcal{M}$ satisfies $\varphi(a_1, a_2, a_3)$. If the $\mathcal{L}$-structure in question is a graph G, we are asking whether a_1, a_2, a_3 are the vertices of a copy of C_3, a triangle, in G.

Sometimes, we might want to treat some of the free variables as parameters and ask about which elements of our structure will make the formula true given certain values for the parameters. Perhaps, rather than asking whether any particular a_1, a_2, a_3 form a triangle in a given graph G, we want to know which $a \in V(G)$ form a triangle with some particular b_1 and b_2 in $V(G)$. This would be the set $\{a \in V(G) | G \vDash \varphi(a, b_1, b_2)\}$. In general, for an $\mathcal{L}$-structure $\mathcal{M}$, if there is an $\mathcal{L}$-formula $\varphi(x, \overline{y})$ (where $\overline{y}$ is an n-tuple of distinct variables for some $n \in \mathbb{N}$) and a tuple $\overline{b} \in M^n$ such that $X = \{a \in M | M \vDash \varphi(a, \overline{b})\}$, then we say that $\varphi(x, \overline{b})$ *defines* X and that X is *definable.*

In algebra, we say that $a \in K$ for a field K is algebraic over a subfield F of K if there is some non-zero polynomial $p(x) \in F[x]$ such that a is a root of the polynomial. Notice that the zero set of $p(x) \in F[x] \setminus \{0\}$ can be defined by $\varphi(x, \overline{b})$ where $\varphi(x, \overline{y})$ is a formula in the language of fields and $\overline{b}$ is a tuple of elements of F (giving the coefficients of the polynomial). Since the number of roots of $p(x)$ must be less than or equal to the degree of $p(x)$, the size of the set X defined by $\varphi(x, \overline{b})$ must be finite. We'll define what it means for an element $a \in M$ for some $\mathcal{L}$-structure $\mathcal{M}$ to be algebraic over a set $A \subseteq M$ similarly.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. An element $a \in M$ is *algebraic* over A if there is an $\mathcal{L}$-formula $\varphi(x, \overline{y})$ and a tuple $\overline{b} \in A^n$ such that $M \vDash \varphi(a, \overline{b})$ and the set X defined by $\varphi(x, \bar{b})$ is finite. The *algebraic closure* of A, denoted $acl(A)$, is the set $\{a \in M | a \text{ is algebraic over } A\}$.

For a language $\mathcal{L}$, we can define embeddings, isomorphisms, and substructures of $\mathcal{L}$-structures.

Definition 3.1.1. Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An $\mathcal{L}$-*embedding* of $\mathcal{M}$ into $\mathcal{N}$ is an injective map $\eta : M \to N$ such that

- for each constant symbol c in $\mathcal{L}$, $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ (i.e., η takes the interpretation of c in $\mathcal{M}$ to the interpretation of c in $\mathcal{N}$),

- for each relation symbol R in $\mathcal{L}$ with arity n_R and each $(a_1, a_2, ..., a_{n_R}) \in M^{n_R}$, $(a_1, a_2, ..., a_{n_R}) \in R^{\mathcal{M}}$ if and only if $(\eta(a_1), \eta(a_2), ..., \eta(a_{n_R})) \in R^{\mathcal{N}}$, and

- for each function symbol f in $\mathcal{L}$ with arity n_f and each $(a_1, a_2, ..., a_{n_f}) \in M^{n_f}$,

$$\eta\left(f^{\mathcal{M}}(a_1, a_2, ..., a_{n_f})\right) = f^{\mathcal{N}}\left(\eta(a_1), \eta(a_2), ..., \eta(a_{n_f})\right).$$

An $\mathcal{L}$-*isomorphism* is a bijective $\mathcal{L}$-embedding and an $\mathcal{L}$-*automorphism* is an isomorphism from $\mathcal{M}$ to itself. We denote the set of all automorphisms of $\mathcal{M}$ by $Aut(\mathcal{M})$. If $M \subseteq N$ and the inclusion map from M to N is an $\mathcal{L}$-embedding, then $\mathcal{M}$ is a *substructure* of $\mathcal{N}$.

It is important to note that a subgraph of a graph G may not be a substructure of G in the language of graphs. If H is a substructure of a graph G, then for any $x, y \in V(H)$ we must have an edge between x and y if and only if there was an edge between x and y in G. In other words, H must be the induced subgraph of G on $V(H)$.

3.2 Amalgamation Classes

Let $\mathcal{L}$ be a countable language (that is, the total number of symbols in $\mathcal{L}$ is at most countable). An amalgamation class is a class of finitely generated $\mathcal{L}$-structures which satisfies certain properties that guarantee the existence of a *generic model* – a unique countable $\mathcal{L}$-structure $\mathcal{M}$ such that the finite substructures of $\mathcal{M}$ are (up to isomorphism) precisely the elements of the class and any isomorphism between two finite substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$. The precise properties the class needs to satisfy depend on the type of amalgamation class. In this thesis, we are primarily interested in the type of amalgamation class called a Fraïssé class.

Definition 3.2.1. Let $\mathcal{L}$ be a countable language and $\mathbf{K}$ a class of finitely generated $\mathcal{L}$-structures. We call $\mathbf{K}$ a *Fraïssé class* if it satisfies the following three properties.

Hereditary Property (HP): If $A \in \mathbf{K}$ and B is a substructure of A, then B

is isomorphic to some element of $\mathbf{K}$.

Joint Embedding Property (JEP): If $A, B \in \mathbf{K}$, then there exists some $C \in \mathbf{K}$ such that both A and B can be embedded in C. (When we can do this so that the images of A and B are disjoint, we say that $\mathbf{K}$ has the *Disjoint Joint Embedding Property.*)

Amalgamation Property (AP): Suppose $A, B, C \in \mathbf{K}$ and $f_1 : A \to B$, $g_1 : A \to C$ embeddings. Then there exists some $D \in \mathbf{K}$ and embeddings $f_2 : B \to D$, $g_2 : C \to D$, such that $f_2 \circ f_1 = g_2 \circ g_1$. (When we can do this so that $f_2(B) \cap g_2(C) = f_2 \circ f_1(A) = g_2 \circ g_1(A)$, we say that $\mathbf{K}$ has the *Disjoint Amalgamation Property.*)

It is tempting to think that the Joint Embedding Property is just a special case of the Amalgamation Property, but while that is true for many classes, it is not true in general. For instance, the class of all finite fields satisfies the Amalgamation Property, but it doesn't have JEP because we cannot jointly embed two fields with different characteristic.

For a proof that Fraïssé classes have generic models, see Section 6.1 in *A Shorter Model Theory* by Wilfrid Hodges. [4]

The reason we are interested in Fraïssé classes specifically is that the class of finite graphs is an example of a Fraïssé class. The generic model of this class is the random graph (or Rado graph), which is the unique countable graph, G, such that for any disjoint $A, B \subset_{\text{fin}} V(G)$, there exists an $x \in V(G) \setminus (A \cup B)$ such that x is adjacent in G to every element of A and is not adjacent to each element of B.

Graphs are an example of what is sometimes called a society (but more often called a hypergraph).

Definition 3.2.2. Let $\mathcal{L}$ be a finite relational language (i.e., it contains no constant or function symbols). For each $R \in \mathcal{L}$, if n_R is the arity of R, define φ_R to be the sentence

$$\forall x_0, x_1, ..., x_{n_R-1} \ \left(R(\overline{x}) \to \wedge_{\sigma \in Sym(n_R)} R(x_{\sigma(0)}, x_{\sigma(1)}, ..., x_{\sigma(n_R-1)})\right)$$
$$\wedge \forall x_0, x_1, ..., x_{n_R-1} \ \left(R(\overline{x}) \to \wedge_{i<j<n_R} x_i \neq x_j\right).$$

Let $T_{\mathcal{L}} = \{\varphi_R | R \in \mathcal{L}\}$. We define $\mathbf{Soc}_{\mathcal{L}}$ to be the class of all finite models of $T_{\mathcal{L}}$ and call each element of $\mathbf{Soc}_{\mathcal{L}}$ a *society* (or *hypergraph*). If $A \in \mathbf{Soc}_{\mathcal{L}}$ and $\overline{a} \in R^A$ for some $R \in \mathcal{L}$, then we say that $\overline{a}$ is an *edge* of A.

For any finite relational language, $\mathbf{Soc}_{\mathcal{L}}$ is a Fraïssé class. That it satisfies HP is clear. For JEP, we can just take copies of the two structures and not add any edges between them. Similarly, for AP, we can find a structure D in $\mathbf{Soc}_{\mathcal{L}}$ which consists of copies of B and C, agreeing on the image of A, and does not add any edges which consist of elements from both B and C that was not already an edge in B or C.

$\mathbf{Soc}_{\mathcal{L}}$, then, has both Disjoint Joint Embedding and Disjoint Amalgamation. But the fact that we can put the copies of structures together without adding any edges says something stronger. For any amalgamation class $\mathbf{K}$, for every $A, B, C \in \mathbf{K}$, if A embeds into both B and C and we can find a $D \in \mathbf{K}$ which consists of copies of B and C, agreeing on the image of A, and doesn't add any relations which were not already present in B or C, we call D the *free amalgam* of B and C over A.

An example of a Fraïssé class which we will return to in the next section is the class of all finite metric spaces with distances in $\{0, 1, 2, 3\}$. This class can be described in either the language $\mathcal{L} = \{R_1, R_2, R_3\}$ or the language $\mathcal{L}' = \{R_1, R_3\}$. In either case, for a finite metric space A, we interpret R_i to contain any pair of elements $(a, b) \in A^2$ such that the distance between a and b is i. These structures are basically societies except that if our language is $\mathcal{L}$, then every pair of distinct elements must satisfy exactly one of the relations, and, in either case, the triangle inequality must hold. Again, HP is clear. For JEP, the fact that the distances are in $\{0, 1, 2, 3\}$ means that we can always assign a distance of 2 between a point in the copy of one structure and a point in the copy of the other structure without violating the triangle inequality. It follows that the class of all finite metric spaces has JEP in either $\mathcal{L}$ or $\mathcal{L}'$. The amalgamation property will work similarly.[3]

[3]This is precisely why our example uses distances in $\{0, 1, 2, 3\}$. If we limited the distances

Another well known Fraïssé class is the class of finite fields of characteristic p.

Something common to all of these examples except the last is that if $\mathcal{M}$ is the generic model, then for any finite subset A of $\mathcal{M}$, A will be the universe of a substructure. This isn't true in the class of finite fields of characteristic p, since there is no guarantee that either 0 or 1 will appear in A. So for this property to hold, we will certainly need to be in a purely relational language (i.e., a language with no constant or function symbols). But being in a purely relational language will still not guarantee the desired result. The condition we need is for the generic model of **K** to be algebraically trivial.

Definition 3.2.3. Let **K** be a Fraïssé class and $\mathcal{M}$ its generic model. We say that **K** is *algebraically trivial* if for all $A \subset_{\text{fin}} M$, $A = acl(A)$.

Since the class of finite graphs is algebraically trivial and we are trying to generalize properties of graphs, we will restrict our attention in this thesis to algebraically trivial Fraïssé classes.

---

further to $\{0, 1, 2\}$, then we can describe the structures in the language $\mathcal{L} = \{R_1\}$ and we essentially just have a graph. If the permitted distances are $\{0, 1, 2, ..., n\}$ for some $n > 3$, we will have trouble assigning distances for JEP and AP.

3.3 An Existing Generalization

In fact, there is already a known generalization of tree decomposition which will work for any algebraically trivial Fraïssé class.

Definition 3.3.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure for a purely relational language $\mathcal{L}$ (i.e., $\mathcal{L}$ contains no constant or function symbols. The *Gaifman graph* of $\mathcal{M}$ is the graph $G_{\mathcal{M}}$ such that $V(G_{\mathcal{M}}) = M$ and

$$E(G) = \left\{ \{a,b\} \in \binom{V(G_{\mathcal{M}})}{2} \middle| \text{for some } R \in \mathcal{L} \text{ with } n_R = r \text{ there exists a tuple } (a_1, a_2, ..., a_r) \in R^{\mathcal{M}} \text{ such that } a, b \in \{a_1, a_2, ..., a_r\} \right\}$$

that is, we put an edge between two elements of M if and only if they appear together in a tuple contained in some relation.

We can define the tree decompositions of $\mathcal{M}$ to be the tree decompositions of its Gaifman graph and, moreover, Courcelle's Theorem will still apply. (See Chapter 11 of *Parametrized Complexity Theory* by Flum and Grohe [3].)

Why then are we looking for other generalizations of tree decomposition and the Cops and Robbers game? There are two weaknesses of this generalization which are both demonstrated when considering $\mathcal{M}$ to be the generic model of the class of finite metric spaces with distances in $\{0, 1, 2, 3\}$.

If our language is $\mathcal{L} = \{R_1, R_2, R_3\}$ where for any distinct elements in $a, b \in M$, $(a, b) \in R_i$ if and only if the distance between a and b is i, then *every* pair of distinct vertices in M will be contained in one of the relations. It follows that the Gaifman graph of any $A \subset_{\text{fin}} M$ will be the complete graph on A.

This is not an uncommon issue and it makes it impossible to find a class of arbitrarily large (but finite) structures with bounded tree width.

On the other hand, we can equally well consider the class of finite metric spaces with distances in $\{0, 1, 2, 3\}$ in the language $\mathcal{L}' = \{R_1, R_3\}$. In this language, the distance between distinct $a, b \in M$ will be two if and only if $(a, b) \notin R_1^{\mathcal{M}}$ and $(a, b) \notin R_3^{\mathcal{M}}$. When something like this happens, when for two languages $\mathcal{L} \supseteq \mathcal{L}'$ we can define the interpretation of each symbol of $\mathcal{L}$ in $\mathcal{M}$ using only the interpretations of the symbols of $\mathcal{L}'$, we call $\mathcal{M}$ in the language $\mathcal{L}$ an *expansion by definitions* of $\mathcal{M}$ in the language $\mathcal{L}'$. When $\mathcal{M}_1$ is an expansion by definitions of $\mathcal{M}$ (here we use subscripts to avoid confusion about which language we are in, but $M_1 = M$), we are really talking about the same structure. For this reason, we like properties to be language invariant, i.e., we want them to hold for the same structure regardless of the language we use to talk about it.

But tree width as defined through the Gaifman graph will not be language invariant for finite metric spaces. As we have seen, the tree width of any finite metric space A with distances in $\{0, 1, 2, 3\}$ will be $|A| - 1$ if we use $\mathcal{L}$. But in $\mathcal{L}'$, for a finite metric space B such that the distance between b_1 and b_2 is 2 for all distinct $b_1, b_2 \in B$, the Gaifman graph G_B will have no edges and, therefore,

B will have tree width 0.

We would like to find a generalization of tree decomposition that is perhaps less likely to result in the tree width of each structure being the size of the structure and which will be language invariant.

3.4 Abstract Free Amalgamation Relations

At heart, the idea of adding edges between elements that appear in the same relation in the Gaifman graph of a structure is a way of indicating that these two elements are connected somehow, that they are not independent of each other. But it is doing this in a very binary way and connections among elements may be much more complicated than this. We would like to try to build our generalization of tree decomposition using a relation that allows us to capture a higher level of complexity among the connections between elements.

Relations indicating some kind of independence or freeness among substructures of a given structure arise often in model theory. If $\mathcal{M}$ is the random graph, we can define a ternary independence relation on $\mathcal{M}$ as follows: for $A, B, C \subset_{\text{fin}} \mathcal{M}$, write $A \mathop{\smile\!\!\!\!\!\mid}^{\circ}_{C} B$ (read as "A is independent of B over C") when $A \cap B \subseteq C$ and there are no edges from $A \setminus C$ to $B \setminus C$. This relation satisfies a number of properties which we take as the axioms of an independence relation which we will call an abstract free amalgamation relation.

Definition 3.4.1. Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class. An *abstract free amalgamation relation* on $\mathcal{M}$ is a ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ which satisfies the following:

Notation. In the list of properties below, all A, B, C are finite subsets/substructures of $\mathcal{M}$. $A' \equiv_C A$ means that there is an automorphism g of $\mathcal{M}$ fixing C such that $g(A) = A'$. We read $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$ as "A is independent of B over C."

- $Aut(\mathcal{M})$-invariance.
- Regularity 1: $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B \Rightarrow A \cap B \subseteq C$
- Regularity 2: $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B \Leftrightarrow A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B \cup C$
- Symmetry: $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B \Rightarrow B \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C A$
- Existence: For any A, B, C, there is an $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$
- Monotonicity: If $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$ and $A_0 \subseteq A$, then $A_0 \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$
- Transitivity: $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B_1 \cup B_2 \Leftrightarrow A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B_1 \wedge A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C \cup B_1} B_2$.
- Stationarity: If $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$, $A' \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$, and $A' \equiv_C A$, then $A' \equiv_{B \cup C} A$.
- Freeness: If $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$ and $C \cap (A \cup B) \subseteq D \subseteq C$, then $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_D B$.

This list of axioms is not minimal. Regularity 2 can be seen to be a consequence of existence and transitivity: if we take $A = C$, we can use existence to show that for any $B, C \subset_{\text{fin}} M$, $C \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{C} B$. Then setting $B_1 = C$ and $B_2 = B$ in transitivity, we see that $A \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{C} B \cup C \Leftrightarrow A \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{C} C \wedge A \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{C \cup C} B$, which, in light of the result from existence and symmetry, just says that $A \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{C} B \cup C \Leftrightarrow A \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{C} B$. Nevertheless, the form of Regularity 2 as stated is useful enough that we keep it in our list of axioms. Not listed among our axioms, but also useful, will be the following fact.

Fact 3.4.2. For any $B, C \subset_{\text{fin}} M$, $\emptyset \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{C} B$.

This is a direct consequence of existence when we take $A = \emptyset$. We will use this fact freely without referencing it directly. Similarly, we will generally not comment on any applications of symmetry.

Our abstract free amalgamation relations are closely related to Conant's free amalgamation relations in [1]. Since he does not limit his relations to algebraically trivial Fraïssé classes, he has an axiom related to closure which will be trivially true for any abstract free amalgamation relation. The other difference is that he does not include either of our regularity axioms. As we have seen, Regularity 2 is a consequence of other axioms, so not including it is inconsequential. Regularity 1 is more significant, but is assumed in the following definition.

Definition 3.4.3 (Definition 3.1 of [1])**.** Assume $\mathcal{L}$ is relational. Given an $\mathscr{L}$-structure $\mathcal{M}$ and $A, B, C \subseteq \mathcal{M}$, we set $A \mathop{\smile\!\!\!\!\!\!\,|}^{\text{fa}}_{C} B$ (in $\mathcal{M}$) if $A \cap B \subseteq C$ and, for all $R \in \mathcal{L}$ and $\overline{a} \in (A \cup B \cup C)^{n_R}$, if $\overline{a} \in R^A$ then either $\overline{a} \in (A \cup C)^{n_R}$ or

$\overline{a} \in (B \cup C)^{n_R}$.

That is, we define A to be independent of B over C if the intersection of A and B is in C and there is no relation $R \in \mathcal{L}$ such that some n_R-tuple $\overline{a} \in R$ contains elements of both $A \setminus C$ and $B \setminus C$.

Conant shows that $\mathop{\smile\hspace{-0.9em}\mid}^{\text{fa}}$ is a free amalgamation relation for the generic model of certain Fraïssé classes. It will follow immediately that $\mathop{\smile\hspace{-0.9em}\mid}^{\text{fa}}$ is also an abstract free amalgamation relation for the generic model of these classes.

Theorem 3.4.4 (Theorem 3.4 of [1])**.** *Let $\mathcal{L}$ be a finite relational language and let $\mathcal{M}$ be the generic model of a Fraïssé class $\mathbf{K}$ such that for all $A, B, C \in \mathbf{K}$ such that $A \cap B \subseteq C$, there are $D, A', B' \in \mathbf{K}$ such that $D = A' \cup B'$, $A' \equiv_C A$, $B' \equiv_C B$, and $A' \mathop{\smile\hspace{-0.9em}\mid}^{\text{fa}}_C B'$ (in D). Then $\mathop{\smile\hspace{-0.9em}\mid}^{\text{fa}}$ is a free amalgamation relation.*

Corollary 3.4.5. *Let $\mathcal{L}$ be a finite relational language and let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class $\mathbf{K}$ such that for all $A, B, C \in \mathbf{K}$ such that $A \cap B \subseteq C$, there are $D, A', B' \in \mathbf{K}$ such that $D = A' \cup B'$, $A' \equiv_C A$, $B' \equiv_C B$, and $A' \mathop{\smile\hspace{-0.9em}\mid}^{\text{fa}}_C B'$ (in D). Then $\mathop{\smile\hspace{-0.9em}\mid}^{\text{fa}}$ is an abstract free amalgamation relation.*

This means that there are many algebraically trivial Fraïssé classes for which there exists an abstract free amalgamation on the generic model relation, including our chief examples from Section 3.2 – the class of finite graphs (where $\mathop{\smile\hspace{-0.9em}\mid}^{\text{fa}}$ is the relation mentioned at the beginning of this section), $\mathbf{Soc}_{\mathcal{L}}$, and the class of finite metric spaces with distances in $\{0,1,2,3\}$ in the language $\mathcal{L}' = \{R_1, R_3\}$.

What about the class of finite metric spaces with distances in $\{0,1,2,3\}$ in the language $\mathcal{L} = \{R_1, R_2, R_3\}$? This class doesn't satisfy the hypothesis of the corollary for the same reason that we get the complete graph as the Gaifman graph of any element of this class: any pair of distinct elements must satisfy one of the three relations. However, we can see that the definition of $\mathop{\smile\hspace{-0.9em}\vert}^{\text{fa}}$ for the generic model of this class in $\mathcal{L}'$ can be translated to say that $A \mathop{\smile\hspace{-0.9em}\vert}^{\text{fa}}_C B$ if $A \cap B \subseteq C$ and for all $a \in A \setminus C$ and $b \in B \setminus C$, the distance between a and b is 2. Because the generic model in $\mathcal{L}$ is a definitional expansion of the generic model in $\mathcal{L}'$, we can then take that translation as the definition of $\mathop{\smile\hspace{-0.9em}\vert}^{\circ}$ for the generic model in $\mathcal{L}$ and the result will still be an abstract free amalgamation relation.

For similar reasons, we can find another abstract free amalgamation of the random graph taking $\mathop{\smile\hspace{-0.9em}\vert}^{\circ}$ to be defined by $A \mathop{\smile\hspace{-0.9em}\vert}^{\circ}_C B$ if $A \cap B \subseteq C$ and for each $a \in A \setminus C$ and each $b \in B \setminus C$, a and b *are* adjacent in the random graph.

We will refer to the abstract free amalgamation $\mathop{\smile\hspace{-0.9em}\vert}^{\text{fa}}$ of $\mathcal{M}$ in $\mathcal{L}$ as the *standard abstract free amalgamation relation* of $\mathcal{M}$ in $\mathcal{L}$.

As we will see, having any abstract free amalgamation relation, whether or not it is the standard one, will allow us to define components for the substructures of the generic model of an algebraically trivial Fraïssé class, in a language invariant way, which in turn will allow us to extend the definitions of tree decomposition, tree width, and the Cops and Robbers game.

Chapter 4

Generalizing Cops and Robbers

The definition of the Cops and Robbers game for graphs involves paths, a concept which does not have an obvious generalization in every algebraically trivial Fraïssé class. So in order to define Cops and Robbers in this more general context, we must find an appropriate way of replacing the idea of paths in the definition.

4.1 Components in Graphs

As noted in Section 2.3, we could easily phrase the definition of the Cops and Robbers game in terms of components, rather than paths. This is a useful observation because the idea of components fits naturally with an independence relation – these are the subsets which are independent of each other but which cannot be broken into further independent pieces. More precisely, we have the following:

Observation. In a graph G with the standard abstract free amalgamation relation, if $C \subseteq V(G)$, there is a unique partition $\{H_0, H_1, ..., H_{m-1}\}$ of the vertices of $G - C$ such that

(i) for any disjoint $I, J \subseteq m$, $\left(\bigcup_{i \in I} H_i\right) \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} \left(\bigcup_{j \in J} H_j\right)$ and

(ii) for each i, for any bipartition $\{H_i', H_i''\}$ of H_i, $H_i' \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} H_i''$.

$H_0, H_1, ..., H_{m-1}$ are the vertex sets of the components of $G - C$.

Certainly the vertex sets of the components of $G - C$ form such a partition, since there are no edges between distinct components and if the vertices of any component are bipartitioned, there must be a path from each vertex in one part to each vertex in the other part containing only vertices of that component, hence there must be an edge between the two parts.

On the other hand, given such a partition of the vertices of $G - C$, consider one particular H_i. If $J = m \setminus \{i\}$, we have $H_i \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} \left(\bigcup_{j \in J} H_j\right)$, so there is no edge from any vertex in H_i to any vertex in $\bigcup_{j \in J} H_j$ and, hence, there cannot be a path from any vertex in H_i to any vertex not in H_i. So if $v \in H_i$, then H_i must contain every vertex of the component containing v. But if H_i contained the vertices of multiple components, then we could bipartition it into H_i' and H_i'' by letting H_i' contain the vertices of exactly one component and H_i'' all the other vertices in H_i and we would have $H_i' \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_C H_i''$ – a contradiction. So H_i must contain the vertices of exactly one component. Thus the elements of the partition must indeed be the vertex sets of the components and the partition is unique.

We will show that each finite subset of the generic model of an algebraically trivial Fraïssé class with an abstract free amalgamation relation can similarly be broken uniquely into components. This will give us a natural generalization of the cops and robbers game.

4.2 Components in Fraïssé Classes

First, to simplify the discussion, we make the following definition.

Definition 4.2.1. Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\;|}^{\circ}$ and let $A, C \subset_{\text{fin}} M$. We say that A is *weakly* $\circ$*-irreducible* over C if for each bipartition $\{A_1, A_2\}$ of A, $A_1 \not\mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_C A_2$.

Why is this *weakly* irreducible rather than just irreducible? In a graph, we have seen that the components of the graph are weakly irreducible over the empty set (or the components of $G - C$ are weakly irreducible over C). Between any two vertices in a component, we must have a path. But there is a stronger way a subset of vertices may be connected: there may be an edge between each pair of vertices, that is, the vertices may form a clique. We reserve the term "irreducible" to apply to the analogue of this stronger connection, which we will see in the next chapter.

Using this terminology, our claim becomes:

Proposition 4.2.2. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$. If $C \subseteq A \subset_{fin} \mathcal{M}$, then there is a unique partition $\{B_0, B_1, ..., B_{m-1}\}$ of $A \setminus C$ such that*

(i) for any disjoint $I, J \subseteq m$, $\left(\bigcup_{i\in I} B_i\right) \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C \left(\bigcup_{j\in J} B_j\right)$ and

(ii) for each $i \in m$, B_i is weakly $^\circ$-irreducible over C.

Proof. Consider the following algorithm:

- Set $\mathscr{F} = \{A \setminus C\}$, $\mathscr{G} = \emptyset$, and $k = 0$;
- **while** $\mathscr{F} \neq \emptyset$:

  Set $k = k + 1$;

  Choose $X \in \mathscr{F}$ arbitrarily and set $\mathscr{F} := \mathscr{F} \setminus \{X\}$;

  if X is weakly $^\circ$-irreducible over C, **then** set $\mathscr{G} := \mathscr{G} \cup \{X\}$;

  else let $\{X_k, Y_k\}$ be a bipartition of X such that $X_k \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C Y_k$, and set $\mathscr{F} := \mathscr{F} \cup \{X_k, Y_k\}$;

- **return** $\mathscr{G}$.

We must show that the algorithm ends in finite time, that the set $\mathscr{G}$ that it produces is a partition of $A \setminus C$ satisfying both (i) and (ii), and that this is the unique partition of $A \setminus C$ which satisfies both.

At each step, we remove from $\mathscr{F}$ an element X which is a subset of $A \setminus C$. If X is weakly $\circ$-irreducible over C, it is not replaced, otherwise, we replace X with two new elements, subsets of $A \setminus C$ which form a partition of X. Since $A \setminus C$ is finite and single-element subsets of $A \setminus C$ are (vacuously) weakly $\circ$-irreducible over C, this process must end in finite time.

The resulting $\mathscr{G}$ will clearly be a set of disjoint subsets of $A \setminus C$ each of which is weakly $\circ$-irreducible over C. Suppose there were some $x \in A \setminus C$ such that x is not an element of any $B \in \mathscr{G}$. Since $x \in A \setminus C$, x is an element of some element of $\mathscr{F}$ when $k = 0$. Consider what happens to x on each run of the while loop of the algorithm.

At the beginning of each step, we choose some $X \in \mathscr{F}$ and remove it from $\mathscr{F}$. If $x \notin X$, then x must be an element of some set still in $\mathscr{F}$. If x *is* in X, then X must not be weakly $\circ$-irreducible over C, since if X were weakly $\circ$-irreducible over C, then it would be added to $\mathscr{G}$ and x does not appear in any element of $\mathscr{G}$. So X will be bipartitioned into X_k and Y_k, one of which must contain x, and both of those sets will be added back into $\mathscr{F}$.

So in either case, at the end of *each* step, x must be an element of some element of $\mathscr{F}$. But this is a contradiction since the process ends in finite time and it only stops when $\mathscr{F} = \emptyset$. Thus $\mathscr{G}$ must be a partition of $A \setminus C$ satisfying (ii).

Let $\mathscr{G} = \{B_0, B_1, ..., B_m\}$. Observe that for any $i, j \in m$ with $i \neq j$, since $B_i, B_j \in \mathscr{G}$, there must be some k such that (without loss of generality) $B_i \subseteq X_k$ and $B_j \subseteq Y_k$ with $X_k \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C Y_k$. Therefore, by montonicity, we must have $B_i \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C B_j$. As a step towards showing that for any disjoint $I, J \subseteq m$ we must have $\left(\bigcup_{i\in I} B_i\right) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C \left(\bigcup_{j\in J} B_j\right)$, we show first that for any $i \in m$ and any $J \subseteq m \setminus \{i\}$, $B_i \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C \left(\bigcup_{j\in J} B_j\right)$. We will fix $i \in m$ and proceed by induction on $|J|$.

Base case: If $|J| = 0$, then $\bigcup_{j\in J} B_j = \emptyset$, so certainly $B_i \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C \left(\bigcup_{j\in J} B_j\right)$.

Inductive step: Assume that for some $l \geq 0$, for each $J \subseteq m \setminus \{i\}$ with $|J| \leq l$, $B_i \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C \left(\bigcup_{j\in J} B_j\right)$. Let $J \subseteq m \setminus \{i\}$ such that $|J| = l + 1$. We must show that we still get $B_i \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C \left(\bigcup_{j\in J} B_j\right)$.

By the observation above, for each $j \in J$, there exists some k_j such that (without loss of generality) $B_i \subseteq X_{k_j}$ and $B_j \subseteq Y_{k_j}$. Let $K_J = \{k_j | j \in J\}$ and let k_* be the minimum element of K_J, $J_* = \{j \in J | k_j = k_*\}$. Let $J' = J \setminus J_*$.

Since J_* is not empty, $|J'| < |J| = l + 1$, so by the inductive assumption we have that

$$B_i \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C \left(\bigcup_{j\in J'} B_j\right). \qquad (*)$$

By the definition of k_*, we must have $B_i \subseteq X_{k_*}$ and $B_j \subseteq Y_{k_*}$ for each $j \in J_*$. Since k_* is the minimal element of K_J, for each $j \in J'$ we must have $k_j > k_*$, so for each $j \in J'$, $B_j \subseteq X_{k_*}$ (it isn't separated from B_i until a later step). Thus we have $B_i \cup \big(\bigcup_{j\in J'} B_j\big) \subseteq X_{k_*}$. By monotonicity, $X_{k_*} \mathop{\downarrow}^{\circ}_C Y_{k_*}$ then gives us

$$B_i \cup \left(\bigcup_{j\in J'} B_j\right) \mathop{\downarrow}^{\circ}_C \left(\bigcup_{j\in J_*} B_j\right).$$

Applying transitivity, we get

$$B_i \mathop{\downarrow}^{\circ}_{C\cup\left(\bigcup_{j\in J'} B_j\right)} \left(\bigcup_{j\in J_*} B_j\right). \qquad (**)$$

Since we have both $*$ and $**$, by transitivity we get

$$B_i \mathop{\downarrow}^{\circ}_C \left(\bigcup_{j\in J'} B_j\right) \cup \left(\bigcup_{j\in J_*} B_j\right),$$

i.e., $B_i \mathop{\downarrow}^{\circ}_C \big(\bigcup_{j\in J} B_j\big)$, as desired.

So we have that for any $i \in m$ and $J \subseteq m \setminus \{i\}$, $B_i \mathop{\downarrow}^{\circ}_C \big(\bigcup_{j\in J} B_j\big)$. We will use this to show that for any disjoint $I, J \subseteq m$, $\big(\bigcup_{i\in I} B_i\big) \mathop{\downarrow}^{\circ}_C \big(\bigcup_{j\in J} B_j\big)$. For ease of notation, we will denote $\bigcup_{i\in I} B_i$ by B_I and $\bigcup_{j\in J} B_j$ by B_J. We proceed by induction on $|I|$.

Base case: If $|I| = 0$, then $B_I = \emptyset$, and $\emptyset \mathop{\downarrow}^{\circ}_C B_J$ for any $J \subseteq m$.

Inductive step: Assume that for some $l \geq 0$, whenever $I \subseteq m$ with $|I| \leq l$, for any $J \subseteq m \setminus I$ we have $B_I \mathop{\downarrow}^{\circ}_C B_J$. Let $I \subseteq m$ such that $I = l + 1$. We must show that for each $J \subseteq m \setminus I$ we have $B_I \mathop{\downarrow}^{\circ}_C B_J$.

Since $|I| \geq 1$, $I \neq \emptyset$. Let i be any element of I and set $I' = I \setminus \{i\}$. Then by the previous result we have

$$B_i \mathop{\smile\!\!\!\!\!\!\!\mid}\nolimits^{\circ}_{C} \left(\bigcup_{j \in J \cup I'} B_j \right),$$

i.e., $B_i \mathop{\smile\!\!\!\!\!\!\!\mid}\nolimits^{\circ}_{C} B_J \cup B_{I'}$. Using transitivity, this gives us

$$B_i \mathop{\smile\!\!\!\!\!\!\!\mid}\nolimits^{\circ}_{C \cup B'_I} B_J.$$

We also have

$$B_{I'} \mathop{\smile\!\!\!\!\!\!\!\mid}\nolimits^{\circ}_{C} B_J$$

by inductive assumption. Combining these and applying transitivity yields

$$B_i \cup B_{I'} \mathop{\smile\!\!\!\!\!\!\!\mid}\nolimits^{\circ}_{C} B_J$$

or $B_I \mathop{\smile\!\!\!\!\!\!\!\mid}\nolimits^{\circ}_{C} B_J$, as desired.

Finally, we must show that $\mathscr{G}$ is the unique partition of $A \setminus C$ which satisfies both (i) and (ii). Suppose $\mathscr{G}' = \{B'_0, B'_1, ..., B'_{m'}\}$ is another partition satisfying both (i) and (ii). Let $i \in m$ and let $x \in B_i$. Since $\mathscr{G}'$ is a partition of $A \setminus C$, there must be some $j \in m'$ such that $x \in B'_j$. We claim that $B_i \subseteq B'_j$.

For the sake of contradiction, suppose not. Then there exists a $y \in B_i$ such that $y \notin B'_j$. So there must be some $l \neq j$ such that $y \in B'_l$. Let $\{J, L\}$ be any bipartition of m' such that $j \in J$ and $l \in L$. Let $B'_J = \bigcup_{t \in J} B'_t$, $B'_L = \bigcup_{t \in L} B'_t$. Since $\mathscr{G}'$ satisfies (i), we must have $B'_J \mathop{\smile\!\!\!\!\!\!\!\mid}\nolimits^{\circ}_{C} B'_L$. By monotonicity, we have $B'_J \cap B_i \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C B'_L \cap B_i$. But $\{B'_J \cap B_i, B'_L \cap B_i\}$ is a bipartition of B_i, contradicting that $\mathscr{G}$ satisfies (ii).

So indeed we must have $B_i \subseteq B'_j$. Now the same argument in the other direction will show that $B'_j \subseteq B_i$ and, hence, we have $B_i = B'_j$. It follows that each element of $\mathscr{G}$ is also an element of $\mathscr{G}'$. Since the elements of $\mathscr{G}$ contain all of the elements of $A \setminus C$, there cannot be any additional elements in $\mathscr{G}'$. Therefore we must have $\mathscr{G} = \mathscr{G}'$. □

We have thus shown that each finite subset of the generic model of an algebraically trivial Fraïssé class with an abstract free amalgamation relation has a unique partition which mimics certain properties of the partition into components in graphs. It makes sense, then, to refer to the elements of this partition as the components of our structure.

Definition 4.2.3. Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}$. If $C \subseteq A \subset_{\text{fin}} \mathcal{M}$, then we denote by $\mathcal{D}^\circ(A/C)$ the unique partition of $A \setminus C$ given by Proposition 4.2.2. We call this the $\circ$-*decomposition* of A over C and refer to the elements of $\mathcal{D}^\circ(A/C)$ as the $\circ$-*components* of A over C.

An observation which will prove useful in the future is that, as with graphs, if $C' \subseteq C \subset A$, then each $\circ$-component of A over C must be a subset of a $\circ$-component of A over C'.

Lemma 4.2.4. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\circ}$ and $A \subset_{fin} M$. If $C' \subseteq C \subseteq A$, $\mathcal{D}^{\circ}(A/C) = \{X_0, X_1, ..., X_{m-1}\}$, and $\mathcal{D}^{\circ}(A/C') = \{Y_0, Y_1, ..., Y_{n-1}\}$, then for each $i \in m$ there exists some $j_i \in n$ such that $X_i \subseteq Y_{j_i}$.*

Proof. For the sake of contradiction, suppose that for some $i \in m$, $X_i \nsubseteq Y_j$ for any $j \in n$. Without loss of generality, we may assume $i = 0$. Since $C' \subseteq C$, $A \setminus C$ must be a subset of $A \setminus C'$. By definition, $X_0 \subseteq A \setminus C$, so we must have $X_0 \subseteq A \setminus C' = \bigcup_{j \in n} Y_j$. Since $X_0 \nsubseteq Y_j$ for any $j \in n$, there must be distinct $j_1, j_2 \in n$ such that $X_0 \cap Y_{j_l} \neq \emptyset$ for $l = 1, 2$. Let $\{J_1, J_2\}$ be a bipartition of n such that $j_1 \in J_1$ and $j_2 \in J_2$. Then $\{X_0 \cap \bigcup_{j \in J_1} Y_j, X_0 \cap \bigcup_{j \in J_2} Y_j\}$ is a bipartition of X_0.

By definition of the $\circ$-decomposition of A over C', we must have

$$\bigcup_{j \in J_1} Y_j \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{C'} \bigcup_{j \in J_2} Y_j.$$

Note that $C \setminus C' \subseteq \bigcup_{j \in n} Y_j$, so by applying transitivity twice, we can move the points of $C \setminus C'$ down from each side to the base to get

$$\left(\bigcup_{j \in J_1} Y_j \right) \setminus C \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{C} \left(\bigcup_{j \in J_2} Y_j \right) \setminus C.$$

$X_0 \cap C = \emptyset$, so for $l = 1, 2$, $X_0 \cap \bigcup_{j \in J_l} Y_j$ does not contain any points of C and

$$X_0 \cap \bigcup_{j \in J_l} Y_j \subseteq \left(\bigcup_{j \in J_l} Y_j \right) \setminus C.$$

Thus, by monotonicity, we must have

$$X_0 \cap \bigcup_{j \in J_1} Y_j \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_C X_0 \cap \bigcup_{j \in J_2} Y_j,$$

but this contradicts X_0 being a $\circ$-component of A over C. □

4.3 Cops and Robbers in Fraïssé Classes

Note that for a graph G with $C \subseteq V(G)$ and the standard free amalgamation relation, if we abuse notation slightly and conflate the graph G with its set of vertices, the $\circ$-components of G over C are the components of $G - C$. We can now take the version of the graph definition of the Cops and Robbers game rephrased in terms of components as our definition of the game on finite subsets of the generic model of an algebraically trivial Fraïssé class.

For the remainder of this chapter, let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}$ and let $A \subset_{\text{fin}} \mathcal{M}$.

Definition 4.3.1. The $\circ$-Cops and Robbers game with k cops is played on A in discrete rounds as follows:

Round 0**:** The cops choose a set $C_0 \in \binom{A}{\leq k}$, then the robber chooses an element $z_0 \in A \setminus C_0$.

Round $n+1$**:** The cops choose a set $C_{n+1} \in \binom{A}{\leq k}$, then the robber chooses an element $z_{n+1} \in A \setminus C_{n+1}$ such that z_{n+1} is in the same $\circ$-component of A over $C_n \cap C_{n+1}$ as z_n.

The cops win if the robber is ever unable to choose an appropriate element. The robber wins if he can force the game to continue forever.

We also define robber-winning runs, robber strategies, and cop strategies as one would expect.

Definition 4.3.2. For the $\circ$-Cops and Robbers game with k cops on A, a *robber-winning run* of length $m+1$ is a pair of sequences $(C_0, C_1, ..., C_m)$ and $(z_0, z_1, ..., z_m)$ where for each i, C_i is a subset of A such that $|C_i| \leq k$, $z_i \in A \setminus C_i$, and if $i > 0$, then z_i is in the same $\circ$-component of A over $C_{i-1} \cap C_i$ as z_{i-1}. We will generally write $(C_i, z_i)_{i \leq m}$ as shorthand for the pair of sequences $(C_0, C_1, ..., C_m)$ and $(z_0, z_1, ..., z_m)$. We consider the "empty run," ε, to be the unique robber-winning run of length 0.

We denote the set of all robber-winning runs of length n by $RW_k(A, n)$. The set of all robber-winning runs of any finite length is $RW_k(A) = \bigcup_{n \in \mathbb{N}} RW_k(A, n)$.

Definition 4.3.3. A *robber strategy* for the $\circ$-Cops and Robbers game with k cops on A is a function

$$\varrho : RW_k(A) \times \binom{A}{\leq k} \longrightarrow A.$$

A run $(C_i, z_i)_{i \leq m}$ *respects* ϱ if for each $i \leq m$,

$$z_i = \varrho\left((C_j, z_j)_{j<i}, C_i\right).$$

A robber strategy ϱ is *winning* if for every $(C_i, z_i)_{i \leq m}$ in $RW_k(A)$ that respects ϱ and every $C_{m+1} \in \binom{A}{\leq k}$, if $z_{m+1} = \varrho\left((C_i, z_i)_{i \leq m}, C_{m+1}\right)$, then $(C_i, z_i)_{i \leq m+1}$ is a robber winning run.

Definition 4.3.4. A *cop strategy* for the °-Cops and Robbers game with k cops on A is a function

$$\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}.$$

A run $(C_i, z_i)_{i \leq m}$ *respects* κ if for each $i \leq m$,

$$C_i = \kappa\left((C_j, z_j)_{j<i}\right).$$

A cop strategy κ is *winning* if there is a number N such that for each robber winning run $(C_i, z_i)_{i \leq m}$ that respects κ, $m \leq N$.

Since our argument for the orginal Cops and Robbers game that for any finite graph G either the cops have a winning strategy or the robber has a winning strategy (proof of Lemma 2.3.5) did not use the properties of the graph besides its finiteness, precisely the same argument can be used for the generalized game.

As we previously observed, for a graph G and set of vertices C, the °-components of G over C are exactly the same as the components of $G - C$. It follows that the robber-winning runs are the same whether we consider the original version of the game or this new one and, hence, k cops have a winning strategy on G in the Cops and Robbers game if and only if they have a winning strategy on G in the $^\circ$-Cops and Robbers game.

Chapter 5

Generalizing Tree Decomposition

We saw with the definition of the Cops and Robbers game for graphs that there was a natural way to rephrase the definition in terms of an idea that fit easily with our independence relation. Ideally, we would hope that something similar would be true for the definition of tree decomposition. In the first section of this chapter, we will see that there is indeed an equivalent definition of tree decomposition for graphs in terms of a graph concept, clique, that can be described easily with the standard abstract free amalgamation relation for graphs. But we will also see that the natural generalization of tree decomposition to Fraïssé classes which arises from this definition may not be the "right" defintion. In the second section, we will offer another generalization which we will take as our primary definition of tree decomposition for the elements of an algebraically trivial Fraïssé class with an abstract free amalgamation relation and show that it is also equivalent to the original definition for graphs.

5.1 Irreducibles

Consider the following definition:

Definition 5.1.1. An *alt tree decomposition* of a graph, G, is a tree T such that each $t \in V(T)$ is a subset of $V(G)$ and

- $\bigcup V(T) = V(G)$,
- if $C \subseteq V(G)$ such that the induced subgraph of G on C is complete (i.e., C is the set of vertices of a clique in G), then there is some $t \in V(T)$ such that $C \subseteq t$, and
- for all $t_1, t_2 \in V(T)$, if $t \in V(T)$ is on the unique path between t_1 and t_2, then $t_1 \cap t_2 \subseteq t$.

We have already observed in Section 2.2 that for any graph G, T is an "alt" tree decomposition of G if and only if it is a tree decomposition of G. This suggests a natural way to generalize tree decomposition. While edges are a very binary idea and it is not immediately clear that there should be any concept equivalent to edges in a Fraïssé class where the relations are not binary, cliques are a concept that can be described easily in terms of the standard abstract free amalgmation relation for graphs. For a graph G, a clique is an induced subgraph H such that for any disjoint $A, B, C \subseteq V(H)$ with $A, B \neq \emptyset$ and $A \cup B \cup C = V(H)$, $A \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$. We can generalize this definition to other algebraically trivial Fraïssé classes with an abstract free amalgamation relation.

Definition 5.1.2. Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$. A finite substructure $I \subset_{\text{fin}} M$ is $\circ$-*irreducible* if for any disjoint $A, B, C \subseteq I$ with $A, B \neq \emptyset$ and $A \cup B \cup C = I$, $A \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$.

The second bullet point in the definition of alt tree decompositions for graphs then says that each $\circ$-irreducible subset of G is contained in some vertex of T. Consider the following definition for alt $\circ$-tree decompositions in any algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$:

Definition 5.1.3. Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$. If $A \subset_{\text{fin}} M$, an *alt* $\circ$*-tree decomposition* of A is a tree T such that each $t \in V(T)$ is a subset of A and

- $\bigcup V(T) = A$,
- if $I \subseteq A$ is $\circ$-irreducible, then there is some $t \in V(T)$ such that $I \subseteq t$, and
- for all $t_1, t_2 \in V(T)$, if $t \in V(T)$ is on the unique path between t_1 and t_2, then $t_1 \mathop{\smile\!\!\!\!\!\!|}^{\circ}_t t_2$.

Note that there are two changes here from the defintion we gave for an alt tree decomposition of a graph. The first is that we have changed the second bullet point as previously discussed to generalize from the concept of cliques in graphs to irreducibles in any algebraically trivial Fraïssé class. We have also changed the third bullet point.

If $t_1 \mathop{\downarrow\!\!\!\!\smile}^{\circ}_{t} t_2$, then by Regularity 1 we must have $t_1 \cap t_2 \subseteq t$. So clearly if a tree T satisfies the third bullet point of the definition for an alt °-tree decomposition, then it will also satisfy the third bullet point from the original alt tree decomposition definition.

On the other hand, as we have seen in Lemma 2.4.2, if G is a graph and T is a tree decomposition of G, then whenever $x, y \in V(G)$ are in the same component of $G - t$ for some $t \in V(T)$, each $s_x, s_y \in V(T)$ such that $x \in s_x$, $y \in s_y$ must be in the same component of $T - \{t\}$. The contrapositive of this is that if t lies on the unique path from s_1 to s_2 in T, then for each $x \in s_1 \setminus t$ and each $y \in s_2 \setminus t$, x and y must be in separate components of $G - t$. In particular, this implies that there can be no edge from $s_1 \setminus t$ to $s_2 \setminus t$, so $s_1 \mathop{\downarrow\!\!\!\!\smile}^{\circ}_{t} s_2$.

So for graphs, this change to the third bullet point makes no difference. We choose to go with the stronger version for Fraïssé classes in general since we expect that this is something we will want to be true (we will almost certainly want the analogue of Lemma 2.4.2 to hold), even if it means our axioms may not be the weakest possible.[1]

Since alt °-tree decompositions are equivalent to tree decompositions for graphs, why the "alt"? That is, why don't we take this as our generalization of tree decomposition to all algebraically trivial Fraïssé class with an abstract free amalgamation relation? Ultimately, it isn't entirely clear that alt °-tree decomposition is the right generalization of tree decomposition for all algebraically trivial Fraïssé classes.

Our proof of the "backwards direction" of Seymour and Thomas's result in Section 2.4 does not contain any direct reference to paths or edges and could, therefore, be used in the more general context, but being able to apply it depends on having an analogue of Lemma 2.4.4 which originally required us to prove Lemmas 2.4.2 and 2.4.3. While the statements of these lemmas are in terms of components (any edges or paths they refer to are in the tree decomposition, T) and can, thus, be translated to our more general setting, the proofs all made heavy use of the edges and paths in the graph. Can we prove them another way?

To do so we would clearly need to make use of the fact that each °-irreducible set is contained in some vertex of the tree decomposition, so it would seem

[1] In fact, this is certainly not a minimal list of axioms, since the first bullet point is completely unnecessary in the presence of the second. Single point sets are vacuously °-irreducible, so the second bullet is sufficient to guarantee that each element of A appears in some vertex of T.

that we need to be able to make a connection between °-components and °-irreducibles. The key question is the following:

Question 5.1.4. *Suppose $\mathcal{M}$ is the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\mid}^{\circ}$. Is it true that for each $A \subset_{fin} M$ and $C \subseteq A$, for each °-component H of A over C and each bipartition $\{H_1, H_2\}$ of H, there must be some °-irreducible $I \subseteq H \cup C$ such that $H_i \cap I \neq \emptyset$ for $i = 1, 2$?*

That is, if we split a °-component in two, must there always be some °-irreducible connecting the two parts the way there would be an edge connecting the two parts of a component in a graph? If this were true, we could use °-irreducibles and sequences of °-irreducibles in basically the same way we use edges and paths in the original proofs.

Note that if $W \subseteq H$ is weakly °-irreducible over C, then we must have W as the unique component of $W \cap C \subset_{\text{fin}} M$ over C. So answering Question 5.1.4 is equivalent to determining whether for all disjoint $W_1, W_2, C \subset_{\text{fin}} M$ if $W_1 \cup W_2$ is weakly °-irreducible over C there must be some °-irreducible $I \subseteq W_1 \cup W_2 \cup C$ such that $I \cap W_i \neq \emptyset$ for $i = 1, 2$.

But, more than that, we can show that whenever $A, B, C \subset_{\text{fin}} M$, $A \not\mathop{\smile\!\!\!\!\!\mid}^{\circ}_{C} B$ if and only if there is some $W \subseteq A \cup B$ such that $W \cap A \neq \emptyset$, $W \cap B \neq \emptyset$, and W is weakly °-irreducible over C.

Lemma 5.1.5. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ and let $A, B, C \subset_{fin} M$. $A \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} B$ if and only if there exists $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cup B'$ is weakly $\circ$-irreducible over C.*

Proof.

(⇐) Suppose there exists $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cup B'$ is weakly $\circ$-irreducible over C. We must show that $A \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} B$. Since $A' \cup B'$ is weakly $\circ$-irreducible over C, certainly $A' \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} B'$. By applying the contrapositive of monotonicity, we see first that $A \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} B'$ and then that $A \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} B$.

(⇒) Suppose that $A \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} B$. We need to show that there exist $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cup B'$ is weakly $\circ$-irreducible over C. First, we note that if there is an $x \in A \cap B \setminus C$, then $\{x\}$ is weakly $\circ$-irreducible over C and we can set $A', B' = \{x\}$. So we may assume that $A \cap B \setminus C = \emptyset$. Let I be a subset of $A \cup B$ of minimal size such that $A \cap I \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} B \cap I$. Let $A' = A \cap I$ and $B' = B \cap I$. Observe that $C \cap I$ must be empty by the minimality of I and (the contrapositive of) Regularity 2, so A', B', and C are disjoint. We claim that $I = A' \cup B'$ is weakly $\circ$-irreducible over C.

Let $\{X, Y\}$ be any bipartition of I. We must show that $X \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C} Y$. If $\{X, Y\} = \{A', B'\}$, then we are done, so we may assume not. Let

$$A'_X = A' \cap X \qquad A'_Y = A' \cap Y = A' \setminus X$$

$$B'_X = B' \cap X \qquad B'_Y = B' \cap Y = B' \setminus X.$$

so we have $A' = A'_X \cup A'_Y$ and $B' = B'_X \cup B'_Y$. Since $\{X, Y\} \neq \{A', B'\}$, at least one of X or Y must intersect *both* A' and B'. We assume without loss of generality that X intersects both.

We begin with $A'_X \cup A'_Y \not\mathop{\smile\!\!\!\!\!|}^{\circ}_C B'$ and use tranisitivity to move A'_Y to the other side. Since $A'_X \cup A'_Y \not\mathop{\smile\!\!\!\!\!|}^{\circ}_C B'$, by the contrapositive of transitivity, that we must have either

$$A'_Y \not\mathop{\smile\!\!\!\!\!|}^{\circ}_C B' \text{ or } A'_X \not\mathop{\smile\!\!\!\!\!|}^{\circ}_{C \cup A'_Y} B'.$$

$A'_Y \subsetneq A'$ since X must contain some elements of A', so by the minimality of I we must have $A'_Y \mathop{\smile\!\!\!\!\!|}^{\circ}_C B'$. Therefore

$$A'_X \not\mathop{\smile\!\!\!\!\!|}^{\circ}_{C \cup A'_Y} B'.$$

Applying (the contrapositive of) transitivity again gives us

$$A'_X \not\mathop{\smile\!\!\!\!\!|}^{\circ}_C B' \cup A'_Y.$$

Since $B' \cup A'_Y = B'_X \cup B'_Y \cup A'_Y = B'_X \cup Y$, we have

$$A'_Y \not\mathop{\smile\!\!\!\!\!|}^{\circ}_C B'_X \cup Y.$$

Now we apply transitivity again similarly to move B'_X to the other side. Since $A'_Y \mathop{\not\perp}^{\circ}_C B'_X \cup Y$, we must have either

$$A'_X \mathop{\not\perp}^{\circ}_C B'_X \text{ or } A'_X \mathop{\not\perp}^{\circ}_{C \cup B'_X} Y.$$

Since $\{X, Y\}$ is a bipartition of I, Y is not empty and $A'_X \cup B'_X = X \subsetneq I$. Hence, by the minimality of I, we must have $A'_X \mathop{\perp}^{\circ}_C B'_X$. Therefore

$$A'_X \mathop{\not\perp}^{\circ}_{C \cup B'_X} Y.$$

Applying (the contrapositive of) transitivity one last time, we get $A'_X \cup B'_X \mathop{\not\perp}^{\circ}_C Y$, which we can rewrite as

$$X \mathop{\not\perp}^{\circ}_C Y.$$

Thus we have shown that for any bipartition $\{X, Y\}$ of $I = A' \cup B'$ we must have $X \mathop{\not\perp}^{\circ}_C Y$, so $A' \cup B'$ is indeed weakly $\circ$-irreducible over C.

$\square$

So in fact, what we are actually asking in Question 5.1.4 is whether $\circ$-irreducibles play the same role in determining whether two sets are not independent over a third as edges do in graphs for the standard free amalgamation relation. In graphs, for disjoint sets A, B, C of the vertices of the random graph, $A \mathop{\not\perp}^{\circ}_C B$ if and only if there is an edge from some vertex of A to some vertex of B. If $\mathcal{M}$ is the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\!\perp}^{\circ}$, for $A, B, C \subset_{\text{fin}} M$, do we have $A \not\mathop{\smile\!\!\!\!\!\!\!\perp}^{\circ}_{C} B$ if and only if there is a $\circ$-irreducible $I \in A \cup B \cup C$ intersecting both A and B?

We have shown in the proof of Lemma 5.1.5 that if $A \not\mathop{\smile\!\!\!\!\!\!\!\perp}^{\circ}_{C} B$, then there are minimal $A' \subseteq A$ and $B' \subseteq B$ such that $A' \cup B'$ is weakly $\circ$-irreducible over C. If we now also take $C' \subseteq C$ minimal such that $A' \not\mathop{\smile\!\!\!\!\!\!\!\perp}^{\circ}_{C'} B'$, then $A' \cup B'$ is also weakly $\circ$-irreducible over C'.[2] $A' \cup B' \cup C'$ is our primary candidate for a $\circ$-irreducible $I \subseteq A \cup B \cup C$ intersecting both A and B.

In fact, when $C' = \emptyset$ or $|C'| = 1$, $A' \cup B' \cup C'$ *is* $\circ$-irreducible. But it is not clear when $|C'| \geq 2$. In particular, when $\{C'_1, C'_2\}$ is a bipartition of C', we have yet to find a reason why we cannot have $C'_1 \mathop{\smile\!\!\!\!\!\!\!\perp}^{\circ}_{A' \cup B'} C'_2$.

While this does not guarantee that alt $\circ$-tree decomposition is the wrong notion of generalized tree decomposition, it does suggest that the definition is not as easy to work with as we might hope. Therefore, although we will revisit alt $\circ$-tree decompositions later, for now we look for a different definition to be our main generalization of tree decomposition to the elements of an algebraically trivial Fraïssé class with an abstract free amalgamation relation.

[2] By the previous lemma, there are $A'' \subseteq A'$ and $B'' \subseteq B'$ such that $A'' \cup B''$ is weakly $\circ$-irreducible over C'. By the contrapositive of the Freeness axiom, since A', B' and C are disjoint, we would then have $A'' \cup B''$weakly $\circ$-irreducible over C. By the minimality of A' and B', this implies $A'' = A'$ and $B'' = B'$.

5.2 Another Definition

In evaluating whether alt °-tree decomposition was the "right" definition of generalized tree decomposition, we considered whether we could prove analogues of Lemmas 2.4.2, 2.4.3, and 2.4.4 using it, with the goal being to be able to apply the same method of creating a winning cop strategy from a tree decomposition as we did for graphs. We take this as a jumping-off point in our search for a different definition – we want to create a definition that will definitely allow us to be able to use the same method. In particular, this means we want to find a definition that ensures a generalization of Lemma 2.4.4 holds.

Removing the part about uniqueness of t_{n+2} (which can easily be proved separately), Lemma 2.4.4 says:

Lemma 5.2.1 (Shortened version of 2.4.4)**.** *Let G be a graph and T a tree decomposition of G. Let t_n, t_{n+1} be adjacent in T and let $x \in V(G) \setminus t_n$ such that t_{n+1} is on the unique path from t_n to s for some $s \in V(T)$ such that $x \in s$. For each $y \in V(G) \setminus t_{n+1}$ which is in the same component of $G - (t_n \cap t_{n+1})$ as x, there exists a neighbor $t_{n+2} \neq t_n$ of t_{n+1} such that t_{n+2} is on the unique path from t_{n+1} to s' for some $s' \in V(T)$ such that $y \in s'$.*

The contrapositive is then

Lemma 5.2.2 (Contrapositive of 5.2.1)**.** *Let G be a graph and T a tree decom-*

position of G. Let t_n, t_{n+1} be adjacent in T and let $x \in V(G) \setminus t_n$ such that t_{n+1} is on the unique path from t_n to s for some $s \in V(T)$ such that $x \in s$. For every $y \in V(G) \setminus t_{n+1}$ and $s' \in V(T)$ with $y \in s'$, if there is no neighbor $t_{n+2} \neq t_n$ of t_{n+1} such that t_{n+2} is on the unique path from t_{n+1} to s', then y is not in the same component of $G - (t_n \cap t_{n+1})$ as x.

Since $y \notin t_{n+1}$, saying that there is no neighbor $t_{n+2} \neq t_n$ of t_{n+1} such that t_{n+2} is on the unique path from t_{n+1} to s' is equivalent to saying that t_n is on the unique path from t_{n+1} to s'. Since t_{n+1} is on the unique path from t_n to s, this is also equivalent to saying that the edge $\{t_n, t_{n+1}\}$ is on the unique path from s' to s. So Lemma 5.2.2 can be rephrased as

Lemma 5.2.3 (Rephrasing of 5.2.2)**.** *Let G be a graph and T a tree decomposition of G. Let t_n, t_{n+1} be adjacent in T and let $x \in V(G) \setminus t_n$ such that t_{n+1} is on the unique path from t_n to s for some $s \in V(T)$ such that $x \in s$. For every $y \in V(G) \setminus t_{n+1}$ and $s' \in V(T)$ with $y \in s'$, if the edge $\{t_n, t_{n+1}\}$ is on the unique path from s' to s, then y is not in the same component of $G - (t_n \cap t_{n+1})$ as x.*

Finally, we note that if we take any edge $\{t, t'\} \in E(T)$ and any $s, s' \in V(T)$ such that $\{t, t'\}$ lies on the unique path from s to s', then for any $x \in s \setminus (t \cap t')$, $y \in s' \setminus (t \cap t')$, x and y will satisfy Lemma 5.2.3 for an appropriate identification of t and t' with t_n and t_{n+1} (depending on the order in which t and t' appear in the s, s'-path). This allows us to reformulate the lemma one last time to get

Lemma 5.2.4. *Let G be a graph and T a tree decomposition of G. Let t_1, t_2 be adjacent in T. If $s_1, s_2 \in V(T)$ such that the edge $\{t_1, t_2\}$ lies on the unique path from s_1 to s_2, then for any $x \in s_1 \setminus (t_1 \cap t_2)$ and any $y \in s_2 \setminus (t_1 \cap t_2)$, x and y are not in the same component of $G - (t_1 \cap t_2)$.*

This rewriting allows us to better see what condition our definition of $^\circ$-tree decomposition will need to enforce to make sure we have the analogue of 2.4.4 and can apply the same method of creating a winning cop strategy. We will need to make sure that for any $^\circ$-tree decomposition, T, of a structure A, if the edge $\{t_1, t_2\}$ lies on the unique path from s_1 to s_2 in T and $\mathcal{D}^\circ(A/(t_1 \cap t_2)) = \{B_0, B_1, ..., B_{m-1}\}$ is the unique decomposition into $^\circ$-components of A over $t_1 \cap t_2$, then there is a set $I \subseteq m$ such that $s_1 \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in I} B_i$ and $s_2 \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in m \setminus I} B_i$.

While we did make some attempt to find an appropriate condition that would imply this (and we will see such a definition in Section 6.5), the easiest solution was just to take this condition as one of our axioms for $^\circ$-tree decompostion. This results in the following definition.

Definition 5.2.5. Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\mid}\limits^{\circ}$. If $A \subset_{\text{fin}} M$, a *$^\circ$-tree decomposition* of A is a tree T such that each $t \in V(T)$ is a subset of A and

- $\bigcup V(T) = A$,

- for all $s_1, s_2, t_1, t_2 \in V(T)$ such that t_1 and t_2 are adjacent in T and the edge $\{t_1, t_2\}$ lies on the unique path from s_1 to s_2, if $\mathcal{D}^\circ(A/(t_1 \cap t_2)) = \{B_0, B_1, ..., B_{m-1}\}$, then there is a set $I \subseteq m$ such that $s_1 \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in I} B_i$ and $s_2 \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in m \setminus I} B_i$, and

- for all $t_1, t_2 \in V(T)$, if $t \in V(T)$ is on the unique path between t_1 and t_2, then $t_1 \mathop{\smile\!\!\!\!\!\!\!\;|}^\circ_t t_2$.

We can then define the width of a $^\circ$-tree decomposition and the $^\circ$-tree width of a structure exactly as one would expect.

Definition 5.2.6. Let M be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\!\;|}^\circ$ and $A \subset_{\text{fin}} M$. If T is a $^\circ$-tree decomposition of A, then the *width* of T is

$$w(T) = \max\{|t| - 1 | t \in T\}.$$

The $^\circ$-tree width of A is

$$tw^\circ(A) = \min\{w(T) | T \text{ is a } ^\circ\text{-tree decomposition of } A\}.$$

We have chosen a definition for $^\circ$-tree decomposition which should make it easy to construct a winning strategy for k cops from a $^\circ$-tree decomposition of width less than k. However, since we did not develop this definition directly from the definition of tree decompositions for graphs, it is not obvious that for a graph G with the standard abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\!\;|}^\circ$, a tree, T, is a a tree decomposition of G if and only if it is a $\circ$-tree decomposition of G. We verify this below.

Proposition 5.2.7. *Let G be a graph and T a tree. T is a $\circ$-tree decomposition of G if and only if T is a tree decomposition of G.*

Proof. Suppose T is a $\circ$-tree decomposition of G. Then, by definition, $\bigcup V(T) = V(G)$.

Suppose $s_1, t, s_2 \in V(T)$ such that t is on the unique s_1, s_2-path in T. Since T is a $\circ$-tree decomposition, $s_1 \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_{t} s_2$. So $s_1 \cap s_2 \subseteq t$ by Regularity 1.

It remains to show that each edge of G is contained in some vertex of T. Suppose $\{x, y\} \in E(G)$. Since T is a $\circ$-tree decomposition, there exists some element of $V(T)$ containing x and some element containing y. Choose $s_x, s_y \in V(T)$ with $x \in s_x$, $y \in s_y$ such that the length of the unique v_x, v_y-path in T is minimal.

Claim. $s_x = s_y$.

For the sake of contradiction, suppose $s_x \neq s_y$. Then there exist $t_1, t_2 \in V(T)$ on the unique s_x, s_y-path such that t_1 and t_2 are adjacent. Let $\{B_0, ..., B_{m-1}\} = \mathcal{D}^{\circ}\left(V(G)/\left(t_1 \cap t_2\right)\right)$.

Since s_x, s_y were chosen so that the unique path between them was of minimal length, for $i = 1, 2$, $x \in t_i$ if and only if $t_i = s_x$ and $y \in t_i$ if and only if $t_i = s_y$. $t_1 \neq t_2$ then implies neither x nor y can be in both t_1 and t_2, i.e.,

$\{x,y\} \cap (t_1 \cap t_2) = \emptyset$. Because $\{x,y\} \in E(G)$, we must then have $\{x\} \not\mathop{\smile}\limits^{\circ}_{t_1 \cap t_2} \{y\}$. By the (contrapositive of) the monotonicity axiom, this gives us $s_x \not\mathop{\smile}\limits^{\circ}_{t_1 \cap t_2} s_y$. But then there cannot be any $I \subseteq n$ such that $s_x \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in I} B_i$, $s_y \setminus (t_1 \cap t_2) \subseteq \bigcup_{j \in n \setminus I} B_j$, contradicting T being a °tree decomposition.

Since we have reached a contradiction, we must have $s_x = s_y$ as claimed. Thus $\{x,y\} \subseteq s_x$, so there is indeed a vertex of T containing the edge $\{x,y\}$.

Thus if T is a °-tree decomposition of G, it also satisfies the necessary conditions to be a tree decomposition of G.

The other direction, that whenever T is a tree decomposition of G it is also a °-tree decomposition of G, mostly follows from arguments given earlier in this chapter. We will, however, give an alternate proof here.

Suppose T is a tree decomposition of the graph G. Again, it is clear that $\bigcup V(T) = V(G)$ and the vertices of G are considered to be the elements of G when treating G as a substructure of the random graph.

Suppose that $s_1, t, s_2 \in V(T)$ such that t is on the unique s_1, s_2-path in T. We must show that $s_1 \mathop{\smile}\limits^{\circ}_t s_2$. For the sake of a contradiction, suppose not.

If $s_1 \not\mathop{\smile}\limits^{\circ}_t s_2$, then there exists some $x \in s_1 \setminus t$, $y \in s_2 \setminus t$ such that $\{x,y\} \in E(G)$. Since T is a tree decomposition of G, there must be some $t' \in V(T)$ such that $x, y \in t'$. t must be on either the unique s_1, t'-path in T or the unique t', s_2-path in T, since otherwise there would be a path from s_1 to s_2 not containing t, contradicting T being a tree. But then T being a tree decomposition of G

implies that either $s_1 \cap t' \subseteq t$ or $s_2 \cap t' \subseteq t$ and, thus, that either x or y is in t. This is a contradiction, since $x \in s_1 \setminus t$ and $y \in s_2 \setminus t$. So we do indeed have $s_1 \mathop{\downarrow}^{\circ}_t s_2$.

It remains to show that for all $s_1, s_2, t_1, t_2 \in V(T)$ such that t_1 and t_2 are adjacent in T and the edge $\{t_1, t_2\}$ lies on the unique path from s_1 to s_2, if $\mathcal{D}^{\circ}(G/(t_1 \cap t_2)) = \{B_0, B_1, ..., B_{m-1}\}$, then there is a set $I \subseteq m$ such that $s_1 \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in I} B_i$ and $s_2 \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in m \setminus I} B_i$.

Suppose that $\{t_1, t_2\} \in E(T)$. Since T is a tree, the graph T' obtained from T by removing the edge $\{t_1, t_2\}$ will consist of two components; let H_1 be the component containing t_1 and H_2 be the component containing t_2. Let $C = t_1 \cap t_2$ and $\mathcal{D}^{\circ}(G/C) = \{B_0, ...B_{m-1}\}$.

Claim. If $B_0 \cap \bigcup_{v \in V(H_1)} v \neq \emptyset$, then $B_0 \cap \bigcup_{w \in V(H_2)} w = \emptyset$.

Let $x_0 \in B_0 \cap \bigcup_{v \in V(H_1)} v$. We want to show that for each $x \in B_0$, $x \notin w$ for each $w \in V(H_2)$. We proceed by induction on the length of the shortest x_0, x-path in B_0.

Base case: Suppose $x = x_0$. There exists $v \in V(H_1)$ such that $x_0 \in v$. For any $w \in V(H_2)$, $\{t_1, t_2\}$ must lie on the unique v, w-path in T. Since T is a tree decomposition of G, we would then have $v \cap w \subseteq t_1, t_2$ and, hence, $v \cap w \subseteq t_1 \cap t_2 = C$. Thus, x_0 cannot be an element of w for any $w \in V(H_2)$, since that would contradict $x \in B_0 \in \mathcal{D}^{\circ}(G/C)$.

Inductive step: Assume that if the length of the shortest path from x_0 to x in B_0 is n, then $x \notin w$ for each $w \in V(H_2)$. We must show that when $x \in X_0$ such that the length of the shortest path from $x_0 to x$ is $n+1$, then $x \notin w$ for each $w \in V(H_2)$.

Suppose that the length of the shortest path from x_0 to x in B_0 is $n+1$. Let x_n be the neighbor of x on an x_0, x-path of length $n+1$. Then the shortest path from x_0 to x_n in B_0 must be of length n, so by inductive assumption, $x_n \notin w$ for each $w \in V(H_2)$. Since $\{x_n, x\} \in E(G)$ and T is a tree decomposition of G, there exists some $t \in V(T)$ such that $x_n, x \in t$. Since $x_n \notin w$ for each $w \in V(H_2)$, $t \in V(H_1)$. Thus, as in the base case, for any $w \in V(H_2)$, $\{t_1, t_2\}$ must lie on the unique path from t to w, so T a tree decomposition implies that $t \cap w \subseteq t_1 \cap t_2 = C$. Since $x \in t$ and $x \in B_0$ (so $x \notin C$), this implies that $x \notin w$ for each $w \in V(H_2)$.

Clearly, the choice of B_0 among the components of G over C was arbitrary and, by symmetry, we can reverse the roles of H_1 and H_2. It follows that for any $s_1 \in V(H_1)$, $s_2 \in V(H_2)$, $B_i \cap s_1 \neq \emptyset$ implies that $B_i \cap s_2 = \emptyset$ for each $i \in [n]$, and vice versa.

Now let $s_1, s_2 \in V(T)$ such that $\{t_1, t_2\}$ lies on the unique s_1, s_2-path in T. We may assume without loss of generality that $s_1 \in V(H_1)$ and $s_2 \in V(H_2)$. Let $I = \{i \in m | B_i \cap s_1 \neq \emptyset\}$. Then, by the above, $s_1 \setminus C \subseteq \bigcup_{i \in I} B_i$ and

$s_2 \setminus C \subseteq \bigcup_{j \in m \setminus I} B_j$.

This shows that for any $s_1, t_1, t_2, s_2 \in V(T)$ such that t_1, t_2 are adjacent and the edge $\{t_1, t_2\}$ lies on the unique path from s_1 to s_2, if $C = t_1 \cap t_2$ and $\mathcal{D}^\circ(G/C) = \{B_0, ..., B_{m-1}\}$, then there exists an $I \subseteq m$ such that $s_1 \setminus C \subseteq \bigcup_{i \in I} B_i$ and $s_2 \setminus C \subseteq \bigcup_{j \in m \setminus I} B_j$.

Thus if T is a tree decomposition of G, it also satisfies the necessary conditions to be a $^\circ$-tree decomposition of G. □

We make one more useful observation about $^\circ$-tree decompositions before proceeding to the next chapter. As noted in the previous section, the contrapositive of Lemma 2.4.2 says that for a graph G with tree decomposition T, if t lies on the unique path from s_1 to s_2 in T, then for each $x \in s_1 \setminus t$ and each $y \in s_2 \setminus t$, x and y must be in separate components of $G - t$. In Chapter 2, we used Lemma 2.4.2 to prove Lemma 2.4.4. We have effectively taken Lemma 2.4.4 as an axiom in our definition of $^\circ$-tree decomposition, but we can now use that definition to prove the analogue of Lemma 2.4.2 (in contrapositive form).

Lemma 5.2.8. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\downarrow^\circ$, $A \subset_{fin} M$, and T a $^\circ$-tree decomposition of A. If $s_1, s_2, t \in V(T)$ such that t is on the unique path from s_1 to s_2 and $\mathcal{D}^\circ(A/t) = \{B_0, B_1, ..., B_{m-1}\}$, then there is a set $I \subseteq m$ such that $s_1 \setminus t \subseteq \bigcup_{i \in I} B_i$ and $s_2 \setminus t \subseteq \bigcup_{i \in m \setminus I} B_i$.*

Proof. The case where $s_1 = t = s_2$ is vacuously true. If $s_1 = t \neq s_2$ or $s_1 \neq t = s_2$, then the claim is also obvious. So we assume that $s_1 \neq t$ and $s_2 \neq t$.

Let t' be the unique neighbor of t on the t, s_2-path. If $\mathcal{D}^\circ(A/(t \cap t')) = \{X_0, X_1, ..., X_n\}$, then since T is a $^\circ$-tree decomposition and the edge $\{t, t'\}$ is on the path from s_1 to s_2, there must be some $J \subseteq n$ such that $s_1 \setminus (t \cap t') \subseteq \bigcup_{j \in J} X_i$ and $s_2 \setminus (t \cap t') \subseteq \bigcup_{j \in n \setminus J} X_j$.

$t \cap t'$ is a subset of t, so by Lemma 4.2.4, each B_i must be a subset of X_j for some $j \in n$. Let $I = \{i \in m | B_i \subseteq \bigcup_{j \in J} X_j\}$. Note that if $i \notin I$, then $B_i \subseteq \bigcup_{j \in n \setminus I'} X_i$, so any elements of $\bigcup_{j \in J} X_j$ which are *not* in $\bigcup_{i \in I} B_i$ must be in t. We then have

$$s_1 \setminus t = (s_1 \setminus (t \cap t')) \setminus t \subseteq \left(\bigcup_{j \in J} X_j \right) \setminus t = \bigcup_{i \in I} B_i.$$

Similarly, we have

$$s_2 \setminus t = (s_2 \setminus (t \cap t')) \setminus t \subseteq \left(\bigcup_{j \in n \setminus J} X_j \right) \setminus t = \bigcup_{i \in m \setminus I} B_i.$$

□

Chapter 6

Main Result

Having found generalizations of tree decomposition and the Cops and Robbers game for finite subsets of the generic model of an algebraically trivial Fraïssé class with an abstract free amalgamation, the main result that we want is an analogue of Seymour and Thomas's Theorem. That is, we would like to show that for A a finite subset of the generic model of an algebraically trivial Fraïssé class with free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\circ}$, k cops have a winning strategy in the $\circ$-Cops and Robbers game on A if and only if A has a $\circ$-tree decomposition of width less than k. This result would help assure us that we have chosen the correct generalizations and might be useful in further work.

In the first section of this chapter, we will prove one direction of the desired result. Sections 2 and 3 will lay the groundwork for the approach which will allow us, in Section 4, to prove the full analogue of Seymour and Thomas's

Theorem. In Sections 5 and 6, respectively, we will revisit the defintion of °-tree decomposition and the Gaifman graph. Finally, in Section 7, we include material on our initial approach to the problem; although this approach did not lead to the full result, we include it for historical reasons and because it may still be relevant for future work.

6.1 One Direction

As shown in the previous chapter, we already had the goal of proving the analogue of Seymour and Thomas's result in mind when we were trying to define °-tree decomposition. We chose to define °-tree decomposition in such a way that proving one direction of the desired result becomes straightforward.

Proposition 6.1.1. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ and $A \subset_{fin} M$. If A has °-tree width less than k, then k cops have a winning strategy in the °-Cops and Robbers game on A.*

We can prove this exactly as we proved the equivalent statement for graphs in Section 2.4 if we substitute the following lemma for Lemma 2.4.4.

Lemma 6.1.2. *Let M be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ and $A \subset_{fin} M$. Let T be a °-tree decomposition of A and $\{t_n, t_{n+1}\} \in E(T)$. If $x \in A \setminus t_n$ such that t_{n+1}*

is on the unique path from t_n to s for some $s \in V(T)$ such that $x \in s$, then for each $y \in A \setminus t_{n+1}$ which is in the same $\circ$-component of A over $t_n \cap t_{n+1}$ as x, there exists a unique neighbor $t_{n+2} \neq t_n$ of t_{n+1} such that t_{n+2} is on the unique path from t_{n+1} to s' for each $s' \in V(T)$ such that $y \in s'$.

Proof. Since this is the analogue of Lemma 2.4.4, the proof follows from reversing the sequence of equivalences in Section 5.2 along with an argument for uniqueness.

Let $x \in A \setminus t_n$ such that t_{n+1} is on the unique path from t_n to s for some $s \in V(T)$ such that $x \in s$ and suppose $y \in A \setminus t_{n+1}$ is in the same $\circ$-component of A over $t_n \cap t_{n+1}$ as x. Since T is a $\circ$-tree decomposition of A, there must be some $s' \in V(T)$ such that $y \in s'$. Then because y is in the same $\circ$-component of A over $t_n \cap t_{n+1}$ and T is a $\circ$-tree decomposition of A, the edge $\{t_n, t_{n+1}\}$ cannot lie on the unique path from s to s'. $y \notin t_{n+1}$ then implies that there must be some neighbor of t_{n+1} not equal to t_n that lies on the t_{n+1}, s'-path in T; call this vertex t_{n+2}.

Now suppose $y \in s''$ for some $s'' \in V(T)$. Let t'_{n+2} be the unique neighbor of t_{n+1} that lies on the t_{n+1}, s''-path in T. If $t'_{n+2} \neq t_{n+2}$, then combining the s', t_{n+1}-path and the t_{n+1}, s''-path gives an s', s''-path. But then, since T is a $\circ$-tree decomposition we would have $s' \mathop{\downarrow\!\!\!\!\smile}^{\circ}_{t_{n+1}} s''$ and, hence, $y \in s' \cap s'' \subseteq t$. So t_{n+2} must be the unique neighbor of t_{n+1} such that t_{n+2} is on the unique path from t_{n+1} to any vertex of $V(T)$ containing y. $\square$

Proof of Proposition 6.1.1. As already noted, this proof will be virtually identical to the one given for the equivalent result for graphs. We construct a cop strategy from a $^\circ$-tree decomposition and then show that strategy is winning.

Assume that A has $^\circ$-tree width less than k. Then there exists a $^\circ$-tree decomposition of A having width less than k. Let T be such a $^\circ$-tree decomposition. Note that for each $t \in V(T)$, $t \in \binom{A}{\leq k}$. We define a cop strategy κ as follows:

- Let t_0 be any vertex of T. Define $\kappa(\varepsilon)$ to be t_0.

- Let (C_0, z_0) be a robber-winning run of length one. If $C_0 \neq t_0$, define $\kappa((C_0, z_0)) = t_0$. If $C_0 = t_0$, then we must have $z_0 \in A \setminus t_0$. Since T is a $^\circ$-tree decomposition of A, z_0 must be an element of some vertex of T. Since $z_0 \notin t_0$, some neighbor of t_0 must lie on the unique path from t_0 to a vertex of T containing z_0. That neighbor must be unique, since if t was a neighbor of t_0 on the unique path from t_0 to s for some s containing z_0 and $t' \neq t$ was a neighbor of t_0 on the unique path from t_0 to s' for some s' containing z_0, then t_0 would be on the unique path from s to s' and, by T a tree decomposition, we would have $z_0 \in s \cap s' \subseteq t_0$, a contradiction. Define $\kappa((C_0, z_0))$ to be this unique neighbor of t_0.

- Let $(C_i, z_i)_{i \leq m}$ be a robber-winning run of length $m + 1$. If $C_0, C_1, ..., C_m$ is a path in T and C_m is on the unique path from C_{m-1} to s for some $s \in V(T)$ such that $z_{m-1} \in s$, then by Lemma 6.1.2, since z_m must be in the same component of $A - (C_{m-1} \cap C_m)$ as z_{m-1} (because $(C_i, z_i)_{i \leq m}$ is a robber-winning run), there exists a unique neighbor $t_{m+1} \neq C_{m-1}$ of C_m such that t_{m+1} is on the unique path from C_m to s' for each $s' \in V(T)$ such that $z_m \in s'$. In this case, define $\kappa\left((C_i, z_i)_{i \leq m}\right) = t_{m+1}$. Otherwise, define $\kappa\left((C_i, z_i)_{i \leq m}\right) = C_0$.

We need to show that κ is a winning cop strategy. That is, we must show that each robber-winning run which respects κ must have length less than or equal to N for some $N \in \mathbb{N}$. We claim that if $(C_i, z_i)_{i \leq m}$ is a robber-winning run of length at least two (i.e., $m \geq 1$) which respects κ, then C_m is on the unique path from C_{m-1} to s for some $s \in V(T)$ such that $z_{m-1} \in s$ and $C_0, C_1, ..., C_m$ must be a path in T.

By induction on m:

Base Case: Since $(C_i, z_i)_{i \leq m}$ respects κ, we must have $C_0 = \kappa(\varepsilon) = t_0$ for some vertex $t_0 \in V(T)$. If $m = 1$, then we must have $C_1 = \kappa\left((C_0, z_0)\right)$. Since $C_0 = t_0$, $\kappa\left((C_0, z_0)\right)$ must be the unique neighbor of t_0 which lies on a path from t_0 to some vertex $s \in V(T)$ containing z_0 and C_0, C_1 is a path in T.

Inductive Step: Suppose for some $l \in \mathbb{N}$, for each $1 \leq m \leq l$ and each κ respecting robber-winning run $(C_i, z_i)_{i \leq m}$, C_m is on the unique path from C_{m-1} to s for some $s \in V(T)$ such that $z_{m-1} \in s$ and $C_0, C_1, ..., C_m$ is a path in T. We need to show that the same holds for robber-winning runs

$(C_i, z_i)_{i \leq m}$ with $m = l + 1$.

Let $(C_i, z_i)_{i \leq l+1}$ be a robber-winning run which respects κ. Then $(C_i, z_i)_{i \leq l}$ is a robber-winning run which respects κ and, by assumption, $C_0, C_1, ..., C_l$ is a path in T with C_l on the unique path from C_{l-1} to s for some $s \in V(T)$ such that $z_{l-1} \in s$. We must have $C_{l+1} = \kappa\left((C_i, z_i)_{i<l+1}\right)$. By the third bullet point of the definition of κ, $\kappa\left((C_i, z_i)_{i<l+1}\right)$ must be the unique neighbor of C_l which lies on the path from C_l to s' for every $s' \in V(T)$ such that $z_l \in s'$. Therefore C_{l+1} does indeed lie on the unique path from C_l to s' for some $s' \in V(T)$ such that $z_l \in s'$ and $C_0, C_1, ..., C_l, C_{l+1}$ is a path in T.

Since $C_0, C_1, ..., C_m$ is a path in T for any κ respecting robber-winning run $(C_i, z_i)_{i \leq m}$ of length at least two , we cannot have a robber-winning run of length greater than $N = \max\{1, |V(T)|\}$. Thus, κ is a winning cop strategy. □

So we have one direction of the desired result, although one could say that we "rigged the game" by choosing a definition of °-tree decomposition in such a way that we knew this direction would hold. In any case, showing the other direction would prove more challenging.

6.2 °paths

Throughout this section and the next, assume that $\mathcal{M}$ is the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ and $A \subset_{\text{fin}} M$.

In a graph, G, two vertices are in the same component of G over some $C \subseteq V(G)$ if and only if there is a path between those two vertices which doesn't include any point of C. Since we are able to define °-components in our structure based only on the abstract free amalgamation relation, it is natural to wonder whether there is also an associated notion of path such that two elements of A are in the same °-component of A over some $C \subseteq A$ if and only there is a path between them which doesn't intersect C.

If we consider an x, y-path P as a subgraph of G, then P is the unique component of G over $V(G) \setminus V(P)$. On the other hand, if we take away any vertex $z \neq x, y$ from P, the remaining vertices will split into two components (one containing x, the other containing y) of G over $(V(G) \setminus V(P)) \cup \{z\}$.

So, similarly, we define an x, y-°path in A as follows:

Definition 6.2.1. Let $x, y \in A$. An x, y-°path in A is a subset P of A such that $x, y \in P$, P is weakly irreducible over $A \setminus P$, and for any $z \in P$ which is not equal to x or y, there exists a biparition $\{Q_1, Q_2\}$ of $P \setminus \{z\}$ with $x \in Q_1$,

$y \in Q_2$, and $Q_1 \mathop{\smile\!\!\!\!\!\!\!\bot}^{\circ}_{(A \setminus P) \cup \{z\}} Q_2$.

As we would hope, this definition guarantees that two elements $x, y \in A$ are in the same $\circ$-component of A over C for some $C \subseteq A$ if and only if there is an x, y-$\circ$path in A which doesn't intersect C.

Lemma 6.2.2. *Let $C \subseteq A$. $x, y \in A$ are in the same $\circ$-component of A over C if and only if there exists an x, y-$\circ$path in A which doesn't intersect C.*

Proof. For the forward direction, let $H_0 \in \mathcal{D}^\circ(A/C)$ such that $x, y \in H_0$. By the contrapositive of freeness, since H_0 is weakly irreducible over C, H_0 must be weakly irreducible over $A \setminus H_0$. Let P be a minimal subset of H_0 such that $x, y \in P$ and P is weakly irreducible over $A \setminus P$. If $P = \{x, y\}$, then P is an x, y-$\circ$path in A and we're done. Otherwise, let $z \in P$ such that $z \neq x, y$.

Let $\{X_0, X_1, ..., X_n\} = \mathcal{D}^\circ(A/((A \setminus P) \cup \{z\}))$ and assume without loss of generality that $x \in X_0$. $y \notin X_0$ since X_0 is weakly irreducible over $(A \setminus P) \cup \{z\}$ and, hence, over $A \setminus X_0$ by the contrapositive of freeness, so $y \in X_0$ would contradict the minimality of P. Let $Q_1 = X_0$ and $Q_2 = \bigcup_{i=1}^{n} X_i$. Then $\{Q_1, Q_2\}$ is a bipartition of $P \setminus \{z\}$ with $x \in Q_1$ and $y \in Q_2$, and, by the definition of the decomposition, $Q_1 \mathop{\smile\!\!\!\!\!\!\!\bot}^{\circ}_{(A \setminus P) \cup \{z\}} Q_2$.

For the other direction, suppose there exists P, an x, y-$\circ$path in A, which doesn't intersect C. We want to show that x and y are in the same $\circ$-component of A over C. For the sake of contradiction, suppose they aren't.

Let $\mathcal{D}^\circ(A/C) = \{H_0, ..., H_n\}$ and assume without loss of generality that $x \in H_0$, $y \in H_n$. Then $H_0 \mathop{\smile\!\!\!\!\!\!\downarrow}^\circ_C \bigcup_{i=1}^n H_i$. Applying transitivity twice, we get

$$P \cap H_0 \mathop{\smile\!\!\!\!\!\!\downarrow}^\circ_{C \cup (H_0 \setminus P) \cup \left(\bigcup_{i=1}^n H_i \setminus P\right)} P \cap \bigcup_{i=1}^n H_i.$$

But since $C \cap P = \emptyset$, the base above is really $A \setminus P$ and $\{P \cap H_0, P \cap \bigcup_{i=1}^n H_i\}$ is a bipartition of P, a contradiction of P being an x, y-$^\circ$path. □

This definition of $^\circ$path also preserves certain other useful properties of paths in graphs.

Lemma 6.2.3. *If P is an x, y-$^\circ$path in A and $z \in P$, then P contains an x, z-$^\circ$path in A.*

Proof. Since P is weakly irreducible over $A \setminus P$, P is the unique component of A over $A \setminus P$. Since $x, z \in P$, by the previous lemma there exists an x, z-$^\circ$path in A which doesn't intersect $A \setminus P$ (and, therefore, is contained in P). □

So if we have P an x, y-$^\circ$path in A, for any $w, z \in P$ we can find a w, z-$^\circ$path in A which is contained in P. More precisely, if we take any point $z \in P$ which is not an endpoint of P, we can find a $^\circ$path from x to z in A which is contained in $P \setminus \{y\}$.

Lemma 6.2.4. *If P is an x, y-$^\circ$path in A with $x \neq y$ and $z \in P$ such that $z \neq y$, then there exists an x, z-$^\circ$path P' in A with $P' \subseteq P \setminus \{y\}$.*

Proof. If $z = x$, then $P' = \{x\}$ is the desired x, z-$^\circ$path, so we may assume $z \neq x$.

Let $\mathcal{D}^\circ(A/((A \setminus P) \cup \{z\})) = \{X_0, X_1, ..., X_n\}$ with $x \in X_0$. By the definition of P being an x, y-$^\circ$path in A, there exists a bipartition $\{Q_1, Q_2\}$ of $P \setminus \{z\}$ such that $x \in Q_1$, $y \in Q_2$ and $Q_1 \downfree^\circ_{(A\setminus P)\cup\{z\}} Q_2$. By monotonicity, $X_0 \cap Q_1 \downfree^\circ_{(A\setminus P)\cup\{z\}} X_0 \cap Q_2$, so we must have $X_0 \cap Q_2 = \emptyset$ (since otherwise $\{X_0 \cap Q_1, X_0 \cap Q_2\}$ would be a bipartition of X_0). So $y \notin X_0$ (and $n > 0$).

By definition of the decomposition into $^\circ$-components,

$$X_0 \downfree^\circ_{(A\setminus P)\cup\{z\}} \bigcup_{i=1}^{n} X_i.$$

On the other hand, since P is an x, y-$^\circ$path in A and $\{X_0, (\bigcup_{i=1}^{n} X_i) \cup \{z\}\}$ is a bipartition of P,

$$X_0 \not\downfree_{A\setminus P} \left(\bigcup_{i=1}^{n} X_i\right) \cup \{z\}.$$

So by transitivity, we must have $X_0 \not\downfree_{A\setminus P} \{z\}$.

Also, by the definition of the decomposition into $^\circ$-components, for any bipartition $\{Y_1, Y_2\}$ of X_0 we have $Y_1 \not\downfree_{(A\setminus P)\cup\{z\}} Y_2$. Transitivity, then, gives us $Y_1 \not\downfree_{A\setminus P} Y_2 \cup \{z\}$. Thus $X_0 \cup \{z\}$ is weakly irreducible over $A \setminus P$. By the contrapositive of freeness, we then have that $X_0 \cup \{z\}$ is also weakly irreducible over $A \setminus (X_0 \cup \{z\})$. Therefore $X_0 \cup \{z\}$ is the unique $^\circ$-component of A over $A \setminus (X_0 \cup \{z\})$ and by Lemma 6.2.2, there exists P', an x, z-$^\circ$path in A which doesn't intersect $A \setminus (X_0 \cup \{z\})$ (so $P' \subseteq X_0 \cup \{z\} \subseteq P \setminus \{y\}$). $\square$

We can also, as we could in graphs, show that if we have a path from x to y and a path from y to z, then we must have a path from x to z:

Lemma 6.2.5. *If P_1 is an x, y-$^\circ$path in A and P_2 is a y, z-$^\circ$path in A, then $P_1 \cup P_2$ contains an x, z-$^\circ$path in A.*

Proof. We show first that $P_1 \cup P_2$ is weakly irreducible over $A \setminus (P_1 \cup P_2)$:

Suppose not. Then there exists a bipartition $\{Q_1, Q_2\}$ of $P_1 \cup P_2$ such that

$$Q_1 \mathop{\smile\hskip-0.9em\vert\,}^{\circ}_{A\setminus(P_1\cup P_2)} Q_2.$$

Assume without loss of generality that $y \in Q_1$. Since $\{Q_1, Q_2\}$ is a bipartition, $Q_2 \neq \emptyset$, so either $P_1 \cap Q_2 \neq \emptyset$ or $P_2 \cap Q_2 \neq \emptyset$. Without loss of generality, we may assume $P_1 \cap Q_2 \neq \emptyset$. Then $\{P_1 \cap Q_1, P_1 \cap Q_2\}$ is a bipartition of P_1. So by the definition of P_1 being a path

$$P_1 \cap Q_1 \not\mathop{\smile\hskip-0.9em\vert\,}^{\circ}_{A\setminus P_1} P_1 \cap Q_2.$$

But since

$$Q_1 \mathop{\smile\hskip-0.9em\vert\,}^{\circ}_{A\setminus(P_1\cup P_2)} Q_2,$$

by transitivity (applied twice) we have

$$P_1 \cap Q_1 \mathop{\smile\hskip-0.9em\vert\,}^{\circ}_{(A\setminus(P_1\cup P_2))\cup(Q_1\setminus P_1)\cup(Q_2\setminus P_1)} P_1 \cap Q_2.$$

Since

$$\begin{aligned}(Q_1 \setminus P_1) \cup (Q_2 \setminus P_1) &= (Q_1 \cup Q_2) \setminus P_1 \\ &= (P_1 \cup P_2) \setminus P_2 \\ &= P_2 \setminus P_1,\end{aligned}$$

the base here is really

$$(A \setminus (P_1 \cup P_2)) \cup P_2 \setminus P_1 = A \setminus P_1.$$

So we can rewrite the above as

$$P_1 \cap Q_1 \mathop{\perp\!\!\!\!\!\!\!\smile}^{\circ}_{A \setminus P_1} P_1 \cap Q_2,$$

a contradiction.

Since $P_1 \cup P_2$ is weakly irreducible over $A \setminus (P_1 \cup P_2)$, $P_1 \cup P_2$ is the single °-component of A over $A \setminus (P_1 \cup P_2)$. Thus x and z lie in the same °-component of A over $A \setminus (P_1 \cup P_2)$ and, by Lemma 6.2.2, there exists an x, z-°path in A which doesn't interset $A \setminus (P_1 \cup P_2)$. Therefore, $P_1 \cup P_2$ contains an x, z-°path in A. □

It is important to note, however, that for graphs, °paths are not precisely equivalent to paths. For instance, if G is a 3-cycle with vertices x, y, z, then x, z, y is an x, y-path, so there is an x, y-path containing all three vertices. However, $P = \{x, z, y\}$ is *not* an x, y-°path in G, since $\{x\} \not\mathop{\perp\!\!\!\!\!\!\!\smile}^{\circ}_{\{z\}} \{y\}$. The difference is one of subgraphs versus induced subgraphs: the vertices of a path P in G form a $^{\circ}$path in G if and only if the induced subgraph of G on those vertices is a path, that is, if the only edges between vertices of P in G are those included in P. This distinction is perhaps not surprising since in a model theory context subgraph generally means induced subgraph due to the definition of substructure.

Nevertheless, for consistency with the usage in graphs, we will define the *length* of a $^{\circ}$path P to be $|P| - 1$.

6.3 Abstract Free Amalgamation Graphs

Having a definition of path naturally gives rise to a definition of adjacency in our structure – we will say that two elements of A are adjacent if there is a path of length one between them.

Definition 6.3.1. Let $x, y \in A$. x is $^{\circ}$adjacent to y in A if $x \neq y$ and $\{x, y\}$ is weakly irreducible over $A \setminus \{x, y\}$.

This in turn lets us create an associated graph:

Definition 6.3.2. Let A be an element of an algebraically trivial Fraïssé class with an abstract free amalgamation relation, $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$. We define the *abstract free amalgamation graph of* A to be the graph G_A° with $V(G_A^{\circ}) = A$ and $E(G_A^{\circ}) = \{\{x, y\} \subseteq A | x$ is $^{\circ}$adjacent to y in $A\}$.

We will ultimately show that T is a tree decomposition of this graph G_A° if and only if T is a $^\circ$tree decomposition of A and that k cops have a winning strategy on G_A° if and only if k cops have a winning strategy on A. Towards this end, we make a few useful observations about $^\circ$paths in A and the relationship between A and G_A°.

First, we see that any proper subset of an x, y-$^\circ$path cannot also be a $^\circ$path from x to y.

Lemma 6.3.3. *If $x, y \in A$, P an x, y-$^\circ$path in A, and $P' \subsetneq P$, then P' is not an x, y-$^\circ$path in A.*

Proof. This is clear if either x or y is not in P', so we may assume $x, y \in P'$.

$P' \subsetneq P$, so there exists $z \in P \setminus P'$. Note that $z \neq x, y$. Since P is an x, y-$^\circ$path in A, there is a bipartition $\{Q_1, Q_2\}$ of $P \setminus \{z\}$ with $x \in Q_1$, $y \in Q_2$, and $Q_1 \mathop{\downarrow\!\!\!\!\smile}^{\circ}_{(A\setminus P)\cup\{z\}} Q_2$. We can then apply transitivity twice to get

$$P' \cap Q_1 \mathop{\downarrow\!\!\!\!\smile}^{\circ}_{((A\setminus P)\cup\{z\})\cup(Q_1\setminus P')\cup(Q_2\setminus P')} P' \cap Q_2.$$

$P' \subseteq P \setminus \{z\} = Q_1 \cup Q_2$, so $((A \setminus P) \cup \{z\}) \cup (Q_1 \setminus P') \cup (Q_2 \setminus P') = A \setminus P'$ and we have

$$P' \cap Q_1 \mathop{\downarrow\!\!\!\!\smile}^{\circ}_{A\setminus P'} P' \cap Q_2.$$

With $x \in Q_1$, $y \in Q_2$, and $x, y \in P' \subseteq P \setminus \{z\}$, $\{P' \cup Q_1, P' \cup Q_2\}$ is a bipartition of P', so P' is not weakly irreducible over $A \setminus P'$ and, therefore, cannot be an x, y-$^\circ$path in A. $\square$

Lemma 6.3.4. *If P is an x, y-$^\circ$path in A, with $x \neq y$, then there exists $z \in P$ such that x is $^\circ$adjacent to z in A.*

Proof. Fix A and x; let y vary. We proceed by induction on $|P|$.

Base case: If $|P| = 2$, then $P = \{x, y\}$ and, by the definition of P being an x, y-$^\circ$path in A, $\{x\} \not\angle_{A\setminus\{x,y\}} \{y\}$.

Inductive step: Assume that if $2 \leq |P| \leq m$ for P a $^\circ$path in A with x as an endpoint, there must exist $z \in P$ such that z is $^\circ$adjacent to x in A. We must show that when P is a x, y-$^\circ$path in A with $|P| = m + 1$, then there exists $z \in P$ such that z is $^\circ$adjacent to x in A.

Suppose P is a x, y-$^\circ$path in A with $|P| = m + 1$. Since $m \geq 2$, $m + 1 \geq 3$, so there exists $y' \in P$ such that $y' \neq x, y$. By Lemma 6.2.4, there exists $P' \subseteq P \setminus \{y\}$, an x, y'-$^\circ$path in A. $|P'| \leq |P| - 1 = m$, so we may apply the inductive assumption to find $z \in P' \subsetneq P$ such that z is $^\circ$adjacent to x in A.

□

Combining the previous two lemmas will allow us to show that we can write $^\circ$paths in A as sequences of adjacent points (all distinct), just as we can with paths in graphs.

Lemma 6.3.5. *If $x, y \in A$ such that $x \neq y$ and P an x, y-$^\circ$path in A with $|P| = n+1$, then $P = \{x_0, ...x_n\}$ where the x_i are all distinct, $x_0 = x$, $x_n = y$, and for each $1 \leq i \leq n$, x_i is $^\circ$adjacent to x_{i-1} in A.*

Proof. We'll show first that if $x \neq y$, any x, y-$^\circ$path in A, P, must contain a sequence of distinct points $x_0, ...x_m \in P$ for some $2 \leq m \leq |P|$ with $x_0 = x$, $x_m = y$, and x_i adjacent to x_{i-1} for $1 \leq i \leq m$. We'll do this using induction on $|P|$.

Base case: If $|P| = 2$, then $P = \{x, y\}$ and x is $^\circ$adjacent to y in A. Let $x_0 = x$, $x_1 = y$.

Inductive step: Assume that for any $x, y \in A$ with $x \neq y$, if there is an x, y-$^\circ$path in A, P, such that $|P| \leq r$, then there exists a sequence of distinct points $x_0, ...x_m \in P$ for some $2 \leq m \leq |P|$ with $x_0 = x$, $x_m = y$, and x_i $^\circ$adjacent to x_{i-1} in A for $1 \leq i \leq m$. We must show that for any $x, y \in A$ with $x \neq y$, if there is an x, y-$^\circ$path in A, P, such that $|P| = r+1$, then there exists a sequence of distinct points $x_0, ...x_m \in P$ for some $2 \leq m \leq |P|$ with $x_0 = x$, $x_m = y$, and x_i $^\circ$adjacent to x_{i-1} in A for $1 \leq i \leq m$.

Let $x, y \in A$ with $x \neq y$ and P an x, y-$^\circ$path in A with $|P| = r+1$. By Lemma 6.3.4 (and symmetry), there exists $y' \in P$ such that y' is $^\circ$adjacent to y in A. If $y' = x$, let $x_0 = y' = x$ and $x_1 = y$ and we're done. Otherwise, $y' \neq x, y$.

Since $y' \neq y$, by Lemma 6.2.4, there exists $P' \subseteq P \setminus \{y\}$ an x, y'-°path in A. $|P'| < |P| = r + 1$, so P' is an x, y'-°path in A with $x \neq y'$ and $|P| \leq r$. Then, by inductive assumption, for some $2 \leq m' \leq r$, there exists a sequence of distinct points $x_0, ..., x_{m'} \in P'$ such that $x_0 = x$, $x_{m'} = y'$, and x_i is °adjacent to x_{i-1} in A for $1 \leq i \leq m'$. Let $m = m' + 1 \leq r + 1$ and let $x_m = y$. Then $x_0, ..., x_m \in P$ are distinct since $x_i \neq y$ for $i < m$, $x_0 = x$, $x_m = y$, and x_i is °adjacent to x_{i-1} in A for $1 \leq i \leq m$.

So we can find such a sequence of distinct points within any x, y-°path P, but how do we know that the sequence must contain all points of the path (i.e., that $m = |P|$)? This will follow immediately from Lemma 6.3.3 if we can show that the sequence of points itself forms an x, y-°path.

Claim. Let P be an x, y-°path in A for some $x \neq y$. Let $P' = \{x_0, ..., x_m\} \subseteq P$ such that the x_i are distinct, $x_0 = x$, $x_m = y$, and x_i is °adjacent to x_{i-1} for $1 \leq i \leq m$. Then P' is a x, y-°path in A.

We must show that P' is weakly irreducible over $A \setminus P'$. Let $\{P'_1, P'_2\}$ be a bipartition of P'. Assume without loss of generality that $x_0 \in P'_1$. Since $P'_2 \neq \emptyset$, there must exist some $1 \leq i \leq m$ such that $x_{i-1} \in P'_1$, $x_i \in P'_2$. x_i is adjacent to x_{i-1}, so

$$\{x_{i-1}\} \not\angle_{A \setminus \{x_{i-1}, x_i\}} \{x_i\}.$$

By (the contrapositive of) transitivity, then,

$$P'_1 \not\mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{A\setminus P'} P'_2.$$

So P' is indeed weakly irreducible over $A \setminus P'$.

We must also show that for any $0 < i < m$, there exists a bipartition $\{Q'_1, Q'_2\}$ of $P' \setminus \{x_i\}$ such that $x_0 \in Q'_1$, $x_m \in Q'_2$, and

$$Q'_1 \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{(A\setminus P')\cup\{x_i\}} Q'_2.$$

Let $0 < i < m$ (if no such i exists, we're done); since P is an x_0, x_m-°path and $x_i \neq x_0, x_m$, there exists a bipartition $\{Q_1, Q_2\}$ of $P \setminus \{x_i\}$ with $x_0 \in Q_1$ and $x_m \in Q_2$ such that

$$Q_1 \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{(A\setminus P)\cup\{x_i\}} Q_2.$$

Applying transitivity (twice) gives us

$$P' \cap Q_1 \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{((A\setminus P)\cup\{x_i\})\cup(Q_1\setminus P')\cup(Q_2\setminus P')} P' \cap Q_2.$$

Since $P' \setminus \{x_i\} \subseteq P \setminus \{x_i\}$, the base above is actually $A \setminus (P' \setminus \{x_i\}) = (A \setminus P') \cup \{x_i\}$ and $\{P' \cap Q_1, P' \cap Q_2\}$ is a bipartition of $P' \setminus \{x_i\}$ (since $x_0 \in P' \cap Q_1$, $x_m \in P' \cap Q_2$). Thus, if we let $Q'_1 = P' \cap Q_1$ and $Q'_2 = P' \cap Q_2$, then $\{Q'_1, Q'_2\}$ is a bipartition of $P' \setminus \{x_i\}$ with $x_0 \in Q'_1$, $x_m \in Q'_2$, and

$$Q'_1 \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{(A\setminus P')\cup\{x_i\}} Q'_2,$$

as desired.

Since P' is an x, y-$^\circ$path in A, by Lemma 6.3.3, it cannot be a proper subset of any other x, y-$^\circ$path in A. Therefore, we must have $P' = P$. $\square$

Note that this result means that any x, y-$^\circ$path in A gives (the vertices of) an x, y-path in G_A°. It follows that if x and y are in the same $^\circ$-component of A over C then they are also in the same component of G_A° over C. The reverse also holds.

Lemma 6.3.6. *Let $x, y \in A$, $C \subseteq C$. x and y are in the same $^\circ$-component of A over C if and only if they are in the same component of G_A° over C.*

Proof. Suppose that x and y are in the same $^\circ$-component of A over C. By Lemma 6.2.2, there exists P, an x, y-$^\circ$path in A not intersecting C. So Lemma 6.3.5, there exist distinct $x_0, ..., x_n \in P \subseteq A \setminus C$ with $x_0 = x$, $x_n = y$ and for each $1 \leq i \leq n$, x_i is $^\circ$adjacent to x_{i-1} in A. Whenever x_i is $^\circ$adjacent to x_{i-1} in A, there is an edge from x_i to x_{i-1} in G_A°. Since all the x_i are distinct, they form a path from x to y in G_A° which doesn't intersect C. Since there is a path from x to y in G_A° which doesn't intersect C, x and y are in the same component of G_A° over C.

On the other hand, suppose that x and y are in the same component of G_A° over C, then there exists a path from x to y in G_A° which doesn't intersect C. If the length of this path is n, then the vertices can be labeled $x_0, ..., x_n$ such that the x_i are distinct, $x_0 = x$, $x_n = y$, and for each $1 \leq i \leq n$, x_i is adjacent to x_{i-1}

in G°_A. If two vertices are adjacent in G°_A, then they are °adjacent in A, so for each $1 \leq i \leq n$, $\{x_{i-1}\} \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.2ex\hbox{$|$}}^\circ_{A\setminus\{x_{i-1},x_i\}} \{x_i\}$. Thus, for each $1 \leq i \leq n$, $\{x_{i-1}, x_i\}$ is a °path in A not intersecting C and, by Lemma 6.2.2, x_i and x_{i-1} must be in the same °-component of A over C. This implies that all of the x_i must be in the same °-component of A over C, so, in particular, $x = x_0$ and $y = x_n$ must be in the same °-component of A over C. □

6.4 Proof of Full Result

We have seen that a tree T is a tree decomposition of a graph G if and only if T is a °tree decomposition of G. Since for a graph A where the abstract free amalgamation relation is the standard one, $G^\circ_A = A$, this shows that in the particular case of graphs with the standard abstract free amalgamation relation, T is a °tree decomposition of A if and only if T is a tree decomposition of G°_A. But this isn't only true for graphs.

Theorem 6.4.1. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.8em\lower0.2ex\hbox{$|$}}^\circ$ and let $A \subset_{fin} \mathcal{M}$. T is a °tree decomposition of A if and only if T is a tree decomposition of G°_A.*

Proof. (⇒) The proof of the forward direction is nearly identical to the argument we gave in the graph case. We suppose that T is a °-tree decomposition of A and verify that it satisfies each of the bullet points in the defintion of tree decompositon.

For the first bullet point, since T is a $^\circ$-tree decomposition of A,

$$\bigcup V(T) = A = V(G^\circ_A).$$

The third bullet point also follows easily from the definition of $^\circ$-tree decomposition. Suppose $s_1, t, s_2 \in V(T)$ such that t is on the unique s_1, s_2-path in T. Since T is a $^\circ$tree decomposition of A, $s_1 \mathop{\downarrow}^\circ_t s_2$. So $s_1 \cap s_2 \subseteq t$ by Regularity 1. So it only remains to verify the second bullet point.

Suppose $\{x, y\} \in E(G^\circ_A)$. We want to show that there is some $t \in V(T)$ such that $\{x, y\} \subseteq t$. Since T is a $^\circ$tree decomposition of A, there exists some element of $V(T)$ containing x and some element containing y. Choose $v_x, v_y \in V(T)$ with $x \in v_x$, $y \in v_y$ such that the length of the unique v_x, v_y-path in T is minimal.

Claim. $v_x = v_y$.

For the sake of contradiction, suppose $v_x \neq v_y$. Then there exist $t_1, t_2 \in V(T)$ on the unique v_x, v_y-path such that t_1 and t_2 are adjacent. Let $\{X_0, ..., X_{n-1}\} = \mathcal{D}^\circ\left(V(G)/\left(t_1 \cap t_2\right)\right)$.

Since v_x, v_y were chosen so that the unique path between them was of minimal length, for $i = 1, 2$, $x \in t_i$ if and only if $t_i = v_x$ and $y \in t_i$ if and only if $t_i = v_y$. $t_1 \neq t_2$ then implies neither x nor y can be in both t_1 and

t_2, i.e., $\{x,y\} \cap (t_1 \cap t_2) = \emptyset$. Since $\{x,y\} \in E(G_A^\circ)$, $\{x,y\}$ is an x,y-°path in A which doesn't intersect $t_1 \cap t_2$, thus, by Lemma 6.2.2, x and y must be in the same °-component of A over $t_1 \cap t_2$. But then there can be no $I \in n$ such that $v_x \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in I} X_i$, $v_y \setminus (t_1 \cap t_2) \subseteq \bigcup_{j \in n \setminus I} X_j$, contradicting T being a °tree decomposition.

Since $v_x = v_y$, there exists $v_x \in V(T)$ such that $\{x,y\} \subseteq v_x$.

Thus if T is a °tree decomposition of A, it also satisfies the necessary conditions to be a tree decomposition of G_A°.

($\Leftarrow$) On the other hand, suppose T is a tree decomposition of G_A°. We need to verify that T satisfies each of the bullet points in the definition of °-tree decomposition. Again, the first bullet point is clear, since $\bigcup V(T) = V(G_A^\circ) = A$.

The third bullet point will require somewhat more effort than in the argument for the forward direction. Suppose $s_1, t, s_2 \in V(T)$ such that t is on the unique path in T from s_1 to s_2. We want to show that $s_1 \mathop{\smile\!\!\!\!\!\!\!\;|}^{\circ}_{t} s_2$. This is clear if $t = s_i$ for some $i \in \{1,2\}$, so we may assume that $t \neq s_1, s_2$ (and, hence, $s_1 \neq s_2$). By the Regularity 2 axiom, it suffices to show that $s_1 \setminus t \mathop{\smile\!\!\!\!\!\!\!\;|}^{\circ}_{t} s_2 \setminus t$. For the sake of a contradiction, suppose not.

If $s_1 \setminus t \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{\circ}_{t} s_2 \setminus t$, then by (the contrapositive of) freeness, $s_1 \not\mathop{\smile\!\!\!\!\!\!\!\;|}^{\circ}_{t \cup (A \setminus (s_1 \cup s_2))} s_2$. Thus there must exist some $x \in s_1$, $y \in s_2$, such that x and y are in the same $^\circ$-component of A over $t \cup (A \setminus (s_1 \cup s_2))$. By Lemma 6.3.6, x and y must also be in the same component of G_A° over $t \cup (A \setminus (s_1 \cup s_2))$, so there exists an x, y-path $P \subseteq s_1 \cup s_2$.

Since T is a tree decomposition of G_A°, $s_1 \cap s_2 \subseteq t$. This implies that $s_1 \setminus t \cap s_2 \setminus t$ is empty. It follows that P has length at least one and, hence, that there must be some $x' \in s_1 \setminus t$, $y' \in s_2 \setminus t$ such that $\{x', y'\} \in E(G_A^\circ)$.

Because T is a tree decomposition of G_A°, there must be some $t' \in V(T)$ such that $x', y' \in t'$. Then $x' \in s_1, t'$ but $x' \notin t$, so t cannot be on the unique s_1, t'-path in T. Similarly, $y' \in t', s_2$, but $y' \notin t$, so t cannot be on the unique t', s_2-path in T. But the existence of both an s_1, t'-path and a t', s_2-path neither of which contains t implies that there is also an s_1, s_2-path in T which doesn't contain t, a contradiction. Thus we must have $s_1 \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_{t} s_2$ whenever t lies on the unique s_1, s_2-path in T.

Finally, we address the second bullet point in the definition of $^\circ$-tree decomposition. Suppose $s_1, t_1, t_2, s_2 \in V(T)$ such that t_1 an t_2 are adjacent in T and the edge $\{t_1, t_2\}$ lies on the unique s_1, s_2-path in T. Let $C = t_1 \cap t_2$ and $\mathcal{D}^\circ(A/C) = \{X_0, ..., X_{n-1}\}$. We must show that there is an $I \subseteq n$ such that $s_1 \setminus C \subseteq \bigcup_{i \in I} X_i$ and $s_2 \setminus C \subseteq \bigcup_{j \in n \setminus I} X_j$.

By Lemma 6.3.6, the $^\circ$-components of A over C are the same as the components of G_A° over C. Since T is a tree decomposition of G_A°, by Proposition

5.2.7, it must also be a ^{s}-tree decomposition for G_A°, with $\mathop{\smile\hskip-0.9ex\raise0.3ex\hbox{$|$}}^{s}$ the standard abstract free amalgamation relation for graphs, and the $^\circ$-components of G_A° over C with respect to this relation are the same as the components of the graph. Thus we must have some $I \subseteq n$ such that $s_1 \setminus C \subseteq \bigcup_{i \in I} X_i$ and $s_2 \setminus C \subseteq \bigcup_{j \in n \setminus I} X_j$. Therefore, whenever $s_1, t_1, t_2, s_2 \in V(T)$ such that t_1 an t_2 are adjacent in T and the edge $\{t_1, t_2\}$ lies on the unique s_1, s_2-path in T, if $C = t_1 \cap t_2$ and $\mathcal{D}^\circ(A/C) = \{X_0, ..., X_{n-1}\}$, then there exists an $I \in n$ such that $s_1 \setminus C \subseteq \bigcup_{i \in I} X_i$ and $s_2 \setminus C \subseteq \bigcup_{j \in n \setminus I} X_j$.

Thus, if T is a tree decomposition of G_A°, then it must also be a $^\circ$tree decomposition of A.

□

As an immediate corollary, the $^\circ$-tree width of A must equal the tree width of G_A°.

Corollary 6.4.2. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\hskip-0.9ex\raise0.3ex\hbox{$|$}}^{\circ}$ and let $A \subset_{fin} \mathcal{M}$. $tw^\circ(A) = tw(G_A^\circ)$.*

Proof. By Theorem 6.4.1, T is a $^\circ$-tree decomposition of A if and only if it is a tree decomposition of G_A°. So

$$\{T | T \text{ is a } ^\circ\text{-tree decomposition of } A\} = \{T | T \text{ is a tree decomposition of } G_A^\circ\}.$$

Since the width of T is the same whether we consider T as a $\circ$-tree decomposition of A or as a tree decomposition of G_A°,

$$\{w(T)|\text{T is a }\circ\text{-tree decomposition of } A\} = \{w(T)|\text{T is a tree decomposition of } G_A^\circ\}.$$

Thus

$$\begin{aligned} tw^\circ(A) &= \min\{w(T)|\text{T is a }\circ\text{-tree decomposition of } A\} \\ &= \min\{w(T)|\text{T is a tree decomposition of } G_A^\circ\} \\ &= tw(G_A^\circ). \end{aligned}$$

□

This ensures that A will have a $\circ$-tree decomposition of width less than k if and only if G_A° has a tree decomposition of width less than k. We know that G_A° has a tree decomposition of width less than k if and only if k cops have a winning strategy on G_A° from Seymour and Thomas's Theorem (2.4.1). Thus to show that k cops have a winning strategy on A if and only if A has a $\circ$tree decomposition of width less than k, it only remains to show that k cops have a winning strategy on A if and only if they also have a winning strategy on G_A°.

Theorem 6.4.3. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^\circ$ and let $A \subset_{fin} \mathcal{M}$. k cops have a winning strategy on A if and only if they also have a winning strategy on G_A°.*

Proof. Since the $\circ$-components of A are precisely the same as the components of G°_A, $(C_i, z_i)_{i\leq m}$ will be a robber-winning run for the $\circ$-Cops and Robbers game for k cops on A if and only if it is a robber-winning run for the Cops and Robbers game with k cops on G°_A. It follows that $RW_k(A) = RW_k(G^\circ_A)$ and that κ is a cop strategy for the $\circ$-Cops and Robbers game with k cops on A if and only if it is also a cop strategy for the Cops and Robbers game with k cops on G°_A. Since $RW_k(A) = RW_k(G^\circ_A)$, for any cop strategy κ, a robber-winning run $(C_i, z_i)_{i\leq m}$ respects κ as a cop strategy on A if and only if it respects κ as a cop strategy on G°_A, thus the same maximum length of a robber-winning run which respects κ will apply for both A and G°_A. Hence, κ is a winning cop strategy for the $\circ$-Cops and Robbers game with k cops on A if and only if it is a winning cop strategy for the Cops and Robbers game with k cops on G°_A. $\square$

We now have all the pieces for the full analogue of Seymour and Thomas's result.

Theorem 6.4.4. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\downarrow}^\circ$ and let $A \subset_{fin} \mathcal{M}$. k cops have a winning strategy in the $\circ$-Cops and Robbers game on A if and only if A has $\circ$-tree width less than k.*

Proof. By Theorem 6.4.3, k cops have a winning strategy in the $\circ$-Cops and Robbers game on A if and only if they have a winning strategy in the Cops and

Robbers game on G_A°. By Theorem 2.4.1, k cops have a winning strategy in the Cops and Robbers game on G_A° if and only if G_A° has tree width less than k. Finally, by Corollary 6.4.2, G_A° has tree width less than k if and only if A has °tree width less than k. □

6.5 Definition of °-Tree Decomposition Revisited

Finding a suitable definition of °-tree decomposition was difficult. At first we tried to define it by generalizing the idea of cliques in graphs as °-irreducibles resulting in "alt °-tree decompositions," but the definition seemed to be difficult to work with. Then we chose to just use a condition which we had proved to be true in the graph case and which made it easy to derive a winning cop strategy from a °-tree decomposition. Fortunately, this turned out to be the right definition in the sense of being equivalent to tree decomposition for graphs with the standard abstract free amalgamation relation and, as we have shown in this chapter, having the same relationship with the °-Cops and Robbers game as regular tree decompositions have with the Cops and Robbers game on graphs.

Our work in this chapter suggests another definition which would be equivalent:

Definition 6.5.1 (Alternate definition of °-tree decomposition equivalent to Definition 5.2.5)**.** Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!\mid}^{\circ}$. If $A \subset_{\text{fin}} M$, a $^\circ$*-tree decomposition* of A is a tree T such that each $t \in V(T)$ is a subset of A and

- $\bigcup V(T) = A$,
- if $x, y \in A$ such that x and y are $^\circ$adjacent in A (i.e., $x \neq y$ and $\{x\} \not\mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{A\setminus\{x,y\}} \{y\}$), then there is some $t \in V(T)$ such that $x, y \in t$, and
- for all $t_1, t_2 \in V(T)$, if $t \in V(T)$ is on the unique path between t_1 and t_2, then $t_1 \mathop{\smile\!\!\!\!\!\!\mid}^{\circ}_{t} t_2$.

This is equivalent because asking for each pair of $x, y \in A$ which are $^\circ$adjacent to appear in some vertex of T means that we are, in fact, asking for a tree decomposition of G°_A and we have seen in Theorem 6.4.1 that T is a tree decomposition of G°_A if and only if it is a $^\circ$-tree decomposition.

One advantage of this definition over our main definition of $^\circ$-tree decomposition is that it is much clearer to see how this relates to the usual definition of tree decomposition – we are basically asking that the "endpoints of each edge" in our structure belong to a vertex of T just as we do for tree decompositions of graphs. The disadvantage is also one of the reasons that we didn't start by looking for "paths" and "edges" in our structure: these are all very binary ideas and may therefore be hard to generalize further to other kinds of structures.

What about alt $^\circ$-tree decompositions? Has our further work given us any insight as to whether this definition might also be equivalent?

It is true that A is $^\circ$-irreducible if and only if every pair of distinct elements of A is $^\circ$-adjacent.

Lemma 6.5.2. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^\circ$. $A \subset_{fin} M$ is $^\circ$-irreducible if and only if for all $a_1, a_2 \in A$,*

$$\{a_1\} \not\mathop{\smile\!\!\!\!\!\!|}_{A\setminus\{a_1,a_2\}} \{a_2\}.$$

Proof. ($\Rightarrow$) The forward direction is clear. By definition, if A is $^\circ$-irreducible, $A_1 \not\mathop{\smile\!\!\!\!\!\!|}_C A_2$ for any disjoint $A_1, A_2, C \subseteq A$ such that $A_1, A_2 \neq \emptyset$ and $A_1 \cup A_2 \cup C = A$. So, in particular, we must have $\{a_1\} \not\mathop{\smile\!\!\!\!\!\!|}_{A\setminus\{a_1,a_2\}} \{a_2\}$ whenever $a_1 \neq a_2$. In the case that $a_1 = a_2$, $a_1 \notin A \setminus \{a_1, a_2\}$ gives the desired result by Regularity 1.

($\Leftarrow$) Let $A_1, A_2, C \subseteq A$ be disjoint with A_1 and A_2 nonempty and $A_1 \cup A_2 \cup C = A$. We want to show that $A_1 \not\mathop{\smile\!\!\!\!\!\!|}_C A_2$.

Let $a_1 \in A_1$ and $a_2 \in A_2$. $\{a_1\} \not\mathop{\smile\!\!\!\!\!\!|}_{A\setminus\{a_1,a_2\}} \{a_2\}$ by assumption. Note that we can rewrite this as:

$$\{a_1\} \not\mathop{\smile\!\!\!\!\!\!|}_{(A\setminus(A_1\cup A_2))\cup(A_1\setminus\{a_1\})\cup(A_2\setminus\{a_2\})} \{a_2\}.$$

By (the contrapositive of) transitivity, we can move the $A_1 \setminus \{a_1\}$ from the base to the left side, so we have

$$A_1 \not\mathop{\smile\!\!\!\!\!\!|}_{(A\setminus(A_1\cup A_2))\cup(A_2\setminus\{a_2\})} \{a_2\}.$$

Applying (the contrapositive of) transivity again to move $A_2 \setminus \{a_2\}$ from the base to the right side gives us

$$A_1 \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\raise-0.3ex\hbox{$|$}}^{\circ}_{A\setminus(A_1\cup A_2)} A_2.$$

Finally, we observe that since A_1, A_2, and C are disjoint and their union is A, we have $A \setminus (A_1 \cup A_2) = C$, so $A_1 \not\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\raise-0.3ex\hbox{$|$}}^{\circ}_C A_2$, as desired.

□

What this shows is that the abstract free amalgamation graph of any $\circ$-irreducible set will be complete and that if I is an $\circ$-irreducible set contained in a set A then the elements of I will be the vertices of a clique in G°_A. This implies that any $\circ$-tree decomposition of A will also be an alt $\circ$-tree decomposition of A.

Lemma 6.5.3. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstrract free amalgamation relation $\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.9em\raise-0.3ex\hbox{$|$}}^{\circ}$ and $A \subset_{fin} M$. If T is a $\circ$-tree decomposition of A then T is an alt $\circ$-tree decomposition of A.*

Proof. Let T be a $\circ$-tree decomposition of A. By Theorem 6.4.1, T must also be a tree decomposition of G°_A and, by Fact 2.2.4, T must also be an alt tree decomposition of G°_A.

Let $I \subseteq A$ be $\circ$-irreducible. By Lemma 6.5.2, for each $a_1, a_2 \in I$ such that $a_1 \neq a_2$, a_1 and a_2 are $\circ$adjacent in A. Thus for each pair of distinct a_1, a_2 in A, there is an edge between a_1 and a_2 in G°_A, i.e., the elements of I are the vertices of a clique in G°_A. Since T is an alt tree decomposition of G°_A, there must be some $t \in V(T)$ such that $I \subseteq t$.

Since we have shown that for each $^\circ$-irreducible $I \subseteq A$ that there is some $t \in T$ containing I (and since the first and third bullet points are the same in the defintions of $^\circ$-tree decomposition and alt $^\circ$-tree decomposition), T must be an alt $^\circ$-tree decomposition of A. $\square$

The problem with trying to reverse this proof is that it is not clear that the vertices of every clique in G°_A form a $^\circ$-irreducible set in A. We would need to be able to show that if $\{a_1\} \not\mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{A \setminus \{a_1,a_2\}} \{a_2\}$, then there must be some $^\circ$-irreducible $I \subseteq A$ containing both $\{a_1\}$ and $\{a_2\}$. This is similar to what we saw we would want to show in Section 5.1 and we still don't have a way to prove this.

We do, however, have an *almost* counterexample:

Example 6.5.4. Let $\mathcal{L} = \{E, R\}$ where E is a binary relation symbol and R is a 4-ary relation symbol. Let T be the theory consisting of the following five $\mathcal{L}$-sentences.

- $\forall x, y\, (E(x, y) \to x \neq y)$
- $\forall x, y\, (E(x, y) \leftrightarrow E(y, x))$
- $\forall x_1, x_2, x_3, x_4\, (R(x_1, x_2, x_3, x_4) \to \bigwedge_{1 \leq i < j \leq 4} x_i \neq x_j)$
- $\forall x_1, x_2, x_3, x_4\, (R(x_1, x_2, x_3, x_4) \leftrightarrow R(x_4, x_3, x_2, x_1))$

- $\forall x_1, x_2, x_3, x_4 \, (R(x_1, x_2, x_3, x_4) \to (E(x_1, x_2) \wedge E(x_1, x_3)))$

In an $\mathcal{L}$-structure A, we can think of the pairs in E^A as edges (note that the first two bullet points are the same conditions for an edge in a graph) and the 4-tuples in R^A as arcs – we can reverse the indices, but not necessarily apply other permutations – with edges from each endpoint to each interior point.

Let $\mathbf{K}$ be the class of all finite $\mathcal{L}$ structures which are models of T. Then $\mathbf{K}$ is an algebraically trivial Fraïssé class. (HP is clear and $\mathbf{K}$ has both Disjoint Joint Embedding and Disjoint Amalgamation.) Let $\mathcal{M}$ be the generic model of $\mathbf{K}$. For $A, B, C \subset_{\text{fin}} M$, define $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$ when each of the following conditions are met:

- $A \cap B \subseteq C$,
- there are no $a \in A \setminus C$ and $b \in B \setminus C$ such that $(a, b) \in E^{\mathcal{M}}$, and
- for any $a \in A \setminus C$ and $b \in B \setminus C$, there are no $c_1, c_2 \in A \cup B \cup C$ such that $(a, c_1, c_2, b) \in R^{\mathcal{M}}$.

Since $\mathbf{K}$ is a Fraïssé class, there must be some $A = \{a, b, c_1, c_2\} \subset_{\text{fin}} M$ such that

$$E^A = \{(a, c_1), (a, c_2), (b, c_1), (b, c_2), (c_1, a), (c_2, a), (c_1, b), (c_2, b)\}$$

and

$$R^A = \{(a, c_1, c_2, b), (b, c_2, c_1, a)\}.$$

We then have $\{a\} \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{A \setminus \{a,b\}} \{b\}$ by the third bullet point in the definition of $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$. Clearly $A \setminus \{a, b\} = \{c_1, c_2\}$ is the minimal base in A over which $\{a\}$ is not independent of $\{b\}$, so if there is some $\circ$-irreducible set in A containing both a and b, it would have to be A itself. But $\{c_1\} \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{A \setminus \{a,b\}} \{c_2\}$, so A is not $\circ$-irreducible.

In this example, $\mathbf{K}$ is an algebraically trivial Fraïssé class and A is a finite subset of the generic model. The problem is that the ternary relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ is not quite an abstract free amalgamation relation. It satisfies all of the axioms except, as shown by the following counterexample, stationarity.

Example 6.5.5. Let $\mathcal{L}$, T, $\mathbf{K}$, $\mathcal{M}$, and $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ be as above. Since $\mathbf{K}$ is a Fraïssé class, there is a substructure $D = \{a, a', b, c_1, c_2\}$ of $\mathcal{M}$ such that

$$E^D = \{(a, c_1), (a, c_2), (a', c_1), (a', c_2), (b, c_1), (b, c_2), (c_1, a), (c_1, a'), (c_1, b), (c_2, a), (c_2, a'), (c_2, b)\}$$

and

$$R^D = \{(c_1, a, b, c_2), (c_2, b, a, c_1)\}.$$

Let $A = \{a\}$, $A' = \{a'\}$, $B = \{b\}$, and $C = \{c_1, c_2\}$. Now $A \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$, $A' \mathop{\smile\!\!\!\!\!\!|}^{\circ}_C B$, and $A' \equiv_C A$, but $A' \not\equiv_{B \cup C} A$.

This suggests that if it is true that $\{a_1\} \not\mathop{\smile\!\!\!\!\!\!|}^{\circ}_{A \setminus \{a_1,a_2\}} \{a_2\}$ if and only if there is some $\circ$-irreducible $I \subseteq A$ containing both $\{a_1\}$ and $\{a_2\}$, proving it will probably involve the stationarity axiom, which is the one axiom of abstract free amalgamation relations we have not used at all in our work so far.

6.6 Gaifman Graph Versus Abstract Free Amalgamation Graph

By Theorem 6.4.1 and its corollary, we could define the $^\circ$-tree decompositions and $^\circ$-tree width of a structure as the tree decompositions and tree width of the abstract free amalgamation graph of the structure. Since the previously known generalization of tree decomposition was similarly defined in terms of the Gaifman graph, this raises the question of the relationship between the abstract free amalgamation graph of a structure and the Gaifman graph of the same structure.

In the case where the abstract free amalgamation relation $\mathop{\perp\!\!\!\perp}^{\circ}$ is the standard one for the language, $\mathcal{L}$, in which our structure is described, the abstract free amalgamation graph will in fact be the same as the Gaifman graph. With $\mathop{\perp\!\!\!\perp}^{\circ}$ the standard abstract free amalgamation relation on the generic model $\mathcal{M}$ of an algebraically trivial Fraïssé class, for $A, B, C \subset_{\text{fin}} M$, we have $A \not\mathop{\perp\!\!\!\perp}^{\circ}_C B$ precisely when there is some $a \in A \setminus C$ and $b \in B \setminus C$ such that for some $R \in \mathcal{L}$ there is an $\overline{a} \in R^{A \cup B \cup C}$ such that a and b both appear in $\overline{a}$. So, in particular, if $A \subset_{\text{fin}} M$ with a and b distinct elements of A, then $\{a\} \not\mathop{\perp\!\!\!\perp}^{\circ}_{A \setminus \{a,b\}} \{b\}$ if and only if there is some relation $R \in \mathcal{L}$ and some $\overline{a} \in R^A$ such that a and b both appear in $\overline{a}$. Thus there is an edge between a and b in the abstract free amalgamation relation graph G°_A if and only if there is an edge between a and b in the Gaifman graph of A.

Of course, we also have examples of abstract free amalgamation relations which are not the standard one for the language in which the structure is described. For instance, when $\mathcal{M}$ is the generic model of the class of finite metric spaces with distances in $\{0, 1, 2, 3\}$ in the language $\mathcal{L} = \{R_1, R_2, R_3\}$ and $\mathop{\smile\!\!\!\!\!\!|}^\circ$ is defined by $A \mathop{\smile\!\!\!\!\!\!|}^\circ_C B$ when $A \cup B \subseteq C$ and the distance between each $a \in A \setminus C$ and $b \in B \setminus C$ is 2. Or, in the case of the random graph, when we define $\mathop{\smile\!\!\!\!\!\!|}^\circ$ by $A \mathop{\smile\!\!\!\!\!\!|}^\circ_C B$ precisely when $A \cup B \subseteq C$ and for each $a \in A \setminus C$ and $b \in B \setminus C$, a and b are adjacent in the random graph.

In both of these examples, we could change the language in which our structure is described to one for which the given abstract free amalgamation relation is the standard one. This raises the question of whether there is always a language in which our algebraically trivial Fraïssé class can be described to guarantee that the abstract free amalgamation graph of any substructure of the generic model will be the same as its Gaifman graph. Even if this turns out to be true, the abstract free amalgamation graph construction should still be useful as it doesn't require us to find this appropriate language – so long as we have an abstract free amalgamation relation, we can find the associated abstract free amalgamation graph, regardless of the language in which our structure is described.

6.7 Result for Restricted Cop Strategies

This section details our initial approach to trying to show the analogue of Seymour and Thomas's result. As we will show shortly, with some minor tweaks to the definition for κ from Section 6.1, we can produce a particularly "nice" winning cop strategy for k cops on A from a $^\circ$-tree decomposition of A with width k. Our original intent was to show that it is also true that if k cops have a "nice" winning strategy on A, then A has a $^\circ$-tree decomposition of width less than k. We could then complete the result by showing that k cops have a "nice" winning strategy on A whenever they have any winning strategy on A.

We were able to accomplish the first part of this plan. Following a number of definitions and technical lemmas, Propositions 6.7.21 and 6.7.24 will show that when k cops have a "nice" winning strategy on A, then A has a $^\circ$-tree decomposition of width less than k. The difficulty has been in showing directly that whenever k cops have any winning strategy on A, they must have a "nice" winning strategy on A. Ultimately, this lead to a different approach to proving the analogue of Seymour and Thomas's theorem which we have discussed in Sections 6.2, 6.3, and 6.4. While that result supercedes the work done in this section (and, therefore, this section may be skipped), we nevertheless preserve this work with the thought that it might be more widely applicable than our later approach as it does not rely as heavily on our structures being graph-like.

Suppose $\mathcal{M}$ is the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ and $A \subset_{\text{fin}} M$ has $\circ$-tree width k. Here is a slightly different construction of a winning cop strategy for k cops in the $\circ$-Cops and Robbers game on A:

- Choose $t_0 \in V(T)$ and set $\kappa'(\varepsilon) = t_0$.
- For each $m \in \mathbb{N}$, for each robber-winning run $(C_i, z_i)_{i \leq m} \in RW_k(A)$,
  - if $C_m = t_m$ for some $t_m \in V(T)$, set $\kappa'\left((C_i, z_i)_{i \leq m}\right)$ to be t_{m+1} where t_{m+1} is the unique neighbor of t_m in T such that $z_m \in s$ for some $s \in V(T)$ such that t_{m+1} is on the t_m, s-path in T, and
  - if $C_m \notin V(T)$, set $\kappa'\left((C_i, z_i)_{i \leq m}\right) = t_0$.

Assuming $\kappa(\varepsilon) = \kappa'(\varepsilon)$ (i.e., we choose the same t_0 in both constructions), a robber-winning run $(C_i, z_i)_{i \leq m}$ will respect κ' if and only if it respects the cop strategy κ from the previous section: this is clear for $m = 0$ and will follow for robber-winning runs with $m = l + 1$ if we assume it is true for robber-winning runs with $m \leq l$. κ' must therefore be winning too, since any robber-winning run which respects κ' must also have length less than or equal to $\max\{1, |V(T)|\}$.

We are interested in κ' because it satisfies certain nice properties which we will detail shortly (some, but not all, of which are also satisfied by κ). Since we constructed κ' from a $\circ$-tree decomposition of A, we see that we can always construct a cop strategy with these properties. We will also see that when we have a winning cop strategy for k cops on A which satisfies these properties, we can find a $\circ$-tree decomposition of A with width less than k.

The first property that we are interested in is coarseness, defined just as it was for graphs.

Definition 6.7.1. Let $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ be a cop strategy for the $\circ$-Cops and Robbers game with k cops on A. We say that κ is *coarse* if for any two robber-winning runs of length $m+1$, $(C_i, z_i)_{i \leq m}$ and $(C_i', z_i')_{i \leq m}$, which respect κ and satisfy

- $C_i = C_i'$ for all $i \leq m$ and
- z_i, z_i' are in the same $\circ$-component of A over C_i for all $i \leq m$,

we have $\kappa\left((C_i, z_i)_{i \leq m}\right) = \kappa\left((C_i', z_i')\right)$.

In other words, the cops will respond the same way to all possible choices a robber could make within a particular $\circ$-component of A over the cops' previous position. For graphs, we used this to help show that our version of the Cops and Robbers game is equivalent to the two Seymour and Thomas versions, where the robber chooses a component, rather than a single vertex.

While Seymour and Thomas basically had coarseness "baked into" their games, another property of cop strategies that they explicitly referenced is monotonicity.

Definition 6.7.2. Let $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ be a cop strategy for the $\circ$-Cops and Robbers game with k cops on A. We say that κ is *monotone* if whenever $(C_i, z_i)_{i \leq m}$ is a robber-winning run which respects κ and $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$, for all $0 \leq i_1 \leq i_2 \leq i_3 \leq m+1$, $C_{i_1} \cap C_{i_3} \subseteq C_{i_2}$.

So a cop strategy is monotone if the cops never return to any element of A once they leave it. There is a clear connection between this property and the third bullet point in the definition of tree decomposition. Since third bullet point in our definition of $\circ$-tree decomposition is stronger, it is perhaps not surprising that we will ultimately be interested in a stronger version of this property which we will call "fully monotone." We will also need to be somewhat careful when saying that something is true "by monotonicity" as to whether we are referring to a property of a cop strategy or an axiom of a free amalgamation relation.

A third property of cop strategies that will be of interest to us is marginality.

Definition 6.7.3. Let $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ be a cop strategy for the $\circ$-Cops and Robbers game with k cops on A. We say that κ is *marginal* if for any robber-winning run $(C_i, z_i)_{i \leq m} \in RW_k(A)$, $\kappa\left((C_i, z_i)_{i \leq m}\right)$ depends only on (C_m, z_m). That is, if $(C'_i, z'_i)_{i \leq n} \in RW_k(A)$ and $C'_n = C_m$, $z'_n = z_m$, then $\kappa\left((C'_i, z'_i)_{i \leq n}\right) =$

$\kappa\left((C_i, z_i)_{i\leq m}\right)$.

When a cop strategy is marginal, the cops don't care about the particulars of how they and the robber got into their current positions, they choose their new position based solely on where they and the robber currently stand.

Proposition 6.7.4. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^{\circ}$ and $A \subset_{fin} M$. If A has $\circ$-tree width less than k, then k cops have a coarse, monotone, marginal, winning strategy in the $\circ$-Cops and Robbers game on A.*

Proof. We have already seen that κ' is a winning cop strategy for the $\circ$-Cops and Robbers game with k cops on A. We, therefore, need only show that κ' is coarse, montone, and marginal.

Coarse: Let $(C_i, z_i)_{i\leq m}$ and $(C'_i, z'_i)_{i\leq m}$ be robber-winning runs that respect κ such that

- $C_i = C'_i$ for all $i \leq m$ and
- z_i, z'_i are in the same $\circ$-component of A over C_i for all $i \leq m$.

We must show that $C_{m+1} = \kappa'\left((C_i, z_i)_{i\leq m}\right) = \kappa'\left((C'_i, z'_i)_{i\leq m}\right) = C'_{m+1}$.

Because $(C_i, z_i)_{i\leq m}$ respects κ', we must have $C'_0 = C_0 = t_0$ for some $t_0 \in V(T)$. Hence, $C'_i = C_i$ must be in $V(T)$ for all $i \leq m$. By the definition of κ', C_{m+1} is the unique neighbor of C_m such that C_{m+1} lies on the unique path from C_m to s for some $s \in V(T)$ such that $z_m \in s$. Similarly, C'_{m+1} must be the unique neighbor of C_m such that C'_{m+1} lies on the unique path from C_m to s' for some $s' \in V(T)$ such that $z'_m \in s'$.

If C_{m+1} were not equal to C'_{m+1}, then C_m would be on the unique path from s to s' in T and, by Lemma 6.1.2, for each $x \in s \setminus C_m$ and $x' \in s' \setminus C_m$ we would have x and x' in different $\circ$-components of A over C_m. As $z \in s \setminus C_m$ and $z' \in s' \setminus C_m$ must be in the same $\circ$-component of A over C_m, this is impossible. So we must have $C_{m+1} = C'_{m+1}$.

Monotone: Let $(C_i, z_i)_{i \leq m}$ be a robber-winning run which respects κ' and $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$. Let $0 \leq i_1 \leq i_2 \leq i_3 \leq m + 1$.

We have already argued that $(C_i, z_i)_{i \leq m}$ will respect κ' if and only if it also respects κ, so we must have $C_0, C_1, ..., C_m$ a path in T by the proof of Proposition 6.1.1. Lemma 6.1.2 guarantees that we can append C_{m+1} to this and we will still have a path in T. Since T is a $\circ$-tree decomposition of A and C_{i_2} lies on the unique path from C_{i_1} to C_{i_3} in T, we must have $C_{i_1} \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C_{i_2}} C_{i_3}$. Thus, by Regularity 2, $C_{i_1} \cap C_{i_3} \subseteq C_{i_2}$.

Marginal: This follows directly from the definition of κ'.

□

Note that coarseness and monotonicity are properties which only relate to robber-winning runs which respect the cop strategy in question. Since κ' is coarse and monotone and a robber-winning run $(C_i, z_i)_{i \leq m}$ respects the cop strategy κ from the proof of Proposition 6.1.1 if and only if it respects κ', κ is also coarse and monotone. However, κ is not marginal.

Having shown that when A has $\circ$-tree width less than k, k cops have a coarse, monotone, marginal, winning strategy in the $\circ$-Cops and Robbers game on A, we would like to show that the converse is also true. To do this, we will need to use the cop strategy to create a $\circ$-tree decomposition of A. Specifically, we will create a tree in which the vertices are the cops' positions in a winning strategy as its vertices and following a path from the root will give the cop sequence of a robber-winning run.

Given $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ a coarse, monotone, marginal, winning cop strategy for the $\circ$-Cops and Robbers game with k cops on A, define V_κ to be the set of $C \in \binom{A}{\leq k}$ such that either $C = \kappa(\varepsilon)$ or $C = \kappa\left((C_i, z_i)_{i \leq m}\right)$ for some κ-respecting robber-winning run $(C_i, z_i)_{i \leq m}$.

In the special case where there is no robber-winning run (C_0, z_0) which respects κ, V_κ will consist of a single element, $C = \kappa(\varepsilon)$. Since no robber-winning run respects κ, there must be no $z \in A \setminus C$ for the robber to choose as his initial position, so C must equal A. In this case, the tree $T = (\{C\}, \emptyset)$ will be a $\circ$-tree decomposition for A. So from here onward we will assume that there is some robber-winning run (C_0, z_0) which respects κ.

We will make use of the following fact to create a tree whose set of vertices is V_κ:

Fact 6.7.5. If $\leq$ is a partial ordering on a finite set V such that

- $\leq$ has a minimum element x_0 and
- for each $y \in V$, the set $\{x \in V | x \leq y\}$ is linearly ordered by $\leq$,

then $T = (V, E)$ where

$$E = \left\{ \{x, y\} \in \binom{V}{2} \middle| y \text{ is an immediate successor of } x \right\}$$

is a rooted tree with root x_0.

Clearly we will need a partial ordering on V_κ to do this. Define $\leq_\kappa$ as follows:

> $C \leq_\kappa C'$ if there is a κ-respecting run $(C_i, z_i)_{i \leq m}$ with $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$ such that for some $0 \leq i \leq j \leq m+1$, $C = C_i$ and $C' = C_j$.

After a series of preparatory lemmas and definitions, we will show in Proposition 6.7.21 that (for a nice enough cop strategy κ) $\leq_\kappa$ satisfies the hypotheses of Fact 6.7.5 and then, in Proposition 6.7.24, that the resulting tree is a °-tree decomposition of A. For the rest of this section, unless otherwise stated, assume that $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ is a coarse, monotone, marginal, winning cop strategy, $(C_l, z_l)_{l \leq m}$ a robber-winning run which respects κ, and $C_{m+1} = \kappa\left((C_l, z_l)_{l \leq m}\right)$.

We'll start by making some observations about how the cops' positions and robber's choices behave in such a strategy.

First, we note that the cops can never stay in exactly the same position for two turns in a row.

Lemma 6.7.6. $\kappa\left((C_l, z_l)_{l\leq m}\right) \neq C_m$.

Proof. If $\kappa\left((C_l, z_l)_{l\leq m}\right) = C_m$, then $(C_l, z_l)_{l\leq m}, C_m, z_m$ is a robber-winning run which respects κ. Since κ is marginal, we must have $\kappa\left((C_l, z_l)_{l\leq m}, C_m, z_m\right) = C_m$. Then the robber could again choose z_m and the cops would have to choose C_m, ad infinitum. But this contradicts that κ is a winning cop strategy. □

Next, we see that if the robber is able to choose a particular position in Round m of the °-Cops and Robbers game on A, then that particular element of A cannot have been occupied by the cops at any time in the past.

Lemma 6.7.7. *$z_m \notin C_l$ for any $l \leq m$.*

Proof. Since $(C_l, z_l)_{l\leq m}$ is robber-winning, $z_m \notin C_m$. Since κ is a winning cop strategy, there is a limit to the possible length of a robber-winning run, so the cops must do something to prevent the robber from just staying at z_m forever. But for any cop position, C, if the robber's previous choice was z_m and $z_m \notin C$, then z_m will be an acceptable choice for the robber's new position (clearly it is in the same component as z_m over C, and, hence, over any subset of C).

So eventually the cops' chosen position in response to the robber choosing z_m will have to contain z_m. Thus, for some $m' \geq m$, there must be a κ-respecting robber-winning run $(C_l, z_l)_{l \leq m'}$ with $z_l = z_m$ for all $l \geq m$ such that $z_m \in C_{m'+1} = \kappa\left((C_l, z_l)_{l \leq m'}\right)$. Since κ is monotone, we must have $C_l \cap C_{m'+1} \subseteq C_m$ for all $l \leq m$. Thus, since $z_m \in C_{m'+1}$, but $z_m \notin C_m$, we must have $z_m \notin C_l$ for all $l \leq m$. □

As a result, the robber must never be able to choose as a position an element of A which has appeared in any previous cop position.

Lemma 6.7.8. *Let* $0 \leq i < j \leq m$ *and* $Y_0 \in \mathcal{D}^\circ\left(A/\left(C_{j-1} \cap C_j\right)\right)$ *such that* $z_{j-1}, z_j \in Y_0$. *Then* $Y_0 \cap C_i = \emptyset$.

Proof. Suppose not; suppose $z \in Y_0 \cap C_i$. Since $z \in Y_0$, then either $z \in C_j$ or $(C_l, z_l)_{l<j}, C_j, z$ is robber-winning. By the contrapositive of the previous lemma, since $z \in C_i$, $(C_l, z_l)_{l<j}, C_j, z$ cannot be a robber-winning run. So we must have $z \in C_j$.

If $z \in C_j$ and $z \in Y_0$, then $z \notin C_{j-1}$. But κ monotone implies that $C_i \cap C_j \subseteq C_{j-1}$, so $z \in C_j$ and $z \in C_i$ implies $z \in C_i \cap C_j \subseteq C_{j-1}$, a contradiction. □

It follows that, in fact, the robber will never gain access to any choices for his position that he didn't already have. That is, if $z \in A$ is a position the robber could choose in Round j, then he could have chosen z for his position in Round i for any $i \leq j$.

Lemma 6.7.9. *For $0 \leq i < m$, if $X_0 \in \mathcal{D}^{\circ}(A/(C_i \cap C_{i+1}))$ such that $z_i \in X_0$, then for all $i \leq j \leq m$, $z_j \in X_0$.*

Proof. This is clear for $j = i$.

Towards a contradiction, suppose $z_j \notin X_0$ for some $j > i$. We may assume without loss of generality that j is the lowest such index.

$z_j \notin C_l$ for $l \leq j$ by Lemma 6.7.7, hence, $z_j \notin C_i \cap C_{i+1}$. We must then have z_j contained in some element of $\mathcal{D}^{\circ}(A/(C_i \cap C_{i+1})) = \{X_0, X_1, ..., X_n\}$. $z_j \notin X_0$, by assumption.

Let $Y_0 \in \mathcal{D}^{\circ}(A/(C_{j-1} \cap C_j))$ such that $z_{j-1}, z_j \in Y_0$. Let $Z_0 = Y_0 \cap X_0$ and $Z_1 = Y_0 \setminus X_0$. By assumption, for all $i \leq l < j$ we have $z_l \in X_0$, so $z_{j-1} \in Z_0$, while $z_j \in Z_1$. $\{Z_0, Z_1\}$ is, then, a bipartition of Y_0. By the definition of the decomposition into components, we must have $Z_0 \not\perp^{\circ}_{C_{j-1} \cap C_j} Z_1$. Then, since $Y_0 \cap (C_i \cap C_{i+1}) = \emptyset$ by Lemma 6.7.8, by (the contrapositive of) the freeness axiom, we have $Z_0 \not\perp^{\circ}_{(C_{j-1} \cap C_j) \cup (C_i \cap C_{i+1})} Z_1$.

We can then divide the part of $C_{j-1} \cap C_j$ which is not in $C_i \cap C_{i+1}$ in two – splitting it into the points that are in X_0 and those that are in $\bigcup_{t=1}^{n} X_t$. Let $Z_0' = X_0 \cap ((C_{j-1} \cap C_j) \setminus (C_i \cap C_{i+1}))$ and let $Z_1' = X_0 \setminus ((C_{j-1} \cap C_j) \setminus (C_i \cap C_{i+1}))$. This allows us to write $Z_0 \not\perp^{\circ}_{Z_0' \cup Z_1' \cup (C_i \cap C_{i+1})} Z_1$. Transitivity then tells us that $Z_0 \cup Z_0' \not\perp^{\circ}_{C_i \cap C_{i+1}} Z_1 \cup Z_1'$.

However, $Z_0 \cup Z_0' \subseteq X_0$, while $Z_1 \cup Z_1' \subseteq \bigcup_{t=1}^{n} X_t$. So, since $X_0 \perp^{\circ}_{C_i \cap C_{i+1}} \bigcup_{t=1}^{n} X_t$, by definition, monotonicity of the independence relation gives us

$$Z_0 \cup Z_0' \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_{C_i \cap C_{i+1}} Z_1 \cup Z_1',$$

a contradiction. □

Using a nearly identical argument (letting $i = 0$ and replacing $C_i \cap C_{i+1}$ with C_0 wherever it occurs), we can also prove that the following Lemma holds.

Lemma 6.7.10. *If $X_0 \in \mathcal{D}^\circ (A/C_0)$ such that $z_0 \in X_0$, then for all $0 \leq j \leq m$, $z_j \in X_0$.*

Together, these two lemmas show that with a coarse, monotone, marginal, winning cop strategy, the robber's available options at a given step are a subset of his choices at any previous step. In fact, the °-component containing the robber's choices at a given step must be a subset of the °-component containing the robber's choices at any previous step.

Lemma 6.7.11. *For $0 \leq i < m$, if $X_0 \in \mathcal{D}^\circ (A/(C_i \cap C_{i+1}))$ such that $z_i \in X_0$, then for all $i < j \leq m$, if $Y_0 \in \mathcal{D}^\circ (A/(C_{j-1} \cap C_j))$ such that $z_{j-1} \in Y_0$, $Y_0 \subseteq X_0$.*

Proof. Let $0 \leq i < m$ be given and, for the sake of contradiction, suppose that the statement fails to hold for some $i < j \leq m$. Without loss of generality, we may assume that j is the lowest index greater than i for which this fails.

By assumption, $Y_0 \setminus X_0 \neq \emptyset$. $Y_0 \cap X_0$ is also non-empty, since $z_{j-1} \in X_0$ by Lemma 6.7.9. So $\{Y_0 \setminus X_0, Y_0 \cap X_0\}$ is a bipartition of Y_0.

By Lemma 6.7.9, every z such that $(C_l, z_l)_{l<j}, C_j, z$ is robber-winning must be an element of X_0. Since for $z \in Y_0$, $(C_l, z_l)_{l<j}, C_j, z$ is *not* robber-winning if and only if $z \in C_j$, we must have $Y_0 \setminus X_0 = Y_0 \cap C_j$ by the following argument.

Let $y \in Y_0 \cap C_j$, then $y \notin C_{j-1}$. By the monotonicity of κ, $C_i \cap C_j \subseteq C_{j-1}$. Thus $y \notin C_{j-1}$ implies that $y \notin C_i$. So $y \notin C_i \cap C_{i+1}$ and $y \notin X_0$. Therefore, if $\mathcal{D}^\circ(A/(C_i \cap C_{i+1})) = \{X_0, X_1, ..., X_n\}$, we must have

$$Y_0 \setminus X_0 = Y_0 \cap C_j \subseteq \bigcup_{t=1}^{n} X_t.$$

By definition, $X_0 \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C_i \cap C_{i+1}} \bigcup_{t=1}^{n} X_t$. Applying the montonicity axiom allows us to write

$$(Y_0 \cap X_0) \cup ((C_{j-1} \cap C_j) \cap X_0) \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C_i \cap C_{i+1}} (Y_0 \setminus X_0) \cup \left((C_{j-1} \cap C_j) \cap \left(\bigcup_{t=1}^{n} X_t \right) \right).$$

Then using transitivity twice, we can bring the two parts of $C_{j-1} \cap C_j$ down into the base to get

$$Y_0 \cap X_0 \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{(C_{j-1} \cap C_j) \cup ((C_i \cap C_{i+1}) \setminus (C_{j-1} \cap C_j))} Y_0 \setminus X_0.$$

If $x \in (C_i \cap C_{i+1}) \setminus (C_{j-1} \cap C_j)$, then, in particular, $x \in C_i$, so $(C_l, z_l)_{l<j}, C_j, x$ can't be robber-winning by Lemma 6.7.7. So either $x \notin Y_0$ or $x \in Y_0$ *and* $x \in C_j$. It can't be the latter, since $x \in Y_0$ and $x \in C_j$ would imply that $x \notin C_{j-1}$ which would contradict the montonicity of κ (we must have $C_i \cap C_j \subseteq C_{j-1}$). So $x \in (C_i \cap C_{i+1}) \setminus (C_{j-1} \cap C_j)$ implies that $x \notin Y_0$. We can, therefore, use freeness to get

$$Y_0 \cap X_0 \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{(C_{j-1} \cap C_j)} Y_0 \setminus X_0.$$

But this is a contradiction since $\{Y_0 \cap X_0, Y_0 \setminus X_0\}$ is a bipartition of the $\circ$-component Y_0 of A over $C_{j-1} \cap C_j$. □

Again, running the same argument but with $i = 0$ and replacing all occurrences of $C_i \cap C_{i+1}$ with C_0 allows us to prove the equivalent result for $X_0 \in \mathcal{D}^\circ(A/C_0)$:

Lemma 6.7.12. *If $X_0 \in \mathcal{D}^\circ(A/C_0)$ such that $z_0 \in X_0$, then for all $0 < j \leq m$, if $Y_0 \in \mathcal{D}^\circ(A/(C_{j-1} \cap C_j))$ such that $z_{j-1} \in Y_0$, $Y_0 \subseteq X_0$.*

Several of the lemmas given so far involve the $\circ$-components of A over an intersection of consecutive cop positions. It will often be more convenient to consider the $\circ$-components of A over a single cop position, rather than an intersection. In general, if $C, C' \subseteq A$ and $Y_0 \in \mathcal{D}^\circ(A/C)$, we only know that Y_0 is a subset of X_0 for some $X_0 \in \mathcal{D}^\circ(A/(C \cap C'))$. But in the case where C and C' are consecutive cop positions in a coarse, monotone, marginal, winning cop strategy, we can do better.

Lemma 6.7.13. *For $0 \leq i \leq m$, if $X_0 \in \mathcal{D}^\circ(A/(C_i \cap C_{i+1}))$ such that $z_i \in X_0$ and $\mathcal{D}^\circ(A/C_i) = \{Y_0, Y_1, ..., Y_n\}$ such that $z_i \in Y_0$ (such Y_0 exists since $z_i \notin C_i$ by the definition of $(C_l, z_l)_{l \leq m}$ being a robber-winning run), then $X_0 = Y_0$.*

Proof. $C_i \cap C_{i+1} \subseteq C_i$ and $X_0 \cap Y_0 \neq \emptyset$, so we must have $Y_0 \subseteq X_0$. We will show that $Y_0 \mathop{\downarrow}^{\circ}_{C_i \cap C_{i+1}} X_0 \setminus Y_0$ and, hence, $X_0 \setminus Y_0$ must be empty by the definition of the decomposition into $\circ$-components.

We note first that $X_0 \cap C_i = \emptyset$:

- $X_0 \cap (C_i \cap C_{i+1}) = \emptyset$ by definition.

- $X_0 \cap (C_i \setminus C_{i+1}) = \emptyset$ by Lemma 6.7.7 since for all $z \in X_0$, if $z \notin C_{i+1}$, $(C_l, z_l)_{l \leq i}, C_{i+1}, z$ is robber-winning and respects κ.

This implies that $X_0 \setminus Y_0 \subseteq \bigcup_{l=1}^{n} Y_l$. By the definition of the decomposition into $\circ$-components, $Y_0 \mathop{\downarrow}^{\circ}_{C_i} \bigcup_{l=1}^{n} Y_l$. Applying the monotonicity axiom we have $Y_0 \mathop{\downarrow}^{\circ}_{C_i} X_0 \setminus Y_0$. Then by freeness (since $X_0 \cap C_i = \emptyset$), $Y_0 \mathop{\downarrow}^{\circ}_{C_i \cap C_{i+1}} X_0 \setminus Y_0$.

As previously noted, this implies that $X_0 \setminus Y_0$ must be empty (else $\{Y_0, X_0 \setminus Y_0\}$ is a bipartition of the $\circ$-component X_0 of A over $C_i \cap C_{i+1}$). So $X_0 = Y_0$, as desired. $\square$

We have seen that in a coarse, monotone, marginal, winning cop strategy, the robber's choices behave nicely in the sense that the robber's available options at a given step are a subset of his choices at any previous step. What we would like to show next is that the cops' choices also behave nicely. Specifically, we would like to say that at any given step, the cops' chosen position is a subset of the union of their prior position at any earlier step and the $\circ$-component containing the robber's available choices in response to that position, i.e., for any $0 \leq i < j \leq m$, if $X_0 \in \mathcal{D}^{\circ}(A/(C_i \cap C_{i+1}))$ such that $z_i \in X_0$, then $C_j \subseteq (C_i \cap C_{i+1}) \cup X_0$ (and, as with previous lemmas, the equivalent for the initial step case). This is true when we make a further assumption about the cops winning strategy.

Definition 6.7.14. If κ is a cop strategy, $(C_i, z_i)_{i<m}$ a robber-winning run which respects κ and $C_m = \kappa\left((C_i, z_i)_{i<m}\right)$, then we say C_m is *cost effective with respect to* κ if whenever $C'_m \subsetneq C_m$ then either

- there exists z'_m such that $(C_i, z_i)_{i<m}, C'_m, z'_m$ is robber-winning but $(C_i, z_i)_{i<m}, C_m, z'_m$ is not *or*

- there exists z'_{m+1} such that for some z_m such that $(C_i, z_i)_{i<m}, C_m, z_m$ is robber-winning and $C_{m+1} = \kappa\left((C_i, z_i)_{i<m}, C_m, z_m\right)$,

$$(C_i, z_i)_{i<m}, C'_m, z_m, C_{m+1}, z'_{m+1}$$

  is robber-winning but

$$(C_i, z_i)_{i<m}, C_m, z_m, C_{m+1}, z'_{m+1}$$

  is not.

We say that a cop strategy κ is *stingy* if for each robber-winning run $(C_i, z_i)_{i<m}$ which respects κ and $C_m = \kappa\left((C_i, z_i)_{i<m}\right)$, C_m is cost effective with respect to κ.

By the way the game is defined, replacing any cop position with a subset at a given step can only give the robber new options overall if it gives the robber new options in that round or the next. So a cop strategy is stingy when a minimal set of elements of A are occupied by cops at each step to give the robber the same options. Note that this does not mean that the strategy is using the fewest possible cops of the k available to give the cops the win. For instance, when $k \geq |A|$, the cop strategy where the cops choose all of A as their initial position and win immediately is always stingy, but it will not in general be the stategy where the fewest possible cops are used (e.g., for a path of length greater than two, the cops have winning strategies which use only two cops).

Stinginess is not hard to guarantee. In fact, as we will prove shortly, whenever the cops have a coarse, monotone, marginal, winning strategy they must have a coarse, monotone, marginal, *stingy*, winning strategy which uses (at most) the same number of cops. The following lemma will be useful in that proof and also serves as an illustration of how we can use cost effectiveness.

Lemma 6.7.15. *Suppose that for some $n \in \mathbb{N}$, for each κ-respecting robber-winning run $(C_i, z_i)_{i<m}$ with $m \geq l$, $C_m = \kappa\left((C_i, z_i)_{i<m}\right)$ is cost effective with respect to κ. Let $(C_i, z_i)_{i<m}$ be a κ-respecting robber-winning run with $m \geq n$ and $0 \leq j < m$. If $x \in C_m \setminus C_{m-1}$ and X_0 is the $^\circ$-component of A over C_j containing z_j, then $x \in X_0$.*

Proof. Let $C'_m = C_m \setminus \{x\}$. Since C_m is cost effective with respect to κ, either

(1) there exists z'_m such that $(C_i, z_i)_{i<m}, C'_m, z'_m$ is robber-winning but

$(C_i, z_i)_{i<m}, C_m, z'_m$ is not, or

(2) there exists z'_{m+1} such that for some z_m such that $(C_i, z_i)_{i<m}, C_m, z_m$ is robber-winning and $C_{m+1} = \kappa\left((C_i, z_i)_{i<m}, C_m, z_m\right)$,

$$(C_i, z_i)_{i<m}, C'_m, z_m, C_{m+1}, z'_{m+1}$$

is robber-winning but

$$(C_i, z_i)_{i<m}, C_m, z_m, C_{m+1}, z'_{m+1}$$

is not.

If (1) holds, observe that since $x \notin C_{m-1}$, $C_{m-1} \cap C'_m = C_{m-1} \cap C_m$. Since the components of A over $C_{m-1} \cap C'_m$ are the same as those of $C_{m-1} \cap C_m$, any position which is available to the robber when the cops choose C'_m instead of C_m must be in $A \setminus C'_m$, but not in $A \setminus C_m$. Thus we must have $z'_m = x$ and x must be in the same component of A over $C_{m-1} \cap C_m$ as z_{m-1}. Call this component Y_0. Then by Lemmas 6.7.11 and 6.7.13, $x_0 \in Y_0 \subseteq X_0$.

On the other hand, suppose that (2) holds and fix a particular z_m and $C_{m+1} = \kappa\left((C_i, z_i)_{i<m}, C_m, z_m\right)$ such that for some z'_{m+1}, $(C_i, z_i)_{i<m}, C'_m, z_m, C_{m+1}, z'_{m+1}$ is robber-winning but $(C_i, z_i)_{i<m}, C_m, z_m, C_{m+1}, z'_{m+1}$ is not. Then we must have $x \in C_{m+1}$ since otherwise we would have $C'_m \cap C_{m+1}$ and the robber would have no new choices.

Let $\mathcal{D}^\circ(A/C_m \cap C_{m+1}) = \{Y_0, Y_1, ..., Y_t\}$ and assume without loss of generality that $z_m \in Y_0$, let $Y'_0 \in \mathcal{D}^\circ(A/C'_m \cap C_{m+1})$ such that $z_m \in Y'_0$. Since $C'_m \cap$

$C_{m+1} \subseteq C_m \cap C_{m+1}$ and $Y_0 \cap Y_0' \neq \emptyset$, $Y_0 \subseteq Y_0'$ by Lemma 4.2.4. Since $(C_i, z_i)_{i<m}, C_m', z_m, C_{m+1}, z_{m+1}'$ is robber-winning, we must have $z_{m+1}' \in Y_0'$ and $z_{m+1}' \in A \setminus C_{m+1}$. If z_{m+1}' were in Y_0, $(C_i, z_i)_{i<m}, C_m, z_m, C_{m+1}, z_{m+1}'$ would also be robber-winning; as it isn't, we must have $z_{m+1}' \in Y_0' \setminus Y_0$. Thus $\{Y_0, Y_0' \setminus Y_0\}$ is a bipartition of Y_0'.

From the defintion of $\circ$-components, $Y_0 \not\perp\!\!\!\perp^{\circ}_{C_m' \cap C_{m+1}} Y_0' \setminus Y_0$. Note that if $x \notin Y_0'$, then by the contrapositive of freeness we would have $Y_0 \not\perp\!\!\!\perp^{\circ}_{(C_m' \cap C_{m+1}) \cup \{x\}} Y_0' \setminus Y_0$ and $Y_0' \setminus Y_0 \subseteq \bigcup_{i \neq 0} Y_i$. But since $(C_m' \cap C_{m+1}) \cup \{x\} = C_m \cap C_{m+1}$, by the contrapositive of the monotonicity axiom we would get $Y_0 \not\perp\!\!\!\perp^{\circ}_{C_m \cap C_{m+1}} \bigcup_{i \neq 0} Y_i$, contradicting the definition of $\circ$-components. So $x \in Y_0' \setminus Y_0$.

$Y_0' \setminus Y_0$ can be written as $\left(Y_0' \cap \left(\bigcup_{i \neq 0} Y_i\right)\right) \cup \{x\}$, which gives us

$$Y_0 \not\perp\!\!\!\perp^{\circ}_{C_m' \cap C_{m+1}} \left(Y_0' \cap \left(\bigcup_{i \neq 0} Y_i\right)\right) \cup \{x\}.$$

Since $Y_0 \perp\!\!\!\perp^{\circ}_{C_m \cap C_{m+1}} Y_0' \cap \left(\bigcup_{i \neq 0} Y_i\right)$ by the definition of $\circ$-components and the monotonicity axiom, the contrapositive of transitivity implies that $Y_0 \not\perp\!\!\!\perp^{\circ}_{C_m' \cap C_{m+1}} \{x\}$. By Lemma 6.7.8, $Y_0 \cap C_m = \emptyset$, so we can add $C_m \setminus C_{m+1}$ to the base by the contrapositive of freeness. Thus we have $Y_0 \not\perp\!\!\!\perp^{\circ}_{C_m'} \{x\}$.

By Lemma 6.7.13, Y_0 must be the $\circ$-component of A over C_m containing z_m. Since $C_m' \subset C_m$, Y_0 must also be a subset of the $\circ$-component of A over C_m' containing z_m. Then by the definition of $\circ$-components and the (contrapositive of) the montonicity axiom, $Y_0 \not\perp\!\!\!\perp^{\circ}_{C_m'} \{x\}$ implies that x is in the same $\circ$-component of A over C'_m as z_m. But since $C'_m \supseteq C_{m-1} \cap C_m$, this means x is in the same $^\circ$-component of A over $C_{m-1} \cap C_m$ as z_m. That same $^\circ$-component must also contain z_{m-1}, since $(C_i, z_i)_{i<m}, C_m, z_m$ is robber-winning. $x \in X_0$ then follows from Lemmas 6.7.11 and 6.7.13. $\square$

Lemma 6.7.16. *If κ is a coarse, monotone, marginal, winning cop strategy for k cops on A then there exists a winning cop strategy for k cops on A, κ', which is coarse, monotone, marginal, and stingy.*

Proof. Let n be the maximal length of a robber-winning run that respects κ, i.e., each such robber-winning run is of the form $(C_l, z_l)_{l<n}$ with $C_n = \kappa\left((C_l, z_l)_{l<n}\right)$. Since this will be a long proof with a lot of moving parts, we provide an outline below.

1. Create a coarse, monotone, marginal, winning cop strategy κ'_n for k cops on A such that the maximum length of a robber-winning run which respects κ'_n is less than or equal to n and for each robber-winning run $(C_i, z_i)_{i<n}$ of length n which respects κ', $C_n = \kappa\left((C_l, z_l)_{l<n}\right)$ is cost effective with respect to κ'_n.

   (a) Define a cop strategy κ'_n for k cops on A.

   (b) Show that κ_n is coarse, monotone, marginal, and winning and that for each robber-winning run $(C_i, z_i)_{i<n}$ of length n which respects κ',

$C_n = \kappa\left((C_l, z_l)_{l<n}\right)$ is cost effective with respect to κ'_n Note that the maximum length of a robber-winning run which respects κ'_n must be less than or equal to n.

2. Recursively define, for each $1 \leq j < n$, a coarse, monotone, marginal, winning cop strategy κ'_j such that the maximum length of a robber-winning run which respects κ'_j is less than or equal to n and for each $m \geq j$ and each robber-winning run $(C_i, z_i)_{i<m}$ which respects κ'_j, $C_m = \kappa'_j((C_i, z_i)_{i<m})$ is cost effective with respect to κ'_j.

   (a) Assume that κ'_{j+1} is a coarse, monotone, marginal, winning cop strategy for k cops on A such that the maximum length of a robber-winning run which respects κ'_{j+1} is less than or equal to n and for each $m \geq j+1$ and each robber-winning run $(C_i, z_i)_{i<m}$ which respects κ_{j+1}, $C_m = \kappa'_{j+1}((C_i, z_i)_{i<m})$ is cost effective with respect to κ'_{j+1}. Use this to define a cop strategy κ_j for k cops on A.

   (b) Show that κ_j is coarse, monotone, and winning.

   (c) Show that for each $m \geq j$ and each robber-winning run $(C_i, z_i)_{i<m}$ which respects κ_j, $C_m = \kappa_j((C_i, z_i)_{i<m})$ is cost effective with respect to κ_j.

   (d) Modify the definition κ_j to create a marginal cop strategy κ^*_j for k cops on A.

(e) Argue that these modifications do not alter coarseness, monotonicity, or winning-ness.

(f) Note that the maximum length of a robber-winning run which respects κ_j^* must be less than or equal to n and that for each $m \geq j+1$ and each robber-winning run which respects κ_j^*, $C_m = \kappa_j^*\left((C_i, z_i)_{i<m}\right)$ is cost effective with respect to κ_j^*.

(g) Show that iterating the process of creating κ_j and κ_j^* must eventually result in the creation of a cop strategy with *all* of the desired properties. Call this cop strategy κ_j'.

3. Argue that κ_1' is the desired coarse, monotone, marginal, stingy, winning cop strategy for k cops on A.

We now begin the proof, labeling the steps according to the outline.

1. (a) Observe that if $(C_i, z_i)_{i<n}$ and $(D_i, y_i)_{i<n}$ are both robber-winning runs that respect κ, such that $C_{n-1} = D_{n-1}$ and z_{n-1} is in the same $\circ$-component of A over $C_{n-1} = D_{n-1}$ as y_{n-1}, then we must have

$$\kappa\left((C_i, z_i)_{i<n}\right) = \kappa\left((D_i, y_i)_{i<n}\right)$$

by the coarseness and marginality of κ. We partition the set of robber-winning runs of length n which respect κ into equivalence classes with

$(C_i, z_i)_{i<n}$ and $(D_i, y_i)_{i<n}$ equivalent if and only if $C_{n-1} = D_{n-1}$ and z_{n-1} is in the same $^\circ$-component of A over $C_{n-1} = D_{n-1}$ as y_{n-1}.

For each equivalence class, choose a representative run $(C_i^*, z_i^*)_{i<n}$ and set $C_n^* = \kappa\left((C_i^*, z_i^*)_{i<n}\right)$. Choose $C_n' \subseteq C_n^*$ minimal such that for all z, $(C_i^*, z_i^*)_{i<n}, C_n', z$ is not robber-winning, i.e., such that replacing C_n^* with C_n' still doesn't give the robber any possible choices for Round n. This means that if X_0 is the component of A over $C_{n-1}^* \cap C_n'$ containing z_{n-1}^*, then $X_0 \setminus C_n'$ must be empty. It follows that for any $(C_i, z_i)_{i<n}$ in the same equivalence class, $(C_i, z_i)_{i<n}, C_n', z$ is also not robber-winning for any z and C_n' must be a minimal subset of $\kappa\left((C_i, z_i)_{i<n}\right)$ such that this is true.

Define

$$\mathcal{C}_n = \{(C, z) \in \binom{A}{\leq k} \times A | C = C_{n-1}, z = z_{n-1}$$
$$\text{for some } \kappa\text{-respecting } (C_i, z_i)_{i<n} \in RW_k(A, n)\}.$$

For each κ-respecting robber-winning run $(C_i, z_i)_{i<m}$ of length m for any $m < n$, if $(C_j, z_j) \in \mathcal{C}_n$ for some $0 \leq j < m$, then, by the coarseness and marginality of κ, we must have $\kappa\left((C_i, z_i)_{i<j}\right) = C_n^*$ for the appropriate equivalence class. Since $(C_i, z_i)_{i<m}$ respects κ and there can be no robber-winning runs $(C_i, z_i)_{i<j}, C_n^*, z$ for any z, we must have $j = m - 1$. We see again, that C_n' is a minimal subset of

$\kappa\left((C_i, z_i)_{i<m-1}\right)$ such that if the cops choose C'_n in Round m, then the robber will have no valid choices in Round m.

With this in mind, we define a new strategy κ'_n for k cops on A as follows:

- If $(C_i, z_i)_{i<n}$ is a κ-respecting robber-winning run of length n, define $\kappa'_n\left((C_i, z_i)_{i<n}\right) = C'_n$.
- If $(C_i, z_i)_{i<m}$ is a (not necessarily κ-respecting) robber-winning run of length m (for *any* m) such that $(C_{m-1}, z_{m-1}) \in \mathcal{C}$, define $\kappa\left((C_i, z_i)_{i<m}\right) = C'_n$ (for the C'_n associated with the equivalence class of a κ-respecting robber-winning run $(D_i, y_i)_{i<n}$ such that $D_{n-1} = C_{m-1}$ and $y_{n-1} = z_{m-1}$).
- For any other elements of $RW_k(A)$, define κ'_n of that run to be the same as κ of that run.

(b) We claim that κ'_n is a coarse, monotone, marginal, winning cop strategy for k cops on A such that for any robber-winning run $(C_i, z_i)_{i<n}$ of length n, $\kappa'_n\left((C_i, z_i)_{i<n}\right)$ is cost effective with respect to κ'_n.

κ'_n was clearly defined to be coarse and marginal and such that for any robber-winning run $(C_i, z_i)_{i<n}$ of length n, $\kappa'_n\left((C_i, z_i)_{i<n}\right)$ is cost effective with respect to κ'_n. Note that any robber-winning run which respects κ'_n must also respect κ, since we only changed the value of

κ on $(C_i, z_i)_{i<m}$ such that, if $C_m = \kappa\left((C_i, z_i)_{i<m}\right)$ there is no robber-winning run $(C_i, z_i)_{i<m}, C_m, z$ for any z and we did so in such a way that, if $C'_m = \kappa'_n\left((C_i, z_i)_{i<m}\right)$, then there is no robber-winning run $(C_i, z_i)_{i<m}, C'_m, z$ for any z. It follows that the maximum length of a robber-winning run which respects κ'_n will be equal to the maximum length of a robber-winning run which respects κ. Thus, κ'_n is winning and the maximal length of a robber-winning run which respects κ'_n is n. It also follows that κ'_n is monotone, since κ itself is monotone and, since for robber-winning runs which respect κ, we are only replacing final cop positions with subsets of the same position, we aren't creating any opportunities for the cops to return to a point they have previously occupied.

2. For each $1 \leq j < n$, we will now recursively define a coarse, monotone, marginal, winning cop strategy κ'_j such that for each $m \geq j$ and each robber-winning run $(C_i, z_i)_{i<m}$ which respects κ'_j, $C_m = \kappa'_j\left((C_i, z_i)_{i<m}\right)$ is cost effective with respect to κ'_j.

   (a) Assume that κ'_{j+1} is a coarse, monotone, marginal, winning cop strategy for k cops on A such that for each $m \geq j+1$ and each robber-winning run $(C_i, z_i)_{i<m}$ which respects κ_{j+1}, $C_m = \kappa'_{j+1}\left((C_i, z_i)_{i<m}\right)$ is cost effective with respect to κ'_{j+1}. We will define the desired κ'_j in

two steps. First we define a coarse, monotone, winning cop strategy κ_j for k cops on A such that for each $m \geq j$ and each robber-winning run $(C_i, z_i)_{i<m}$ which respects κ_j, $C_m = \kappa_j\left((C_i, z_i)_{i<m}\right)$ is cost effective with respect to κ_j. Then we will modify that strategy to get a κ'_j which is marginal without losing the other relevant properties.

Our definition of κ_j will be similar to our definition of κ'_n, except that we won't worry yet about changing the strategy for runs of length less than j or which don't respect κ'_{j+1} to make it marginal. We can again partition the robber winning runs of length j into equivalence classes such that $(C_i, z_i)_{i<j}$ and $(D_i, y_i)_{i<j}$ are equivalent precisely when $C_{j-1} = D_{j-1}$ and z_{j-1}, y_{j-1} are in the same component of A over $C_{j-1} = D_{j-1}$. For each equivalence class, choose a representative run $(C^*_i, z^*_i)_{i<j}$ and set $C^*_j = \kappa\left((C^*_i, z^*_i)_{i<j}\right)$. Choose $C'_j \subseteq C^*_j$ minimal such that for each $z \in A$,

- $(C^*_i, z^*_i)_{i<j}, C'_j, z$ is robber-winning if and only if $(C^*_i, z^*_i)_{i<j}, C{*}_j, z$ is robber-winning, and
- for each z^*_j such that $(C^*_i, z^*_i)_{i<j+1}$ is robber-winning, if $C^*_{j+1} = \kappa\left((C^*_i, z^*_i)_{i<j+1}\right)$, then

$$(C^*_i, z^*_i)_{i<j}, C'_j, z^*_j, C^*_{j+1}, z$$

is robber-winning if and only if

$$(C_i^*, z_i^*)_{i<j+1}, C_{j+1}^*, z$$

is robber-winning.

That is, choose C_j' such that the robber is given no new options in either Round j or Round $j+1$.

We define κ_j as follows:

- For any robber-winning run of length strictly less than j, define κ_j of that run to be κ_{j+1}' of that run.
- For any robber-winning run $(C_i, z_i)_{i<j}$ of length j which respects κ_{j+1}', define $\kappa_j\left((C_i, z_i)_{i<j}\right) = C_j'$.
- Suppose $(C_i, z_i)_{i<m}$ is a robber-winning run such that $m > j$, $(C_i, z_i)_{i<j}$ respects κ_{j+1}', $C_j = C_j'$ for the equivalence class of κ-respecting robber-winning runs of length j containing $(C_i, z_i)_{i<j}$, and there is a κ_{j+1}'-respecting robber-winning run $(D_i, z_i)_{i<m}$ with $D_j = C_j^*$ for that equivalence class and $D_i = C_i$ for $i \neq j$. (That is, suppose we could find $(C_i, z_i)_{i<m}$ by taking a κ_{j+1}'-respecting robber-winning run $(D_i, y_i)_{i<m}$ and replacing D_j with C_j' for the equivalence class of $(D_i, z_i)_{i<j}$.) Define

$$\kappa_j\left((C_i, z_i)_{i<m}\right) = \kappa_{j+1}'\left((D_i, z_i)_{i<m}\right).$$

- For all other robber-winning runs $(C_i, z_i)_{i<m}$, define

$$\kappa_j \left((C_i, z_i)_{i<m}\right) = \kappa' \left((C_i, z_i)_{i<m}\right).$$

(b) We now show that κ_j is coarse, winning, and monotone.

Coarse: That κ_j is coarse follows immediately from the definition of κ_j and from κ'_{j+1} being coarse.

Winning: For any robber-winning run $(C_i, z_i)_{i<m}$ that respects κ_j, if $m \leq j$, then that run also respects κ'_{j+1}, while if $m > j$, there is a robber-winning run $(D_i, z_i)_{i<m}$ which respects κ'_{j+1} with $D_j = C^*_j$ for the equivalence class of $(C_i, z_i)_{i<j}$ and $D_i = C_i$ for $i \neq j$. Since for each robber-winning run which respects κ_j there is a corresponding robber-winning run of the same length which respects κ'_{j+1} and κ'_{j+1} is a winning strategy, there must be some $N \in \mathbb{N}$ such that every robber-winning run which respects κ_j has length less than N. So κ_j is winning.

Monotone: To show that κ_j is monotone, it suffices to show that for each robber-winning run $(C_i, z_i)_{i<m}$ of length $m > j$ which respects κ'_{j+1}, if we replace C_j with C'_j, then we have not removed any points from a cop position which appear in an immediately following cop position.

For the sake of a contradiction, suppose we have. That is, suppose that for some robber-winning run $(C_i, z_i)_{i<j}$ which respects κ'_{j+1}, with $C_j = \kappa'_{j+1}\left((C_i, z_i)_{i<j}\right)$, $C'_j = \kappa_j\left((C_i, z_i)_{i<j}\right)$, for some z_j such that $(C_i, z_i)_{i<j}, C_j, z_j$ is robber-winning and $C_{j+1} = \kappa'_{j+1}\left((C_i, z_i)_{i<j+1}\right) = \kappa_j\left((C_i, z_i)_{i<j}, C'_j, z_j\right)$ there exists $c \in C_j \cap C_{j+1}$ such that $c \notin C'_j$.

Note that since $c \in C_{j+1}$ and C_{j+1} is cost effective with respect to κ'_{j+1}, either $c \in Y_0 \in \mathcal{D}^\circ\left(A/\left(C_j \cap \left(C_{j+1} \setminus \{c\}\right)\right)\right)$ such that $z_j \in Y_0$ or for some z such that $(C_i, z_i)_{i<j+1}, C_{j+1}, z$ is robber-winning and $D_{j+2} = \kappa'_{j+1}\left((C_i, z_i)_{i<j+1}, C_{j+1}, z\right)$, there exists a y such that $(C_i, z_i)_{i<j+1}, C_{j+1} \setminus \{c\}, z, D_{j+2}, y$ is robber-winning but $(C_i, z_i)_{l<j+1}, C_{j+1}, z, D_{j+2}, y$ is not. We will show that neither of these can be true.

Claim. $c \notin Y'_0 \in \mathcal{D}^\circ\left(A/\left(C'_j \cap C_{j+1}\right)\right)$ such that $z_j \in Y'_0$

Suppose, for the sake of contradiction, that $c \in Y'_0$.

Let $Z_0 \in \mathcal{D}^\circ\left(A/C_{j-1} \cap C_j\right)$ such that $z_{j-1} \in Z_0$ and let

$$\mathcal{D}^\circ\left(A/C_{j-1} \cap C'_j\right) = \{Z'_0, Z'_1, ..., Z'_m\}$$

such that $z_{j-1} \in Z'_0$. Since $C_{j-1} \cap C'_j \subseteq C_{j-1} \cap C_j$, $Z_0 \subseteq Z'_0$. Also, by our choice of C'_j, the robber has no new options at the j level. Since $z \in Z'_0$ is not an available choice for the robber if and only if $z \in C'_j$ and $c \in C_j$ means that c is not an available choice for the robber under κ'_{j+1}, we must have $c \notin Z'_0$.

Since $z_j \in Z_0 \subseteq Z'_0$ and $c \notin Z'_0$, we can bipartition Y'_0 into $Y'_0 \cap Z'_0$ and $Y'_0 \setminus Z'_0$. Then by the defintion of the decomposition into $\circ$-components, we must have

$$Y'_0 \cap Z'_0 \not\perp\!\!\!\perp_{C'_j \cap C_{j+1}} Y'_0 \setminus Z'_0.$$

We note that if $x \in C_{j-1} \cap C'_j \subseteq C_{j-1} \cap C_j$ and $x \notin C_{j+1}$, then $x \notin Y'_0$: since κ'_{j+1} is a coarse, monotone, marginal, winning cop strategy, $(C_l, z_l)_{l<j+1}, C_{j+1}, x$ can't be robber-winning by Lemma 6.7.7, so by our choice of C'_j, $(C_l, z_l)_{l<j}, C'_j, z_j, C_{j+1}, x$ is not robber-winning. Since $x \notin C_{j+1}$, $(C_l, z_l)_{l<j}, C'_j, z_j, C_{j+1}, x$ is not robber-winning if and only if $x \notin Y'_0$. This enables us to use the (contrapositive of) the freeness axiom to expand the base set to include all of $C_{j-1} \cap C'_j$.

$$Y'_0 \cap Z'_0 \not\perp\!\!\!\perp_{\left(C'_j \cap C_{j+1}\right) \cup \left(C_{j-1} \cap C'_j\right)} Y'_0 \setminus Z'_0.$$

Now we observe that if $x \in C'_j \cap C_{j+1} \subseteq C'_j$, but $C \notin C_{j-1}$, then x must be in some component of A over $C_{j-1} \cap C'_j$. If we let $B_0 = Z'_0 \cap \left(\left(C'_j \cap C_{j+1}\right) \setminus C_{j-1}\right)$ and we let $B_1 = \bigcup_{i=1}^m Z'_i \cap \left(\left(C'_j \cap C_{j+1}\right) \setminus C_{j-1}\right)$, then applying (the contrapositive of) transitivity twice gives us

$$B_0 \cup \left(Y'_0 \cap Z'_0\right) \not\perp\!\!\!\perp_{C_{j-1} \cap C'_j} B_1 \cup \left(Y'_0 \setminus Z'_0\right).$$

But $B_0 \cup (Y_0' \cap Z_0') \subseteq Z_0'$ and $B_1 \cup (Y_0' \setminus Z_0') \subseteq \bigcup_{i=1}^m Z_i'$, so by the definition of the decomposition into components and the monotonicity axiom, we must have

$$B_0 \cup (Y_0' \cap Z_0') \mathop{\smile\!\!\!\!\!\!\;\!|}^{\circ}_{C_{j-1} \cap C_j'} B_1 \cup (Y_0' \setminus Z_0'),$$

a contradiction.

So $c \notin Y_0'$ and, hence, not in $Y_0 \subseteq Y_0'$. In particular, this means that $Y_0 \in \mathcal{D}^\circ (A/C_j \cap C_{j+1})$, i.e., removing the c from C_{j+1} does not change the component containing z_j.

By the cost effectiveness of C_{j+1} with respect to κ_{j+1}', then, there must exist a y such that for some z with $(C_l, z_l)_{l<j+1}, C_{j+1}, z$ robber-winning and $D_{j+2} = \kappa_{j+1}' \left((C_l, z_l)_{l<j+1}, C_{j+1}, z \right)$,

$$(C_l, z_l)_{l<j+1}, C_{j+1} \setminus \{c\}, z, D_{j+2}, y$$

is robber-winning but

$$(C_l, z_l)_{l<j+1}, C_{j+1}, z, D_{j+2}, y$$

is not.

Note that we must have $c \in D_{j+2}$, since otherwise $C_{j+1} \cap D_{j+2} = (C_{j+1} \setminus \{c\}) \cap D_{j+2}$ and the robber would have no new options at the $j+2$ level. So if $X_0 \in \mathcal{D}^\circ (A/C_{j+1} \cap D_{j+2})$ such that $z \in X_0$, $c \notin X_0$. On the other hand, if $X_0' \in \mathcal{D}^\circ (A/(C_{j+1} \setminus \{c\}) \cap D_{j+1})$

such that $z \in X_0'$, then we must have $c \in X_0'$. ($\{X_0, X_0' \setminus X_0\}$ is a bipartition of X_0', so $X_0 \not\perp\!\!\!\perp^{\circ}_{(C_{j+1} \setminus \{c\}) \cap D_{j+1}} X_0' \setminus X_0$. If $c \notin X_0'$, then by freeness $X_0 \not\perp\!\!\!\perp^{\circ}_{C_{j+1} \cap D_{j+1}} X_0' \setminus X_0$, but by the definition of the decomposition into components and monotonicity, we must have $X_0 \perp\!\!\!\perp^{\circ}_{C_{j+1} \cap D_{j+1}} X_0' \setminus X_0$.)

We can, therefore, bipartition X_0' into $\{X_0, X_0' \setminus X_0\}$, with $z \in X_0$ and $c \in X_0' \setminus X_0$. So we have

$$X_0 \not\perp\!\!\!\perp^{\circ}_{(C_{j+1} \setminus \{c\}) \cap D_{j+2}} (X_0' \setminus X_0) \setminus \{c\} \cup \{c\}$$

by the definition of the decomposition into $\circ$-components. Since

$$((C_{j+1} \setminus \{c\}) \cap D_{j+2}) \cup \{c\} = C_{j+1} \cap D_{j+2},$$

using transitivity, we must have either

$$X_0 \not\perp\!\!\!\perp^{\circ}_{(C_{j+1} \setminus \{c\}) \cap D_{j+2}} \{c\}$$

or

$$X_0 \not\perp\!\!\!\perp^{\circ}_{C_{j+1} \cap D_{j+2}} (X_0' \setminus X_0) \setminus \{c\}.$$

The latter cannot be true by the definition of the decomposition into $\circ$-components (and the monotonicity axiom), so it must be the former which holds.

If $x \in C_j \cap (C_{j+1} \setminus \{c\})$, then $x \notin Y_0$ and, thus, $x \notin X_0$ since $X_0 \subseteq Y_0$ by Lemma 6.7.11. So by (the contrapositive of) freeness we have

$$X_0 \not\mathrel{\raise{0.2ex}{\perp}}^{p}_{((C_{j+1}\setminus\{c\})\cap D_{j+2})\cup(C_j\cap(C_{j+1}\setminus\{c\}))}\{c\}.$$

Now observe that if $x \in (C_{j+1} \setminus \{c\}) \cap D_{j+2}$ but $x \notin C_j$, then x must be in some component of A over $C_j \cap (C_{j+1} \setminus \{c\})$. We can, then, split the elements of $(C_{j+1} \setminus \{c\}) \cap D_{j+2}$ which aren't in $C_j \cap (C_{j+1} \setminus \{c\})$ into two sets, A_0 and A_1, where A_0 consists of the elements which are in Y_0 and A_1 consists of the elements in any other component. Applying (the contrapositive of) transitivity twice gives us

$$A_0 \cup X_0 \not\mathrel{\raise{0.2ex}{\perp}}^{p}_{C_j\cap(C_{j+1}\setminus\{c\})}\{c\} \cup A_1.$$

But this is a contradiction, since by the definition of the decomposition into $\circ$-components and monotonicity, we must have

$$A_0 \cup X_0 \mathrel{\raise{0.2ex}{\perp}}^{\circ}_{C_j\cap(C_{j+1}\setminus\{c\})}\{c\} \cup A_1.$$

So κ_j is monotone.

(c) We must now consider the cost effectiveness of $\kappa_j\left((C_i, z_i)_{i<m}\right)$ for each robber-winning run of length $m \geq j$ which respects κ_j. Clearly, κ_j was defined so that $C'_j = \kappa_j\left((C_i, z_i)_{i<m}\right)$ is cost effect with respect to κ_j.

For any robber-winning run $(C_i, z_i)_{i<m}$ of length $m > j$, there must, as previously observed, be some κ'_{j+1}-respecting robber-winning run

$(D_i, z_i)_{i<m}$ such that $D_j = C_j^*$ for the equivalence class of $(C_i, z_i)_{i<j}$ and $D_i = C_i$ for all $i \neq j$. So for any $m \geq j+2$, we must have $D_{m-1} = C_{m-1}$ and $C_m = \kappa_j\left((C_i, z_i)_{i<m}\right) = \kappa'_{j+1}\left((D_i, z_i)_{i<m}\right)$. The cost effectiveness of $\kappa_j\left((C_i, z_i)_{i<m}\right)$ will then follow from the cost effectiveness of $\kappa'_{j+1}\left((D_i, z_i)_{i<m}\right)$ and the fact that for any z_m, $(C_i, z_i)_{i<m+1}$ is a robber-winning run which respects κ_j if and only if $(D_i, z_i)_{i<m}, C_m, z_m$ is a robber-winning run which respects κ'_{j+1}.

For robber-winning runs $(C_i, z_i)_{i<m}$ of length $m = j+1$, we must, effectively, make sure that replacing D_j with C'_j would not change the cost effectiveness of $D_{j+1} = \kappa'_{j+1}\left((D_i, z_i)_{i<m}\right)$ with respect to κ'_{j+1}. Clearly, any point in D_{j+1} whose removal would give the robber new options in the next round, rather than immediately, must still have that effect. What of the points whose removal would give the robber new options immediately? For any such $x \in D_{j+1}$, we must have x in the same component of $D_j \cap (D_{j+1} \setminus \{x\})$ as z_j. This implies that $(C_i, z_i)_{i<j}, C'_j, z_j, D_{j+1} \setminus \{x\}, x$ will be robber-winning, but $(C_i, z_i)_{i<j}, C'_j, z_j, D_{j+1}, x$ is not. So it is still true that removing any point of D_{j+1} would give the robber new options. Thus $D_{j+1} = \kappa_j\left((C_i, z_i)_{i<j+1}\right)$ is cost effective with respect to κ_j.

(d) We have shown that κ_j is a coarse, monotone, winning cop strategy for k cops on A such that for each robber-winning run $(C_i, z_i)_{i<m}$ of length $m \geq j$ which respects κ, $\kappa_j\left((C_i, z_i)_{i<m}\right)$ is cost effective with respect to κ_j. Unfortunately, κ_j is not marginal. However, we can adjust the definition of κ_j to create a κ'_j which will be marginal while still being coarse, monotone, and winning and with the maximum length of a robber-winning run which respects κ'_j less than or equal to n. (It may not preserve the cost effectiveness, but we will present a way to deal with that.)

We need to deal with two issues that are keeping κ_j from being marginal. First, we need to make sure (as we did when defining κ'_n) that if

$$\mathcal{C}_j = \left\{(C, z) \in \binom{A}{\leq k} \times A \middle| C = C_{j-1}, z = z_{j-1} \right.$$
$$\left. \text{for some } \kappa\text{-respecting } (C_i, z_i)_{i<n} \in RW_k(A, j)\right\},$$

then κ'_j of any robber-winning run ending in (C, z) for some $(C, z) \in \mathcal{C}_j$ is C'_j for the appropriate equivalence class of robber-winning runs of length j. We will need to extend this further than we did in the n case, since, in general, the cops choosing C'_j need not end the game.

The other issue is that there may already have been robber-winning runs which respect κ'_{j+1} containing C'_j (for some equivalence class of robber-winning runs of length j) as a cop position. So we will also need to make sure the responses to any such run are appropriate for marginality. This was not an issue for κ'_n because we did not introduce any new responses to a pair C'_j, z – each C'_j we added ended the game. In trying to deal with both of these, we will need to be careful that we don't introduce unaccounted for new occurrences of either issue.

Note that if $(C, z) \in \mathcal{C}_j$ and there exists a κ_j-respecting robber-winning run $(C_i, z_i)_{i<m}$ with $m > j$ such that $C_{m-1} = C$ and $z_{m-1} = z$, then there exists a κ'_{j+1}-respecting robber-winning run $(D_i, z_i)_{i<m}$ such that $D_j = C^*_j$ for the for the equivalence class of κ-respecting robber-winning runs of length j containing $(C_i, z_i)_{i<j}$ and $D_i = C_i$ for all $i \neq j$. $C_m = \kappa_j\left((C_i, z_i)_{i<m}\right) = \kappa'_{j+1}\left((D_i, z_i)_{i<m}\right)$ must be cost effective with respect to κ'_{j+1}. Since κ'_{j+1} is marginal, for any robber-winning run of length j in the equivalence class corresponding to (C, z), we must have κ'_{j+1} of that run be C_m. But by the marginality and cost effectiveness of κ'_{j+1}, that means κ'_{j+1} of each run in that equivalence class must be cost effective with respect to κ'_{j+1} and, hence, we would not have changed its value when creating κ_j. So we do not need to do anything to robber-winning runs ending in such a pair (C, z) to make our new strategy marginal. That is, we only need to worry about pairs $(C, z) \in \mathcal{C}_j$ if they do not appear as C_{m-1} and z_{m-1} for some κ_j-respecting robber-winning run of length

m. This will help us avoid introducing extra copies of such pairs.

(Due to time constraints, the remainder of the proof is more of a sketch than a full argument, but it does hold.)

Somewhat similarly, suppose that $(C_i, z_i)_{i<j+1}$ is any κ_j-respecting robber-winning run of length $j+1$. If there is any κ_j-respcting robber winning run $(D_i, y_i)_{i<m}$ for some $m > j+1$, such that $D_{m-1} = C_j$, $y_{m-1} = z_j$. Then (considering the appropriate κ'_{j+1}-respecting run) the marginality of κ'_{j+1} ensure that the value of κ_j on any κ_j-respecting robber-winning run of length other than $j+1$ ending in the pair must be the same. We will fix the issue by having the cops respond to the runs of length exactly $j+1$ precisely as they do for some/any longer run ending in that pair.

Define $\kappa_{j,0}$ as follows:

- If $(C_i, z_i)_{i<m}$ is a robber-winning run with $m = j+1$ and there is some κ_j-respecting robber winning run $(D_i, y_i)_{i<l}$ with $l > j+1$ such that $D_{l-1} = C_{m-1}$ and $y_{l-1} = z_{m-1}$, define $\kappa_{j,0}\left((C_i, z_i)_{i<m}\right) = \kappa_j\left((D_i, y_i)_{i<l}\right)$.
- If $(C_i, z_i)_{i<m}$ is a robber-winning run with $m > j+1$ and there is some κ_j-respecting robber winning run $(D_i, y_i)_{i<l}$ with such that $D_{l-t} = C_{m-t}$ and $y_{l-t} = z_{m-t}$ for $1 \leq t \leq m-j$, define

$\kappa_{j,0}\left((C_i, z_i)_{i<m}\right) = \kappa_j\left((D_i, y_i)_{i<l}\right)$.

- For all other robber-winning runs, let $\kappa_{j,0}$ of that run equal κ_j of that run.

Define $\kappa_{j,1}$ by

- Let $(C_i, z_i)_{i<m}$ be a robber-winning run such that for some $\kappa_{j,0}$-respecting robber-winning run $(C'_i, z'_i)_{i<j+1}$,$C'_j = C_{m-1}$ and $z'_j = z_{m-1}$, and for $i \neq m-1$, C_i, z_i do not appear as the Round j cop and robber positions for any $\kappa_{j,0}$-respecting robber-winning run. If there exists a κ_j-respecting robber winning run $(D_i, y_i)_{i<l}$ with $l > j+1$ such that $D_{l-1} = C'_j$ and $y_{l-1} = z'_j$, define $\kappa_{j,1}\left((C_i, z_i)_{i<m}\right) = \kappa_{j,0}\left((C_i, z_i)_{i<m}\right)$ (i.e., don't change its value). Otherwise, define $\kappa_{j,1}\left((C_i, z_i)_{i<m}\right) = \kappa_{j,0}\left((C'_i, z'_i)_{i<j+1}\right)$.
- Similarly to the second bullet point in our definition of $\kappa_{j,0}$, extend $\kappa_{j,1}$ for robber-winning runs which extend the one in the first bullet point and respect $\kappa_{j,1}$ as defined so far.
- For all other robber-winning runs, let $\kappa_{j,1}$ of that run equal $\kappa_{j,0}$ of that run.

Finally, define κ_j^* by now replacing the strategy following occurrences of a pair $(C, z) \in \mathcal{C}j$ by copying $\kappa_{j,1}$, analogously to the way we copied $\kappa_{j,0}$ following occurences of a pair that arises as the *Roundj* cop and robber position in defining $\kappa_{j,1}$ and otherwise have κ_j^* equal to $\kappa_{j,1}$.

(e) κ_j^* is defined to be marginal – we correct each of the potential problem pairs without introducing new ones. Each step of the definition also preserves coarseness and, since we may make robber-winning runs shorter, but we never lengthen them, κ_j^* is winning and the maximum length of a robber-winning run respecting κ_j^* must be less than or equal to n.

Each step also maintains monotonicity. We will give an argument for the step from κ_j to $\kappa_{j,0}$, but the others proceed somewhat similarly. We are effectively pasting two strategies together – we take a robber-winning run $(C_i, z_i)_{i<j+1}$ that respects κ_j and then tacking on the tail of some robber winning run $(D_i, y_i)_{i<l}$ that respects κ_j starting at some $i_0 > j$ so that $D_{i_0} = C_j$ and $y_{i_0} = z_j$. Since κ_j is monotone, the only way this could go wrong is if there is some $i_C < j$ and some $i_D > i_0$ such that for some x, $x \in C_{i_C}$ and $x \in D_{i_D}$, but it doesn't appear in any of the inbetween C_i and D_i. For the sake of a contradiction, suppose this does happen.

If the robber-winning run $(D_i, y_i)_{i<l}$ satisfies $D_i = C_i$ and z_i in the same $\circ$-component of A over $C_i = D_i$ as y_i for $0 \leq i \leq j$, then the monotonicity of κ_j will guarantee that this does not occur. By the

coarseness of κ_j (and since we must have $C_0 = D_0$), then, there must be some minimal $i_1 < j$, such that $C_{i_1} = D_{i_1}$, but z_{i_1} and y_{i_1} are not in the same $\circ$-component of A over $C_{i_1} = D_{i_1}$.

Note that there must be a κ_{j+1}-respecting robber-winning run for which the cop and robber positions agree with $(D_i, z_i)_{i<l}$ for all $i \neq j$. Since $i_D > i_0 > j$, this mean that (by the cost effectiveness of κ'_{j+1} for runs of length at least $j+1$ and the definition of κ_j) $\kappa_j\left((D_i, z_i)_{i<i_D}\right)$ must be cost effective with respect to κ_j. Since $x \in D_{i_D}$ but $x \notin D_{i_D-1}$, we can apply Lemma 6.7.15 to get that x must be in the same $\circ$-component of A over D_{i_0} as y_{i_0}. But then x must be in the same $\circ$-component of A over C_j as z_j. Thus, $(C_i, z_i)_{i<j}, C_j, x$ must be a robber-winning run which respects κ.

$C_j = C'_j$ for an equivalence class of robber-winning runs of length j that respect κ'_{j+1}. It was chosen to be cost effective, so replacing C^*_j with C_j gave the robbers no new options. This implies that $(C_i, z_i)_{i<j}, C^*_j, x$ is a robber winning run that respects κ'_{j+1}. But then Lemma 6.7.7 tells us that $x \notin C_i$ for any $i < j$, a contradiction. Thus $\kappa_{j,0}$ must be monotone.

(In the other two cases, we'll end up with a different kind of contradiction – showing that x must be an element of two different components of A over some C. There are arguments similar to this in some of the proofs that use stinginess.)

(f) We have already seen that the maximum length of a robber-winning run that respects κ_j^* must be less than or equal to n. If $(C_i, z_i)_{i<m}$ is a robber-winning run that respects κ_j^*, then we must have a κ'_{j+1}-respecting robber-winning run, $(D_l, y_l)_{l<n}$ such that for some $l_0 \geq j$, we get $D_{l_0+t} = C_{j+t}$ and $y_{l_0+t} = C_{j+t}$ for $0 \leq t \leq m-j$. It follows from the cost effectiveness of κ'_{j+1} of robber-winning runs of length at least $j+1$ with respect to κ'_{j+1}, that κ_j^* of any robber-winning run of length at least $j+1$ will be cost effective with respect to κ_j^*.

    What about for runs of length equal to j which we had in κ_j? The only place where this might have gone wrong is when we created $\kappa_{j,0}$. Even though we chose all of the C'_j to be minimal at the time, we may have changed the following cop position when we pasted the strategies together and that might impact the cost effectiveness.

(g) Fortunately, there is a solution. Since κ_j^* fits the criteria we assumed when creating κ_j, we can go back and repeat this process from a new starting point. We end the process when going back through it produces no changes from the last run. It must end in a finite number of iterations because we never increase the maximum length of the robber-winning runs which respect the cop strategies and A is finite, since each time we go through the creation of κ_j, we must either remove at least one point from a Round j cop position or the process will stop when we reach the end of the creation of κ_j^* because we have made no changes. The final κ_j^* must then also satisfy the cost effectiveness condition in addition to all the others for the desired κ_j'

κ_1' will be the desired coarse, monotone, marginal, stingy, winning cop strategy.

□

Henceforth, unless otherwise stated, assume that κ, in addition to being coarse, monotone, marginal and winning, is stingy. $(C_l, z_l)_{l\leq m}$ remains a robber-winning run which respects κ and $C_{m+1} = \kappa\left((C_l, z_l)_{l\leq m}\right)$.

Lemma 6.7.17. *For all $0 \leq i < j \leq m$, if $X_0 \in \mathcal{D}^\circ\left(A/\left(C_i \cap C_{i+1}\right)\right)$ such that $z_i \in X_0$, then $C_j \subseteq (C_i \cap C_{i+1}) \cup X_0$.*

Proof. [1] Let i be given and, for the sake of contradiction, assume the statement fails. Let j be the lowest index such that it fails. Let $C_j' = C_j \cap$

[1]This lemma actually follows fairly easily from the proof of Lemma 6.7.15, though not quite from the statement. However, since the proof given here significantly predates the one for Lemma 6.7.15, we preserve it for posterity.

$(X_0 \cup (C_i \cap C_{i+1}))$. We will show that replacing C_j with C_j' doesn't give the robber any new options, contradicting that κ is stingy.

Since j is the smallest index such that the hypothesis fails, either $C_{j-1} \subseteq (C_i \cap C_{i+1}) \cup X_0$ or $j - 1 = i$. In either case, we must have $C_{j-1} \cap C_j \subseteq (C_i \cap C_{i+1}) \cup X_0$. So $C_{j-1} \cap C_j = C_{j-1} \cap C_j'$. Let $Y_0 \in \mathcal{D}^\circ (A/(C_{j-1} \cap C_j)) = \mathcal{D}^\circ \left(A/\left(C_{j-1} \cap C_j'\right)\right)$ such that $z_{j-1} \in Y_0$. By Lemma 6.7.11, $Y_0 \subseteq X_0$, so $C_j' \cap Y_0 = C_j \cap Y_0$ and $(C_l, z_l)_{l<j}, C_j', z$ is robber-winning if and only if $(C_l, z_l)_{l<j}, C_j, z$ is robber-winning. That is, the robber has no new options at the j level.

It remains to show that the robber also has no new options at the $j + 1$ level. Let z_j' be any element of A such that $(C_l, z_l)_{l<j}, C_j, z_j'$ is robber-winning. Let $C_{j+1}' = \kappa\left((C_l, z_l)_{l<j}, C_j, z_j'\right)$. We must show that for all $z \in A$, $(C_l, z_l)_{l<j}, C_j', z_j', C_{j+1}', z$ is robber-winning if and only if $(C_l, z_l)_{l<j}, C_j, z_j', C_{j+1}', z$ is robber-winning.

Let $\mathcal{D}^\circ \left(A/\left(C_j \cap C_{j+1}'\right)\right) = \{Z_0, Z_1, ..., Z_n\}$ with $z_j' \in Z_0$ and let $\mathcal{D}^\circ \left(A/\left(C_j' \cap C_{j+1}'\right)\right) = \{Z_0', Z_1', ..., Z_{n'}'\}$ with $z_j' \in Z_0'$. Clearly, $Z_0 \subseteq Z_0'$. If $Z_0 = Z_0'$, then we're done.

Note that it suffices to show that $Z_0 \mathop{\perp\!\!\!\!\perp}^\circ_{C_j' \cap C_{j+1}'} Z_0' \setminus Z_0$, since, by the definition of the decomposition into $\circ$-components, this implies that $Z_0' \setminus Z_0$ must be empty. $Z_0' \subseteq A \setminus \left(C_j' \cap C_{j+1}'\right) = \left(\bigcup_{l=0}^n Z_l\right) \cup \left(C_j \setminus C_j'\right)$. So, by monotonicity, $Z_0 \mathop{\perp\!\!\!\!\perp}^\circ_{C_j' \cap C_{j+1}'} Z_0' \setminus Z_0$ is a consequence of the following claim:

Claim. $Z_0 \mathop{\perp\!\!\!\!\perp}^\circ_{C_j' \cap C_{j+1}'} \left(C_j \setminus C_j'\right) \cup \left(\bigcup_{l=1}^n Z_l\right)$.

Since $Z_0 \subseteq X_0$ by Lemma 6.7.11 and $\left(C_j \setminus C'_j\right) \cap X_0 = \emptyset$, $\left(C_j \setminus C'_j\right) \cap Z_0 = \emptyset$. So $C_j \setminus C'_j \subseteq \left(C_j \cap C'_{j+1}\right) \cup \left(\bigcup_{l=1}^{n} Z_l\right)$. We can re-write the claim as

$$Z_0 \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_{C'_j \cap C'_{j+1}} \left(\left(C_j \setminus C'_j\right) \cap C'_{j+1}\right) \cup \left(\bigcup_{l=1}^{n} Z_l\right)$$

and then prove it using transitivity by showing that

(a) $Z_0 \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_{C'_j \cap C'_{j+1}} \left(C_j \setminus C'_j\right) \cap C'_{j+1}$ and

(b) $Z_0 \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_{\left(C'_j \cap C'_{j+1}\right) \cup \left(\left(C_j \setminus C'_j\right) \cap C'_{j+1}\right)} \bigcup_{l=1}^{n} Z_l$.

For (b), note that

$$\left(C'_j \cap C'_{j+1}\right) \cup \left(\left(C_j \setminus C'_j\right) \cap C'_{j+1}\right) = C_j \cap C'_{j+1}.$$

So, in fact, (b) just says $Z_0 \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_{C_j \cap C'_{j+1}} \bigcup_{l=1}^{n} Z_l$, which comes directly from the definition of the decomposition into $\circ$-components.

For (a), we have already observed that $Z_0 \subseteq X_0$. Since $C_j \setminus C'_j$ is precisely the set of elements of C_j which are not in X_0 and also not in $C_i \cap C_{i+1}$, if $\mathcal{D}^{\circ}\left(A / \left(C_i \cap C_{i+1}\right)\right) = \{X_0, X_1, ..., X_r\}$, then $C_j \setminus C'_j \subseteq \bigcup_{l=1}^{r} X_l$.

From the definition of the decomposition into $\circ$-components, we have

$$X_0 \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_{C_i \cap C_{i+1}} \bigcup_{l=1}^{r} X_l.$$

Applying monotonicity we can then get

$$Z_0 \cup \left(\left(C_j \cap C'_{j+1}\right) \cap X_0\right) \mathop{\smile\!\!\!\!\!\!\;|}^{\circ}_{C_i \cap C_{i+1}} C_j \setminus C'_j.$$

Since, by the definition of the decomposition into $\circ$-components, $(C_i \cap C_{i+1}) \cap X_l = \emptyset$ for any l, the freeness axiom tells us that we can remove any points we wish to from the base. In particular, we have

$$Z_0 \cup \left(\left(C_j \cap C'_{j+1}\right) \cap X_0\right) \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_{(C_i \cap C_{i+1}) \cap \left(C_j \cap C'_{j+1}\right)} C_j \setminus C'_j.$$

Finally, we can use transitivity to move the $\left(C_j \cap C'_{j+1}\right) \cap X_0$ into the base. Since

$$\begin{aligned}\left(\left(C_j \cap C'_{j+1}\right) \cap X_0\right) \cup \left((C_i \cap C_{i+1}) \cap \left(C_j \cap C'_{j+1}\right)\right) \\ &= \left(C_j \cap C'_{j+1}\right) \cap (X_0 \cup (C_i \cap C_{i+1})) \\ &= (C_j \cap (X_0 \cup (C_i \cap C_{i+1}))) \cap C'_{j+1} \\ &= C'_j \cap C'_{j+1},\end{aligned}$$

this gives us $Z_0 \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_{C'_j \cap C'_{j+1}} \left(C_j \setminus C'_j\right)$ and (a) follows by monotonicity.

So we have proven the claim and, by monotonicity, we have $Z_0 \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_{C'_j \cap C'_{j+1}} Z'_0 \setminus Z_0$. Since Z_0 is nonempty, we must have $Z'_0 \setminus Z_0$ empty (since otherwise $\{Z_0, Z'_0 \setminus Z_0\}$ would be a bipartition of Z'_0) and $Z'_0 = Z_0$.

But this means that $(C_l, z_l)_{l<j}, C'_j, z'_j, C'_{j+1}, z$ is robber-winning if and only if $(C_l, z_l)_{l<j}, C_j, z'_j, C'_{j+1}, z$ is robber-winning. Since we already have $(C_l, z_l)_{l<j}, C'_j, z$ robber-winning if and only if $(C_l, z_l)_{l<j}, C_j, z$ is robber-winning, this contradicts the assumption that κ is stingy. $\square$

Applying Lemma 6.7.13 will give us the following useful corollary (which includes the initial step case).

Corollary 6.7.18. *For all $0 \leq i < j \leq m$, if $Y_0 \in \mathcal{D}^\circ(A/C_i)$ such that $z_i \in Y_0$, then $C_j \subseteq C_i \cup Y_0$.*

Proof. Let X_0 be the $\circ$-component of A over $C_i \cap C_{i+1}$ containing z_i. By Lemma 6.7.17, $C_j \subseteq (C_i \cap C_{i+1}) \cup X_0$. By Lemma 6.7.13, $X_0 = Y_0$. Since $C_i \cap C_{i+1} \subseteq C_i$, we have $C_j \subseteq C_i \cup Y_0$. □

So we have shown that at any step the cops' chosen position will be a subset of their previous position at an earlier step and the $\circ$-component containing the robber's choices in response to that position. Must it necessarily contain elements of both? In particular, we know that the $\circ$-component containing the robber's choices at any given step must be a subset of the $\circ$-component containing the robber's choices at any previous step (Lemmas 6.7.11 and 6.7.12); must the cops be eating away at the robber's choices at each step?

Not quite. Recall that in Seymour and Thomas's version of the cops and robber's game, the cops are allowed to choose a subset of their previous position. Some such winning strategies will still satisfy our criteria (so long as the cops don't choose subsets two turns in a row – that would fail to be stingy). So the cops will not necessarily eliminate options for the robbers at each step. They will, however, eliminate options at least at every other step.

Lemma 6.7.19. *Let $0 \leq i < m-1$. If $X_0 \in \mathcal{D}^\circ(A/(C_i \cap C_{i+1}))$ such that $z_i \in X_0$, then for all $i+1 < j \leq m$, $C_j \cap X_0 \neq \emptyset$.*

Proof. Let $0 \leq i < m - 1$ be given. For the sake of contradiction, suppose the hypothesis fails and let j be the lowest index greater than $i+1$ for which it fails.

We know by Lemma 6.7.17 that $C_j \subseteq (C_i \cap C_{i+1}) \cup X_0$. Since $C_j \cap X_0 = \emptyset$, we must have $C_j \subseteq C_i \cap C_{i+1}$. We consider two cases:

(a) $C_{j-1} \cap X_0 = \emptyset$. (Note that by the minimality of j, this implies $j = i + 2$.)

(b) $C_{j-1} \cap X_0 \neq \emptyset$.

(a) If $j = i + 2$ and $C_{i+1} \cap X_0 = C_{j-1} \cap X_0 = \emptyset$, then by Lemma 6.7.17 we have $C_j \subseteq C_{i+1} \subseteq C_i$. Since κ is marginal and winning, we can't have $C_{l+1} = C_l$ for any l by Lemma 6.7.6, so $C_j \subsetneq C_{i+1} \subsetneq C_i$. We'll show that this contradicts κ being stingy, since we could replace C_{i+1} with C_j without giving the robber any new options.

Let $X_0' \in \mathcal{D}^\circ (A/(C_i \cap C_j)) = \mathcal{D}^\circ (A/C_j)$ such that $z_i \in X_0'$. We want to show that $X_0 \setminus C_{i+1} = X_0' \setminus C_j$ to show that the robber has no additional choices at the $i+1$ level when C_{i+1} is replaced with C_j.

Let $Y_0 \in \mathcal{D}^\circ (A/(C_{i+1} \cap C_j)) = \mathcal{D}^\circ (A/C_j)$ such that $z_{i+1} \subseteq Y_0$. By Lemma 6.7.11, $Y_0 \subseteq X_0$. On the other hand, since $C_j \subset C_{i+1} = C_i \cap C_{i+1}$ and $X_0 \cap Y_0 \neq \emptyset$, we must have $X_0 \subseteq Y_0$. Thus $Y_0 = X_0$. This implies that $z_i \in Y_0$, so we must have $X_0' = Y_0 = X_0$. And since $C_{i+1} \cap X_0 = \emptyset = C_j \cap X_0$,

$$X_0 \setminus C_{i+1} = X_0 = X_0 \setminus C_j = X_0' \setminus C_j.$$

Therefore, for all $z \in A$, $(C_l, z_l)_{l \leq i}, C_j, z$ is robber-winning if and only if $z \in A$, $(C_l, z_l)_{l \leq i}, C_{i+1}, z$ is robber-winning.

The robber also doesn't pick up any new choices at the j level: since $C_i \cap C_{i+1} = C_{i+1}$, $X_0 \in \mathcal{D}^\circ (A/C_{i+1})$. Because κ is coarse, we must have $\kappa\left((C_l, z_l)_{l \leq i}, C_{i+1}, z'_{i+1}\right) = C_j$ for each $z'_{i+1} \in X_0 \setminus C_{i+1} = X_0$. Note that $\mathcal{D}^\circ (A/(C_{i+1} \cap C_j)) = \mathcal{D}^\circ (A/C_j) = \mathcal{D}^\circ (A/(C_j \cap C_j))$. Therefore, for any $z'_{i+1} \in X_0$ (i.e., any z'_{i+1} such that $(C_l, z_l)_{l \leq i}, C_{i+1}, z'_{i+1}$ is robber-winning), $(C_l, z_l)_{l \leq i}, C_{i+1}, z'_{i+1}, C_j, z$ is robber-winning if and only $(C_l, z_l)_{l \leq i}, C_j, z'_{i+1}, C_j, z$ is robber-winning.

But this contradicts that κ is stingy.

(b) If $C_{j-1} \cap X_0 \neq \emptyset$, let $z \in C_{j-1} \cap X_0$. Note that $z \notin C_j$. Let $Y_0 \in \mathcal{D}^\circ (A/(C_{j-1} \cap C_j))$ such that $z_{j-1} \in Y_0$. $z_{j-1} \in X_0$ (by Lemma 6.7.9) and $C_{j-1} \cap C_j \subseteq C_j \subseteq C_i \cap C_{i+1}$ implies that $X_0 \subseteq Y_0$ and, therefore, $z \in Y_0$. Thus $(C_l, z_l)_{l \leq j-1}, C_j, z$ is a robber-winning run which respects κ. But then $z \notin C_{j-1}$ by Lemma 6.7.7, a contradiction.

Since we have reached a contradiction either way, there cannot be a $j > i+1$ such that $C_j \cap X_0 = \emptyset$. □

The goal of making all of these observations was to help us show that $\leq_\kappa$ satisfies the hypotheses of Fact 6.7.5 and that the resulting tree is a $^\circ$-tree decomposition of A. The next lemma is directly applicable to these goals, allowing us to show the transitivity and antisymmetry of $\leq_\kappa$.

Lemma 6.7.20. *If $C, C', C'' \in V_\kappa$ such that $C \leq_\kappa C'$ and $C' \leq_\kappa C''$, then there exists a κ-respecting robber-winning run $(C_l, z_l)_{l<m}$ with $C_m = \kappa\left((C_l, z_l)_{l<m}\right)$ such that for some $0 \leq i_0 \leq i_1 \leq i_2 \leq m$, $C_{i_0} = C$, $C_{i_1} = C'$, and $C_{i_2} = C''$.*

Proof. If $C = C'$ or $C' = C''$, this is clear, so we may assume not.

Since $C <_\kappa C'$, there exists a κ-respecting robber-winning run $(C_l, z_l)_{l<n}$ with $C_n = \kappa\left((C_l, z_l)_{l<n}\right)$ such that for some $i < j \leq n$, $C = C_i$ and $C' = C_j$. Similarly, since $C' <_\kappa C''$, there exists a κ-respecting robber-winning run $(D_l, y_l)_{l<n'}$ with $D_{n'} = \kappa\left((D_l, y_l)_{l<n'}\right)$ such that for some $j' < l' \leq n'$, $C' = D_{j'}$ and $C'' = D_{l'}$. If $D_l = C_i$ for any $l < j'$, then $(D_l, y_l)_{l<n'}$ is the desired robber-winning run, so suppose not.

Since both of these robber-winning runs respect κ, we must have $C_0 = D_0$. But $D_l \neq C_i$ for any $l < j'$, so there must exist a maximum $l_0 < i$ such that for all $l \leq l_0$, $C_l = D_l$, i.e., $l_0 + 1$ is the lowest index such that the cops' positions don't match. Because κ is coarse, the fact that $C_{l_0+1} \neq D_{l_0+1}$ implies that there exists some $l \leq l_0$ such that z_l and y_l don't lie in the same component of A over C_l. Let l_1 be the least such index.

Let $\mathcal{D}^\circ\left(A/C_{l_1}\right) = \{Z_0, Z_1, ..., Z_{r_{l_1}}\}$. We may assume that $z_{l_1} \in Z_0$ and $y_{l_1} \in Z_1$. For each $1 \leq l \leq j$, let $X_{0,l} \in \mathcal{D}^\circ\left(A/\left(C_{l-1} \cap C_l\right)\right)$ such that $z_l \in X_{0,l}$.

Similarly, for each $1 \leq l \leq j'$, let $Y_{0,l} \in \mathcal{D}^\circ(A/(D_{l-1} \cap D_l))$ such that $y_l \in Y_{0,l}$. Then by Lemma 6.7.13, $X_{0,l_1+1} = Z_0$ and $Y_{0,l_1+1} = Z_1$.

Since $l_1 \leq l_0 < i < j$, $l_1+1 < j$, so by Lemma 6.7.19, $C_j \cap Z_0 = C_j \cap X_{0,l_1+1} \neq \emptyset$. Let $z \in C_j \cap Z_0$.

Note that either $z \in C_{l_1+1}$, in which case $z \notin C_{l_1}$ by the definition of X_{0,l_1+1}, or $(C_l, z_l)_{l \leq l_1}, C_{l_1+1}, z$ is a κ-respecting robber-winning run, in which case $z \notin C_{l_1}$ by Lemma 6.7.7. So $z \notin C_{l_1}$.

By Lemma 6.7.17, $D_{j'} \subseteq (C_{l_1} \cap D_{l_1+1}) \cup Y_{0,l_1+1} \subseteq C_{l_1} \cup Z_1$. Since $z \in Z_0$, $z \notin Z_1$, so $z \notin C_{l_1} \cup Z_1$ and, thus, $z \notin D_{j'}$. But $D_{j'} = C' = C_j$ and $z \in C_j$, a contradiction!

So we must have $D_l = C_i$ for some $l < j'$ and the robber-winning run $(D_l, y_l)_{l<n'}$ has the desired properties. □

We are now prepared to show that $\leq_\kappa$ satisfies the hypotheses of Fact 6.7.5.

Proposition 6.7.21. *Suppose that $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ a coarse, monotone, marginal, stingy, winning cop strategy for the $^\circ$-Cops and Robbers game with k cops on A. $\leq_\kappa$ is a partial ordering of V_κ with a unique minimal element, and for each $C^* \in V_\kappa$, $\{C \in V_\kappa | C \leq_\kappa C^*\}$ is linearly ordered by $\leq_\kappa$.*

Proof. We show first that $\leq_\kappa$ is a partial ordering of V_κ.

Reflexivity: Let $C \in V_\kappa$. Then, by definition, either $C = \kappa(\varepsilon)$ or $C = \kappa((C_i, z_i)_{i\leq m})$ for some robber-winning run $(C_i, z_i)_{i\leq m}$ which respects κ.

In the latter case, since $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$, we have $C = C_i$ and $C = C_j$ where $i = j = m+1$, so $C \leq_\kappa C$. By assumption, there must be some robber-winning run (C_0, z_0) which respects κ and, therefore, satisfies $C_0 = \kappa(\varepsilon)$. So in the former case we will have $C = C_i$ and $C = C_j$ where $i = j = 0$ and, again, $C \leq_\kappa C$.

Transitivity: This is precisely what Lemma 6.7.20 shows. If $C, C', C'' \in V_\kappa$ such that $C \leq C'$ and $C' \leq C''$, then by 6.7.20 there exists a κ-respecting robber-winning run $(C_l, z_l)_{l<m}$ with $C_m = \kappa\left((C_l, z_l)_{l<m}\right)$ such that for some $0 \leq i_0 \leq i_1 \leq i_2 \leq m$, $C_{i_0} = C$, $C_{i_1} = C'$, and $C_{i_2} = C''$. Setting $i = i_0$ and $j = i_2$ gives us $C \leq_\kappa C''$.

Antisymmetry: Suppose for some $C, C' \in V_\kappa$ we have $C \leq_\kappa C'$ and $C' \leq_\kappa C$. By Lemma 6.7.20, we would then have a κ-respecting robber-winning run $(C_l, z_l)_{l<m}$ with $C_m = \kappa\left((C_l, z_l)_{l<m}\right)$ such that for some $0 \leq i_0 \leq i_1 \leq i_2 \leq m$, $C_{i_0} = C$, $C_{i_1} = C'$, and $C_{i_2} = C$. Since κ is monotone, this would imply that $C = C \cap C \subseteq C'$. On the other hand, since $C' \leq C$ and $C \leq C'$, Lemma 6.7.20 also gives us a κ-respecting robber-winning run $(C'_l, z'_l)_{l<m}$ with $C'_m = \kappa\left((C'_l, z'_l)_{l<m}\right)$ such that for some $0 \leq i_0 \leq i_1 \leq i_2 \leq m$, $C'_{i_0} = C'$, $C'_{i_1} = C'$, and $C'_{i_2} = C$. So by the monotonicity of κ, we must have $C' = C' \cap C' \subseteq C$. Since $C \subseteq C'$ and $C' \subseteq C$, we must have $C = C'$.

For each κ-respecting robber-winning run $(C_i, z_i)_{i \leq m}$ and $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$, we must have $C_0 = \kappa(\varepsilon)$, so $\kappa(\varepsilon) \leq_\kappa C$ for all $C \in V_\kappa$. By antisymmetry, $\kappa(\varepsilon)$ is the unique minimum element of V_κ.

Let $C^* \in V_\kappa$ be given and suppose $C, C' \in \{C \in V_\kappa | C \leq_\kappa C^*\}$. We must show that either $C \leq_\kappa C'$ or $C' \leq_\kappa C$. If either C or C' is equal to C^*, then the result follows immediately from the fact that $C, C' \in \{C \in V_\kappa | C \leq_\kappa C^*\}$, so we may assume that neither C nor C' is equal to C^*.

Since $C \leq_\kappa C^*$, there must be some κ-respecting robber-winning run $(C_l, z_l)_{l \leq m}$ with $C_{m+1} = \kappa((C_l, z_l)_{l \leq m})$ such that $C = C_i$, $C^* = C_j$ for some $0 \leq i < j \leq m+1$. Similarly, $C' \leq_\kappa C^*$ and there must be some κ-respecting robber-winning run $(C'_l, z'_l)_{l \leq m'}$ with $C'_{m+1} = \kappa((C'_i, z'_i)_{i \leq m'})$ such that $C' = C'_{i'}$, $C^* = C'_{j'}$ for some $0 \leq i' < j' \leq m'+1$. Assume without loss of generality that $i \leq i'$.

Since both runs respect κ, we must have $C_0 = C'_0$. We proceed as in the proof of Lemma 6.7.20, letting l_0 be the maximum index such that for all $l \leq l_0$, $C_l = C'_l$ and $l_1 \leq l_0$ the least index l such that z_l and z'_l are not in the same $^\circ$-component of A over C_l. If $i \leq l_0$, then $C'_i = C_i$ and $(C'_l, z'_l)_{l \leq m'}$ is a robber-winning run in which $C = C_i$ and $C' = C'_{i'}$ both appear. In this case we have $C \leq_\kappa C'$.

Suppose $i > l_0 \geq l_1$ and note that this means that $i' > l_0 \geq l_1$ as well. Let $\mathcal{D}^\circ(A/C_{l_1}) = \{Z_0, Z_1, ..., Z_r\}$. Without loss of generality, we may assume that $z_{l_1} \in Z_0$ and $z'_{l_1} \in Z_1$. By Lemma 6.7.13, if X_0 is the $^\circ$-component of A over $C_{l_1} \cap C_{l_1+1}$ containing z_{l_1}, we must have $X_0 = Z_0$. Similarly, the $^\circ$-component of

A over $C_{l_1} \cap C'_{l_1+1}$ containing z'_{l_1} must be equal to Z_1.

Since $j > i > l_1$, by Lemma 6.7.19 we must have $C_j \cap Z_0 = C_j \cap X_0 \neq \emptyset$. Since $C_j = C^*$, this implies that there exists some $z \in C^* \cap Z_0$. Note that $z \in Z_0$ implies that $z \notin C_{l_1}$ and $z \notin Z_1$. But by Lemma 6.7.17, since $j' > i' > l_1$, we must have $C^* = C'_{j'} \subseteq (C_{l_1} \cap C_{l_1+1}) \cup Z_1 \subseteq C_{l_1} \cup Z_1$, a contradiction!

Having reached a contradiction, it must not be possible to have $i > l_0$. Thus, $i \leq l_0$ and, as we have seen, this implies $C \leq_\kappa C'$.

Since we have shown that for all $C, C' \in \{C \in V_\kappa | C \leq_\kappa C^*\}$ either $C \leq_\kappa C'$ or $C' \leq_\kappa C$, $\leq_\kappa$ is a linear ordering on $\{C \in V_\kappa | C \leq_\kappa C^*\}$. □

So we have that $\leq_\kappa$ satisfies the hypotheses of Fact 6.7.5. If we define

$$E_\kappa = \{\{C, C'\} | C, C' \in V_\kappa \text{ and either } C' \text{ an immediate } \leq_\kappa\text{-successor of } C \text{ or vice versa}\},$$

then we will have a tree $T = (V_\kappa, E_\kappa)$. The question is, must T be a $^\circ$-tree decomposition of A?

Before attempting to answer that question, let's define one further property a cop strategy might satisfy.

Definition 6.7.22. Let $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ be a cop strategy for the $^\circ$-Cops and Robbers game with k cops on A. We say that κ is *fully monotone* if for any robber-winning run $(C_i, z_i)_{i \leq m}$ which respects κ, if $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$, then

- $C_i \mathop{\smile\!\!\!\!\!\!\!\!\!\;\;|}^{\circ}_{C_j} C_l$ whenever $i \leq j \leq l \leq m+1$ and

- for $i \leq j < j+1 \leq l \leq m+1$, if $\mathcal{D}^\circ\left(A/(C_j \cap C_{j+1})\right) = \{X_0, X_1, ..., X_{n-1}\}$, then there exists $I \subseteq n$ such that $C_i \setminus (C_j \cap C_{j+1}) \subseteq \bigcup_{i \in I} X_i$ and $C_l \setminus (C_j \cap C_{j+1}) \subseteq \bigcup_{i \in n \setminus I} X_i$.

Note that, by the first bullet point and Regularity 1, any strategy which is fully monotone will also be monotone. Obviously, full monotonicity is meant to help ensure that $T = (V_\kappa, E_\kappa)$ will satisfy the second and third bullet points of the definition of $^\circ$-tree decomposition. Fortunately, asking for a fully monotone, coarse, marginal, stingy, winning cop strategy when we already have a monotone, coarse, marginal, stingy, winning cop strategy is not a difficult requirement. In fact, it turns out that any monotone, coarse, marginal, stingy, winning cop strategy, will also be fully monotone.

Lemma 6.7.23. *If $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ is a monotone, coarse, marginal, stingy, winning, cop strategy for the $^\circ$-Cops and Robbers game with k cops on A, then κ is fully monotone.*

Proof. Let $(C_i, z_i)_{i \leq m}$ be a robber-winning run which respects κ and let $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$.

- Suppose $i \leq j \leq l \leq m+1$. We must show that $C_i \mathop{\smile\!\!\!\!\!\!|}^{\circ}_{C_j} C_l$. This is clear (by Regularity 2) if either $i = j$ or $l = j$, so we may assume $i < j < l$.

  If $X_0 \in \mathcal{D}^\circ\left(A/(C_j \cap C_{j+1})\right)$ such that $z_j \in X_0$ and $\mathcal{D}^\circ(A/C_j) = \{Y_0, Y_1, ..., Y_{n-1}\}$ with $z_j \in Y_0$, then by Lemma 6.7.13, $X_0 = Y_0$. From Lemma 6.7.17, we then have

$$C_l \subseteq (C_j \cap C_{j+1}) \cup X_0 = (C_j \cap C_{j+1}) \cup Y_0 \subseteq C_j \cup Y_0.$$

So $C_l \setminus C_j \subseteq Y_0$.

By Lemma 6.7.8, $C_i \cap Y_0 = C_i \cap X_0 = \emptyset$. So we must have $C_i \setminus C_j \subseteq \bigcup_{i \neq 0} Y_i$. By the definition of $\mathcal{D}^\circ(A/C_j)$, $Y_0 \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_{C_j} \bigcup_{i \neq 0} Y_i$. Applying the monotonicity axiom will give us

$$C_l \setminus C_j \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_{C_j} C_i \setminus C_j.$$

Using Regularity 2 and the monotonicity axiom, we can add all of C_j to the right side, then remove the points of C_j which are not in C_i, so we have

$$C_l \setminus C_j \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_{C_j} C_i.$$

Switching the two sides by symmetry before applying a similar procedure to get all of C_l to appear one one side results in the desired outcome:

$$C_i \mathop{\smile\!\!\!\!\!\!\downarrow}^{\circ}_{C_j} C_l.$$

- Suppose $i \leq j < j+1 \leq l$ and $\mathcal{D}^\circ(A/(C_j \cap C_{j+1})) = \{X_0, X_1, ..., X_{n-1}\}$. We need to find $I \subseteq n$ such that $C_i \setminus (C_j \cap C_{j+1}) \subseteq \bigcup_{i \in I} X_i$ and $C_l \setminus (C_j \cap C_{j+1}) \subseteq \bigcup_{i \in n \setminus I} X_i$.

  Assume without loss of generality that X_0 is the $^\circ$-component of A over $C_j \cap C_{j+1}$ containing z_j. Then $C_l \setminus (C_j \cap C_{j+1}) \subseteq X_0$ by Lemma 6.7.17 and

$C_i \cap X_0 = \emptyset$ by Lemma 6.7.8. Letting $I = n \setminus \{0\}$, we have $C_i \setminus (C_j \cap C_{j+1}) \subseteq \bigcup_{i \neq 0} X_i = \bigcup_{i \in I} X_i$ and $C_l \setminus (C_j \cap C_{j+1}) \subseteq X_0 = \bigcup_{i \in n \setminus I} X_i$.

□

We can now show that $T = (V_\kappa, E_\kappa)$ is a $^\circ$-tree decomposition of A.

Proposition 6.7.24. *If $\kappa : RW_k(A) \longrightarrow \binom{A}{\leq k}$ is a coarse, monotone, marginal, stingy, winning, cop strategy for the $^\circ$-Cops and Robbers game with k cops on A, then the tree $T = (V_\kappa, E_\kappa)$ is a $^\circ$-tree decomposition of A of width less than k.*

Proof. By definition, each element of V_κ is in $\binom{A}{\leq k}$. We need to show that

- $\bigcup V(T) = A$,
- for all $s_1, s_2, t_1, t_2 \in V(T)$ such that t_1 and t_2 are adjacent in T and the edge $\{t_1, t_2\}$ lies on the unique path from s_1 to s_2, if $\mathcal{D}^\circ(A/(t_1 \cap t_2)) = \{B_0, B_1, ..., B_{m-1}\}$, then there is a set $I \subseteq m$ such that $s_1 \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in I} B_i$ and $s_2 \setminus (t_1 \cap t_2) \subseteq \bigcup_{i \in m \setminus I} B_i$, and
- for all $t_1, t_2 \in V(T)$, if $t \in V(T)$ is on the unique path between t_1 and t_2, then $t_1 \mathop{\downarrow}^{\circ}_{t} t_2$.

We treat each bullet point in order:

1. For the first bullet point, towards a contradiction, suppose $\bigcup V(T) \neq A$. Then there exists some $z \in A$ such that $z \notin C$ for any $C \in V_\kappa$. We claim

that for each $m \in \mathbb{N}$, there exists a robber-winning run $(C_i, z_i)_{i \leq m}$ which respects κ with $z_i = z$ for all $i \leq m$. By induction:

Base case: Let $m = 0$. We must produce a robber-winning run (C_0, z_0) which respects κ and has $z_0 = z$. Define $C_0 = \kappa(\varepsilon)$. Since $C_0 \in V_\kappa$, $z \notin C_0$. So (C_0, z) is a robber-winning run which respects κ.

Inductive step: Suppose that whenever $m \leq l$ for some $l \in \mathbb{N}$ there exists a robber-winning run $(C_i, z_i)_{i \leq m}$ which respects κ with $z_i = z$ for all $i \leq m$. We must show that the statement also holds when $m = l + 1$. By our inductive assumption, there is a robber-winning run $(C_i, z_i)_{i \leq l}$ which respects κ and has $z_i = z$ for all $i \leq l$. Let $C_{l+1} = \kappa\left((C_i, z_i)_{i \leq l}\right)$. By definition, $C_{l+1} \in V_\kappa$, so $z \notin C_{l+1}$. Since $z_i = z$, z is certainly in the same $\circ$-component of A over $C_l \cap C_l + 1$, so $(C_i, z_i)_{i \leq l}, C_{l+1}, z$ is a robber-winning run. It also respects κ and satisfies $z_i = z$ for all $i \leq l + 1$. This is the desired run with $m = l + 1$.

So the claim holds. But then there is no $N \in \mathbb{N}$ such that for any robber-winning run $(C_i, z_i)_{i \leq m}$ that respects κ, m must be less than N. This contradicts κ being a winning cop strategy. Thus each $z \in A$ must be in C for some $C \in V(T)$.

2. To show the second bullet point, let $C, C', D, D' \in V(T)$ such that $\{D, D'\} \in E(T)$ and lies on the unique path from C to C'. We must show that if

$\mathcal{D}^{\circ}(A/(D \cap D')) = \{X_0, X_1, ..., X_{n-1}\}$, then there is some $I \subseteq n$ such that $C \setminus (D \cap D') \subseteq \bigcup_{i \in I} X_i$ and $C' \setminus (D \cap D') \subseteq \bigcup_{i \in n \setminus I} X_i$.

Since D, D' are adjacent in T, one must be an immediate $\leq_\kappa$-successor of the other. We assume, without loss of generality, that $D <_\kappa D'$. Note that this means that for a robber-winning run $(C_i, z_i)_{i \leq m}$ with $C_{m+1} = \kappa((C_i, z_i)_{i \leq m})$, if $C_i = D$ and $C_j = D'$ for some $i, j \in [m+2]$, then $i = j - 1$.[2]

In the special case that $C \leq_\kappa C'$, we must have $C \leq_\kappa D <_\kappa D' \leq_\kappa C'$. From the proof of Lemma 6.7.20, we must have a κ-respecting robber-winning run $(C_i, z_i)_{i \leq m}$ with $C_{m+1} = \kappa((C_i, z_i)_{i \leq m})$ such that for some $0 \leq i_1 \leq i_2 < i_3 \leq m+1$ we have $C_{i_1} = C$, $C_{i_2} = D$, $C_{i_2+1} = D'$, and $C_{i_3} = C'$. The desired result then follows immediately from κ being fully monotone. Since we can treat the case where $C' \leq_\kappa C$ similarly, we now assume that C and C' are not comparable.

Let $t_0 = \kappa(\varepsilon)$, the $\leq_\kappa$ minimum element of V_κ. For every $t \in V(T)$ on the t_0, C-path in T, we must have $t \leq_\kappa C$. Similarly, for each $t \in V(T)$ on the t_0, C'-path, we must have $t \leq_\kappa C'$. Let C^* be the vertex of maximal

[2]By the antisymmetry of $\leq_\kappa$, since $D \neq D'$, we must have $i < j$. If there were an l such that $i < l < j$, then we would have $D \leq_\kappa C_l$ and $C_l \leq_\kappa D'$. Since C_l is not equal to either C_i or C_j (by some combination of Lemma 6.7.6 and the antisymmetry of $\leq_\kappa$), this would contradict D' being an immediate $\leq_\kappa$-successor of D.

distance from t_0 that appears on both the t_0, C-path and the t_0, C'-path. Then $C^* \leq_\kappa C$, $C^* \leq_\kappa C'$, and combining the C, C^*-path and the C^*, C'-path must give the unique C, C'-path.

Since the edge $\{D, D'\}$ lies on the C, C'-path, it must lie on either the C, C^*-path or the C^*, C'-path. Without loss of generality, we assume that it lies on the C^*, C'-path. Then, as in the special case above, we get a κ-respecting robber-winning run $(C'_i, z'_i)_{i \leq m'}$ with $C'_{m'+1} = \kappa\left((C'_i, z'_i)_{i \leq m'}\right)$ such that for some $0 \leq i_1 \leq i_2 < i_3 \leq m' + 1$ we have $C'_{i_1} = C^*$, $C_{i_2} = D$, $C'_{i_2+1} = D'$, and $C'_{i_3} = C'$. z_{i_2} must be contained in some element of $\mathcal{D}^\circ\left(A/(D \cap D')\right)$; without loss of generality, assume $z_{i_2} \in X_0$. Then by Lemma 6.7.17, we must have $C' \subseteq (D \cap D') \cup X_0$, so $C' \setminus (D \cap D') \subseteq X_0$. Since κ is fully monotone, we must then have $C^* \setminus (D \cap D') \subseteq \bigcup_{i=1}^{n-1} X_i$.

There is also a κ-respecting robber-winning run $(C_j, z_j)_{j \leq m}$ with $C_{m+1} = \kappa\left((C_j, z_j)_{i \leq m}\right)$ such that for some $0 \leq j_1 \leq j_2 \leq m+1$, $C_{j_1} = C^*$ and $C_{j_2} = C$. Since C^* is the vertex of maximal distance from t_0 appearing on both the t_0, C- and t_0, C'-paths, we must have $\kappa\left((C_j, z_j)_{j \leq j_1}\right) \neq \kappa\left((C'_i, z'_i)_{i \leq i_1}\right)$. Because κ is coarse and marginal, this implies that z_{j_1} and z'_{i_1} must be in different $^\circ$-components of A over C^*. Let $\mathcal{D}^\circ(A/C^*) = \{Y_0, Y_1, ..., Y_r\}$ and, without loss of generality, assume $z'_{i_1} \in Y_0$, $z_{j_1} \in Y_1$.

By Corollary 6.7.18, we must have $C' \subseteq C^* \cup Y_0$ and $C \subseteq C^* \cup Y_1$. Since

$X_0 \subseteq Y_0$ by Lemmas 6.7.11 and 6.7.13, $X_0 \cap Y_1 = \emptyset$ and, thus, $Y_1 \setminus (D \cap D') \subseteq \bigcup_{i=1}^{n-1} X_i$. Having already observed above that $C^* \setminus (D \cap D') \subseteq \bigcup_{i=1}^{n-1} X_i$, this gives us

$$C \setminus (D \cap D') \subseteq (C^* \cup Y_1) \setminus (D \cap D') = (C^* \setminus (D \cap D')) \cup (Y_1 \setminus (D \cap D')) \subseteq \bigcup_{i=1}^{n-1} X_i.$$

Therefore, setting $I = n \setminus \{0\}$ we get $C \setminus (D \cap D') \subseteq \bigcup_{i \in I} X_i$ and $C' \setminus (D \cap D') \subseteq \bigcup_{i \in n \setminus I} X_i$, as desired.

3. For the third bullet point, let $C, C', D \in V(T)$ such that D is on the unique C, C'-path in T. We need to show that $C \mathop{\downarrow\!\!\!\!\!\;{}^{\circ}}_D C'$.

   If $C \leq_\kappa C'$, then we must have $C \leq_\kappa D \leq_\kappa C'$ and so there is a κ-respecting robber-winning run $(C_i, z_i)_{i \leq m}$, with $C_{m+1} = \kappa\left((C_i, z_i)_{i \leq m}\right)$ and $0 \leq i \leq j \leq l \leq m+1$ such that $C_i = C$, $C_j = D$, and $C_l = C'$. In this case, the desired result follows immediately from κ being fully monotone. The case where $C' \leq_\kappa C$ is similar. So we now assume that C and C' are not comparable (and, hence, there is no κ-respecting robber-winning run featuring all three of C, C', and D among its cop positions).

   We define C^*, as we did above, to be the vertex of maximal distance from $t_0 = \kappa(\varepsilon)$ that appears on both the t_0, C-path and the t_0, C'-path. Again, $C^* \leq_\kappa C$, $C^* \leq_\kappa C'$, and combining the C, C^*-path and the C^*, C'-path must give the unique C, C'-path.

Consider the special case where $C^* = D$. Let $(C_i, z_i)_{i<m}$ be a κ-respecting robber-winning run with $C_{m+1} = \kappa\left((C_i, z_i)_{i<m}\right)$ and $0 \le j \le l \le m+1$ such that $C_j = C^* = D$ and $C_l = C$. Let $(C'_i, z'_i)_{i<m'}$ be a κ-respecting robber-winning run with $C'_{m+1} = \kappa\left((C'_i, z'_i)_{i<m'}\right)$ and $0 \le j' \le l' \le m'+1$ such that $C'_{j'} = C^* = D$ and $C'_{l'} = C$. Since $C^* = D$ is the vertex of maximal distance from t_0 appearing on both the t_0, C- and t_0, C'-paths, we must have $\kappa\left((C_i, z_i)_{i\le j}\right) \neq \kappa\left((C'_i, z'_i)_{i\le j'}\right)$. Because κ is coarse and marginal, this implies that z_j and $z'_{j'}$ must be in different $\circ$-components of A over $C^* = D$. Let $\mathcal{D}^\circ(A/D) = \{Y_0, Y_1, ..., Y_r\}$ and, without loss of generality, assume $z_j \in Y_0$, $z_{j'} \in Y_1$.

By the definition of the decomposition into $\circ$-components, $Y_0 \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{D} Y_1$. Then by Regularity 2, $D \cup Y_0 \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{D} D \cup Y_1$. Using Corollary 6.7.18, we have $C \subseteq D \cup Y_0$ and $C' \subseteq D \cup Y_1$. So applying the monotonicity axiom of the abstract free amalgamation relation allows us to conclude that $C \mathop{\smile\!\!\!\!\!\!\,|}^{\circ}_{D} C'$, as desired.

Finally, suppose $D \neq C^*$. Then D must lie on either the path from C^* to C or the path from C^* to C'. Without loss of generality, we assume that D lies on the path from C^* to C. We begin similarly to the special case.

Let $(C_i, z_i)_{i<m}$ be a κ-respecting robber-winning run with $C_{m+1} = \kappa\left((C_i, z_i)_{i<m}\right)$ and $0 \le j_1 \le j_2 \le j_3 \le m+1$ such that $C_{i_1} = C^*$, $C_{i_2} = D$, and $C_{i_3} =$

C. Let $(C'_i, z'_i)_{i<m'}$ be a κ-respecting robber-winning run with $C'_{m+1} = \kappa\left((C'_i, z'_i)_{i<m'}\right)$ and $0 \leq j \leq l \leq m'+1$ such that $C'_j = C^* = D$ and $C'_l = C$. Since C^* is the vertex of maximal distance from t_0 appearing on both the t_0, C- and t_0, C'-paths, we must have $\kappa\left((C_i, z_i)_{i \leq i_1}\right) \neq \kappa\left((C'_i, z'_i)_{i \leq j}\right)$. Because κ is coarse and marginal, this implies that z_{i_1} and z'_j must be in different $\circ$-components of A over C^*. Let $\mathcal{D}^\circ(A/C^*) = \{Y_0, Y_1, ..., Y_r\}$ and, without loss of generality, assume $z_{i_1} \in Y_0$, $z_j \in Y_1$.

From Corollary 6.7.18, we have that $C \subseteq C^* \cup Y_0$, $D \subseteq C^* \cup Y_0$, and $C' \subseteq C^* \cup Y_1$. Since $Y_0 \mathop{\downarrow}^{\circ}_{C^*} Y_1$, by Regularity 2 we get $C^* \cup Y_0 \mathop{\downarrow}^{\circ}_{C^*} C^* \cup Y_1$. From the monotonicity axiom, we have $C \cup D \mathop{\downarrow}^{\circ}_{C^*} C'$. Transitivity then gives us $C \mathop{\downarrow}^{\circ}_{C^* \cup D} C'$. Since $C \mathop{\downarrow}^{\circ}_{D} C^*$ by κ being fully monotone, we can apply transitivity again to get $C \mathop{\downarrow}^{\circ}_{D} C' \cup C^*$. Applying the monotonicity one more time, we have $C \mathop{\downarrow}^{\circ}_{D} C'$, as desired.

Having satisfied each point of the definition, we see that T is indeed a $\circ$-tree decomposition of A. $\square$

Combining Propositions 6.7.4 and 6.7.24, we get the following partial analogue of the Seymour and Thomas result,

Theorem 6.7.25. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\downarrow}^{\circ}$ and $A \subset_{fin} M$. A has $\circ$-tree* *width less than k if and only if k cops have a coarse, monotone, marginal, winning strategy in the $^\circ$-Cops and Robbers game on A.*

Proof. The forward direction is precisely Proposition 6.7.4. For the backward direction, we need Lemma 6.7.16 to produce a coarse, monotone, marginal, *stingy*, winning cop strategy κ. Then we apply Proposition 6.7.24 to produce a $^\circ$-tree decomposition of width less than k. □

Ideally, we hoped to show directly that whenever k cops have any winning strategy on A, they must have a coarse, monotone, marginal, winning strategy. If we could, combining that result with Theorem 6.7.25 and Proposition 6.1.1 would have given us the full analogue of Seymour and Thomas's result. Unfortunately, a direct proof that we can construct such a strategy from any winning strategy proved elusive, so we turned to other methods to achieve the full desired result.

As we have seen, that other approach was successful and allowed us to prove the full analogue of Seymour and Thomas's Theorem. As a consequence, we now know that if k cops have any winning strategy on A then they must have a coarse, monotone, marginal, winning strategy on A, even if we still don't know directly how to construct such a strategy.

Corollary 6.7.26. *Let $\mathcal{M}$ be the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\!\!\!\!\!\!|}^\circ$ and let $A \subset_{fin} \mathcal{M}$. k cops have a winning strategy in the $^\circ$-Cops and Robbers game on A if and only if they have* *a coarse, monotone, marginal, winning strategy in the* $\circ$*-Cops and Robbers game on* A*.*

Proof. Clearly, if k cops have a coarse, monotone, marginal, stingy winning strategy in the $\circ$-Cops and Robbers game on A, then they have a winning strategy. So we need only concern ourselves with the other direction of the statement.

Suppose k cops have a winning strategy in the $\circ$-Cops and Robbers game on A. Then by Theorem 6.4.4, A has $\circ$-tree width less than k. It follows from Theorem 6.7.25 that k cops have a coarse, monotone, marginal, winning strategy for the $\circ$-Cops and Robbers game on A. □

Chapter 7

Menger's Theorem

The proof that Seymour and Thomas use to show that k cops can catch the robber on the graph G if and only if G has a tree decomposition of width less than k uses Menger's Theorem, a classic result of graph theory. We did not even attempt to imitate Seymour and Thomas's proof because initially it was not clear that certain techniques in their proof which depend heavily on graph theoretic ideas such as paths and edges could be reasonably translated to our more general setting.

For instance, Menger's Theorem is as follows:

Theorem 7.0.1 (Menger's Theorem). *Let G be a graph, $X, Y \subseteq V(G)$. The maximum number of disjoint X,Y-paths is equal to the minimum size of a set $C \subseteq V(G)$ which separates X from Y.*

Here, an X, Y-path is an x, y-path such that $x \in X$, $y \in Y$ and x is the only vertex belonging to X on the path, y is the only vertex belonging to Y on the path. A set $C \subseteq V(G)$ separates X from Y if every X, Y-path must contain some point of C. Proofs of Menger's Theorem can be found in both [2] and [7].

In light of our findings in Section 6.2 that there *is* a natural generalization of paths, it became reasonable to wonder whether there might be an analogue for Menger's Theorem that holds on the finite substructures of the generic model of an algebraically trivial Fraïssé class with an abstract free amalgamation relation. The answer, as we will see, is yes.

For the rest of this chapter, assume that $\mathcal{M}$ is the generic model of an algebraically trivial Fraïssé class with abstract free amalgamation relation $\mathop{\smile\kern-0.55em\raise0.9ex\hbox{$\scriptstyle|$}}^{\circ}$ and $A \subset_{\text{fin}} M$. We define an X, Y-$^\circ$path in the expected way:

Definition 7.0.2. Let $X, Y \subseteq A$. An X, Y-$^\circ$path in A, P, is an x, y-$^\circ$path in A such that $x \in X$, $y \in Y$, and $P \cap X = \{x\}$, $P \cap Y = \{y\}$.

The appropriate definition of what it means for a set C to separate X from Y also seems fairly intuitive. If every $^\circ$path from $x \in X$ to $y \in Y$ must contain a point of C, then x and y cannot be in the same component of A over C by Lemma 6.2.2. On the other hand, if x and y are in different components of A over C (or at least one of x or y is in C), then any x, y-$^\circ$paths in A must contain a point of C, by the following argument.

Suppose not. Let P be an x,y-$^\circ$path such that $P \cap C = \emptyset$. Let $\mathcal{D}^\circ(A/C) = \{H_0, ..., H_n\}$ and assume without loss of generality that $x \in H_0$, $y \in H_n$. Then $H_0 \mathop{\smile\!\!\!\!\!\!\downarrow}^\circ_C \bigcup_{i=1}^n H_i$. Applying transitivity twice, we get

$$P \cap H_0 \mathop{\smile\!\!\!\!\!\!\downarrow}^\circ_{C \cup (H_0 \setminus P) \cup \left(\bigcup_{i=1}^n H_i \setminus P\right)} P \cap \bigcup_{i=1}^n H_i.$$

But since $C \cap P = \emptyset$, the base above is really $A \setminus P$ and $\{P \cap H_0, P \cap \bigcup_{i=1}^n H_i\}$ is a bipartition of P, a contradiction of the P being an x,y-$^\circ$path.

This gives rise to the following definition:

Definition 7.0.3. Let $X, Y \subseteq A$. Let $C \subseteq A$ and $\mathcal{D}^\circ(A/C) = \{H_0, ..., H_{n-1}\}$. C *$^\circ$-separates* X from Y in A if there exists $I \subseteq [n]$ such that $X \setminus C \subseteq \bigcup_{i \in I} H_i$, $Y \setminus C \subseteq \bigcup_{i \in [n] \setminus I} H_i$.

We can now state and prove a generalization of Menger's Theorem.

Theorem 7.0.4. *The maximum number of disjoint X, Y-$^\circ$paths in A is equal to the minimum size of a set $C \subseteq A$ which $^\circ$-separates X from Y.*

Proof. As is the case with graphs, it is easy to show that the minimum size of a separating set must be greater than or equal to the maximum size of a set of disjoint X, Y-$^\circ$paths. We have already argued above that C $^\circ$-separates X from Y if and only if every X, Y-$^\circ$path must contain a point of C, so, by the pigeonhole principle, we cannot have a set of disjoint X, Y-$^\circ$paths of size greater than $|C|$ for any $^\circ$-separating set C. Thus the minimum size of a set which $^\circ$-separates X

from Y must be greater than or equal to the maximum size of a set of disjoint X, Y-$^\circ$paths.

It remains to show that the other direction also holds, that the maximum size of a set of disjoint X, Y-$^\circ$paths must be greater than or equal to the minimum size of a $^\circ$-separating set. Our proof will closely follow one of the proofs given by Diestel in [2].

Given A, for $X, Y \subseteq A$, denote the minimum size of a set $^\circ$-separating X from Y by $k(A, X, Y)$. We want to show that we can always find a set of disjoint X, Y-$^\circ$paths of size $k(A, X, Y)$. Note that in the case where $k(A, X, Y) = 0$ there is nothing to prove since the empty set will suffice, so we may assume that $k(A, X, Y) \geq 1$.

We begin by making a definition:

Definition 7.0.5. If $\mathscr{P}$ and $\mathscr{Q}$ are each sets of disjoint X, Y-$^\circ$paths in A, then $\mathscr{Q}$ *exceeds* $\mathscr{P}$ if $X \cap \left(\bigcup_{P \in \mathscr{P}} P\right) \subsetneq X \cap \left(\bigcup_{Q \in \mathscr{Q}} Q\right)$ and $Y \cap \left(\bigcup_{P \in \mathscr{P}} P\right) \subsetneq Y \cap \left(\bigcup_{Q \in \mathscr{Q}} Q\right)$.

That is, $\mathscr{Q}$ exceeds $\mathscr{P}$ if for each endpoint of a $^\circ$path in $\mathscr{P}$ there is a $^\circ$path in $\mathscr{Q}$ with that endpoint and $\mathscr{Q}$ also contains some $^\circ$path with an X endpoint which isn't the endpoint of any $^\circ$path in $\mathscr{P}$ and some $^\circ$path with a Y endpoint which isn't the endpoint of any $^\circ$path in $\mathscr{P}$. Note that since no path in $\mathscr{Q}$ can contain more than one element of X (or Y), $|\mathscr{Q}| > |\mathscr{P}|$.

What we want to show is that so long as we have a set of fewer than $k(A,X,Y)$ disjoint X,Y-$^\circ$paths, we can find another set of disjoint X,Y-$^\circ$paths that exceeds it. It follows that we must be able to find a set of at least $k(A,X,Y)$ disjoint X,Y-$^\circ$paths.

Claim. If $\mathscr{P}$ is any set of n disjoint X,Y-$^\circ$paths in A where $0 \le n < k(A,X,Y)$, then there exists a set $\mathscr{Q}$ of $n+1$ disjoint X,Y-$^\circ$paths in A which exceeds $\mathscr{P}$.

Fix A and X and let Y vary. We will proceed by induction on $|\bigcup_{P\in\mathscr{P}} P|$.

Base case: If $|\bigcup_{P\in\mathscr{P}} P| = 0$, then $\mathscr{P} = \emptyset$ and $n = 0$. Since we have assumed $k(A,X,Y) > n$, we must have $k(A,X,Y) \ge 1$, so $\emptyset$ does not $^\circ$-separate X from Y. Therefore, there must be some $x \in X$, $y \in Y$ such that x and y are in the same $^\circ$-component of A over $\emptyset$. Then by Lemma 6.2.2, there must be an x,y-$^\circ$path Q in A. Setting $\mathscr{Q} = \{Q\}$ gives us the desired set of disjoint X,Y-$^\circ$paths exceeding $\mathscr{P}$.

Inductive step: Assume the statement holds for all A,X,Y and all $0 \le n < k(A,X,Y)$ so long as $0 \le |\bigcup_{P\in\mathscr{P}} P| < m$. We must show that it also holds when $|\bigcup_{P\in\mathscr{P}} P| = m$.

Let Y be given and let $\mathscr{P}$ be a set of n disjoint X,Y-$^\circ$paths in A with $n < k(A,X,Y)$ and $|\bigcup_{P\in\mathscr{P}} P| = m$. We must produce a set $\mathscr{Q}$ of $n+1$ disjoint X,Y-$^\circ$paths in A which exceeds $\mathscr{P}$.

Since $n < k(A,X,Y)$ and $|\left(\bigcup_{P\in\mathscr{P}} P\right)\cap Y| = n$, $\left(\bigcup_{P\in\mathscr{P}} P\right)\cap Y$ (the set of Y

endpoints of the °paths in $\mathscr{P}$) does not °-separate X from Y. Thus, there exist $x \in X$, $y \in Y \setminus \left(\bigcup_{P\in\mathscr{P}} P\right) \cap Y$ which are in the same °-component of A over $\left(\bigcup_{P\in\mathscr{P}} P\right) \cap Y$. By Lemma 6.2.2, then, there exists an x, y-°path in A, R, with $R \cap \left(\left(\bigcup_{P\in\mathscr{P}} P\right) \cap Y\right) = \emptyset$. If $R \cap \left(\bigcup_{P\in\mathscr{P}} P\right) = \emptyset$, (i.e., if R is disjoint from all of the °paths in $\mathscr{P}$), then setting $\mathscr{Q} = \mathscr{P} \cup \{R\}$ gives us the desired set of $n+1$ disjoint X, Y-°paths exceeding $\mathscr{P}$.

Otherwise, consider the set $R \cap \left(\bigcup_{P\in\mathscr{P}} P\right)$. By Lemma 6.2.3 (and symmetry), for each $z \in R \cap \left(\bigcup_{P\in\mathscr{P}} P\right)$, we can find R_z, a z, y-°path in A which is a subset of R. By applying Lemma 6.2.4 repeatedly if necessary, there must be some $z \in R \cap \left(\bigcup_{P\in\mathscr{P}} P\right)$ such that $R_z \cap \left(\bigcup_{P\in\mathscr{P}} P\right) = \{z\}$. Denote by P_0 the unique element of $\mathscr{P}$ such that $z \in P_0$.

For some $x_0 \in X$, $y_0 \in Y$, P_0 is an x_0, y_0-°path in A. By Lemma 6.2.4 (and symmetry), since $z \neq y_0$, there exists $\tilde{P}_0 \subsetneq P_0$, an x_0, z-°path in A. Also, by Lemma 6.2.3, there exists $\hat{P}_0 \subseteq P_0$, a z, y_0-°path in A.

Let $Y' = Y \cup \hat{P}_0 \cup R_z$ and let $\mathscr{P}' = (\mathscr{P} \setminus \{P_0\}) \cup \{\tilde{P}_0\}$. Since for each $P \in \mathscr{P} \setminus \{P_0\}$ $P \cap \left(\hat{P}_0 \cup R_z\right) = \emptyset$ and $\tilde{P}_0 \cap Y' = \{z\}$, $\mathscr{P}'$ is a set of n disjoint X, Y'-°paths in A. Since $Y \subset Y'$, we must have $n < k(A, X, Y) \leq k(A, X, Y')$. Note that since $\tilde{P}_0 \subsetneq P_0$, $|\tilde{P}_0| < |P_0|$ and, hence, $|\bigcup_{P\in\mathscr{P}'} P| < |\bigcup_{P\in\mathscr{P}} P| = m$. So we may apply the inductive hypothesis to get $\mathscr{Q}'$, a set of $n+1$ disjoint X, Y'-°paths in A which exceeds $\mathscr{P}'$.

Since $\mathscr{Q}'$ exceeds $\mathscr{P}'$, there exists $Q \in \mathscr{Q}'$ such that Q is an x_1, z-°path in A for some $x_1 \in X$. There also exists $Q' \in \mathscr{Q}'$ such that Q' is an x', y'-°path in A for some $x' \in X$, $y' \in Y'$ such that $y' \notin \left(\bigcup_{P \in \mathscr{P}'} P\right) \cap Y'$. (In particular, $Q \neq Q'$ and $y' \neq z$.)

We consider three cases, based on where y' is in $Y' = Y \cup \hat{P}_0 \cup R_z$.

- If $y' \in Y$, then no path in $\mathscr{Q}'$ intersects $\hat{P}_0 \setminus \{z\}$ and only Q contains z. By Lemma 6.2.5, there exists $Q_0 \subseteq Q \cup \hat{P}_0$, an x_1, y_0-°path in A. Q_0 must then be an X, Y-°path in A disjoint from all elements of $\mathscr{Q}' \setminus \{Q\}$. Since $\mathscr{Q}'$ exceeds $\mathscr{P}'$ and $y' \in Y$, all °paths in $\mathscr{Q}'$ other than Q must be X, Y-°paths in A. Thus, letting $\mathscr{Q} = (\mathscr{Q}' \setminus \{Q\}) \cup \{Q_0\}$ gives us a set of $n+1$ disjoint X, Y-°paths in A.

  Note that we have replaced an x_1, z-°path in A with an x_1, y_0-°path in A, so
$$\left(\bigcup_{P \in \mathscr{Q}'} P\right) = X \cap \left(\bigcup_{P \in \mathscr{Q}} P\right)$$
  and, since $z \notin Y$,
$$\left(Y \cap \left(\bigcup_{P \in \mathscr{Q}'} P\right)\right) \cup \{y_0\} = Y \cap \left(\bigcup_{P \in \mathscr{Q}} P\right).$$
  Since $\mathscr{Q}'$ exceeds $\mathscr{P}'$, by the definition of $\mathscr{P}'$ we have
$$X \cap \left(\bigcup_{P \in \mathscr{P}} P\right) = X \cap \left(\bigcup_{P \in \mathscr{P}'} P\right) \subsetneq X \cap \left(\bigcup_{P \in \mathscr{Q}'} P\right) = X \cap \left(\bigcup_{P \in \mathscr{Q}} P\right)$$

and, since $y' \in Y$, we have

$$Y \cap \left(\bigcup_{P \in \mathscr{P}} P \right) = \left(Y \cap \left(\bigcup_{P \in \mathscr{P}'} P \right) \right) \cup \{y_0\}$$
$$\subsetneq \left(Y \cap \left(\bigcup_{P \in \mathscr{Q}'} P \right) \right) \cup \{y_0\} = Y \cap \left(\bigcup_{P \in \mathscr{Q}} P \right),$$

so $\mathscr{Q}$ exceeds $\mathscr{P}$.

- If $y' \in R_z$, then no element of $\mathscr{Q}'$ intersects $\hat{P}_0 \setminus \{z\}$, Q' is the only one which intersects $R_z \setminus \{z\}$, and only Q contains z. Since $y' \in R_z \setminus \{z\}$, by Lemma 6.2.4 (and symmetry), there exists $R' \subseteq R_z \setminus \{z\}$, a y', y-°path in A. By Lemma 6.2.5, there exist $\hat{Q} \subseteq Q \cup \hat{P}_0$, an x_1, y_0-°path in A, and $\tilde{Q} \subseteq Q' \cup R'$, an x', y-°path in A. $\hat{Q}$ and $\tilde{Q}$ are then both X, Y-°paths in A which are disjoint from all elements of $\mathscr{Q}' \setminus \{Q, Q'\}$ and from each other. So $\mathscr{Q} = (\mathscr{Q}' \setminus \{Q, Q'\}) \cup \{\hat{Q}, \tilde{Q}\}$ is a set of $n+1$ disjoint X, Y-°paths in A.

  In this case, we are replacing x_1, z- and x', y'-°paths in A with x_1, y_0- and x', y-°paths in A. Thus

$$\left(\bigcup_{P \in \mathscr{Q}'} P \right) = X \cap \left(\bigcup_{P \in \mathscr{Q}} P \right)$$

  and

$$\left(Y \cap \left(\bigcup_{P \in \mathscr{Q}'} P \right) \right) \cup \{y_0, y\} = Y \cap \left(\bigcup_{P \in \mathscr{Q}} P \right).$$

  Again, by the definition of $\mathscr{P}'$ and since $\mathscr{Q}'$ exceeds $\mathscr{P}'$,

$$X \cap \left(\bigcup_{P \in \mathscr{P}} P \right) = X \cap \left(\bigcup_{P \in \mathscr{P}'} P \right) \subsetneq X \cap \left(\bigcup_{P \in \mathscr{Q}'} P \right) = X \cap \left(\bigcup_{P \in \mathscr{Q}} P \right),$$

while

$$\begin{aligned} Y \cap \left(\bigcup_{P \in \mathscr{P}} P \right) &= \left(Y \cap \left(\bigcup_{P \in \mathscr{P}'} P \right) \right) \cup \{y_0\} \\ &\subsetneq \left(Y \cap \left(\bigcup_{P \in \mathscr{Q}'} P \right) \right) \cup \{y_0, y\} = Y \cap \left(\bigcup_{P \in \mathscr{Q}} P \right). \end{aligned}$$

So $\mathscr{Q}$ exceeds $\mathscr{P}$.

- Finally, if $y' \in \hat{P}_0$, then no element of $\mathscr{Q}'$ intersects $R_z \setminus \{z\}$, Q' is the only one which intersects $\hat{P}_0 \setminus \{z\}$, and only Q contains z. Since $y' \in \hat{P}_0 \setminus \{z\}$, by Lemma 6.2.4 (and symmetry), there exists $P_0' \subseteq \hat{P}_0 \setminus \{z\}$, a y', y_0-°path in A. Then by Lemma 6.2.5, there exist $Q_0 \subseteq Q' \cup P_0'$, an x', y_0-°path in A, and $Q_1 \subseteq Q \cup R_z$, an x_1, y-°path in A. Q_0 and Q_1 are both X, Y-°paths in A; they are disjoint from each other and from all elements of $\mathscr{Q}' \setminus \{Q, Q'\}$. So $\mathscr{Q} = (\mathscr{Q}' \setminus \{Q, Q'\}) \cup \{Q_0, Q_1\}$ is a set of $n + 1$ disjoint X, Y-°paths in A.

This time we are replacing x_1, z- and x', y'-°paths in A with x', y_0- and x_1, y-°paths in A. As in the previous case, this gives us

$$\left(\bigcup_{P \in \mathscr{Q}'} P \right) = X \cap \left(\bigcup_{P \in \mathscr{Q}} P \right)$$

and

$$\left(Y \cap \left(\bigcup_{P \in \mathscr{Q}'} P \right) \right) \cup \{y_0, y\} = Y \cap \left(\bigcup_{P \in \mathscr{Q}} P \right).$$

And again we have that

$$X \cap \left(\bigcup_{P \in \mathscr{P}} P \right) = X \cap \left(\bigcup_{P \in \mathscr{P}'} P \right) \subsetneq X \cap \left(\bigcup_{P \in \mathscr{Q}'} P \right) = X \cap \left(\bigcup_{P \in \mathscr{Q}} P \right),$$

and

$$\begin{aligned} Y \cap \left(\bigcup_{P \in \mathscr{P}} P \right) &= \left(Y \cap \left(\bigcup_{P \in \mathscr{P}'} P \right) \right) \cup \{y_0\} \\ &\subsetneq \left(Y \cap \left(\bigcup_{P \in \mathscr{Q}'} P \right) \right) \cup \{y_0, y\} = Y \cap \left(\bigcup_{P \in \mathscr{Q}} P \right), \end{aligned}$$

so $\mathscr{Q}$ exceeds $\mathscr{P}$.

Since in each case we have been able to produce a set $\mathscr{Q}$ of $n+1$ disjoint X, Y-°paths which exceeds $\mathscr{P}$, the induction is complete. □

Chapter 8

Conclusion

The goal of this thesis was to generalize the ideas of tree decomposition and the Cops and Robbers game from graphs to the finite substructures of the generic model of an algebraically trivial Fraïssé class with an abstract free amalgamation relation. We have seen that having an abstract free amalgamation relation allows us to define components in our structure which, in turn, allows us to generalize the definitions of tree decomposition and the Cops and Robbers game. Extending the idea that two elements should be in the same component if and only if there is a path between them, we were able to define paths and edges for our structure and create an associated abstract free amalgamation graph. Finally, we were able to show that playing the generalized version of the Cops and Robbers game on a structure is the same as playing on the associated graph and that generalized tree decompositions of the structure are the same as tree decompositions of the associated graph. This allowed us to prove the analogue of Seymour and Thomas's theorem for finite substructures of the generic model of an algebraically trivial Fraïssé class with an abstract free amalgamation relation by utilizing the original result on graphs.

There are a number of questions that naturally arise from this work. First, as previously mentioned, Courcelle's theorem says that for a class of graphs with bounded tree width, monadic second order properties can be computed in linear time. A significant motivating factor for attempting to extend the definition of tree width to algebraically trivial Fraïssé classes is the hope that doing so would in turn extend this benefit. This in fact seems extremely likely, since the tree decompositions for an element of an algebraically trivial Fraïssé class are just the tree decompositions of the associated abstract free amalgamation graph, but there is still work to be done here.

Several open questions are related to the later sections of Chapter 6. We first introduce alt $^\circ$-tree decompositions in Section 5.1 as a possible generalization of tree decomposition. This definition is more natural than our ultimate definition of $^\circ$-tree decomposition and it would therefore be interesting to know whether the two definitions are actually equivalent. We discuss this issue in Section 6.5, but do not reach any conclusion.

In the following section, we explore the relationship between the Gaifman graph of a structure and its abstract free amalgamation graph. We see that each our main examples of an algebraically trivial Fraïssé class can be described in a language such that the which makes the Gaifman graph identical to the abstract free amalgamation graph. Can this always be done or are there structures for which the abstract free amalgamation graph will not be the same as the Gaifman graph, regardless of the language used to describe the structure.

For each of the previous two questions, it would be helpful to find further examples of algebraically trivial Fraïssé classes with abstract free amalgamation relations.

Another question is whether it is in fact possible to prove the analogue of the Seymour and Thomas theorem without needing to be able to define °paths in our structures. We made some progress on this as shown in Section 6.7, where we tried to use "nice" cop strategies. While this wouldn't give any new results directly, having an alternate proof which doesn't require us to be able to associate a graph with our structure might be an indication that we could further extend the ideas of tree decomposition and the cops and robbers games to other, less graph-like structures.

This raises the question: to what other structures might we be able to extend these concepts? Fraïssé classes are just one of several types of amalgamation classes of structures. It will be interesting to consider whether we can further generalize tree decomposition and cops and robbers to elements of other amalgamation classes.

If not, that may be because these concepts only work on structures that are very graph-like in some way. As noted above, we can associate an abstract free amalgamation graph with each element of an algebraically trivial Fraïssé class with an abstract free amalgamation relation. Additionally, and without using the equivalent result on the abstract free amalgamation graph, we were able to show that there is an analogue of Menger's Theorem which holds on an element of an algebraically trivial Fraïssé class with an abstract free amalgamation relation. How similar to graphs does that mean these structures are? What other results from graph theory can we extend to these structures?

Bibliography

[1] Gabriel Conant. An axiomatic approach to free amalgamation. *J. Symbolic Logic*, 82(2):648–671, 2017.

[2] Reinhard Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg; New York, fourth edition, 2010.

[3] Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006.

[4] Wilfrid Hodges. *A shorter model theory*. Cambridge University Press, Cambridge, 1997.

[5] David Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. An introduction.

[6] Paul D. Seymour and Robin Thomas. Graph searching and a min-max theorem for tree-width. *J. Comb. Theory, Ser. B*, 58(1):22–33, 1993.

[7] Douglas B. West. *Introduction to Graph Theory*. Prentice Hall, 2 edition, September 2000.