Exploiting Wave Transport in $\mathcal{PT}$-Symmetric Media

by

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Abstract

This thesis will present an integrated experimental and theoretical effort toward managing wave transport and introduce novel methods for signal manipulation. To this end, we will develop a new circuitry design that exploits both absorption and gain mechanisms via judicious spatial arrangements of active elements that exhibit space-time reflection, or $\mathcal{PT}$ symmetries. We will show that the resulting structures display intriguing behavior that holds promises for the creation of new synthetic matter with novel functionalities. The playing field will be photonic and electronic circuits with complex potentials, corresponding to complex index of refraction and active impedances, respectively.
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Chapter 1

Introduction

While there is absolutely no doubt as to the usefulness of gain mechanisms in boosting signals and transmitting information, loss, on the other hand, is typically considered an evil — one to be avoided at all if possible — since it degrades the efficiency of the structures employed to perform useful operations on these signals. It is perhaps for this particular reason that researchers have never intentionally explored the combination of gain and loss as a duality of useful ingredients in device and materials engineering.

However, an alternate viewpoint is currently emerging [2] aiming to manipulate absorption and, via a judicious design that involves the combination of delicately balanced amplification and absorption mechanisms, achieve new classes of synthetic structures with altogether new physical behavior and novel functionality. This idea can potentially have a vast range of applicability to wave systems, ranging from acoustic and optical structured materials, to antenna arrays. In fact, optical media with delicately balanced gain and loss, characteristic of systems with joint parity-time ($\mathcal{PT}$) symmetry, have been reported [3], showing intriguing functionality. In electronics, it has been demonstrated that a pair of coupled LRC circuits with active elements, one with amplification and the other with equivalent amount of attenuation, provide an experimental realization of a wide class of systems where gain/loss mechanisms break the Hermiticity while
preserving parity-time $\mathcal{PT}$-symmetry. This class of non-Hermitian Hamiltonians, originally proposed by Bender and colleagues \cite{4-6}, has been used in recent years in order to theoretically describe a wide variety of complex systems. This thesis constitutes one of the first attempts to investigate the scattering properties of $\mathcal{PT}$-symmetric systems in a consistent manner, and aims to develop fundamental concepts that will promote a new era of synthetic matter.

The thesis consists of two parts. The first part, Chapters 2 and 3, will review the theoretical foundations of $\mathcal{PT}$-symmetries and discuss the basic principles of $\mathcal{PT}$-symmetric scattering. The second part, Chapters 4, 5 and 6, present original work \cite{7-9} on $\mathcal{PT}$-symmetric scattering in optical and electronic systems. We have chosen to interlay the two frameworks (i.e. optics and electronics) following a presentation pattern which goes from simple examples and realizations of $\mathcal{PT}$-symmetric systems to the most complicated structures.

In chapter 2, we will lay down the theoretical groundwork for a general scattering formalism and introduce transfer and scattering matrices. We derive conservation relations for Hermitian systems in terms of transmittances and reflectances, and show that left/right transmission/reflection is always reciprocal in linear non-magnetic media. We then introduce the main theoretical model that we will use in the optical framework presented in this thesis — a multi-layered optical slab. Furthermore, we present the mathematical formalism that is needed to understand transport of light in such structures.

In chapter 3, we will review the standard theory of $\mathcal{PT}$ symmetry and introduce the associated scattering formalism. Generalized conservation relations which reflect non-unitary nature of scattering processes are derived. We will also introduce and distinguish between the notions of exact and broken phases in $\mathcal{PT}$-symmetric scattering. An intriguing property of $\mathcal{PT}$ scattering is the existence of special frequencies where both lasing and absorption can occur \cite{10, 11}. We will give a detailed review of this phenomenon, and demonstrate it with a simple theoretical exercise using a double-layered
slab.

In chapter 4, we will extend the notion of $\mathcal{PT}$ symmetry into electronic circuits. Specifically, we will report our experimental work on $\mathcal{PT}$-symmetric scattering in a simple circuitry — a pair of coupled LC circuits, one having an amplifier and the other having a resistor. Our measurements confirm the generalized $\mathcal{PT}$ conservation relations discussed in the previous chapter and identify the phenomenon of unidirectional transparency, first predicted in Ref. [8].

In chapter 5, we will move onto a more complex structure in optics framework — a $\mathcal{PT}$-symmetric Bragg grating [8]. We use coupled mode approximation to calculate the transmission and reflection and find that, under special conditions, the structure becomes unidirectionally transparent, that is, the system has perfect transmission while reflection from one side becomes zero. Additionally, we find that the transmission phase is insensitive to the Bragg grating. Finally, we show that the above phenomena persist in the presence of Kerr nonlinearity.

In chapter 6, we will study the transport properties in optical $\mathcal{PT}$-symmetric media with random index of refraction. In particular, we develop a scaling theory for the localization length of such structures which allows us to write in terms of the localization length of an equivalent random Hermitian system (i.e. without gain or loss) and the so-called amplification/attenuation length (which characterizes an equivalent system with pure gain and loss but without disorder). A similar scaling theory is developed for the left/right reflectances. A peculiarity of our systems is that, at the limit of strong disorder/large system sizes, the reflectance is enhanced from one side and is inversely suppressed from the other, thus allowing such $\mathcal{PT}$-symmetric random media to act as unidirectional coherent absorbers.

Finally, we will give our conclusions and an outlook in the last chapter of the thesis.
Chapter 2

Wave Scattering in
One-dimensional Media

We explore and control our environment and stay in touch with one another via waves. In addition to quantum (electronic) and matter waves, we rely increasingly on classical electromagnetic and sound waves for communication and imaging. The shared wave character of classical waves and electrons is behind the rapid growth of the photonics industry which holds promise for continued device miniaturization and increased speed. In this context, the theory of wave scattering provides a general framework for investigating many interesting and important problems of physics.

In this chapter, we will discuss the essential theoretical groundwork for the general scattering formalism in one dimension. In linear media, which will be the main concern of this section, the Fourier decomposition makes it trivial to evolve the waves in time once we know the spatial modulation of each Fourier component. Thus, it is sufficient to consider the so-called time-independent scattering problem whose solution, as will be shown, is succinctly contained in the transfer matrix, which describes the wave propagation in mode space, from one end of the scattering domain to the other. An equivalent
formulation relies on the scattering matrix which relates incoming with outgoing waves. From these matrices, the general properties of wave transport in linear media, such as flux conservation and reciprocity of transmission, can be readily deduced.

### 2.1 General Considerations

Consider a one-dimensional optical medium of infinite length, where the $z$-axis is defined to lie along the length of the medium. We are primarily concerned with propagation of electromagnetic waves through a finite segment $L$ of inhomogeneous scattering element, located within $|z| \leq L/2$, Fig. 2.1. Although for the sake of simplicity, we mainly consider optical (electromagnetic) waves, it should be emphasized that the general results derived are applicable to other types of waves as well, including quantum probability waves and acoustic waves.

We assume the electric field $\vec{E}$ to be linearly polarized in a direction perpendicular to $z$, say $y$, as well as traveling inside a purely dielectric (non-magnetic) material, in which case the vectorial Maxwell’s equations reduce to one-dimensional Helmholtz equation

$$
\frac{\partial^2 E_y(z,t)}{\partial z^2} = \frac{n(z)^2}{c^2} \frac{\partial^2 E_y(z,t)}{\partial t^2}.
$$

(2.1)

Here, $n(z)$ is the index of refraction of the optical medium, $c$ is the speed of light in vacuum, $E_y$ is the $y$-component of $\vec{E}$ and $E_x = E_z = 0$. For the scattering-type medium we are considering, the optical refractive index takes on a constant value $n_0$ for $z \geq L/2$ (homogeneous leads). Without loss of generality, we can take $n_0$ to be that of air or free space, that is, $n_0 = 1$. For $|z| \leq L/2$, $n(z)$ is an inhomogeneous function of position which can be entirely real (energy or flux conserving) or complex-valued (dissipative/amplifying). A Fourier decomposition of $E_y(z,t)$ gives

$$
E_y(z,t) = \int \frac{dk}{2\pi} \phi(k) E(z;k)e^{-i\omega_k t},
$$

(2.2)
where $E(z; k)$ is a stationary scattering state of the time-independent Helmholtz equation

$$\frac{d^2 E(z; k)}{dz^2} + \frac{n(z)^2 \omega_k^2}{c^2} E(z; k) = 0$$

(2.3)

with the corresponding dispersion relation given by $\omega_k = c k$. $k$ is identified as the free-space wave vector associated with the frequency $\omega_k$.

Meanwhile, $E(z; k)$ is further decomposed into forward and backward-traveling components inside the homogeneous leads

$$E(z; k) = \begin{cases} E_f^{-} e^{ikz} + E_b^{-} e^{-ikz} & \text{for } z \leq -L/2 \\ C_1 f(z; k) + C_2 g(z; k) & \text{for } |z| \leq L/2 \\ E_f^{+} e^{ikz} + E_b^{+} e^{-ikz} & \text{for } z \geq L/2 \end{cases}$$

(2.4)

while $f(z; k)$ and $g(z; k)$ are two independent solutions to Eq. 2.3 whose linear combination establishes the stationary field within the inhomogeneous scattering region at the chosen frequency $\omega_k$ (or equivalently, the free-space wave vector $k$). The actual field is determined by the specific nature of $n(z)$ and appropriate boundary conditions at $|z| = L/2$. In principle, if $n(z)$ and the boundary conditions are specified, $E(z; k)$ can be completely solved using numerical or analytical methods. But, more importantly, we
are interested in the transmission and reflection through the scattering system. For this purpose, it is more convenient to introduce the transfer matrix $M(k)$ which connects the field amplitudes on the left and right ends of the sample. Noting that $E(z; k)$ and its first derivative $E'(z; k)$ are continuous at $z = -\frac{L}{2}$ or $\frac{L}{2}$, we obtain a system of linear equations

\[
\begin{align*}
E_f^- e^{-ikL/2} + E_b^- e^{ikL/2} &= C_1f(-L/2; k) + C_2g(-L/2; k) \\
ike_f^- e^{-ikL/2} - ike_b^- e^{ikL/2} &= C_1f'(-L/2; k) + C_2g'(-L/2; k) \\
E_f^+ e^{ikL/2} + E_b^+ e^{-ikL/2} &= C_1f(L/2; k) + C_2g(L/2; k) \\
ike_f^+ e^{ikL/2} - ike_b^+ e^{-ikL/2} &= C_1f'(L/2; k) + C_2g'(L/2; k).
\end{align*}
\]

Direct elimination of the constants $C_1$ and $C_2$, allow us to rewrite the left/right field amplitudes in a matrix form

\[
\begin{pmatrix}
E_f^+ \\
E_b^+
\end{pmatrix}
= M(k)
\begin{pmatrix}
E_f^- \\
E_b^-
\end{pmatrix},
\tag{2.5}
\]

where

\[
M(k) = \begin{pmatrix}
e^{ikL_2} & e^{-ikL_2} \\
ike e^{ikL_2} & -ike^{-ikL_2}
\end{pmatrix}^{-1}
\begin{pmatrix}
f(L_2; k) & g(L_2; k) \\
f'(L_2; k) & g'(L_2; k)
\end{pmatrix}
\begin{pmatrix}
f(-L_2; k) & g(-L_2; k) \\
f'(-L_2; k) & g'(-L_2; k)
\end{pmatrix}^{-1}
\begin{pmatrix}
e^{-ikL_2} & e^{ikL_2} \\
ike e^{-ikL_2} & -ike^{ikL_2}
\end{pmatrix}.
\]

Here, we have used the basis vectors \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) for $e^{ikz}$, \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) for $e^{-ikz}$, so that, in general, $E_f e^{ikz} + E_b e^{-ikz}$ is represented by the amplitude vector \( \begin{pmatrix} E_f \\ E_b \end{pmatrix} \).

Now we can readily express transmission and reflection in terms of the matrix elements of $M(k)$. For a monochromatic wave incident on the left side of the scattering region, $E_b^+ = 0$ and complex transmission and reflection amplitudes are defined as $t_L = \frac{E_f^+}{E_f}$, $r_L = \frac{E_b}{E_f}$. The real-valued transmittance (or transmission coefficient) and reflectance (or reflection coefficient) are the absolute squares of the complex quantities, that is, $T_L = |t_L|^2$, $R_L = |r_L|^2$. Substituting $E_b^+ = 0$ into Eq. 2.5 and solving for $t_L$ and $r_L$, we
obtain \[13, 14\]

\[ t_L = \frac{\det(M)}{M_{22}} \quad \text{(2.6a)} \]
\[ r_L = -\frac{M_{21}}{M_{22}} . \quad \text{(2.6b)} \]

Likewise for the right-incident wave, we define the corresponding transmission and reflection as \( t_R = \frac{E_f^-}{E_b^-} \), \( r_R = \frac{E_f^+}{E_b^+} \). Substituting \( E_f^- = 0 \) for the right-incident boundary condition, we get

\[ t_R = \frac{1}{M_{22}} \quad \text{(2.7a)} \]
\[ r_R = \frac{M_{12}}{M_{22}} . \quad \text{(2.7b)} \]

An alternative formulation is based on the so-called \( S \)-Matrix which, more intuitively, acts on the input channels to give the amplitudes in the outgoing modes.

\[
\begin{pmatrix}
E_f^+ \\
E_b^-
\end{pmatrix} = S(k) \begin{pmatrix}
E_f^- \\
E_b^+
\end{pmatrix} . \quad \text{(2.8)}
\]

For left incident waves, we can substitute \( E_b^+ = 0 \), \( E_f^+ = t_LE_f^- \) and \( E_b^- = r_LE_f^- \), yielding \((S)_{11} = t_L \) and \((S)_{21} = r_L \). In a similar vein, one can easily deduce using right incident boundary conditions, that \((S)_{12} = r_R \) and \((S)_{22} = t_R \) so that the overall \( S \)-matrix is given by

\[
S(k) = \begin{pmatrix}
t_L & r_R \\
r_L & t_R
\end{pmatrix} . \quad \text{(2.9)}
\]

Each of the above formulations for the transport properties of complex structures has its own advantages and deficiencies. For example, while it is much easier to calculate the transfer matrix, due to its multiplicity properties (see below), the \( S \)-matrix formulation, allows for a better theoretical understanding of transport. In the following chapters we will be frequently switching between the use of the two matrices, depending on the needs and specific problems.
2.2 Conservation Relations

If the optical scattering medium has a real refractive index, that is, without dissipation or amplification, energy is conserved. At steady state, the electromagnetic flux that goes into the scattering region must be either reflected or transmitted. In other words, one expects that

\[ T + R = 1 \tag{2.10} \]

Since \( n(z) \) is real-valued, \( n(z) = n^*(z) \). Therefore, if \( E(z; k) \) (see Eq. 2.4) is a solution to the Helmholtz equation (Eq. 2.3), \( E^*(z; k) \) is also another solution with the identical wave vector \( k \).

\[
E^*(z; k) = E_{b}^{-*} e^{ikz} + E_{f}^{-*} e^{-ikz} \quad \text{for } z \leq -L/2 \\
= E_{b}^{+*} e^{ikz} + E_{f}^{+*} e^{-ikz} \quad \text{for } z \geq L/2 .
\]

Since the system is linear, the same transfer matrix \( M(k) \) in Eq. 2.5 connects the left and right amplitudes of \( E^*(z; k) \), that is,

\[
\begin{pmatrix}
E_{b}^{+*} \\
E_{f}^{+*}
\end{pmatrix} = M(k) \begin{pmatrix}
E_{b}^{-*} \\
E_{f}^{-*}
\end{pmatrix} .
\]

Taking the complex conjugate and multiplying with \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) to switch the rows,

\[
\begin{pmatrix}
E_{f}^{+} \\
E_{b}^{+}
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^*(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix}
E_{f}^{-} \\
E_{b}^{-}
\end{pmatrix} .
\]

Comparing with 2.5

\[
M(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^*(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \tag{2.11}
\]

From this, we can deduce

\[
M_{11} = M_{22}^* \\
M_{12} = M_{21}^* . \tag{2.12}
\]
On the other hand, one can show that the transfer matrix has a unit determinant. By Eq. 2.5,

\[
\det(M(k)) = \frac{f\left(\frac{L}{2}; k\right)g'\left(\frac{L}{2}; k\right) - g\left(\frac{L}{2}; k\right)f'\left(\frac{L}{2}; k\right)}{f\left(-\frac{L}{2}; k\right)g'\left(-\frac{L}{2}; k\right) - g\left(-\frac{L}{2}; k\right)f'\left(-\frac{L}{2}; k\right)}.
\]

The stationary Helmholtz equation ensures that the Wronskian is a constant function of position. Since both \( f \) and \( g \) are valid solutions,

\[
\begin{align*}
\frac{d^2 f(z; k)}{dz^2} + \frac{n(z)^2 \omega^2}{c^2} f(z; k) &= 0 \quad (2.13a) \\
\frac{d^2 g(z; k)}{dz^2} + \frac{n(z)^2 \omega^2}{c^2} g(z; k) &= 0. \quad (2.13b)
\end{align*}
\]

Multiplying \(2.13a\) by \( g \) and \(2.13b\) by \( f \) and subtracting, we have

\[
\begin{align*}
&f(z; k)\frac{d^2 g(z; k)}{dz^2} - g(z; k)\frac{d^2 f(z; k)}{dz^2} = 0 \\
&\frac{d}{dz} \left( f(z; k)g'(z; k) - g(z; k)f'(z; k) \right) = 0 \\
&f(z; k)g'(z; k) - g(z; k)f'(z; k) = \text{constant}.
\end{align*}
\]

Therefore,

\[
\det(M(k)) = 1. \quad (2.14)
\]

Combining Eq. 2.14 with the previous relations for the matrix elements of the M-matrix (Eq. 2.12) we get

\[
M_{11}M_{22} - M_{12}M_{21} = 1 \\
1 - \frac{|M_{21}|^2}{|M_{22}|^2} = \frac{1}{|M_{22}|^2} \\
1 - R_L = T_L
\]

which validate the conservation relation of Eq. 2.10 and \( T + R = 1 \). Furthermore, since \( \det(M) = 1 \), transmission is always reciprocal\(^1\) (\( t_L = t_R \), compare Eqs. 2.6, 2.7). Therefore, the above derivation is equally valid for left or right incident waves.

\(^1\)It should be emphasized that \( t_L = t_R \) always for linear dielectric media regardless of whether the system has active (gain/ loss) ingredients [8, 11, 13, 15].
2.3 Optical Many Layered Model

Now that we have laid down the general framework of the transfer matrix method, it is relevant to introduce a numerical model which we will employ throughout the text, see Fig. 2.2. Consider an optical slab consisting of \( L \) layers having refractive indices \( n_j \) where \( 1 \leq j \leq L \). Each layer \( j \) consists of a homogeneous material of width \( d \); accordingly, the electric field in the \( j^{th} \) layer is given by the superposition of forward and backward components \( C_j^1 e^{in_jkz} + C_j^2 e^{-in_jkz} \). Imposing the condition that the field and its derivative are continuous functions at each interface, as well as taking into consideration the phase accumulation in each layer, we get the following iteration relation connecting the fields at \( j^{th} \) and \((j-1)^{th}\) interfaces

\[
\begin{pmatrix}
E_j^f \\
E_j^b
\end{pmatrix} = Q(j) K(j,j-1) \begin{pmatrix}
E_{j-1}^f \\
E_{j-1}^b
\end{pmatrix},
\]

where

\[
K(j,j-1) = \frac{1}{2n_j} \begin{pmatrix}
n_j + n_{j-1} & n_j - n_{j-1} \\
n_j - n_{j-1} & n_j + n_{j-1}
\end{pmatrix}
\]

\[
Q(j) = \begin{pmatrix}
e^{ikn_jd} & 0 \\
0 & e^{-ikn_jd}
\end{pmatrix}.
\]
The amplitudes of the forward and backward propagating waves outside of the optical slab can now be related through the transfer matrix $M(k)$

$$
\begin{pmatrix}
E_f^+ \\
E_b^+
\end{pmatrix} = P_0 K(L + 1, L) \prod_{j=1}^{L} Q(j) K(j, j - 1) P_0 \begin{pmatrix}
E_f^- \\
E_b^{-}
\end{pmatrix},
$$

(2.15)

where

$$
P_0 = \begin{pmatrix}
e^{-ikL/2} & 0 \\
0 & e^{ikL/2}
\end{pmatrix}
$$

while we assume that the left and right homogeneous leads consist of the same material with refraction index $n_{L+1} = n_0$ which without loss of generality can be taken to be 1.
Chapter 3

$\mathcal{PT}$ Symmetry

Parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetries, as well as their breaking, belong to the most basic notions in physics. Recently there has been much interest in systems which do not obey $\mathcal{P}$ and $\mathcal{T}$ symmetries separately but do exhibit a combined $\mathcal{PT}$ symmetry. A $\mathcal{PT}$-symmetric system can be realized in optics by creating a medium with alternating regions of gain and loss, such that the (complex) refractive index satisfies the condition $n(z) = n^*(-z)$. This condition implies that creation and absorption of photons occur in a balanced manner, so that the net loss or gain is zero. To date, most of the studies on optical realizations of $\mathcal{PT}$ synthetic media have relied on the paraxial approximation which maps the scalar wave equation to the Schrödinger equation, with the axial wave vector playing the role of energy [2, 3]. This formal analogy allows one to investigate experimentally fundamental $\mathcal{PT}$-concepts that may impact several other areas, ranging from quantum field theory and mathematical physics, to solid state and atomic physics. Among the various themes that have fascinated researchers, is the existence of spontaneous $\mathcal{PT}$-symmetry breaking points (exceptional points) where the eigenvalues of the effective non-Hermitian Hamiltonian describing the dynamics of these systems abruptly turn from real to complex [5].

Recently interest in $\mathcal{PT}$-scattering configurations has been revived in connection with
using $\mathcal{PT}$-symmetric devices under a dual role, that of a lasing and a perfect coherent absorbing cavity [10]. In fact, several intriguing features have already been reported regarding the $\mathcal{PT}$-symmetric optical structures [2, 3, 8, 11, 14–25]. These include among others, power oscillations and non-reciprocity of light propagation [2, 3, 20], non-reciprocal Bloch oscillations [21], and unidirectional invisibility [8]. In the nonlinear domain, such pseudo-Hermitian non-reciprocal effects can be used to realize a new generation of on-chip isolators and circulators [19].

In the next sections, we will investigate the transport properties of $\mathcal{PT}$-symmetric optical structures by employing the transfer matrix formalism discussed in the previous chapter. As we enter the non-Hermitian realm of $\mathcal{PT}$ symmetry, we leave the traditional notions of flux conservation. Instead, we will encounter new conservation laws that reveal the underlying symmetries of the scattering target as well as anomalous transport properties including super-unitary transmission, asymmetric reflection, unidirectional transparency and simultaneous laser-absorbers.

### 3.1 Introduction to $\mathcal{PT}$ Symmetry

Parity ($\mathcal{P}$) and Time-reversal ($\mathcal{T}$) transformations are fundamental symmetry operations in physics. Mathematically, they are defined by their actions on the position operator $\hat{x}$ and the momentum operator $\hat{p}$ as well as the time parameter $t$ [5]. $\mathcal{P}$ is a linear operator which inverses space and momentum,

$$\mathcal{P}: \hat{x} \rightarrow -\hat{x}; \quad \hat{p} \rightarrow -\hat{p} .$$  

(3.1)

$\mathcal{T}$, on the other hand, is an anti-linear operator (that is, complex conjugation) which reverses the time $t \rightarrow -t$. One can show that such an operation is equivalent to the following actions [5]

$$\mathcal{T}: \ i \rightarrow -i; \quad \hat{x} \rightarrow \hat{x}; \quad \hat{p} \rightarrow -\hat{p} .$$  

(3.2)
While systems invariant under $\mathcal{P}$ or $\mathcal{T}$ transformations or both have been thoroughly studied, there has also been much interest in systems which do not obey $\mathcal{P}$ or $\mathcal{T}$ symmetries separately but which respect the combined $\mathcal{PT}$ symmetry. In quantum mechanical context, such systems are described by a Hamiltonian $\mathcal{H}$ that commutes with the combined $\mathcal{PT}$ operator, i.e. $[\mathcal{PT}, \mathcal{H}] = 0$. Despite the fact that $\mathcal{PT}$-Hamiltonians can, in general, be non-Hermitian, their spectra can be entirely real. The departure from Hermiticity is due to the presence of various gain/loss mechanisms which occur in a balanced manner, so that the net loss or gain of particles is zero. Furthermore, as some gain/loss parameter $\gamma$ that controls the degree of non-Hermiticity of $\mathcal{H}$ gets a critical value $\gamma_{PT}$, a spontaneous $\mathcal{PT}$ symmetry breaking can occur. For $\gamma > \gamma_{PT}$, the eigenfunctions of $\mathcal{H}$ cease to be eigenfunctions of the $\mathcal{PT}$-operator despite the fact that $\mathcal{H}$ and the $\mathcal{PT}$-operators still commute. This happens because the $\mathcal{PT}$-operator is anti-linear, and thus the eigenstates of $\mathcal{H}$ may or may not be eigenstates of $\mathcal{PT}$. As a consequence, in the broken $\mathcal{PT}$-symmetric phase, the spectrum becomes partially or completely complex. The other limiting case where both $\mathcal{H}$ and $\mathcal{PT}$ share the same set of eigenvectors, corresponds to the so-called exact $\mathcal{PT}$-symmetric phase, in which the spectrum is entirely real.

3.1.1 $\mathcal{PT}$-Symmetric Dimer

A $\mathcal{PT}$-symmetric system can be realized in optics by creating a medium that has alternating regions of gain and loss. The simplest of such a construction can be demonstrated using two coupled $\mathcal{PT}$-symmetric waveguides (see Fig. 3.1). Each of the waveguides supports one propagating mode — one providing gain for the guided light and the other experiencing an equal amount of loss. Light is transferred from one waveguide to the other via optical tunneling. After paraxial approximation [2, 26], the beam dynamics of the optical dimer along the waveguides ($z$-axis) is given by a Schrödinger-like differential
Section 3.1. Introduction to $\mathcal{PT}$ Symmetry

Figure 3.1: An illustration of $\mathcal{PT}$-symmetric dimer waveguide. The green waveguide indicates associated optical loss $\gamma$ while the red waveguide involves an equivalent amount of optical gain $-\gamma$. Light is transferred from one waveguide to the other via evanescent coupling $\kappa$.

The equation,

$$i \frac{d \Psi}{dz} = \mathcal{H} \Psi ,$$

where $\Psi = (a(z), b(z))^T$ describes the optical mode electric field amplitudes $a(z), b(z)$ in the two waveguides and the paraxial distance $z$ plays an analogous role of time $t$ in the Schrödinger picture. The Hamiltonian $\mathcal{H}$ of the dimer is given by a simple $2 \times 2$ matrix [27],

$$\begin{pmatrix} \epsilon_0 + i \gamma & \kappa \\ \kappa & \epsilon_0 - i \gamma \end{pmatrix},$$

in which $\kappa$ is the evanescent coupling between the two waveguides, $\gamma$ is the gain/loss coefficient, and $\epsilon_0$ is the real part of the index of refraction (which we will set to zero without loss of generality). It is easy to see that the dimer is $\mathcal{PT}$-invariant; parity switches between the two waveguides (the diagonal terms in $\mathcal{H}$) while time reversal transforms them back by complex conjugation so that the combined action of $\mathcal{PT}$ leaves the Hamiltonian unchanged.
The eigenvalues of this system is found via a direct diagonalization

$$\mathcal{E}_\pm = \pm \sqrt{\kappa^2 - \gamma^2}.$$  \hspace{1cm} (3.4)

From the above equation, one can see that $\mathcal{E}$ is real for $\gamma < \kappa$ and becomes imaginary for $\gamma > \kappa$. The sharp transition from a real to a complex spectrum that takes place at $\gamma_{PT} = \kappa$ is coined spontaneous $\mathcal{PT}$-symmetry breaking.

The transition from exact to broken phase is also reflected in the eigenvectors. Diagonalizing $\mathcal{H}$, we get

$$\psi_1 = \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix}; \quad \psi_2 = \begin{pmatrix} ie^{-i\alpha/2} \\ -ie^{i\alpha/2} \end{pmatrix},$$  \hspace{1cm} (3.5)

where $\sin(\alpha) = \frac{\gamma}{\kappa}$. Let us consider these eigenvectors for both above and below the phase transition point $\gamma_{PT} = \kappa$. Below this point, $\sin(\alpha) < 1$, giving $\alpha \in \mathbb{R}$. In this case, we see that these eigenvectors are also the eigenvectors of the $\mathcal{PT}$-operator, that is, switching the rows ($\mathcal{P}$ operation) and taking the complex conjugate ($\mathcal{T}$ operation) changes $\psi_1, \psi_2$ only by a factor $\pm 1$. On the other hand, above the phase transition point, we have $\sin(\alpha) > 1$, i.e. $\alpha \in \mathbb{I}$. In this case, they are no longer $\mathcal{PT}$-invariant and are not any more eigenfunctions of the $\mathcal{PT}$-operator. At exactly $\gamma = \gamma_{PT}$, we have that $\alpha = \frac{\pi}{2}$, and the system is having both eigenvalue and eigenvector degeneracy.

This remarkable collapse of Hilbert space turns out to be a signature of $\mathcal{PT}$-symmetry breaking, and is characteristic of systems with Exceptional Point (EP) singularity. Its existence leads to several intriguing phenomena including, among others, $z^2$ evolution of field intensities along the waveguides [20].

Next, we solve analytically the beam dynamics in the waveguides by a Taylor series expansion of the time evolution operator [27] $U = e^{-i\mathcal{H}z}$, yielding

$$\psi(z) = \frac{1}{\cos \alpha} \begin{pmatrix} c_1 \cos(\mathcal{E}z/2 - \alpha) - ic_2 \sin(\mathcal{E}z/2) \\ c_2 \cos(\mathcal{E}z/2 + \alpha) - ic_1 \sin(\mathcal{E}z/2) \end{pmatrix}.$$
Figure 3.2: Temporal evolution of total intensity $I(z)$ along the $\mathcal{PT}$-symmetric coupled waveguides. Grey color indicates passive waveguides without gain or loss while green indicates loss and red indicates gain. In the left (right) panel, the initial beam is launched into the left (right) waveguide. (a) $\gamma = 0$. The beam intensity stays constant over the propagation distance, following the exact same course regardless of whether the beam is launched into left or right waveguide. (b) $0 < \gamma < \gamma_{\mathcal{PT}}$. Finite oscillations in beam intensity are observed along the waveguides. At the same time, the propagation becomes nonreciprocal, depending on the waveguide the beam is initially launched into. (c) $\gamma > \gamma_{\mathcal{PT}}$. The beam power grows exponentially (vertical logarithmic scale) while the propagation is again nonreciprocal. Figure reproduced with permission from Ref \cite{28}. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.2.png}
\caption{Temporal evolution of total intensity $I(z)$ along the $\mathcal{PT}$-symmetric coupled waveguides.}
\end{figure}
The total intensity \( I(z) = |\psi(z)|^2 \) is then given by

\[
I(z) = \frac{1}{\cos^2(\alpha)} \left( c_1^2 \cos^2(\alpha - \frac{\mathcal{E}z}{2}) + c_2^2 \sin^2(\frac{\mathcal{E}z}{2}) \right) \left( c_2^2 \cos^2(\alpha + \frac{\mathcal{E}z}{2}) + c_1^2 \sin^2(\frac{\mathcal{E}z}{2}) \right),
\]

where \( \mathcal{E} = \sqrt{\kappa^2 - \gamma^2} \) and \( c_1, c_2 \) are generic constants associated with initial conditions of the launched beam. For \( \gamma < \gamma_{PT} \), the system exhibits power oscillations due to the non-orthogonality of the eigenvectors Eq. 3.5 whereas, for \( \gamma > \gamma_{PT} \), the intensity grows exponentially. An intriguing characteristic of \( \mathcal{P}\mathcal{T} \)-symmetric beam dynamics is that evolution is no longer reciprocal, that is, the dynamics follows different routes depending on into which waveguide the initial beam is launched, see Fig. 3.2. This nonreciprocal behavior can be exploited by introducing nonlinearity in order to provide for diode action; indeed, the nonlinear dimer has been demonstrated to act as an optical diode which holds promise for on-chip implementation of an optical isolator [19].

Although it seems to be a rather simplified theoretical model, the optical dimer has been experimentally realized by Rüter et al in Ref [3]. Their setup (see Fig. 3.3) makes use of a variable Ar\(^+\) laser input launched into Fe-doped LiNbO\(_3\) waveguides. In particular, this photo-refractive nonlinear material allows for realization of both gain and loss. Loss is inherent in the structure due to optical excitation of Fe electrons whereas optical gain is introduced via nonlinear two-wave mixing between an external laser pump and the input beam [29]. A partial mask covers the sample so that amplification is provided only.
in the gain waveguide. Finally, a CCD camera is used to monitor the output intensity and the phase relation between the two channels (using interference with a reference plane wave). The measurements demonstrate nonreciprocal oscillations in output beam intensity as well as spontaneous symmetry breaking characteristic of $\mathcal{PT}$ structures\[3\]. On the other hand, the spectral features discussed above are not specific only to optical or quantum mechanical systems. In fact, the universality of $\mathcal{PT}$ symmetry is further corroborated in \[30\] where the authors manage to demonstrate theoretically and experimentally the same features of $\mathcal{PT}$ symmetry in active LRC circuits described by completely different Liouvillian rate equations (see also Chapter 4).

3.2 $\mathcal{PT}$-Symmetric Scattering

The optical waveguide geometry considered in the previous section involves modulation of refractive index in a direction perpendicular to the propagation axis. Another possibility is to modulate the refractive index along the propagation direction (cf. Fig. 2.1), the $z$-axis, so that we have $\mathcal{PT}n(z)\mathcal{PT} = n^*(-z) = n(z)$ for $\mathcal{PT}$-symmetric structures. This constraint requires that $\text{Re}[n(z)] = \text{Re}[n(-z)]$ and $\text{Im}[n(z)] = -\text{Im}[n(-z)]$. In other words, the real part of refractive index is an even function of position while the imaginary part is an odd function. According to this scenario, back-scattering mechanisms are now allowed.

3.2.1 Representation of $\mathcal{PT}$

$\mathcal{P}$ and $\mathcal{T}$ defined in section 3.1 are operators that act on the Hilbert space (or the space of solutions to the Maxwell’s equations in optics). In one-dimensional scattering systems, it is more convenient to work with amplitude vectors (cf. Eqs. 2.5, 2.8) than the wave function itself. In this notation, one needs to be more careful about the representation of $\mathcal{P}$ and $\mathcal{T}$. First, we examine the action of these operators on the two basis vectors
\( e^{ikz} \) and \( e^{-ikz} \) [13].

\[ P: Ae^{ikz} \rightarrow Ae^{-ikz} \quad \text{so that} \quad P: \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ A \end{pmatrix}, \]

\[ P: Be^{-ikz} \rightarrow Be^{ikz} \quad \text{so that} \quad P: \begin{pmatrix} 0 \\ B \end{pmatrix} \rightarrow \begin{pmatrix} B \\ 0 \end{pmatrix}, \]

from which we can deduce,

\[ P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.6) \]

Similarly,

\[ T: Ae^{ikz} \rightarrow A^*e^{-ikz} \quad \text{so that} \quad T: \begin{pmatrix} A \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ A^* \end{pmatrix}, \]

\[ T: Be^{-ikz} \rightarrow B^*e^{ikz} \quad \text{so that} \quad T: \begin{pmatrix} 0 \\ B \end{pmatrix} \rightarrow \begin{pmatrix} B^* \\ 0 \end{pmatrix}. \]

Thus,

\[ T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{K}, \quad (3.7) \]

where \( \mathcal{K} \) is the complex conjugation operator. Therefore, in this representation, \( PT = PPK = \mathcal{K} \) [13].

### 3.2.2 Generalized Conservation Relations

For \( PT \)-symmetric structures, scattering signals satisfy a generalized conservation relation which reveal the underlying symmetries of the scattering target. Recall the general solution to the Helmholtz equation (2.3),

\[ E(z; k) = E_f^- e^{ikz} + E_b^- e^{-ikz} \quad \text{for} \quad z \leq -L/2 \]

\[ = C_1 f(z; k) + C_2 g(z; k) \quad \text{for} \quad |z| \leq L/2 \]

\[ = E_f^+ e^{ikz} + E_b^+ e^{-ikz} \quad \text{for} \quad z \geq L/2. \]
Associated with this solution, we can write down the $S$-matrix equation,
\[
\begin{pmatrix}
E^+_f \\
E^-_b
\end{pmatrix} = S(k) \begin{pmatrix}
E^+_f \\
E^-_b
\end{pmatrix}.
\] (3.8)

Since the system is $\mathcal{PT}$ symmetric, the problem admits a $\mathcal{PT}$-transformed solution, 
\[
\tilde{E}(z;k) = \mathcal{PT}E(z;k) = E^*(z; k) = E^+e^{ikz} + E^-e^{-ikz}
\]
for $z \leq -L/2$
\[
= C_1f^*(-z;k) + C_2g^*(-z;k)
\]
for $|z| \leq L/2$
\[
= E^+_fe^{ikz} + E^-be^{-ikz}
\]
for $z \geq L/2$.

Therefore, it is also true that
\[
\begin{pmatrix}
E^-e^{ikz} \\
E^be^{-ikz}
\end{pmatrix} = S(k) \begin{pmatrix}
E^+e^{ikz} \\
E^-e^{-ikz}
\end{pmatrix}.
\] (3.9)

Comparing Eqs. 3.8 and 3.9 we have
\[
S^*(k) = S^{-1}(k).
\] (3.10)

Using Eq. 3.10 and recalling that elements of $S$-matrix are none other than transmission and reflection amplitudes, it is straightforward to show that
\[
r_Lr_R^* = 1 - |t|^2
\] (3.11)
\[
r_Lt^* + r_R^*t = 0
\] (3.12)
\[
r_Rt^* + r_R^*t = 0.
\] (3.13)

Here, we have used the fact that $t_L = t_R = t$ for linear non-magnetic systems (see Chapter 2).

In particular, 3.11 implies that
\[
\sqrt{R_LR_R} = |T-1|,
\] (3.14)

where $R_L$ can in principle be different from $R_R$. Note that Eq. 3.14 is an intriguing generalization of the more familiar conservation relation $R+T = 1$, which applies to unitary processes. In the $\mathcal{PT}$-symmetric case, the geometric mean of the two reflectances,
replace the single reflectance $R$. An important realization about $\mathcal{PT}$ symmetric scattering is that transmittance and reflectance are no longer constrained to sum up to unity. We define as super-unitary the case for which the sum of transmission and reflection is larger than unity while in the opposite case we say that the scattering process is sub-unitary. A curious scenario occurs when we have perfect transmission, that is, $T = |t|^2 = 1$. Although, in this case, each reflection is no longer required to vanish, their product must indeed vanish according to Eq. 3.11. Typically, this constraint can be satisfied by one of the reflections getting to zero while the other remains non-zero. We call this phenomenon **Unidirectional Transparency** since we have reflectionless perfect transparency in one direction but not the other. In Ref [8] (also see Chapter 5), we first theoretically predicted a continuous broadband phenomenon of **Unidirectional Transparency** occurring in periodic $\mathcal{PT}$ symmetric structures.

### 3.2.3 Spectral Properties of $S$ Matrix

In $\mathcal{PT}$-symmetric scattering, we found that $S^* = S^{-1}$ (3.10). In subsection 3.2.1, we worked out the representations of $\mathcal{PT}$ acting on amplitude vector space. Together, we can conclude that

$$S^* = KSK = \mathcal{PT}S\mathcal{PT} = S^{-1}.$$  \hspace{1cm} (3.15)

Following a similar reasoning employed in Ref [10], we investigate the spectral properties of $S$-matrix. Let $\psi_n, n \in \{1, 2\}$, be an eigenvector of $S$ corresponding to the eigenvalue $s_n$. Then,

$$\mathcal{PT}SP\mathcal{T}\psi_n = S^{-1}\psi_n$$

$$SP\mathcal{T}\psi_n = \mathcal{PT} \left( \frac{1}{s_n}\psi_n \right)$$

$$S \left( \mathcal{PT}\psi_n \right) = \frac{1}{s_n} \mathcal{PT}\psi_n.$$  

Therefore, $\mathcal{PT}\psi_n$ is also an eigenvector of $S$. This constraint requires that one of the following must be true:
(i) $\mathcal{PT}$ and $S$ share the same eigenvectors, that is, $\mathcal{PT}\psi_n = \pm \psi_n$. Accordingly, the parameter domain, that is, the set of $\omega, \gamma$ and $L$ values, where the eigenvectors of $S$ are $\mathcal{PT}$-symmetric, is called exact $\mathcal{PT}$ symmetric phase.

(ii) $\mathcal{PT}$ transform the eigenvectors of $S$ into each other, that is, $\mathcal{PT}\psi_n = \pm \psi_{n'}$ where $n' \in \{1, 2\}$ but $n' \neq n$. In other words, the eigenvectors of $S$ cease to be $\mathcal{PT}$-symmetric in this regime, which is appropriately called broken $\mathcal{PT}$ symmetric phase.

(iii) A third scenario can occur where both (i) and (ii) are simultaneously true at the transition between exact and broken phases. This transitional point where the $\mathcal{PT}$ symmetry (of $S$ matrix eigenvectors) is spontaneously broken is called exceptional point or EP. Since

$$\mathcal{PT}\psi_n = \pm \psi_{n'} \text{ from (ii)}$$
$$\psi_n = \psi_{n'} \text{ from (i)},$$

we find that at the exceptional point, the two eigenvectors become degenerate and so do their eigenvalues. It turns out that due to the particular form of $S$, the converse is true as well, that is, if the eigenvalues are degenerate, the eigenvectors are also necessarily degenerate. It is straightforward to prove this fact by diagonalizing $S = \begin{pmatrix} t & r_B \\ r_L & t \end{pmatrix}$ and directly inspecting the eigenvectors and eigenvalues (see below).

Now, let us examine the eigenvalues of $S$. Suppose (i) holds true. Then

$$S(\mathcal{PT}\psi_n) = \frac{1}{s_n^*} \mathcal{PT}\psi_n$$
$$S(\pm \psi_n) = \pm \frac{1}{s_n^*} \psi_n$$
$$s_n \psi_n = \frac{1}{s_n^*} \psi_n$$
$$\Rightarrow |s_n| = 1 .$$

Therefore, the eigenvalues are unimodular. We can examine the converse as well. Let $s_n$
be unimodular, that is, \( s_n = e^{i\phi_n}, \phi_n \in \mathbb{R} \). Suppose that \( \mathcal{P}\mathcal{T}\psi_n = \pm \psi_{n'} \). Then,

\[
S\left( \mathcal{P}\mathcal{T}\psi_n \right) = e^{i\phi_n} \mathcal{P}\mathcal{T}\psi_n \\
S\left( \pm \psi_{n'} \right) = \pm e^{i\phi_n} \psi_{n'} \\
e^{i\phi_{n'}} \psi_{n'} = e^{i\phi_n} \psi_{n'} .
\]

Hence, either the eigenvalues are degenerate, in which case the eigenvectors are also degenerate and the system is at the exceptional point, or the supposition \( \mathcal{P}\mathcal{T}\psi_n = \pm \psi_{n'} \) is wrong, in which case (i) holds true and the system is in exact phase.

Therefore, we can conclude that \( \mathcal{P}\mathcal{T} \) and \( S \) share the same eigenvectors, condition (i), if and only if the eigenvalues of \( S \) are nondegenerate and unimodular. In other words, unimodular eigenvalues correspond to the exact phase whereas non-unimodular eigenvalues, whenever they appear, signify broken phase. In the latter scenario, it can be readily shown that the two eigenvalues are conjugate reciprocals of each other, that is, \( s_n s_{n'} = 1 \). On the other hand, at the exceptional point, the eigenvectors as well as eigenvalues are degenerate.

Having introduced the general properties of \( S \), we can now derive explicit criteria for the spontaneous \( \mathcal{P}\mathcal{T} \)-symmetry breaking transition in terms of the elements (i.e. transmission and reflection) of the scattering matrix. Recall that \( S = \begin{pmatrix} t & r_L \\ r_R & t \end{pmatrix} \) and its eigenvalues are \( s_n = t \pm \sqrt{r_L r_R} \) with the un-normalized eigenvectors given by \( \begin{pmatrix} -\sqrt{r_L} \\ \sqrt{r_R} \end{pmatrix}, \begin{pmatrix} \sqrt{r_L} \\ \sqrt{r_R} \end{pmatrix} \). Using 3.11 and 3.12, we have \( s_n = t \left( 1 \pm i \sqrt{T - 1} \right) \), from which we can deduce that

(i) eigenvalues are unimodular and nondegenerate (exact phase) when \( \frac{1}{T} - 1 > 0 \) or \( T < 1 \).

(ii) eigenvalues are non-unimodular (broken phase) when \( \frac{1}{T} - 1 < 0 \) or \( T > 1 \).

(iii) eigenvalues are degenerate (exceptional point) at exactly \( T = 1 \).

Here, a note of caution is in order. In Ref [10], the authors have used an alternative definition of scattering matrix, which we will denote \( S' = \begin{pmatrix} r_L & t \\ t & r_R \end{pmatrix} \) and come out with
completely different criteria for $\mathcal{PT}$ symmetry breaking, viz, $\frac{(R_L+R_R)}{2} - T > 1$ for broken phase [10]. The root of the discrepancy can be traced to a different definition of $\mathcal{PT}$ than that defined in the subsection 3.2.1. Although it casts doubt on the uniqueness (and the very meaning) of $\mathcal{PT}$ symmetry breaking in scattering systems, we state at this point that the two matrices $S$ and $S'$ serve to measure phases of different interests, in that the criterion associated with $S$ emphasizes the occurrence of anomalous super-unitary transport whereas that of $S'$ dwells on the imbalance of scattering signals in the broken regime.

3.2.4 Simultaneous CPA-Lasers

One unique feature of $\mathcal{PT}$ symmetry is that due to the presence of simultaneous gain and loss, growth and decay modes are tightly linked in a $\mathcal{PT}$-symmetric structure. If the system ever permits lasing at some frequency, $\mathcal{PT}$ symmetry requires that at that same frequency, certain waves having special waveforms and phase relations be entirely absorbed (coherent perfect absorption) [10]. Mathematically, if Eq. 2.3 allows for a purely outgoing signal (lasing) at a particular frequency $\omega_0 = ck_0$

$$E(z; k_0) = E_b^+ e^{-ik_0z} \quad \text{for } z \leq -\frac{L}{2}$$

$$= E_f^+ e^{ik_0z} \quad \text{for } z \geq \frac{L}{2},$$

it immediately follows that the system also allows for a purely incoming signal (perfect absorption with zero reflection) at the same $\omega_0$

$$\mathcal{PT} E(z; k_0) = E^*(-z; k_0) = E_f^{-*} e^{ik_0z} \quad \text{for } z \leq -\frac{L}{2}$$

$$= E_b^{-*} e^{-ik_0z} \quad \text{for } z \geq \frac{L}{2}.$$

We can further analyze the properties of CPA-laser using the transfer matrix approach [11]. Recall from 2.5 that

$$\begin{pmatrix} E_f^+ \\ E_b^+ \end{pmatrix} = M(k) \begin{pmatrix} E_f^- \\ E_b^- \end{pmatrix}. $$

For a laser oscillator, the input amplitudes $E_f^-$ and $E_b^+$ are zero whereas $E_b^-, E_f^+ \neq 0$, which imply $M_{22}(k) = 0$. For a perfect absorber, the reverse scenario holds (absence of reflection) so that $M_{11}(k) = 0$. In general, $M_{22}$ and $M_{11}$ do not simultaneously vanish at the same frequency. Now consider a $\mathcal{PT}$-symmetric laser such that there exists a frequency $\omega_0$ at which $M_{22}(k_0) = 0$. $\mathcal{PT}$ symmetry requires that $M^*(k) = M^{-1}(k)$ from which we can deduce\[ that
\[
M_{22}(k) = M_{11}^*(k) \\
\text{Re}[M_{12}(k)] = 0 \\
\text{Re}[M_{21}(k)] = 0. \tag{3.16}
\]

In particular, $M_{11}(k_0) = 0$ since $M_{22}(k_0) = 0$. Therefore, the $\mathcal{PT}$ laser can also act as an absorber at $k_0$. Furthermore, near the lasing frequency, we can expand the elements of $M$ matrix
\[
M_{11}(k) \approx \kappa(k - k_0) \\
M_{22}(k) \approx \kappa^*(k - k_0) \\
M_{12}(k) \approx i\alpha + i\beta(k - k_0) \\
M_{21}(k) = \frac{M_{11}M_{22} - 1}{M_{12}} \approx \frac{i}{\alpha} - i\frac{\beta}{\alpha^2}(k - k_0),
\]
where $\kappa$ is a complex-valued constant given by $\kappa = M_{11}'(k_0)$ while $\alpha$ and $\beta$ are real-valued constants given by $\alpha = \text{Im}[M_{12}(k_0)]$ and $\beta = \text{Im}[M_{12}'(k_0)]$.

Next, we consider two simultaneously injected signals with a special phase/amplitude relation $\frac{E_b^+}{E_f^+} = \sigma = M_{21}(k)$ \[14\]. We calculate the overall output coefficient $\Theta(k; \sigma)$ defined as the ratio of the total intensity of outgoing (reflected/ transmitted) waves

\[\footnote{The proof follows a similar argument to that given for $S$-matrix, cf. section 3.2.2.}\]
over the total intensity of incoming waves,

$$\Theta(k \approx k_0; \sigma = M_{21}(k)) = \frac{|E^b_0|^2 + |E^b_f|^2}{|E^+_f|^2 + |E^+_b|^2} \approx \frac{|1 + \sigma M_{12}(k)|^2 + |\sigma - M_{21}(k)|^2}{(1 + |\sigma|^2)|M_{22}(k)|^2} \approx \frac{1 + M_{21}(k)M_{12}(k)}{(1 + |M_{21}(k)|^2)|M_{22}(k)|^2} \frac{\beta^4(k - k_0)^2}{(\alpha^2 + \alpha^4 + (\beta^2 - 2\alpha\beta)(k - k_0)^2)\kappa^2}. $$

Therefore, $$\Theta(k_0, M_{21}(k_0)) = 0.$$ It is worth emphasizing that at $$k = k_0,$$ $$\Theta$$ vanishes only for a particular choice of $$\sigma = M_{21}(k_0)$$ but not for others. For example, if we choose left incident signal with $$E^+_b = 0$$ and $$E^-_f = 1,$$ then it turns out that $$\Theta(k; \sigma = 0) \sim (k-k_0)^{-2},$$ which diverges at $$k = k_0,$$ as expected of a lasing point.

At first glance, one might suspect if lasing ever occurs in $$\mathcal{PT}$$ structures at all because of the zero net gain overall. However, lasing does occur in the broken phase ($$T > 1$$) if the asymmetry in $$S$$ matrix eigenmodes is such that the excitation spends more time in the gain part of the medium than the lossy part (see Refs. \[7, 11\]). In the next section, we theoretically demonstrate with a simple example the occurrence of CPA-lasers.

### 3.3 Double-layered Optical Slab

A simple example suffices to demonstrate the many salient features of $$\mathcal{PT}$$-symmetric scattering discussed above. To this end, we study light scattering in double-layered slab, one layer having refractive index $$n_R + i\gamma,$$ and the other having $$n_R - i\gamma$$ embedded in a homogeneous medium of $$n_0 = 1,$$ Fig. 3.4. Using Eq. 2.15, it is straightforward to calculate the transfer matrix $$M$$ which is found to be

$$M(k) = \begin{pmatrix} A \frac{2}{2(\gamma^2 + n_R^2)} & i \frac{B+C}{2(\gamma^2 + n_R^2)} \\ i \frac{-B-C}{2(\gamma^2 + n_R^2)} & A^* \frac{2}{2(\gamma^2 + n_R^2)} \end{pmatrix},$$
where

\[ A = e^{-ikL} \left( 2\gamma^2 \cosh(\gamma kL) + 2n_R^2 \cos(n_R kL) - i(\gamma^3 + \gamma n_R^2 - \gamma) \sinh(\gamma kL) + i(n_R^3 + n_R \gamma^2 + n_R) \sin(n_R kL) \right); \]

\[ B = 2\gamma n_R \left( \cos(n_R kL) - \cosh(\gamma kL) \right); \quad C = (n_R^3 + n_R \gamma^2 - n_R) \sin(n_R kL) - (\gamma^3 + \gamma n_R^2 + \gamma) \sinh(\gamma kL). \]

Knowing the transfer matrix, we can readily calculate transmission and left / right reflections (see the inset of Fig. 3.5), in which case it can be easily checked that they satisfy the $\mathcal{PT}$ conservation relation, Eq. 3.11. On the other hand, we are interested in the behavior of $S$ matrix eigenvalues over a range of $kL$. For the parameters $n_R = 2.5$ and $\gamma = 5 \times 10^{-3}$, an exceptional point is observed around $kL \approx 1445.386$, dashed line in Fig. 3.5 where the two eigenvalues become nonunimodular and their absolute values symmetrically branch out about 1 on a logarithmic scale (since $|s_1 s_2| = 1$). Remarkably, the inspection of transmission and reflection (inset) reveals that at this point, $T = 1$ while $R_L = 0$ and $R_R \neq 0$; therefore, the slab is also unidirectionally transparent at the exceptional point. As we will see later on in Chapter 5 [8], this behaviour can also appear in more complicated structures, like $\mathcal{PT}$-symmetric Bragg gratings resulting in a broadband phenomenon. In Chapter 4 (see also [7]), we will experimentally realize this phenomenon using active LRC circuits.

Next, we identify the CPA-laser action associated with the structure of Fig. 3.4. Around $k_0 L \approx 1445.75$ in Fig. 3.5, a sharp peak in one of the eigenvalues $s_1$ is seen, which signals the appearance of a lasing mode. At the same time, the other eigenvalue $s_2$ vanishes, indicating that a special set of input signals exists, viz $\psi_2$, which get entirely absorbed. Examining the overall output coefficient $\Theta$ (Fig. 3.6) also demonstrates this fact; at
Figure 3.5: S-matrix eigenvalues for the structure of Fig. 3.4. The eigenvalues are plotted on a logarithmic scale against optical path length $kL$, showing transition to non-unimodularity (the grey dashed line). Note that $|s_1s_2| = 1$. The singular peak/dip doublet in eigenvalues denotes the occurrence of a laser-absorber. Inset. Transmission (red), left reflection (green) and right reflection (blue) over the same range of $kL$.

$kL \approx 1445.75$, ordinary signals, for example, the left-sided illumination with $\sigma = 0$, only excite the lasing mode while the specially prepared coherent signal with $\sigma = M_{21}$ undergoes perfect absorption (CPA).

### 3.4 Summary

In this chapter, we have reviewed the basic theory of $\mathcal{PT}$ symmetry and derived the basic properties of the associated scattering formalism. Specifically, we defined the notions of exact/broken phases and exceptional points as well as criteria to distinguish between them. Particularly interesting is the generalized conservation relations, which allows the
scattering system to be unidirectionally transparent, and the possibility to create CPA-lasers which exist as a consequence of the \( PT \) symmetry. Finally, we demonstrated these intriguing features with a simple theoretical example — scattering through an optical slab consisting of two layers with \( PT \)-symmetry.

Figure 3.6: Total output coefficient \( \Theta \) vs optical path length \( kL \), showing lasing for left-sided incidence (red) and perfect absorption for coherently injected signals (green). Parameters are the same as those used for Fig. 3.5.
Chapter 4

\( \mathcal{PT} \) Symmetric Electronics

More recently ideas of \( \mathcal{PT} \) have been extended into the realm of electronic circuitry [30], where it was demonstrated that a pair of coupled \( LRC \) circuits, one with amplification and the other with equivalent amount of attenuation, provide the simplest experimental realization of a \( \mathcal{PT} \) symmetric system. The \( \mathcal{PT} \)-circuitry approach suggested in Ref. [30] opens new avenues for innovative integrated circuitry architectures which will allow for alternative methods of signal manipulation and reduced circuit loss. Moreover, it permits direct contact with cutting edge technological problems appearing in (nano)-antenna theory and split-ring resonator meta-material arrays.

In this chapter, we will begin by reviewing the electronic \( \mathcal{PT} \) dimer demonstrated in Ref. [30] which displays all the phenomena encountered in systems with generalized \( \mathcal{PT} \) symmetries. In particular, this simple circuitry allows a direct observation of spontaneous \( \mathcal{PT} \)-symmetry breaking, that is, a phase transition from a real to a complex eigenfrequency spectrum (cf. chapter 3). Furthermore, we report the first direct experimental results for scattering in \( \mathcal{PT} \)-symmetric systems, utilizing the electronic dimer as a scattering load which we couple to transmission line (TL) leads [7]. Our measurements reveal the signatures of the parity-time symmetry in the conservation relations satisfied by the non-unitary scattering matrix. In the simplest possible scattering set-up where
the target is coupled to a single TL, we find that the reflection signal is non-reciprocal and respects a (non-unimodular) conservation relation $r_L \cdot r_R^* = 1$. Our analysis also reveals a transition from a sub-unitary to a super-unitary scattering process and associates it with the spatial structure of the potential inside the scattering domain. Once a second TL is attached to the $\mathcal{PT}$-scatterer, the system demonstrates unidirectional transparency (see discussion in the previous chapter), where the transmittance is unity and the reflectance is zero, but only for waves incident from a single side. Being free of basic theoretical approximations, and due to its relative simplicity in the experimental implementation, the $LRC$-networks with $\mathcal{PT}$ symmetry can offer new insights into the study of $\mathcal{PT}$-symmetric scattering which is at the forefront of current research in various areas of physics.

Figure 4.1: Electronic implementation of a $\mathcal{PT}$-symmetric dimer. Figure from [30], reproduced with permission.
4.1 Experimental Realization of $\mathcal{P}\mathcal{T}$-Symmetric Active $LRC$ Dimer

The heart of our experimental set-up is a pair of inductively coupled $LC$ resonators (dimer) shown in Fig. 4.1 [30]. Each inductor consists of 75 turns of #24 copper wire wound on 15 cm diameter polyvinyl chloride (PVC) forms in a $6 \times 6$ mm loose bundle for an inductance $L_0 = 2.32 \, mH$. The coils, matched to within 1% by repositioning one of the turns, are mounted coaxially with a bundle separation that can be adjusted to the desired mutual coupling $\mu (= M/L_0)$ ranging over $0.15 \sim 0.4$. The capacitances are $10.3 \sim 10.8 \, nF$ silver-mica in addition to the self capacitance in the coil bundles of $\sim 320 \, pF$. Capacitance balance is trimmed by substituting $\sim 360 \, pF$ of one side with a GR722-M variable capacitance. Loss imposed on the right half of the dimer is a standard carbon resistor, $R$. Gain imposed on the left half of the dimer, symbolized by $-R$, is implemented with an op-amp based negative impedance converter (NIC). The NIC gain is trimmed to oppositely match the value of $R$ used on the loss side, setting the gain/loss parameter $\gamma = R^{-1} \sqrt{L/C} = 1/(\omega_0 RC)$. An additional NIC is included on the loss side so that intrinsic resonator losses on both sides can be compensated for prior to setting the gain/loss parameter.

Overall, the actual experimental circuit deviates from the ideal in the following ways: (1) a resistive component associated with coil wire dissipation is nulled out with an equivalent gain component applied to each coil; (2) a small trim is included in the gain buffer for balancing; and (3) additional LM356 voltage followers are used to buffer the measured voltages.
4.2 Experimental Observation of Spontaneous $\mathcal{PT}$-symmetry Breaking

Application of the first and second Kirchoff’s law, for the coupled circuits of Fig. 4.1, leads to the set of equations

\[ \begin{align*}
\dot{Q}_L - \gamma \omega_0 Q_L + I_M^L &= 0, \\
\dot{Q}_R + \gamma \omega_0 Q_R + I_M^R &= 0, \\
\omega_0^2 Q_L &= \dot{I}_M^L + \mu \dot{I}_M^R, \\
\omega_0^2 Q_R &= \dot{I}_M^R + \mu \dot{I}_M^L,
\end{align*} \tag{4.1} \]

where $Q$ is the charge on capacitor, $I$ is the current, and $\dot{I} = \frac{dI}{dt}$. The superscript $M$ indicates that the quantity is associated with the inductor whereas the subscript $L$, $R$ stand for the left (gain) and right (loss) units of the circuitry.

Taking the time derivatives and eliminating $I^M$s, it is straightforward to recast Eq. 4.1 into a so-called Liouvillian form \[30\]

\[ \frac{d\Psi}{dt} = \mathcal{L} \Psi, \quad \mathcal{L} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{1}{1-\mu^2} & -\frac{\mu}{1-\mu^2} & 0 & 0 \\
\frac{\mu}{1-\mu^2} & -\frac{1}{1-\mu^2} & 0 & -\gamma
\end{pmatrix}. \tag{4.2} \]

Here, $\Psi = (Q_L, Q_R, \dot{Q}_L, \dot{Q}_R)^T$ and the time $t$ is rescaled in units of natural frequency $\omega_0$. The eigenfrequencies $\omega_i$ can be found by a direct diagonalization of the matrix $\mathcal{L}$, which are found to be \[30\]

\[ \omega_{1,4} = \pm \sqrt{\frac{2 + \gamma^2 (\mu^2 - 1) + \sqrt{4 (\mu^2 - 1) + 2 + \gamma^2 (\mu^2 - 1)^2}}{2 (\mu^2 - 1)}} \]

\[ \omega_{2,3} = \pm \sqrt{\frac{2 + \gamma^2 (\mu^2 - 1) - \sqrt{4 (\mu^2 - 1) + 2 + \gamma^2 (\mu^2 - 1)^2}}{2 (\mu^2 - 1)}}. \tag{4.3} \]

For $\gamma = 0$, we have two frequency pairs $\omega_{1,4} = \pm \sqrt{1/(1-\mu)}$ and $\omega_{2,3} = \pm \sqrt{1/(1+\mu)}$.

At $\gamma = \gamma_{P\mathcal{T}} = 1/\sqrt{1-\mu} - 1/\sqrt{1+\mu}$, these eigenmodes undergo a level crossing and
Section 4.2. Experimental Observation of Spontaneous $\mathcal{PT}$-symmetry Breaking

Figure 4.2: Parametric evolution of the experimentally measured eigenfrequencies vs the normalized gain and loss parameter $\gamma/\gamma_{\mathcal{PT}}$. $\omega_0 = 2 \times 10^5 \text{ s}^{-1}$, $\mu = 0.2$. $\gamma$ is set by choosing the loss-side resistance $R$ in the range $1-10 k\Omega$. Figure from [30], reproduced with permission.

branch out into the complex plane (Spontaneous $\mathcal{PT}$-symmetry breaking, cf. section 3.1). In Fig. 4.2 the authors of Ref. [30] have reported the measurements for the frequencies (those with positive real parts) of the experimental circuitry in Fig. 4.1. A comparison with the theoretical results of Eq. 4.1 indeed shows an excellent agreement.

Furthermore, the authors have directly observed the spatio-temporal evolution of the circuitry which also reveals the occurrence of $\mathcal{PT}$ symmetry breaking (cf. section 3.1.1). In particular, the authors chose to study the time dependence of the total capacitance energy $E(\tau) = \frac{1}{2\epsilon_C}(Q_L(\tau)^2 + Q_R(\tau)^2)$, which exquisitely traces the transition of the $\mathcal{PT}$-symmetric system from exact phase to broken phase. For $\gamma < \gamma_{\mathcal{PT}}$, the circuitry
is characterized by bounded power oscillations due to the unfolding of nonorthogonal eigenmodes [20] (see Fig. 4.3). For \( \gamma > \gamma_{PT} \), the dynamics becomes unstable exhibiting exponential growth in stored energy at a rate given by the maximum imaginary eigenfrequency. Remarkably, as \( \gamma \rightarrow \gamma_{PT} \), the \( \tau^2 \) behavior signaling the spontaneous \( \mathcal{PT} \)-symmetry breaking is observed [20], being the first experiment to demonstrate this singular \( \mathcal{PT} \) transition point.

### 4.3 Experimental Observation of \( \mathcal{PT} \)-Symmetric Scattering

The next intuitive step after the study of closed-form \( \mathcal{PT} \) circuitry would be to study an equivalent scattering system. Although a few theoretical results have been published...
in this area \[8, 11, 13–15, 22, 24, 25\], there is a complete lack of experimental investigations on $PT$-scattering. Indeed, given the fact that the additional freedom of the gain/loss parameter provides a fertile ground for developing a wealth of novel scattering phenomena, the current experimental void in scattering studies is quite surprising. In this section, we present the first of such studies \[7\] in the framework of electronic circuitry utilizing the active $LRC$ dimer discussed above.

### 4.3.1 Single Port Scattering

We begin with the following two reciprocal geometries: In the first case, a transmission line (TL) is attached to the left (amplified) circuit of the dimer load while in the second case, the TL is connected to the right (lossy) circuit of the load (see lower right and left insets of Fig. 4.4 respectively). Experimentally, the equivalent of a TL with characteristic impedance $Z_0$ is attached to either side of the dimer at the $LC$ circuit voltage node in the form of a resistance $R_0 = Z_0$ in series with an HP3325A synthesizer. The right and left traveling wave components associated with the TL (see Appendix A) are deduced from the complex voltages on both sides of $R_0$ with an EG&G 7256 lock-in amplifier. With $V_{LC}$ the voltage on the $LC$ circuit, and $V_0$ the voltage on the synthesizer side of the coupling resistor $R_0$, the right (incoming) wave has a voltage amplitude $V^+_L = V_0/2$ and the left (reflected) wave has a voltage amplitude $V^-_L = V_{LC} - V_0/2$. The lock-in is referenced to the synthesizer, defining the phases of the wave components relative to the incoming wave. At any point along a TL, the current and voltage determine the amplitudes of the right and left traveling wave components \[1\]. The forward $V^+_{L/R}$ and backward $V^-_{L/R}$ wave amplitudes, and $V_{L/R}$ and $I_{L/R}$; the voltage and current at the left (L) or right (R) TL-dimer contacts, satisfy the continuity relation

$$V_{L/R} = V^+_{L/R} + V^-_{L/R}; \quad I_{L/R} = \left[ V^+_{L/R} - V^-_{L/R} \right] / Z_0,$$

which connect the wave components to the currents and voltages at the TL-dimer contact points. Note that with this convention, a positive lead current flows into the right
Figure 4.4: Experimental reflectances for a single TL attached to the lossy ($R_R$) or the gain ($R_L$) side of the dimer (see lower insets) for $\mu = 0.29$, $\gamma = 0.188875$, $\eta = 0.0305$ and $\omega_0 = 194.51 \, s^{-1}$. The black line corresponds to $R_L^{-1}$ and confirms the non-reciprocal nature $R_L R_R = 1$ of the $\mathcal{PT}$-scattering. The upper inset shows the measurements for the left (right) reflection phases $\phi_L$ ($\phi_R$). The blue lines are the theoretical results Eq. 4.7.

Application of the first and second Kirchhoff’s laws at the TL leads allow us to find the corresponding wave amplitudes and reflection. For example, the case of the left-attached circuit, but out of the left circuit, and that the reflection amplitudes for left or right incident waves are defined as $r_L \equiv V_L^- / V_L^+$ and $r_R \equiv V_R^+ / V_R^-$ respectively.
lead in the lower right inset of Fig. 4.4 gives
\[ \eta(V_L^+ - V_L^-) = I_L^M - \gamma V_L - i\omega V_L \]
\[ V_L = -i\omega [I_L^M + \mu I_R^M] \quad ; \quad V_R = -i\omega [I_R^M + \mu I_L^M] \]
\[ 0 = I_R^M + \gamma V_R - i\omega V_R \quad , \]
where \( \gamma \) is the gain/loss parameter, \( \eta = \sqrt{L/C}/Z_0 \) is the dimensionless TL impedance, and \( I_{L/R}^M \) are the current amplitudes in the left or right inductors. Here, the dimensionless wave frequency \( \omega \) is in units of \( 1/\sqrt{LC} \). Similar equations apply for the right-attached case shown in the lower left inset of Fig. 4.4.

We are interested in the behavior of the reflectance \( R_{L/R} \equiv |r_{L/R}|^2 \), and spatial profile of the potential \( V_{L/R} \) inside the scattering domain, as the gain/loss parameter \( \gamma \), and the frequency \( \omega \) changes.

For \( \mathcal{PT} \)-symmetric structures, the corresponding scattering signals satisfy generalized unitarity relations which reveal the symmetries of the scattering target. Specifically, in the single-port set up this information is encoded solely in the reflection. To unveil it, we observe that the lower left set-up of Fig. 4.4 is the \( \mathcal{PT} \)-symmetric replica of the lower right one. Assuming therefore that a potential wave at the left lead (lower right inset) has the form \( V_L(x) = \exp(ikx) + r_L \exp(-ikx) \) (we assume \( V_L^+ = 1 \) and \( V_L^- = r_L \) in Eq. 4.4), we conclude that the form of the wave at the right lead associated with the lower left set-up of Fig. 4.4 is \( V_R(x) = \exp(-ikx) + r_r \exp(ikx) = V_L^*(-x) \). Direct comparison leads to the relation
\[ r_L \cdot r_R^* = 1 \rightarrow R_L = 1/R_R \quad \text{and} \quad \phi_L = \phi_R, \]
where \( \phi_{L/R} \) are the left/right reflection phases. Note that Eq. 4.6 differs from the more familiar conservation relation \( R = 1 \), which applies to unitary scattering processes as a result of flux conservation. In the latter case left and right reflectances are equal. Instead in the \( \mathcal{PT} \)-symmetric case we have in general that \( R_L \neq R_R \). For the specific case of the \( \mathcal{PT} \)-symmetric dimer, we can further calculate analytically the exact expression.
Figure 4.5: The $\omega - \gamma$ phase diagram for $\mu = 0.57$, indicating the existence of a sub-unitary ($\log_{10}(R_R) < 0$) and a complimentary super-unitary ($\log_{10}(R_R) > 0$) domain for the set-up shown at the lower left inset of Fig. 4.4. The (white) plane $\log_{10}(R) = 0$ is shown for reference while the boundary Eq. 4.8 is indicated with a red line.

for the complex reflection amplitudes. From Eqs. 4.5, we can eliminate $I_{L/R}^M$ and $V_R$, yielding

$$r_L(\omega) = -f(-\eta, -\gamma)/f(\eta, -\gamma)$$
$$r_R(\omega) = -f(-\eta, \gamma)/f(\eta, \gamma)$$

$$f = 1 - [2 - \gamma m(\gamma + \eta)] \omega^2 + m \omega^4 - i \omega (1 - m \omega^2)$$

with $m = 1/\sqrt{1 - \mu^2}$. 
Note that $r_R$ has been obtained by taking $\gamma \to -\gamma$ in the expression for $r_L$. In the limiting case of $\omega \to 0, \infty$ the reflections becomes $r_R \to \mp 1$ and thus unitarity is restored.

In the main panel of Fig. 4.4 we report representative measurements of the reflection signals for the two scattering configurations and compare them with Eq. 4.7. The synthesizer frequency is slowly swept through the region of interest producing the reflectance $R_L = |V_L^-/V_L^+|^2$ as a function of frequency, resulting in the red squares of Fig. 4.4. A similar procedure is used to obtain the reflectance, $R_R$, from the right (loss) side of the dimer, resulting in the green squares of Fig. 4.4. The measured reflectances $R_L$ and $R_R$ satisfy the generalized conservation relation $R_L \cdot R_R = 1$ while for the reflection phases we have that $\phi_L = \phi_R$ as expected from Eq. 4.6. Therefore, our experiment demonstrates that a $\mathcal{PT}$-symmetric load is a simple electronic Janus device that for the same values of the parameters $\omega, \mu, \gamma$ acts as an absorber as well as a signal amplifier, depending on the direction of incident signal.

Next, we identify the existence of a sub – unitary domain for which $R < 1$ (flux is diminished), and a super – unitary domain for which $R > 1$ (flux is enhanced). At the transition between the two domains $R_L = R_R = 1$, in which case the scattering from both sides conserves flux. Such reflectance degeneracies (RD) occur as a parameter such as the frequency $\omega$ (or $\gamma$) is varied continuously. Requiring that $|r_R| = 1$, we get

$$\gamma^* = \sqrt{-1 + 2 \omega^2 - (1 - \mu^2)\omega^4 \over (1 - \mu^2)\omega^2}$$

and

$$|\omega^2 - 1| \over \omega^2 \leq \mu \leq 1.$$ (4.8)

A panorama of theoretical $R_R(\omega, \gamma)$ are shown in Fig. 4.5. In the same plot we mark the transition line $\gamma^*(\omega)$ where a RD occurs. Inside this domain, a singularity point appears for which $R_R \to \infty$, while a reciprocal point for which $R_R = 0$ is found in the complementary domain. The corresponding $(\omega_s; \gamma_{\infty,0})$ are found from Eq. 4.7 to be $\gamma_{\infty,0} = {1 \over 2} \left( \sqrt{\eta^2 + \frac{4\mu^2}{(1-\mu^2)^2} + \eta} \right)$; $\omega_s = 1 \over \sqrt{1-\mu^2}$. Obviously via Eq. 4.6 we have the reverse

$\text{[1]}$The slight deviation from reciprocity in the vicinity of large reflectances in Fig. 4.4 - see domain around the right peak - is attributed to nonlinear effects.
Section 4.3. Experimental Observation of $\mathcal{PT}$-Symmetric Scattering

Figure 4.6: The spatial potential distribution inside the dimer versus the frequency $\omega$. The TL is coupled to the lossy side. We have used the same parameters as those used in Fig. 4.4. The blue dashed lines indicate the boundaries between sub-unitary to super-unitary scattering where RD occur.

The sub-unitary to super-unitary $\mathcal{PT}$-symmetric transition is also manifest in the spatial structure of the potential $(V_L; V_R)$ inside the dimer. From Eq. 4.5 we get

$$V_L = 2\eta\omega \left[ 1 - m\omega (\omega - i\gamma) \right] / D$$

$$V_R = -2\eta\mu\omega / D$$

$$D = \eta\omega(1 - m\omega^2) + i \left( 1 - \omega^2 \left[ 2 - m(\omega^2 + \gamma\omega + \gamma^2) \right] \right).$$

Typical potential amplitudes ($|V_L|; |V_R|$), for the set-up of the lower left inset of Fig. 4.4 versus the frequency $\omega$ are shown in Fig. 4.6. We observe that they are in general asymmetric. In the super-unitary domain, the gain side is characterized by a larger potential amplitude $|V_L| > |V_R|$ while in the sub-unitary domain the scenario is reversed.
and $|V_L| < |V_R|$. The latter configuration ensures that more power is being consumed than compensated by the gain circuit, while the inverse argument applies for the former configuration. At frequencies where the RD occurs, the potential profiles are spatially symmetric. This is consistent with the intuitive expectation that in order to conserve flux the excitation must on average spend equal amounts of time in the loss and gain circuits of the structure. Obviously, the reverse scenario occurs if we coupled the $\mathcal{PT}$-dimer to the TL from the gain side.

4.3.2 Double Port Scattering

Finally, we analyze the case of double port scattering (see the inset of Fig. 4.7). Application of Kirchoff’s laws again lead to

$$\begin{align*}
\eta(V_L^+ - V_L^-) &= I_L^M - \gamma V_L - i\omega V_L \\
V_L &= -i\omega [I_L^M + \mu I_R^M] ; \quad V_R = -i\omega [I_R^M + \mu I_L^M] \\
\eta(V_R^+ - V_R^-) &= I_R^M + \gamma V_R - i\omega V_R.
\end{align*} \tag{4.10}$$

By eliminating $I_{L/R}^M$, we get a system of linear equations in $V_{L/R}^+$, $V_{L/R}^-$, $V_{L/R}^+$ and $V_{L/R}^-$, which allows us to write down the transfer matrix (Eq. 2.5)

$$M(\omega) = \frac{1}{2\eta \mu \omega} \begin{pmatrix} A & i(B + C) \\
i(B - C) & A^* \end{pmatrix},$$

where

$$\begin{align*}
A &= -2\eta \omega + 2\eta(1 - \mu^2)\omega^3 + i \left[1 - \left(2 - \gamma^2(1 - \mu^2) + \eta^2(1 - \mu^2)\right)\omega^2 + (1 - \mu^2)\omega^4\right] \\
B &= 2(1 - \mu^2)\gamma \eta \omega^2 \\
C &= 1 - (2 - \gamma^2 - \eta^2)\omega^2 - (\gamma^2 + \eta^2)\mu^2 \omega^2 + (1 - \mu^2)\omega^4.
\end{align*}$$

Meanwhile, the elements of $M$ give the desired transmission and reflection (see Eqs. 2.6, 2.7); thereby, we can straightforwardly check that transmittance and reflectance satisfy
the generalized conservation relation \cite{15} derived in section 3.2.2,

\[ \sqrt{R_L R_R} = |T - 1| . \]

Furthermore, it should be realized that Eq. 4.6 is a special case of the more general conservation in which the transmittance \( T = 0 \) as per single port.

Our measurements for \( R_{L/R} \) and \( T \) are shown in Fig. 4.7. The quantity \( R_L R_R + 2T - T^2 \) (blue circles) is evaluated from the experimental data and it is found to be 1 as it is expected from Eq. 3.14. An interesting result of our analysis is that at specific \( \omega \)–values (marked with vertical dashed lines), the transmittance becomes \( T = 1 \), while at the same time one and only one of the reflectances vanishes. Therefore, in our \( \mathcal{PT} \)-symmetric active \( LRC \) dimer, we have for the first time experimentally identified the occurrence of unidirectional transparency theoretically described in section 3.2.2.

With a periodic repetition of the \( \mathcal{PT} \)-symmetric unit, it will result in the creation of unidirectionally transparent frequency bands. The phenomenon was first predicted in \cite{8} and its generalization was discussed in \cite{15}.

### 4.4 Summary

In this chapter, we reviewed the experimental studies of \( \mathcal{PT} \)-symmetric electronic circuitry. In particular, we have presented experimental evidence of the anomalous properties of \( \mathcal{PT} \)-symmetric scattering - unidirectional transparency, conservation relations and spatial wave profiles. On this basis, we propose \( \mathcal{PT} \)-symmetric \( LRC \) circuits as an easily realizable system where many other theoretical ideas can be investigated. Their simplicity and direct accessibility to the dynamical variables enables insight and a more thorough understanding of \( \mathcal{PT} \)-symmetric scattering.
Figure 4.7: Experimental measurements (symbols) of $T$, $R_{L/R}$ for the two-port scattering set-up shown in the inset. The solid lines (with the corresponding colors) are the numerical values of $T$, $R_{L/R}$. The conservation relation Eq. 3.14 $R_L R_R + 2T - T^2 = 1$ extracted from the experimental data is reported with blue filled circles. The horizontal dashed blue line indicate the value 1. The vertical dashed lines indicate the frequencies for which we have unidirectional transparency. We have used the same parameters as those used in Fig. 4.4.
Chapter 5

Broadband Unidirectional
Invisibility induced by $\mathcal{PT}$
Symmetric Periodic Structures

In recent years the subject of cloaking physics has attracted considerable interest, specifically in connection to transformation optics \cite{31,32}. Instead, in this chapter, we explore the possibility of synthesizing $\mathcal{PT}$-symmetric objects which can become unidirectionally invisible under appropriate conditions. In this respect, we have to stress that our notion of invisibility stems from a fundamentally different process. As opposed to surrounding a scatterer with a cloak medium, in our case the invisibility arises because of spontaneous $\mathcal{PT}$-symmetry breaking. This is accomplished via a judicious design that involves a combination of optical gain and loss regions and the process of index modulation. Specifically, we consider scattering from $\mathcal{PT}$-synthetic Bragg structures (see Fig. 5.1) and investigate the consequences of $\mathcal{PT}$ symmetry in the scattering process \cite{8}. It is well known that passive gratings (involving no gain or loss) can act as high efficiency reflectors around the Bragg wavelength. Instead, we find that at the $\mathcal{PT}$ sym-
metric breaking point, the system is reflectionless over all frequencies around the Bragg resonance when light is incident from one side of the structure while from the other side its reflectivity is enhanced (broadband unidirectional transparency). Furthermore, we show that in this same regime the transmission phase vanishes — a necessary condition for evading detectability. Even more surprising is the fact that these effects persist even in the presence of Kerr non-linearities.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.1}
\caption{Unidirectional invisibility of a $\mathcal{PT}$-symmetric Bragg scatterer. The wave entering from the left (upper figure) does not recognize the existence of the periodic structure and goes through the sample entirely unaffected. On the other hand, a wave entering this same grating from the right (lower figure), experiences enhanced reflection.}
\end{figure}

\section{5.1 $\mathcal{PT}$-symmetric Bragg Grating}

We consider an optical periodic structure or grating having a $\mathcal{PT}$-symmetric refractive index distribution

\begin{equation}
 n(z) = n_0 + n_1 \cos(2\beta z) + in_2 \sin(2\beta z) \text{ for } |z| < L/2 . 
\end{equation}

This grating is embedded in a homogeneous medium having a uniform refractive index $n_0$ for $|z| > L/2$ (see Fig. 5.1). Here $n_1$ represents the peak real index contrast and $n_2$ the gain/loss periodic distribution. In practice, these amplitudes are small, e.g. $n_1$, \dots
The grating wavenumber $\beta$ is related to its spatial periodicity $\Lambda$ via $\beta = \pi/\Lambda$ and in the absence of any gain modulation ($n_2 = 0$) the periodic index modulation leads to a Bragg reflection close to the Bragg angular frequency $\omega_\beta = c\beta/n_0$.

Recall Eq. 2.3 that a time-harmonic electric field of frequency $\omega = ck/n_0$ obeys the Helmholtz equation
\[
\frac{\partial^2 E(z; k)}{\partial z^2} + \frac{\omega^2}{c^2}n^2(z)E(z; k) = 0 .
\] (5.2)

For $|z| \geq L/2$, Eq. 5.2 admits the solution $E^\text{\l}_0(z) = E^+_f \exp(ikz) + E^-_b \exp(-ikz)$ for $z < -L/2$ and $E^\text{\l}_0(z) = E^+_f \exp(ikz) + E^+_b \exp(-ikz)$ for $z > L/2$. Recall further Eq. 2.5 that the amplitudes of the forward and backward propagating waves outside of the grating domain are related through the transfer matrix $M$

\[
\begin{pmatrix}
E^+_f \\
E^+_b
\end{pmatrix}
= 
\begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
E^-_f \\
E^-_b
\end{pmatrix} .
\] (5.3)

At the same time, the transmittance and reflectance are given by Eqs. 2.6 2.7.

While the transmission for left or right incidence is the same (see Chapter 2), this is not necessarily the case for the reflection. From the above relations one can deduce the form of the scattering matrix $S$ (Eq. 2.8). For $\mathcal{PT}$-symmetric systems, the eigenvalues of the $S$-matrix either form pairs with reciprocal moduli or they are all unimodular (see section 3.2.3). In the latter case the system is in the exact $\mathcal{PT}$-phase while in the former one it is in the broken-symmetry phase.

## 5.2 Theoretical Analysis

To analyze this structure we decompose the electric field inside the scattering domain $E(z)$, in terms of forward $E^+_f(z)$ and backward $E^-_b(z)$ traveling envelopes as
\[
E(z) = E^+_f(z) \exp(ikz) + E^-_b(z) \exp(-ikz) .
\] (5.4)
Section 5.2. Theoretical Analysis

Here, we will assume slowly varying envelopes for the field i.e. $E_f''(z) = 0$ and $E_b''(z) = 0$. Substituting these expressions in Eq. 5.2 and keeping only synchronous terms while eliminating second order corrections in $n_1$, and $n_2$, we can then express the field at a point $z$ inside the sample in terms of the field at $z = -L/2$. (see appendix B.1 for detail). For $k \approx \beta$ close to the Bragg point, we get

$$\begin{pmatrix} E_f(z) \\ E_b(z) \end{pmatrix} = e^{iz\delta} \hat{U} e^{iL\delta/2} \begin{pmatrix} E_f(-\frac{L}{2}) \\ E_b(-\frac{L}{2}) \end{pmatrix},$$

(5.5)

where $\delta = \beta - k$ is the detuning and the evolution operator $U$ is given by

$$\hat{U} = \cos[\lambda(z + L/2)] \hat{1} - i \sin[\lambda(z + L/2)] \hat{\sigma} \cdot \hat{e}.$$

Here, $\hat{\sigma} \cdot \hat{e}$ is the Pauli matrix vector dotted with the unit vector $\hat{e}$ defined as $\hat{e} = (1/\lambda)(-kn_2/2n_0; -ikn_1/2n_0; \delta)$, while $\lambda = \sqrt{\delta^2 - k^2(n_1^2 - n_2^2)/4n_0^2}$. By imposing continuity of the field at $z = \pm L/2$, Eq. 5.5 becomes equivalent to Eq. 5.3. The transmittance $T \equiv |t|^2$ and reflectance $R_L \equiv |r_L|^2$ and $R_R \equiv |r_R|^2$ are in this case

$$T = \frac{|\lambda|^2}{|\lambda|^2 \cos^2(\lambda L) + |\delta|^2 \sin(\lambda L)|^2},$$

$$R_L = \frac{(n_1 - n_2)^2 k^2/4n_0^2}{\delta^2 + |\lambda \cot(\lambda L)|^2}; \quad R_R = \frac{(n_1 + n_2)^2 k^2/4n_0^2}{\delta^2 + |\lambda \cot(\lambda L)|^2}.$$ 

(5.6a, 5.6b)

For $n_2 = 0$ one recovers the standard scattering features of periodic Bragg structures. Namely, $R_L = R_R$, while close to the Bragg point $\delta = 0$ (that is, within the Bragg bandgap $\sim n_1 \beta$), the reflectance/transmittance becomes unity/zero in the large $L$-limit, see Fig 5.2a. Since the structure is passive (without any gain or loss), the total norm is conserved, that is, $T + R = 1$. Instead if $n_2 \neq 0$, an “asymmetry” in the left/right reflectance starts to develop. For $n_1 > n_2$, the system essentially retains the features of the passive grating except that the norm conservation is now violated, Fig. 5.2b. For $n_1 < n_2$, the asymmetry in the two reflections has become greater and one begins to note transmission/reflection resonances within the former Bragg bandgap, Fig. 5.2d. Recall from section 3.2.3 that sub-unitary transmittance corresponds to exact
Figure 5.2: Transmittance and reflectance for $\mathcal{PT}$ Bragg grating with the refractive index given by Eq. 5.1

(a) $n_2 = 0$, (b) $n_1 > n_2$, (c) $n_1 = n_2$, (d) $n_1 < n_2$

phase whereas super-unitary transmittance signifies broken phase; therefore, by the increasing the gain/loss distribution $n_2$, the grating goes through a $\mathcal{PT}$ transition from exact phase ($n_1 > n_2$) to broken phase ($n_1 < n_2$).

At $n_1 = n_2$, $\lambda = \delta$ and we have $T = 1$, Fig. 5.2. Therefore, the system is at the exceptional point (cf. section 3.2.3). Here, the reflectance asymmetry becomes most pronounced. Remarkably within the frequency window comparable to the former Bragg bandgap, the reflectance for left incident waves vanishes $R_L = 0$ whereas $R_R \neq 0$; therefore, the $\mathcal{PT}$ Bragg structure features a broadband occurrence of unidirectional transparency discussed in section 3.2.2. The scaling of the non-vanishing reflectance is also noteworthy; at the Bragg point $\delta = 0$, the reflectance for right incident waves grows with the size $L$ of the sample as

$$R_R = L^2 \left( \frac{k n_1}{n_0} \right)^2 \left( \frac{\sin(L\delta)}{L\delta} \right)^2 \delta \rightarrow 0 \quad L^2 \left( \frac{k n_1}{n_0} \right)^2 . \quad (5.7)$$

Such quadratic increase of the field intensity is also a hallmark of exceptional point...
Section 5.3. Exact Numerical Model

We will refer to this phenomenon as unidirectional reflectivity. Furthermore, Eqs. 5.6 indicate that a transformation $n_2 \rightarrow -n_2$ reverts the reflectivity of the system, allowing for reflectionless behavior for right incident waves i.e. $R_R = 0$, while the reflectance from the left $R_L$ is now following the prediction of Eq. 5.7. In other words, the phase lag between real and imaginary refractive index dictates the unidirectional reflectivity of the system.

5.3 Exact Numerical Model

![Diagram of an optical slab with PT-symmetric Bragg grating]

Figure 5.3: An illustration of an optical slab with $\mathcal{PT}$-symmetric Bragg grating

The theoretical model considered in the previous section employs a smoothly varying analytic sinusoidal function $n(z)$ that might be impossible to realize in practice. A more realistic model can be created by substituting the sinusoids with the square waveforms. By doing so, one effectively studies a multi-layered optical slab with each layer of width $d$, the layers being arranged so as to simulate the cosine and sine modulations of the analytic model. In other words, we have for $|z| \leq \frac{L}{2}$

$$n(z) = n_0 + n_1 \text{sgn} \{\cos(2\beta z)\} + in_2 \text{sgn} \{\sin(2\beta z)\},$$

where $\beta = \frac{\pi}{4d}$, see Fig. 5.3.

In this case, it is straightforward to numerically compute the transfer matrix and hence the transmission and reflection via the method described in section 2.3. In particular, we have used $n_0 = 1$, $n_1 = 10^{-3}$, $L = 12.5\pi$ and $\beta = 100$, see Fig. 5.4. The exact calculations show that at the exceptional point when $n_2 = n_1$, $R_L$ is diminished (up to
Figure 5.4: Exact numerical evaluation of transmittance $T \equiv |t|^2$ and reflectance $R = |r|^2$ for a multi-layered optical slab with $\mathcal{PT}$-symmetric periodicity.

These results are in excellent agreement with Eqs. 5.6 and 5.7. Also, we report the $L^2$ behavior of reflectivity at the Bragg point, Fig. 5.5.

The multi-layered model also offers a more intuitive understanding of unidirectional reflectivity at Bragg point $k = \beta = \pi/4d$ via phase cancellation effects. Consider a basic $\mathcal{PT}$-symmetric unit of the structure, the four-layered slab, see Fig. 5.6: the entire slab is simply a periodic array of such basic units. The total reflection from each basic unit can be approximated to the first order by the superposition of multiple reflection events at the layer interfaces

$$r \approx r_1 e^{i\theta_1} + r_2 e^{i\theta_2} + r_3 e^{i\theta_3} + r_4 e^{i\theta_4} + r_5 e^{i\theta_5},$$
Figure 5.5: $L^2$ behavior of the right reflectance at $\delta = 0$ and $n_1 = n_2$ for the numerical model of Fig. 5.3. The same parameters are used as those in Fig. 5.4 (see text).

where $r_l \approx \frac{\Delta n_l}{2n_0} = \frac{|\Delta n_l|}{2n_0}e^{i\phi_l}$, $\theta_l$ is the total phase accumulation for the $l$-th reflected wave due to uninterrupted propagation up till the corresponding interface, $\Delta n_l$ is the refractive index contrast across the interface and $\phi_l$ is the phase shift due to reflection.

For the case of $n_1 = n_2$, the index contrasts for intermediate boundaries are the same $|\Delta n_2| = |\Delta n_3| = |\Delta n_4| = 2n_1$. Noting that at Bragg point $k = \pi/4d$, the round trip phase accumulation in each layer is $\pi/2$, we can calculate the total phase change for $r$, yielding for left reflection

$$r_L \approx \frac{n_1|1 + i|}{2n_0}e^{i\frac{\pi}{4}} + \frac{n_1}{n_0}\left(e^{i\frac{\pi}{2}} + e^{i\frac{\pi}{2}} + e^{i\frac{7\pi}{2}}\right) + \frac{n_1|1 - i|}{2n_0}e^{i\frac{11\pi}{4}}$$

$$= 0 .$$

On the other hand, a similar calculation reveals that for each basic unit, $r_R \approx -(3 + \sqrt{2})i\frac{n_1}{n_0}$. In a repetition of such units, $|r_R|$ will grow linearly with the increasing system size, whereas the reflectance will grow quadratically (i.e., $R_R \sim L^2$). Therefore, we recover the phenomenon of unidirectional reflectivity.
Reflectionless potentials in one-dimensional scattering configurations are not in general invisible. This is due to the fact that the phase of the transmitted wave might depend on energy, thus leading to wavepacket distortion after the potential barrier. In this respect, a transparent potential can be detected from simple time-of-flight measurements. It is therefore crucial to examine the phase $\phi_t$ of the transmission amplitude $t = |t| \exp(i\phi_t)$ and compare it with the phase acquired by a wave propagating in a grating-free environment ($\phi_t = 0$)\(^1\). Using Eq. 5.5, we deduce the phase $\phi_t$ close to the

\[^1\text{We do not consider the trivial phase } kL \text{ associated with free propagation} \]
Section 5.4. Unidirectional Invisibility

Figure 5.7: (a) Transmission phase $\phi_t$ as a function of the detuning $\delta$ for the $\mathcal{PT}$-periodic system of Figure 2. For comparison we plot together with the results of the $\mathcal{PT}$-exceptional point ($n_2 = n_1$), also the transmission phase for the passive structure ($n_2 = 0$). (b) The corresponding transmission delay times $\tau_t$ as a function of detuning.

Bragg point

$$\phi_t = \arctan\left(-\frac{\delta}{\lambda}\tan(\lambda L)\right) + L\delta . \quad (5.8)$$

At $n_1 = n_2$ we find that $\delta = \lambda$, which results to a transmission phase $\phi_t = 0$. Thus interference measurements will fail to detect this periodic structure. Although the above theoretical analysis is performed close to the Bragg point $\delta \approx 0$, our numerical results, reported in Fig. 5.7a, indicate that these effects are valid over a very broad range of frequencies. For comparison, we also report in Fig. 5.7a, the transmission phase for the case of a passive ($n_2 = 0$) Bragg grating.

Next, we analyze the dependence of the transmission delay time $\tau_t \equiv d\phi_t/dk$ [34, 35], on the detuning $\delta$. This quantity provides valuable information about the time delay
(or advancement) experienced by a transmitted wavepacket when its average position is compared to the corresponding one in the absence of the scattering medium. Using Eq. [5.8] we find that at the spontaneous $\mathcal{PT}$-symmetry breaking point the transmission delay time is $\tau_t = 0$. In Fig. 5.7b, we show results for a $\mathcal{PT}$-structure at $n_1 = n_2$ together with those expected from the passive case.

### 5.5 Kerr Nonlinearity

It is also interesting to investigate the robustness of the above phenomena in the presence of Kerr nonlinearities. To this end, we assume the presence of a Kerr term in the refractive index profile i.e. $n(z) = n_0 + n_1 \cos(2\beta z) + in_2 \sin(2\beta z) + \chi |E(z)|^2$. As before, we decompose the optical field into two counter-propagating wave envelopes $E(z) = E_f(z) \exp(ikz) + E_b(z) \exp(-ikz)$. In nonlinear problems, it is convenient to work with Stokes variables defined as

$$
S_0 = |E_f|^2 + |E_b|^2 \\
S_1 = |E_f|^2 - |E_b|^2 \\
S_2 = E_f E_b^* + E_b E_f^* \\
S_3 = i(E_f E_b - E_b E_f^*)
$$

Applying slowly varying approximation and considering only synchronous terms [36] (see also appendix B.2), we can then obtain a set of equations describing the dynamics of the Stokes variables,

$$
\dot{S}_0(z) = 2\kappa S_3 \\
\dot{S}_1(z) = 2g S_3 \\
\dot{S}_2(z) = 2\delta S_3 - 3\rho S_0 S_3 \\
\dot{S}_3(z) = 2\delta S_2 - 3\rho S_0 S_2 - 2\kappa S_0 + 2g S_1,
$$

(5.10a-d)
where $\rho = k\chi/n_0$, $\kappa = kn_1/2n_0$ and $g = kn_2/2n_0$. The Stokes variables satisfy the relation $S_1^2 + S_2^2 + S_3^2 = S_0^2$. At the same time the above system Eq. 5.10 conserves the following quantities

$$gS_0 - \kappa S_1 = C_1 \quad (5.11a)$$
$$3\rho gS_0^2 - 4\kappa\delta S_1 + 4\kappa gS_2 = C_2. \quad (5.11b)$$

Using these constants of the motion, one can solve exactly Eqs. 5.10, see appendix B.2 – yielding the transmittance and reflectance

$$T_L = \frac{(\kappa - g)S_0(\frac{L}{2}) - C_1}{(\kappa - g)S_0(\frac{-L}{2}) + C_1}; \quad R_L = \frac{(\kappa - g)S_0(\frac{-L}{2}) + C_1}{(\kappa - g)S_0(\frac{L}{2}) - C_1} \quad (5.12a)$$
$$T_R = \frac{\kappa + g)S_0(\frac{L}{2}) + C_1}{\kappa - g)S_0(\frac{-L}{2}) + C_1}; \quad R_R = \frac{(\kappa + g)S_0(\frac{-L}{2}) - C_1}{(\kappa + g)S_0(\frac{L}{2}) + C_1}. \quad (5.12b)$$

In contrast to the linear case, the transmittance and reflectance are generally bistable at a single input, and we now have that $T_L \neq T_R$ for $n_1 \neq n_2$, indicating a diode action [19] (see Fig. 5.8). However, of interest here is the behavior of the system at the exceptional point $n_1 = n_2$. We find that $T_L = T_R = 1$, while $R_L = 0$, as in the linear case. These results are valid for any input intensity as shown in Fig. 5.8. At the same time, we have found that the additional transmission phase due to the grating is again flat (independent of the detuning $\delta$) and equal to $\phi_t = 0$. We thus conclude that the phenomenon of unidirectional invisibility of a $\mathcal{PT}$-periodic system at the exceptional point is entirely unaffected by the presence of Kerr non-linearities.

5.6 Summary

We have shown that the interplay of Bragg scattering and $\mathcal{PT}$ symmetry allows for a broadband occurrence of unidirectional transparency over a range of frequencies around the Bragg point. Note that this phenomenon is later generalized as anisotropic transmission resonance in [15]. Unique only to our structure, however, is the additional property that the transmission phase be independent of $k$, resulting in a phenomenon termed
Figure 5.8: Transmittance and reflectance in the nonlinear regime vs. left (left column) and right (right column) input intensities for $\delta = 0$ (the same behaviour is observed for other values of $\delta$). The parameters used are $n_0 = 1$, $n_1 = 0.5$, and $L = 7$. Three different values of $n_2$ (below, above and at the exceptional point) are used. If $n_2 \neq n_1$ one can observe the standard bistability behavior of non-linear media. For $n_2 = n_1$, the bistability disappears, signifying the appearance of the spontaneous $\mathcal{PT}$-symmetry breaking point. At this point $R_L = 0$.

unidirectional invisibility [8]. This process was found to be robust against perturbations. In the presence of nonlinearities, this unidirectional invisibility still persists at the exceptional point while non-reciprocal bistable transmission occurs elsewhere.
Chapter 6

Light Transport in Disordered Media with $\mathcal{PT}$ Symmetry

One characteristic of wave propagation that is common to various physical systems as diverse as classical, quantum and atomic-matter waves is wave interference phenomena. Their existence results in an exponentially suppressed transmittance and thus in a complete halt of wave propagation in random media which can be achieved by increasing the randomness of the medium. This phenomenon was predicted fifty years ago in the framework of quantum (electronic) waves by Anderson [37] and its existence has been confirmed in recent years in experiments with matter [38, 39] and classical waves [40–48]. These systems have allowed for a detailed study of the Anderson localization, undisturbed by interactions or other effects which characterize electron propagations.

While the study of localization of quantum and classical wave in random media has been well understood by now [49, 50], only the last decade, the photon propagation in active random media has been pursued intensively [51–60]. Due to the absence of a conservation law for photons, light may be absorbed or amplified in the medium while phase coherence is preserved. This interplay of absorption or amplification and localization
has been studied extensively by using the Helmholtz equation with an imaginary dielectric constant of an appropriate sign. Several interesting effects have been predicted, such as the localization length of a random medium with gain and the dual symmetry of absorption and amplification for the first moment of transmission \[58\ 59\], the sharpness of back scattering coherent peak and the statistics of super-reflection and transmission \[51\ 57\].

Work has also been done on disordered \(\mathcal{PT}\) systems, with main focus on the spectral properties of the corresponding \(\mathcal{PT}\)-Hamiltonians \[61\ 64\]. Results in this direction include the study of the interplay between \(\mathcal{PT}\)-symmetries and Anderson localization, the structure of the localized modes, the statistical properties of the eigenvalues, the investigation of the scenario towards the spontaneous \(\mathcal{PT}\)-symmetry breaking, and the development of a one-parameter scaling theory for the critical value of the gain/loss parameter \(\gamma_{\mathcal{PT}}\).

However, relatively little has been done concerning the transport properties of these systems. In this chapter, we will examine the transmission and reflection through one-dimensional (1D) \(\mathcal{PT}\)-symmetric systems with random index of refraction (see Fig. 6.1). We show that the exponential decay rate of transmission coefficient, which defines the inverse localization length \(\xi^{-1}\), is associated with the harmonic sum of the localization length \(\xi_0(W)\) of a passive system with the same degree of randomness and the amplification length \(\xi_\gamma(0)\) of a periodic \(\mathcal{PT}\)-symmetric system with the same degree of gain/loss. Furthermore, we find that the asymptotic value of the reflection coefficient follows a single parameter scaling law which is dictated by the ratio \(\Lambda = \xi_0(W)/\xi_\gamma(0)\). Finally, we show that while the transmission processes are reciprocal to left and right incident waves, the reflection is enhanced from one side and it is inversely suppressed from the other, thus allowing such \(\mathcal{PT}\)-symmetric random media to act as unidirectional coherent absorbers (see Fig. 6.1).
6.1 Mathematical Model and Theoretical Considerations

We consider a one-dimensional (1D) active disorder sample having a random $\mathcal{PT}$-symmetric refractive index distribution $n(z) = n_0 + n_R(z) + in_I(z)$ in the interval $|z| < L/2$. The system is embedded in a homogeneous medium having a uniform refractive index $n_0$ for $|z| > L/2$ (see Fig. 6.1). Without loss of generality, we will assume below that the refractive index $n_0$ outside the disordered medium is $n_0 = 1$. Here $n_R$ represents the real index contrast and $n_I$ the gain/loss spatial profile. In experimental realization in optics [2, 3], these amplitudes are small, e.g. $n_R, n_I \ll n_0$. For simplicity,
we will assume that the sample is composed of an even number, \( L \), of layers of uniform width \( d \), each with a constant real refractive index \( n^R(z_j - d/2 \leq z \leq z_j + d/2) = n^R_j \) given by a random distribution, while the imaginary part is constant (optical slab model — see section 2.3). Specifically we will assume

\[
    n(z) = n_0 + n^R(z) + i\gamma \quad \text{for} \quad -L/2 < z < 0 \\
    = n_0 + n^R(z) - i\gamma \quad \text{for} \quad 0 < z < L/2, \tag{6.1}
\]

where \( \gamma \geq 0 \) is a fixed gain/loss parameter, which we restrict to nonnegative numbers to eliminate ambiguity.

Although the majority of our simulations below have been done for \( n^I(\pm z) = \mp \gamma \), we have also checked that our results apply for the case that \( n^I(\pm z) = \mp \gamma + \delta n^I \) where \( \delta n^I \) is a random variable given by a uniform distribution centered at zero. Since the qualitative features remain the same, we will not distinguish between these two cases. At the same time, in all cases below, the real refractive index contrast for the \( j \)-th layer \( n^R_j \) is a random variable with a uniform distribution between \((-W, +W)\), which satisfies the \( \mathcal{PT} \)-symmetric constraint \( n^R(z_j) = n^R(-z_j) \). Since the system is \( \mathcal{PT} \)-symmetric, we recall that the scattering signals obey the generalized conservation relation Eq. 3.14.

### 6.2 Scaling Results for Transmittance and Reflectance

Although the transport properties of disordered media and the interplay with gain or loss separately have attracted the attention of researchers during the last decade, the investigation of the combined effect of gain and loss under the constraint of \( \mathcal{PT} \)-symmetry on localization effects is still in its infancy. Most of the existing works address the spectral and eigenvector structure [61][64], without analyzing the transport properties of such systems.
Below we investigate the scaling properties of transmission and reflection from such optical structures [9], with respect to the disordered strength $W$, the gain/loss parameter $\gamma$, and the wave-vector $k$ of the interrogating wave. To this end, we employ the transfer matrix method (see section 2.3) to numerically compute the transmittance $T$ and left and right reflectances $R_L$, $R_R$ for various values of random refractive index contrast $W \in (0,0.5)$, and gain/loss parameter $\gamma \in (0,0.01)$ which is typical for optical media. We use slabs of length between $L = 10$ to $L = 10^4$ number of layers, each having width $d = 1$. The logarithmic averages $\langle \ln T \rangle$ and $\langle \ln R \rangle$ are performed over $10^4$ disorder realizations for each set of parameters.
6.2.1 Transmittance

The transport properties of passive (no gain or loss) disorder systems have been thoroughly studied. At large length-scale, such systems exhibit an exponential decay in transmittance (see black line in Fig. 6.2). The associated inverse decay rate, $\xi_0(W)$, reflects the degree of randomness in the medium and it is defined as

$$1/\xi_0(W) \equiv -\lim_{L \to \infty} \frac{\langle \ln T \rangle}{L}. \quad (6.2)$$

In the case of one-dimensional random media we have that $\xi_0(W) \sim 1/W^2$ [50].

The other limiting case of an ordered $\mathcal{PT}$-symmetric medium, can be also treated analytically. One can explicitly solve for the electric field inside the perfect $\mathcal{PT}$ layered structure subject to scattering boundary conditions (see section [3.3]). The resulting expression for the transmittance reads

$$T = \left(8 (1 + \gamma^2)^2 \right) \left(\gamma^4 (4 + \gamma^2) \cosh(2kL\gamma) + 8 + 4\gamma^2 + 5\gamma^4 - \gamma^6 - \gamma^2 (4 + \gamma^2) \cos(2kL) \right) + 16\gamma^2 \cos(kL) \cosh(kL\gamma) - 4\gamma^3 \left(2 + \gamma^2\right) \sin(kL) \sinh(kL\gamma) \right)^{-1}. \quad (6.3)$$

For sufficiently large $L$, the term involving $\cosh(2kL\gamma)$ becomes dominant and the transmittance decay exponentially as shown in Fig. 6.2 (see red line). For the experimentally relevant case where $\gamma \ll 1$, the asymptotic decay in transmittance can be found from Eq. 6.3 to be

$$T_{\infty} \approx \frac{16e^{-2k\gamma L}}{\gamma^4(4 + \gamma^2)}. \quad (6.4)$$

The corresponding decay rate of transmittance can be calculated to be

$$\frac{1}{\xi_\gamma(0)} \equiv -\lim_{L \to \infty} \frac{1}{L} \ln T \to 2k\gamma, \quad (6.5)$$

which can serve as an operative definition of the so-called attenuation/amplification length $\xi_\gamma(0)$.

On the other hand, for system sizes $L$ smaller than a critical length-scale $L_c$ the transmittance remains approximately constant $T \approx 1$. Near the critical length $L \approx L_c$, large
oscillations in the transmittance emerge (see red line in Fig. 6.2) after which the transmittance decays according to the expression given by Eq. 6.4. The value of $L_c$ can be evaluated approximately by the condition $T_\infty(L = L_c) = 1$ which leads to the following expression for the critical length

$$L_c \approx \frac{1}{2k\gamma} \ln \left( \frac{16}{\gamma^4(4 + \gamma^2)} \right).$$

(6.6)

The existence of a critical length-scale $L_c$ is characteristic of gain media and is associated with the lasing threshold for which $T$ diverges. Below this length stimulated emission enhances transmittance through the gain medium. On larger length scales stimulated emission reduces transmittance. The slopes of $\ln T$ at both sides of the maximum are approximately symmetric. In contrast, in the case of a $\mathcal{PT}$-symmetric refractive index the increase of the transmittance for $L < L_c$, which is due to the gain, is balanced by the equal amount of loss which is symmetrically arranged inside the medium. As a result $T \approx 1$ for $L < L_c$. Nevertheless, this gain/loss balance is not able to smooth out the diverging behavior of the transmittance near the lasing threshold as can be seen from Fig. 6.2.

Let us finally consider the case of $\mathcal{PT}$-symmetric disordered slab geometry. A representative behavior of the transmittance $T$ as a function of the system size $L$ is shown in Fig. 6.2 (see green line). To understand the exponential decay of $T$, one needs to consider the simplified geometry with index of refraction given by Eq. 6.1. For the case of Eq. 6.1 the transfer matrix of the total $\mathcal{PT}$-symmetric system is the product of the transfer matrix $M_l$ associated with the lossy sub-system and $M_g$ associated with the gain sub-system. The corresponding transmittance through the combined system is then given by

$$T = \frac{|T_lT_g|}{1 - r_lr_g^2}.$$

(6.7)

\footnote{We remind again that although the majority of our simulations have been performed using a system consisting of a pair of gain-only and loss-only $\mathcal{P}$-symmetric disordered slabs each having length $L/2$ (see Eq. 6.1), we have checked that our conclusions apply for more complicated disordered $\mathcal{PT}$-symmetric refraction indices.}
Figure 6.3: The numerically extracted localization length $\xi_\gamma(W)$ for various gain/loss parameter $\gamma$ (not indicated in the figure) is plotted after rescaled with $\xi_0(W)$ versus the scaling parameter $\xi_0(W)/\xi_\gamma(0)$. The symbols and colors indicate different wavelengths $k$ of the incident wave, and disorder strengths $W$. The black line indicates the theoretical prediction of Eq. 6.11. The meshed symbols correspond to some typical $\xi_\gamma(W)$, for the scenario where the imaginary part of the refractive index is $n^I = \gamma + \delta n^I$ where $\delta n^I$ is a random variable uniformly distributed around zero. In the inset we report the un-scaled $\xi_\gamma(W)$ versus the gain/loss parameter $\gamma$.

It is, therefore, sufficient to know the scaling behavior of each of the terms on the rhs of Eq. 6.7 in order to predict the scaling behavior of $T$. $T_l$ and $T_g$ have been studied in Ref. [59], where it was found that both absorption and amplification lead to the same exponential decay of the transmittance i.e.

$$\langle \ln T_{l,g} \rangle = -(2k\gamma + \xi_0(W)^{-1})L/2.$$  \hspace{1cm} (6.8)

Note that the above exponential decay rate associated with the transmittance of an am-
plified or attenuating medium is enhanced with respect to a passive disordered medium by the strength of the gain (or loss) rate. Somewhat counter-intuitively, the sample with amplification also exhibits exponentially decaying transmittance due to the enhanced internal reflections from the boundaries. Furthermore, using the duality relation \( r_l r_g^* = 1 \) for the reflection of an amplified or attenuating medium (with the same rate of gain or loss respectively) applied for \( L/2 > \xi_0(W) \) we get

\[
\langle \ln T \rangle = \langle \ln T_l \rangle + \langle \ln T_g \rangle - 2\langle \ln |1 - r_l r_g| \rangle
\]

(6.9)

where \( \theta \) is the phase of the reflection amplitude \( r_l \). Assuming that \( \theta \) is a random variable uniformly distributed on the interval \([0, 2\pi]\) \( (P(\theta) = \frac{1}{2\pi}) \) [60], we get that the last term after performing the average over the random variable \( \theta \) vanishes, that is,

\[
\langle \ln |2(1 - \cos(2\theta))| \rangle = \int_0^{2\pi} \ln |2(1 - \cos(2\theta))| P(\theta) d\theta
\]

\[
= \frac{1}{\pi} \int_0^{\pi} \ln |2(1 - \cos(2\theta))| d\theta = 0 .
\]

Therefore we get

\[
\lim_{L \to \infty} \frac{\langle \ln T \rangle}{L} = \lim_{L \to \infty} \left( \frac{\langle \ln T_l \rangle}{L} + \frac{\langle \ln T_g \rangle}{L} \right)
\]

\[
= -(2k\gamma + \xi_0(W)^{-1}) .
\]

(6.10)

From the above argument, we conclude that the effective localization length for a \( \mathcal{PT} \)-symmetric disorder medium is

\[
\xi_\gamma(W)^{-1} = \xi_\gamma(0)^{-1} + \xi_0(W)^{-1} .
\]

(6.11)

In Fig. 6.3, we show the results of our numerical simulations for a disordered \( \mathcal{PT} \)-symmetric sample. The extracted localization length follows nicely the scaling behavior indicated by our theoretical arguments.
6.2.2 Reflectance

We proceed with the analysis of the reflection coefficient. In the case of random media with only gain or loss it was found in Ref. [59] that the reflection coefficients of two distinct disordered systems, one with amplification strength $-\gamma$ and the other with dissipative strength $\gamma$ satisfy the following reciprocal relation between them

$$R_{\text{gain}} R_{\text{loss}} = 1.$$  \hspace{1cm} (6.12)

Specifically it was found that for gain media $R_{\text{gain}} > 1$, while for lossy media we have via Eq. 6.12 the reciprocal behavior i.e. $R_{\text{loss}} = 1/R_{\text{gain}} < 1$. It is important to stress here that such systems do not distinguish between left and right incidence, that is $R_{L} = R_{R}$ for each of the cases.

On the other hand, $\mathcal{PT}$-symmetric systems distinguish the reflection between left or right incident wave, that is, $R_{L} \neq R_{R}$ in general (see discussion in Section 6.1). This phenomenon has been already observed for periodic $\mathcal{PT}$-symmetric structures in Ref. [8].

Moreover, in the presence of random index of refraction, we have concluded from section 6.2.1 that the transmittance is effectively diminished exponentially with a rate $1/\xi_{\gamma}(W)$ given by Eq. 6.11. In this case therefore, we conclude based on the conservation relation Eq. 3.14 that

$$R_{R} R_{L} \rightarrow 1.$$  \hspace{1cm} (6.13)

Although this relation is similar with Eq. 6.12 it should be emphasized once more that in the case of $\mathcal{PT}$-symmetric disorder media the medium behaves simultaneously as a gain medium (i.e. having $R_{L} > 1$) and as a lossy medium (i.e. it can enhance absorption of incoming coherent waves $R_{R} < 1$). The reciprocity of the left and right reflections is clearly demonstrated for some representative cases in the inset of Fig. 6.4.

A natural question is associated with the scaling behavior of the asymptotic value of the reflectance $R_{L,R}^{\infty}$ as a function of the disorder strength $W$ and the gain/loss parameter $\gamma$. We speculate that a one-parameter scaling law describes the asymptotic reflectance
Figure 6.4: Inset: Typical reflection coefficients $R_{L,R}$ versus the system size (number of layers $L$). The main figure displays the asymptotic value of $|\log R^\infty|$ versus the scaled parameter $\Lambda$. For large values of $\Lambda$ the $|\log R^\infty|$ increases indicating that $R^\infty_L$ (i.e. reflectance for an incident wave entering the sample from the lossy side) diminishes. In this domain, the sample act as a unidirectional absorber. We use the same symbol and color coding for our data as in Fig. 6.3.

i.e.

$$R^\infty_{L,R}(\gamma, W) = f(\Lambda), \quad \text{where} \quad \Lambda = \frac{\xi_0(W)}{\xi_\gamma(0)}.$$

(6.14)

Here, $R^\infty_{L,R}$ is the geometric mean of asymptotic reflectances, that is, $R^\infty_{L,R} = \exp(\langle \ln R^\infty_{L,R} \rangle)$. We have tested our hypothesis numerically. To this end, we have extracted $R^\infty_L$ from our data for various values of $\gamma$ and $W$ and plot them against the scaling variable $\Lambda$. The results are presented in the main part of Fig. 6.4. We find that for realistic values of the gain/loss parameter $\gamma \leq 10^{-2} - 10^{-3}$ the data follow nicely the one-parameter scaling hypothesis Eq. 6.14. As the scaling parameter $\Lambda$ increases (either by decreasing...
the disorder refractive index contrast or by increasing the gain/loss parameter $\gamma$), the asymptotic value $R_{L,R}^\infty$ decreases/increases. Such a behavior allow us to use the proposed structure as a unidirectional coherent absorber that can increase absorption by tuning up the scaling parameter $\Lambda$. Here, we want to mark that our proposed structure, is different from the one suggested in Ref. [65], where it is shown that a disordered system with a single dissipative element causes coherent enhanced absorption at any frequency if the phases of the input field are appropriately manipulated. Instead, we are addressing a different problem, where we have broadband absorption in one direction without a need for phase coherence in the incoming waves.

6.3 Summary

In this chapter, we have investigated the transport properties of one-dimensional $\mathcal{PT}$-symmetric disordered layers. We have found that the localization length $\xi_\gamma(W)$, defined as the inverse decay rate of the transmission coefficient, is smaller than the localization length of the passive disordered system $\xi_0(W)$ and from the absorption/amplification length $\xi_\gamma(0)$ of a periodic $\mathcal{PT}$-symmetric medium. At the same time the reflection depends on the direction of the incident wave: while for incident waves entering the medium from the gain side it is enhanced, it is suppressed if the wave enters the medium from the lossy side. The reduction/enhancement of the reflectance is dictated by a one parameter scaling $\Lambda = \xi_0(W)/\xi_\gamma(0)$ and allow us to use such structures as unidirectional quasi-perfect coherent absorbers.
Chapter 7

Conclusion and Outlook

In this thesis, we have studied wave transport in one-dimensional complex media with parity time (PT) symmetry. These structures incorporate a judiciously balanced gain and loss elements which in the optics framework is reflected in the complex index of refraction satisfying the following relation: \( n(z) = n^*(-z) \). Due to the specific symmetries involved in such structures, the resulting scattering signals show intriguing features which keep promise for novel functionalities. To unveil these properties, two complementary methods have been employed; (a) the transfer matrix method allows us to investigate directly transmittance and reflectance from such structures while (b) the scattering matrix formalism allows us for a detailed theoretical description of the scattering process.

Specifically, new conservation relations for transmitted and reflected signals have been derived which take the form \( \sqrt{R_L R_R} = |1 - T| \) where \( R_L \) and \( R_R \) are left and right reflectances and \( T \) is the transmittance. Typically, \( R_L \neq R_R \) reflecting the fact that PT systems exhibit non-unitary scattering processes. In particular, this asymmetry tends to become most pronounced with one of the reflections going to zero at so-called exceptional points (EP) where the PT-symmetric system transitions from an exact phase (unimodular S-matrix eigenvalues) to a broken phase (non-unimodular S-matrix eigenvalues).
eigenvalues). Another intriguing feature of $\mathcal{PT}$-symmetric scattering is the existence of a special set of input frequencies where incoming signals with special phases get entirely absorbed. For any other phases of the incoming waves, the system exhibits lasing at these specific frequencies.

These anomalous transport properties had found intriguing manifestation in $\mathcal{PT}$-symmetric Bragg gratings [8]. We have found that in the case where the real and imaginary index contrast modulations are the same, a broadband unidirectional transparency occurs. In other words, under this condition, the system exhibits perfect transmission while reflection from one side vanishes. Furthermore, the transmission coefficient and phase are indistinguishable from those expected in the absence of a grating. The phenomenon is robust even in the presence of Kerr non-linearities, and it can also effectively suppress optical bistabilities.

More complex structures involving $\mathcal{PT}$-symmetric disorder were also reviewed in this thesis [9]. When gain and loss are introduced in a $\mathcal{PT}$-symmetric manner, the localization length is found to follow a simple rule involving the harmonic sum of the passive localization length and a pure amplification/absorption length $\xi_\gamma$; in other words, $\xi_\gamma(W)^{-1} = \xi_0(W)^{-1} + \xi_\gamma(0)^{-1}$. This simple rule was predicted by theoretical argumentation and supported by our extensive numerical simulations. Again, as it is common to $\mathcal{PT}$ structures, the right and left reflections are different, following the constrained asymmetry $R_L \cdot R_R \to 1$ in the limit of vanishing transmission at long lengths. As a result, such $\mathcal{PT}$-symmetric random media can act as unidirectional coherent absorbers.

Finally, inspired by the previous results in optics, we have extended the notion of $\mathcal{PT}$-symmetry into electronics. Utilizing a simple experimental setup involving two coupled LC circuits with amplification and attenuation, we were able to exhibit experimentally the generalized conservation relations associated with $\mathcal{PT}$ scattering processes and identify the phenomenon of unidirectional transparency first predicted in the framework of optics [7].
The implications of $\mathcal{PT}$ symmetry and its generalizations in on-chip integrated circuitry has only started being realized. For example, in photonics, there is an ongoing effort to introduce $\mathcal{PT}$-symmetric gain and loss into anisotropic magneto-optic media aiming to create high-quality polarization-independent isolators. At the same time, there is still much to be done in the area of photonically inspired electronics using standard MOS technology. The possibility to manipulate gain and loss together with nonlinearity holds promise for the creation of a new generation of on-chip electronic diodes. Similarly, there is a lot of excitement in the possibility of realizing reconfigurable high-frequency (Tera-Hertz) $\mathcal{PT}$ electronics.

In conclusion, in this thesis we advocated a new approach to synthesize integrated photonic and electronic circuitry. In contrast to the mainstream thinking which seeks to eliminate loss as an ingredient that degrades the efficiency in various devices, we propose to manipulate it together with gain in order to create a new generation of synthetic matter. The goal of this thesis serves to promote the novelty of this new approach. However, there is still a long road ahead. As the saying goes,“the journey is the reward”, so is the scientific endeavor in this direction bound to bring us exciting insights and discoveries.
Appendix A

Waves on a Transmission Line

We consider a right-going or “left-incidence” coaxial cable (transmission line) whose circuit equivalent is given below.

$L_0$ is self-inductance per length; $C_0$ is capacitance per length. Note that the lower terminal is usually grounded and as such $V = V_{\text{upper}} - V_{\text{lower}}$.

$\frac{\partial V}{\partial x} dx$ is the potential difference against the presumed current direction. Therefore, we have $\frac{\partial V}{\partial x} dx = -L_0 dx \frac{\partial I}{\partial t}$. If we assume that after length $dx$, some part of the original

![Circuit diagram of a transmission line](image)

Figure A.1: Circuit diagram of a transmission line. Figure taken from [1]
current $I$ flows down to charge the capacitor, the negative current loss is given by $\frac{\partial I}{\partial x} dx$. Therefore the download flowing current through the capacitor (a positive quantity) is given by $-\frac{\partial I}{\partial x} dx = C_0 dx \frac{\partial V}{\partial t}$.

Combining the two, we get the wave equations for voltage and current:

$$\frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2} \quad (A.1)$$

$$\frac{\partial^2 I}{\partial x^2} = L_0 C_0 \frac{\partial^2 I}{\partial t^2} \quad (A.2)$$

Taking $Z_0 = \sqrt{\frac{L_0}{C_0}}$, we get monochromatic plane wave solutions for $V$ and $I$:

$$V_L(x,t) = V_L^+ e^{(ikx-i\omega t)} + V_L^- e^{(-ikx-i\omega t)} \quad (A.3)$$

$$I_L(x,t) = \frac{V_L^+}{Z_0} e^{(ikx-i\omega t)} - \frac{V_L^-}{Z_0} e^{(-ikx-i\omega t)} \quad (A.4)$$

where $\frac{\omega}{k} = v = \frac{1}{\sqrt{L_0 C_0}}$. 

Appendix B

Coupled Mode Approximation to $\mathcal{PT}$ Bragg Grating

B.1 Linear Problem

Recall the time-independent Helmholtz equation:

$$\frac{d^2 E(z; k)}{dz^2} + k^2 \frac{n^2(z)}{n_0^2(z)} E(z; k) = 0$$

$\mathcal{PT}$-symmetric Bragg grating is given by a refractive index

$$n(z) = n_0 + n_1 \cos(2\beta z) + i n_2 \sin(2\beta z) \quad \text{for } |z| \leq \frac{L}{2}$$

$$= n_0 \quad \text{otherwise.}$$

We assume that $n_1$, $n_2$ are small perturbations compared to the homogeneous $n_0$; therefore $n^2(z) = n_0^2 + n_0(n_1 + n_2)e^{i2\beta z} + n_0(n_1 - n_2)e^{-i2\beta z}$ where we have kept only the first order terms.

We decompose the electric field $E(z; k)$ inside the medium into the forward-travelling and backward-travelling waves with wave vector $k$, $E = E_f(z) e^{ikz} + E_b(z) e^{-ikz}$ where it
Appendix B. Coupled Mode Approximation to PT Bragg Grating

is modulated by the slowly varying envelopes $E_f(z)$ and $E_b(z)$ with the property $E_f'' \approx 0$ and $E_b'' \approx 0$. Substituting in B.1 and averaging over the rapidly oscillating terms $e^{i(2\beta+k)z}$ and $e^{-i(2\beta+k)z}$, we obtain

$$2ikE'_f e^{ikz} - 2ikE'_b e^{-ikz} + k^2 \frac{(n_1 + n_2)}{n_0} E_b e^{i(2\beta-k)z} + k^2 \frac{(n_1 - n_2)}{n_0} E_f e^{-i(2\beta-k)z} = 0$$

If we are close to the Bragg point, i.e., $k \approx \beta$, we can separate the two linearly independent waves as follows:

$$2ikE'_f e^{ikz} + k^2 \frac{(n_1 + n_2)}{n_0} E_b e^{i(2\beta-k)z} = 0$$
$$-2ikE'_b e^{-ikz} + k^2 \frac{(n_1 - n_2)}{n_0} E_f e^{-i(2\beta-k)z} = 0$$

Define $\delta = \beta - k$, $\Sigma = \frac{k(n_1 + n_2)}{2n_0}$, $\Delta = \frac{k(n_1 - n_2)}{2n_0}$ so that

$$E'_f = i\Sigma E_b e^{i2\delta z}$$
$$E'_b = -i\Delta E_f e^{-i2\delta z}$$

Let

$$E_f = u(z) e^{i\delta z}$$
$$E_b = v(z) e^{-i\delta z}$$

so that

$$\frac{d}{dz} \begin{pmatrix} u \\ v \end{pmatrix} = -iH \begin{pmatrix} u \\ v \end{pmatrix} \tag{B.1}$$

where $H = \begin{pmatrix} \delta & \Sigma \\ \Delta & -\delta \end{pmatrix}$. Given the initial values of $E_f$ and $E_b$ at $z = -\frac{L}{2}$, the complete solution is found by

$$\begin{pmatrix} E_f(z) \\ E_b(z) \end{pmatrix} = e^{i\delta z \sigma_3} U(z) e^{i\delta \frac{L}{2} \sigma_3} \begin{pmatrix} E_f(-\frac{L}{2}) \\ E_b(-\frac{L}{2}) \end{pmatrix}$$
where \( e^{i\delta z \hat{\sigma}_3} = \begin{pmatrix} e^{i\delta z} & 0 \\ 0 & e^{-i\delta z} \end{pmatrix} \) is an exponential notation in terms of the third Pauli matrix. Similarly, \( U(z) \) can be expressed as

\[
U(z) = e^{-iH(z+\frac{L}{2})} = e^{-i(z+\frac{L}{2})\lambda \hat{\sigma} \cdot \hat{e}}
\]

\[
= \cos[\lambda(z + L/2)] \hat{1} - i \sin[\lambda(z + L/2)] \hat{\sigma} \cdot \hat{e}
\]

Here, \( \hat{\sigma} \cdot \hat{e} \) is the Pauli matrix vector \((\hat{\sigma}_1; \hat{\sigma}_2; \hat{\sigma}_3)\) dotted with the normalized vector \( \hat{e} \) defined as \( \hat{e} = (1/\lambda)(-kn_2/2n_0; -ikn_1/2n_0; \delta) \), while \( \lambda = \sqrt{\delta^2 - k^2(n_1^2 - n_2^2)/4n_0^2} \).

Taking \( z = L/2 \) gives the transfer matrix \( M = e^{i\delta L \hat{\sigma}_3} U(L/2) e^{i\delta L \hat{\sigma}_3} \). Simple algebra establishes the transmission and reflection

\[
t = \frac{e^{i\delta L \lambda}}{\lambda \cos(\lambda L) + i\delta \sin(\lambda L)}
\]

\[
r_L = \frac{e^{i\delta L k(n_1 - n_2)/2n_0}}{\delta - i\lambda \cot(\lambda L)}
\]

\[
r_R = \frac{e^{i\delta L k(n_1 + n_2)/2n_0}}{\delta - i\lambda \cot(\lambda L)}
\]

In particular, \( \phi_t \) can be calculated from Eq. B.2

\[
\phi_t = \arctan \left(-\frac{\delta}{\lambda \tan \lambda L} \right)
\]

**B.2 Nonlinear Problem**

The refractive index includes a field-dependent term

\[ n(z) = n_0 + n_1 \cos(2\beta_0 z) + in_2 \sin(2\beta_0 z) + \chi |E|^2 \]

Decomposing \( E \) into forward and backward waves as before \( E = E_f(z)e^{ikz} + E_b(z)e^{-ikz} \) and neglecting the higher order terms, we can write

\[ n^2(z) = n_0^2 + n_0(n_1 + n_2)e^{i2\beta_0 z} + n_0(n_1 - n_2)e^{-i2\beta_0 z} + 2n_0 \chi (|E_f|^2 + |E_b|^2 + E_f E_b^* e^{ikz} + E_f^* E_b e^{-ikz}) \]
Appendix B. Coupled Mode Approximation to PT Bragg Grating

Define $\rho = \chi k n_0$, $\kappa = \frac{n_1 k}{2n_0}$, $g = \frac{n_2 k}{2n_0}$, $\delta = \beta_0 - k$. Substituting in B.1 using the slowly varying wave approximations together with $k \approx \beta_0$, and separating the linearly independent waves $e^{ikz}$ and $e^{-ikz}$ we get

\begin{align*}
    iE'_f + \rho(|E_f|^2 + 2|E_b|^2)E_f + (\kappa + g)E_be^{i2\delta z} &= 0 \quad (B.3a) \\
    -iE'_b + \rho(2|E_f|^2 + |E_b|^2)E_b + (\kappa - g)E_fe^{-i2\delta z} &= 0 \quad (B.3b)
\end{align*}

Let

\begin{align*}
    E_f &= u(z)e^{i\delta z} \\
    E_b &= v(z)e^{-i\delta z}
\end{align*}

so that we get the coupled equations

\begin{align*}
    iu' &= \delta u - \rho(|u|^2 + 2|v|^2)u - (\kappa + g)v \quad (B.4a) \\
    iv' &= -\delta v + \rho(2|u|^2 + |v|^2)v + (\kappa - g)u \quad (B.4b)
\end{align*}

Introduce the Stokes parameters

\begin{align*}
    S_0 &= |u|^2 + |v|^2 \quad (B.5a) \\
    S_1 &= |u|^2 - |v|^2 \quad (B.5b) \\
    S_2 &= uv^* + u^*v \quad (B.5c) \\
    S_3 &= i(u^*v - uv^*) \quad (B.5d)
\end{align*}

Note that by virtue of the above definition Eq. B.5, the Stokes’ parameters always satisfy the relation

\begin{align*}
    S_3^2 + S_2^2 + S_1^2 - S_0^2 &= 0 \quad (B.6)
\end{align*}

Differentiating B.5 and using B.4 and their conjugates, we find that

\begin{align*}
    S'_0 &= 2\kappa S_3 \quad (B.7a) \\
    S'_1 &= 2g S_3 \quad (B.7b) \\
    S'_2 &= 2\delta S_3 - 3\rho S_0 S_3 \quad (B.7c) \\
    S'_3 &= -2\delta S_2 + 3\rho S_0 S_2 + 2\kappa S_0 - 2g S_1 \quad (B.7d)
\end{align*}
Appendix B. Coupled Mode Approximation to $\mathcal{P}\mathcal{T}$ Bragg Grating

Manipulating (B.7) we get the conserved quantities:

\[ gS_0 - \kappa S_1 = C_1 \]  \hspace{1cm} (B.8)  
\[ 3\rho gS_0^2 - 4\kappa \delta S_1 + 4\kappa gS_2 = C_2 \]  \hspace{1cm} (B.9)  

Transmission and reflection coefficients can be calculated in terms of $S_0$

\[ T_L = \frac{|u(L/2)|^2}{|u(-L/2)|^2} = \frac{(\kappa + g)S_0(L/2) - C_1}{(\kappa + g)S_0(-L/2) - C_1} \]  \hspace{1cm} (B.10a)  
\[ R_L = \frac{|v(-L/2)|^2}{|u(-L/2)|^2} = \frac{(\kappa - g)S_0(-L/2) + C_1}{(\kappa + g)S_0(-L/2) - C_1} \]  \hspace{1cm} (B.10b)  
\[ T_R = \frac{|v(-L/2)|^2}{|v(L/2)|^2} = \frac{(\kappa - g)S_0(-L/2) + C_1}{(\kappa - g)S_0(L/2) + C_1} \]  \hspace{1cm} (B.10c)  
\[ R_R = \frac{|u(L/2)|^2}{|v(L/2)|^2} = \frac{(\kappa + g)S_0(L/2) - C_1}{(\kappa - g)S_0(L/2) + C_1} \]  \hspace{1cm} (B.10d)  

In the linear problem, if we set $n_1 = n_2$, then $\delta = \lambda$ and we can immediately see that $T_L = T_R = 1, R_L = 0$ while $R_R$ would be a non-zero quantity. This implies that the object becomes transparent on one side while on the other side, its presence could be detected by non-zero reflection. This carries over to the non-linear problem. If $n_1 = n_2$, we have $\kappa = g$. For the left incidence scenario, $|v(L/2)|^2 = S_0(L/2) - S_1(L/2) = 0$. Then by (B.8) $C_1 = 0$. Then $S_0 \equiv S_1$ and by (B.6) $S_2^2 + S_3^2 = 0$; since $S_2$ and $S_3$ are real numbers, this implies that $S_2 \equiv S_3 \equiv 0$ (This can happen when $v \equiv 0$.) This gives $S_0 \equiv$ constant. Thus we have $T_L = 1, R_L = 0$. For the right-incidence scenario, since $\kappa = g$ it is immediately seen that $T_R = 1$ while $R_R$ would be a non-zero number. Furthermore, for the left incidence, since $v \equiv 0$ and $S_0 \equiv$ constant, Eqs. (B.3) reduce to

\[ iE'_f + \rho S_0 E_f = 0, \]

which establishes grating-free propagation (in a non-linear medium).
Bibliography


