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Defending Mathematical Realism

by

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Defending Mathematical Realism[1]

I. Introduction

Mathematics if often considered the paradigm of science, with certitude matched only by logic. Science, however, has identifiable content whereas the questions surrounding the content of mathematics are filled with controversy. Nobody has encountered a number, a set, or a function as he has encountered cells or even atoms. We speak as if numbers exist – Euclid proved that there exist infinitely many primes, for example. All mathematicians make existential claims, such as the infinity of prime numbers. It also appears as though most mathematicians believe that the entities they study really do exist, as the physicist really believes in forces or subatomic particles, which are likewise shielded from perception. If there should be any doubt whether the entities exist or not, a mathematician can agree to condition his claims upon the existence, asserting for example, "if numbers exist, then there exist infinite prime numbers," and thereby avoid philosophical discussion. Not only is this philosophically unsatisfying but it is also unacceptable on a deeper level. It is not a coincidence that many of the great mathematicians were philosophers.

Beginning with Pythagoras, there is a great line of mathematician-philosophers. Descartes, Leibniz, Hilbert, Brouwer, Frege, Russell, and Gödel all have had an impact in both mathematics and philosophy to varying degrees. In fact, many dealt with the questions concerning the foundations of mathematics, which pushed them into a philosophical realm in search of explanations. A look to the history of mathematics shows that it is not a tower that continually builds on itself, but instead that it is a building going through continual renovation at the demands of unsettled architects. The

[1] I would like to thank my advisor, Professor Sanford Shieh for the immense help, clear guidance, thoughtful suggestions, and general encouragement along the way.

ontological questions regarding mathematical entities are important because they carry certain implications within the subject. The surface adherence to realism by mathematicians justifies the continuing adherence to classical logic employed throughout mathematics. Classical logic seems appropriate when dealing with a determinate domain of objects but would seem a worse fit for mentally constructed entities; this simplification of matters explains why some mathematicians use non-classical logic. Some insist on using intuitionist logic instead, despite the difficulties arising from the shift. The difference roughly comes down to the fact that classical logic adheres to the law of the excluded middle and proof by contradiction methods, whereas intuitionist logic does not. This difference is symptomatic of the presumptions in classical logic of an independent mathematical domain and of intuitionists that mathematical entities are mental constructs. There are other non-classical logics but intuitionism is most popular in discussions of mathematics.

Logical considerations aside, the ontological and epistemological questions surrounding mathematical entities are interesting in their own right. The questions tend to generate much prolific philosophical discussion. The motivation for this particular philosophical investigation is attempting to reconcile the prephilosophical realism that many bring to mathematics with the difficulties wrought by the philosophical implications of actually maintaining that realism. I think those familiar with mathematics would recognize the casual realist attitude that is common. Mathematicians speak as if they are exploring deep underlying facts of the world just as an astronomer speaks of his exploration of space. The fascination and romanticization of unknown mathematical territory runs counter to those who claim that the subject I about (possibly arbitrary) manipulations of meaningless symbols. There are the elegant results of mathematics,

$e^{i\pi}$=-1 for one, that seem to strike too deep a chord to be cast aside as meaningless. Few look at this and remark that it is neat that matter happened to turn out in this way. Instead, mathematicians speak of the necessity of this and other results as basic as twice two is four.

It would be nice if the philosophy harmonized with this sentiment to provide for the ease of accepting mathematical realism. Instead, the view is made difficult by antirealist reasoning. If the mathematical entities are invisible, as we tend to think, then it seems difficult to know about them. Invoking parallels with subatomic particles does not get much further as we have indirect evidence for them; they are not acausal like mathematical entities purportedly are. The antirealist might accept drawing a parallel to god, but philosophical defenses of god are not considered strong. We might say that numbers are not abstract after all, but it would then seem difficult to maintain the necessity of mathematical facts. Consideration of the philosophical implications of mathematical realism applies pressure to conceive of mathematics differently or to cease being a realist.

The aim I to defend mathematical realism, but to do so, it is first necessary to give a more detailed description of what mathematical realism is beyond the casual realism embraced by many mathematicians. To do so, I will focus on two realist philosophies of mathematics, both of which I find interesting and promising. The first is neo-Fregeanism championed mainly by Crispin Wright and Bob Hale. The second is methodological naturalism as espoused by Penelope Maddy. The neo-Fregeans develop an account of mathematics through language, which prima facie would seem an odd way of going about defending realism for language is commonly thought of as a human creation. Maddy's theory is a combination of Gödelian mysticism and Quinean indispensability. Again, it would seem odd to defend realism with a view based in perception and science, as it would be difficult to obtain the necessity of the subject through what we consider contingent. Other than the prephilosophical waxings of mathematicians, however, we have no idea what mathematical realism looks like. It is not even clear that realism is the proper view to hold, and so the problem is both to provide an adequately realist account of mathematics while still being able to overcome antirealist challenges – if that is possible.

Though the two philosophies are not straightforwardly or obviously comparable, we would still like to be able to compare them on some level, and to do so we must develop a metric by which to judge any realist philosophical account of mathematics. There are certain canonical antirealist challenges and other difficulties involved in maintaining realism that must be addressed. While it cannot be presupposed what exactly mathematical realism is, there are certain expectations, which include answering to these challenges and difficulties in a particular way. Perhaps the distinctive assertion of realism in mathematics is the independent existence of its entities, and so the challenges cannot be answered by anything which would contradict this. Where mathematics exists and how we know about it can be answered in various ways by the realist, but he cannot betray the independent existence tenet of realism (although this notion of independent existence should be made clearer). The metric by which to judge the philosophies must take these into account.

Primary among the difficulties for a realist philosophy of mathematics is the challenge laid down by Paul Benacerraf[2] and generalized by Hartry Field[3]. The ontological realism pursued often leaves it vulnerable on epistemological ground. How

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[2] Benacerraf (1973)
[3] Field (1989: 25), although the earlier (1984) presents a similar challenge

are we to have any knowledge of mind independent entities (the core of realism), particularly abstract ones? We might be willing to trust our sense to yield a reliable knowledge of the world, but in the case of mathematical entities, which are purportedly unobservable and acausal, we are without sense data. The challenge is to explain the source of knowledge of such entities. Conceding the possibility of mathematical knowledge, while perfectly consistent, goes against much of the motivation for even defending realism in the first place. The first criterion, then, is that a realist philosophy ought to be able to give a satisfying epistemological account of mathematics, making it possible to know about mathematics despite its independence.

On a similar note, we would like to be able to preclude fictionalism- the view that mathematics is a mere story.[4] Fictionalism may allow for some kind of mathematical "knowledge" but does not allow for the truth of mathematics. While false, mathematics may have some less-than-true status, i.e. conservativeness[5], which justifies its continued use in science. To allow for a fictionalist view of mathematics is to move away from the realism and towards antirealism. In order to combat the fictionalist view, the realist philosophy ought to be able to establish that mathematics is truth-apt and provide content about which our knowledge is knowledge of. A reductionist view of mathematics is not necessarily realist or antirealist, but it reduces mathematics to just concrete sensations. Naturalism does not demand that reduction of all of mathematics to the concrete but requires that its content be experienced or implicit in the concrete realm. Structuralism holds that the fundamental content of mathematics is not even numbers or entities at all but structures (for arithmetic the structure would be a progression). And then there is formalism which holds that there is no content to

[4] Field (1980)

[5] Roughly, truth-preserving. The concept of conservativeness will be looked at below.

mathematics at all; instead, it is regarded as just the rules for manipulating meaningless mathematical terms. With so many competing views about the content of mathematics, a good realist philosophy must give an exact account of the nature of mathematical content, and so this is the second criterion.

Without establishing realism in mathematics, the door is open to challenge the logic and methods employed within the discipline. Intutionists and constructivists would reconstruct existential claims to be conditioned on the ability to actually construct the thing asserted to exist (as opposed to denying its nonexistence). The much-used method of deriving a contradiction from the nonexistence of a thing to deduce its existence would be unacceptable along with the Axiom of Choice, and other mathematical instruments and methods which yield many commonly believed results. Without a realist attitude in mathematics, there is less justification, if any to continue using classical logic. Losing nonconstructive reasoning would change the way mathematics is done. Hilbert once remarked that "taking the principle of excluded middle from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists."[6] The concerns have now switched from being philosophical to being mathematical. The concern is that philosophical analysis might leave us with a mathematics that ceases to be recognized as mathematics, or more likely, that our philosophical conception of mathematics will clash with the actual practice of mathematics. There is reason to believe that mathematicians have something to offer to the discussion given their intimate relationship with the discipline. This is not to say that philosophy must uphold mathematics as practiced, but that it ought to be amenable to practice. Philosophy should be mindful of the story and rich history of mathematics, and

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[6] Hilbert (1928)

at the same time look towards the future to provide means by which the discipline can proceed. I do not wish to disallow any revisionist philosophy off the bat; there is no reason to believe that such a theory lacks merit. I do disallow a philosophy that would purport to halt mathematical progress until philosophical questions are resolved, as that, in effect, would filibuster a valuable discipline into eternity. Philosophy ought to regard mathematics as Neurath's boat which must be rebuilt while floating. The concern is less that philosophy might actually impede mathematical progress and more that a philosophy so radical cannot be accepted.

We now have an initial list of criteria which a good philosophy of mathematics should attempt to meet.

1. Epistemological Criterion: If an account chooses to focus on ontology (as realism does), it must not compromise all epistemological accounts of mathematic. Particularly, for the realist who claims that mathematical entities have an independent existence, it must be possible to give an account of the knowledge of these entities.

2. Ontological Criterion: There must be an account of what the content of mathematics is, be it mental constructions, fictions, meaningless symbols, objects, or something else. The realist must give an account of what numbers and other mathematical entities are and where they live. A full ontological account must be given.

3. Pragmatic Criterion: Any account must be amenable to mathematics. It must yield a state where mathematics can continue either in its current form or with explicit revisions. The philosophy must not forget that mathematics is not just a theory, but that it is also a practice.

II. The Neo-Fregean Account

A. Background

Frege's program is understood as part of the foundational movement in mathematics and the philosophy of mathematics of the late 19^{th} Century, which attempted to put mathematics on a firmer base. Frege's aim is to show that mathematical knowledge is on a par with that of logic. Dummett credits him with pioneering the "linguistic turn" in philosophy, dawning a new age of analytic philosophy.[7] His program revolves around careful analysis of mathematical content to achieve an understanding of what mathematics is. It should come as no surprise that the questions, both ontological and epistemological, regarding the status of numbers, find a central place in his inquiry. He attempts to explain the content of mathematics through analysis of certain true mathematical statements. The Fregean innovation is to reverse the order or analysis. Instead of seeing the truth of a statement in compositional terms –the truth of a statement determined by the composition of its parts whose meanings are understood in isolation of any particular context – Frege confers meaning onto the parts of true sentences, giving meaning only within a context. This way of linguistic analysis is demanded by his context principle – "never to ask for the meaning of a word in isolation, but only in the context of a proposition"[8] – which holds a central place in his philosophy. In addition to the context principle, Frege lays down two additional dicta that that determine his method of philosophical analysis: the psychological and the logical must be kept strictly separate, and likewise for objects and concepts.[9] The former aims to bolster mathematical discussion by demanding that the subjective mental images

[7] Dummett (1991: 111)
[8] Wright (1983: 6), originally from Frege (1959: x)
[9] Ibid.

be dispensed in favor of the objective, expressible ideas. Philosophical debate about the foundations of mathematics would be difficult if two people base their statements on their respective mental images, especially if the debate hinges on this picture (as the realism/antirealism debate does). The latter, separating concepts from objects, gets at a distinction in metaphysical behavior between two kinds of things. It will soon be clear that the Fregean account of mathematics will be based on this distinction between concepts and objects.

Frege's mathematical realism entails the belief that numbers exist as objects. This means that contrary to other views, numbers are not just a place in a progression, nor are they predicates that apply to some unit obtained through abstraction away from individual objects, nor are they concrete. Numbers are abstract objects associated with certain concepts. This is the basic ontological claim which runs from Frege to his neo-Fregean successors. Frege argues against the possibility that numbers can be physical or concrete. There is no way to apply number unambiguously to a physical heap. The temptation is to say that one is a property of any single thing. Take, for instance, a dog which can be taken for a single entity and would have the property of oneness. That same dog could be seen as a pair (head and body) or a multitude of millions (the atoms composing it), or even a fraction (of all dogs). There is nothing in the physical heap that is recognized as a dog that would correspond to a unique number property, like there is for other properties of a dog (e.g. fat, soft). Considering the concept of a dog, it is clear that the same dog can only count as one instance of the concept. Concepts delimit how to break up their instances, and so there is a latent relationship between number and the natural divisions defined by a concept. Contra an abstractionist view which applies number to a featureless unit, derived mentally, which was once a dog, Frege supplies the means for finding number a number relationship in the extant world.[10] Much of the realist and antirealist debate hinges on this concept framework. The traditional thought is that anything which purports to be an object must have an identity criterion. If there is an object/concept dichotomy, it is an old thought that concepts are universals which attach to objects, which are self-contained. Many things can be red at once, but Plato cannot be in multiple places (oddities of modern physics aside). We can potentially identify whether some object is *the* Plato or not. Finding the ordinal structure inessential to numbers, Frege focuses on cardinality to lay down the identity criterion of numbers, known as Hume's Principle (hereafter HP).

HP: The number of Fs = The number of Gs $\leftrightarrow$ F$\approx$G[11]

HP does not (logically) entail that numbers themselves are a new kind of object aside from what is given in the world. It does not ensure that among the countably infinite sets of objects, there is one that would be recognized as the natural numbers, nor that any infinite sets exist. HP does not ensure that we can evaluate the truth of the proposition '2=Caesar' and like propositions. Without the ability to have an account of numbers as independently existing objects and without the ability to tell whether they are identical with other objects, Frege turns to another principle – Basic Law V (BLV) to shore up these concerns. BLV equates value ranges of concepts if and only if anything that belongs to one belongs to the other. From BLV, Frege can derive HP. With Frege's theorem, the proof that the Peano Axioms of arithmetic (PA) follow from HP, Frege

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[10] There is a light presumption that the concepts are not themselves mentally constructed in the same way that the featureless units are. It seems uncontroversial to assert a greater likelihood to 'dog' being a real concept in the world than to a featureless, ghost-like unit. Frege ultimately does not even need concrete concepts like 'dog' to apply number to. Instead he provides logical concepts which seem to exist without doubt. He uses the concept 'not equal to itself' which by all logics applies to nothing, and builds the numbers from there.

[11] F and G are sortal concepts which will be explained below. The relationship on the right hand side is the one-one relationship

thought he had an account of mathematics from logic alone. Unfortunately, BLV turned out to be inconsistent as shown by Russell. Devastated, Frege abandoned the project which sat idle until Crispin Wright attempted to pick up the pieces and revitalize it. The neo-Fregeans diverge from Frege at the point of adopting BLV or any principle laid down to further explain HP. They view such a move as unnecessary and attempt to meet the issues with HP head on.

It is not always clear that the neo-Fregean program has the same aim as the original Fregean project did. On the surface, it seems less interested in mathematical consequences and more focused on using mathematics as a theory that yields rich metaphysical discussion.[12] The neo-Fregean program does follow Frege in making three explicit commitments. The first is logicism – the claim that mathematics is on an epistemic par with logic, and so potentially analytic and a priori. Along with logicism is platonism, that natural numbers are full-blooded objects. The platonist claim includes a conception of numbers as existing independently of the human mind, and existing outside of the spatio-temporal realm, i.e. abstract.[13] Thirdly, it is the syntax[14] and logical form of mathematical language which provides for numbers being objects, as the terms standing for numbers function as those of concrete objects do in reference-demanding positions in certain true statements. The third commitment is to there actually being some mathematical propositions which are true in virtue of the state of the world, independent of human thought.

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[12] Wright: "the philosophy even of relatively elementary branches of mathematics is a discipline in which theses and theories about the nature of language and language mastery, knowledge, reference and truth may be sharply focused and tested with clarity. It is, in this way, the fundamental character of the philosophy of mathematics which, in my view, makes it an especially rewarding and important area of philosophical inquiry" (Wright 1988: 433)

[13] There is an obvious connection to Plato's theory the forms, albeit a loose one. We are not meant to have Plato in mind.

[14] Syntax, here, means something more than grammar. It involves the logical structure of propositions.

The general strategy of the neo-Fregeans follows Frege: look to the syntax of language and observe that numbers function as other uncontroversial objects do, at least with regard to the truth or falsity of propositions in which they are seemingly referred to. Propositions such as 'the number of shoes on my feet is 2', and 'the person who stomped on my foot is Greg' seem to have analogous structures. Both propositions contain an occurrence of a concept (number, person respectively) that is in turn instantiated by an object (2, Greg). These particular syntactic structures are seen as reflecting the structure of the reality, and so 2 seeming to fill the role of an object leads to the belief that 2 really is an object. The transition from linguistic singular termhood to objecthood requires adherence to Frege's context principle and the similar syntactic priority thesis – that syntactical analysis is definitive of real structure and there is no further ontological question to be asked once a syntactical analysis has been carried through. Recalling the dicta requiring the psychological to be kept separate from the logical, it is easy to see how the linguistic turn is justified. By placing a premium on the logical, Frege forecloses any considerations of the type which would compel further questions. It is illegitimate to appeal to a difference between the picture given through syntactic analysis and a psychological picture. This is appropriate as it seems as though there is no definite psychological notion of object anyway. This justifies logical redefinition of objects and concepts. To reiterate: on this account, once a question has been settled as to the ontological status of an object candidate via linguistic method, there is no further question as to whether that thing *really* is an object or not. The only acceptable notion of object is to be laid out syntactically.

In order to allow numbers as objects, it is necessary to show that there is a plausible way to define objects syntactically. To do so, syntactic criteria will be proposed and discussed based on loose preconceptions of objects. Before going into the technical aspects of such an analysis, it is important to note that certain sentences of number theory will be accepted as true, on face value. The neo-Fregeans refer to this as number theoretic realism, and it stands in direct contrast to fictionalism. They commit to the truth of at least some number theoretic statements[15] because it is hoped that we can identify the syntactic role of numbers within true sentences. This is not uncontroversial and will have to be defended below, but I will hold it off for the moment. It may not be uncontroversial, but there is a sense in which it is plausible. To deny the truth of any such sentence would require believing that there is a systemic error unknown to all when certain number theoretic statements are asserted, propositions as simple as '2+2=4'. If such as statement is taken to be true, as is the convention, then it is natural to think that the constituent parts of the proposition are involved with or have some relationship to the veracity of it. The syntactic analysis examines the nature of this relationship. It should be clear that platonism and number theoretic realism are compatible, but it is not the case that either entails the other. Providing syntactic grounds for regarding numbers as objects defends against formalist interpretations of mathematical language being meaningless. It does not provide defense against other antirealist attitudes towards numbers that have something to say about the nature of such objects that would counter realist attitudes.

B. Syntactic Criteria for Objects

This part of the neo-Fregean program attempts to show how the notion of object can be defined on logical grounds as opposed to psychological grounds. Unsurprisingly, it hopes to show that numbers qualify as objects under the redefinition

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[15] Number theoretic statements are just statements which have numeric occurrences

of object. As mentioned above, the belief is that numbers lie in some relationship with concepts that clearly delimit individuals falling under them. Such concepts are referred to as sortals. Wright defines the sortal: "a concept is sortal if to instantiate it is to exemplify a certain kind of object that the world contains." He suggests the terms 'person', 'chair', 'duck', and 'elm' as examples which express sortal concepts.[16] These concepts differ from others in relevant ways. They differ from concepts expressed by 'gold' (the metal) and 'water' in that there is no unit of either concept which arises naturally (or unqualified if one has atoms in mind). These sortal concepts differ from properties such as green or light in that there is an identity criterion by which one can distinguish whether two instances of a sortal are the same thing or different things. Sortals distinguish objects from properties by accentuating the fact that only objects have individual differentiability. The individual instances of a sortal concept are said to be the referents of singular terms. The singular terms correspond to objects in the fullest sense. If we consider the sortal 'person', we might say the identity criterion is "person x is the same as person y if and only if x and y have the same brain" (pending scientific progress). It should be clear that it is not always easy to state the identity criterion associated with a sortal. Luckily, HP provides the criterion for the case of number.

In evaluating proposed syntactic criteria, it is necessary to employ tuition about what should and should not count as an object. This is not to presuppose what objects there are, nor to undermine the principle that we must heed to what the syntax dictates. It is merely being cognizant of the fact that in order for a certain set of criteria to be plausible as defining objecthood, it must somewhat reflect our currently vague notion of objects; also the program does not intend to misappropriate or equivocate the meaning

[16] Wright (1983: 2)

of object. The new object is to be understood as having the same ontological gravity as the old object. The goal is to elucidate the notion of object by finding a set of logical guidelines that determine the notion. It must be the case, then, that certain things must remain objects, and others can never be an object even under redefinition. 'Large' cannot correspond to an object, while 'The White House' must. It is this intuition that we hope syntactic analysis will capture and clarify, and so the intuition must serve as a proxy for object determination until the syntactic criteria are fine enough to be accepted. The intuition of object will be discharged in the end, in favor of a syntax-based explicit definition.

Wright sketches several approaches to defining objects through syntax.[17] The first attempt is to say that objects are features substantivally as opposed to adjectivally in propositions. Unfortunately, the adjectival or substantival use of a word can be a matter of aesthetic choice, and therefore has little bearing on its logical form. There is a stylistic choice to be made between the propositions, 'the number of horses there are is four' and 'there are four horses', which seem to express the same idea and would ordinarily be asserted in all of the same contexts. Further, 'sakes' and 'whereabouts' can have substantival occurrences in propositions and intuitively these terms do not correspond to objects of any kind. Much of the substantival/adjectival distinction rests in language practice and happenstance of expression and is not reflective of ontological belief.

The second attempt fails in much the same way. It recognizes singular terms by those terms that are picked out by use of the definite article, which in English is 'the'. The proposition '…the number three…' is an acceptable one, and would purport to pick out a particular instance of number, namely three. Again, where the philosophical stakes

[17] Wright (1983: 53-64)

are high, such an account based on contingent uses of language cannot be accepted. It would also seem circular to pick out singular terms by the correctness of placing the definite article before them, as the definite article might just as well be defined as that which precedes singular terms. There is no way to distinguish apparent versus genuine singular terms, then, and this method does not supply the robust definition sought.

The third attempt is the first that Wright seriously considers and is tied up in the notion of identities. The notion of object demands that there must be some sort of identity criterion associated with anything which is to be accepted as an object. Though object has gone formally undefined, there are paradigmatic objects given by intuition. These objects have associated identity conditions that dictate the indiscernability of two instances of the same object. This motivates the method of defining objects by those things that have individuating features and not those which have only a general or natural kind. This means that for any for any singular term, its reference must be determinate and for pairs of singular terms, it would be determinate whether they refer to the same object. It is required that the identity conditions among singular terms be seen as an equivalence relation that is congruent with respect to every predicate. That is, a relation which is reflexive (xRx), symmetric (xRy → yRx) and transitive (xRy . yRz →xRz). These basic requirements are embedded in the intuition of objects. Wright outlines the big flaw with test three.[18] The first is that if the equivalence relation holds of two singular terms 'a' and 'b', it entails that if a predicate holds of 'a' then it must hold of 'b'. We should only want to restrict the predicates of the congruence to only genuine predicates. In order to make such a restriction, we would want to restrict its arguments to only genuine singular terms. To make this restriction, it would seem we need a prior

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[18] Wright (1983: 55)

knowledge of what counts as a singular term. The proposal is intended to do just that, and so no restriction can be made.

Not to be discouraged, a fourth attempt is made. This attempt is based on the Aristotelian Criterion (hereafter AC), which recognizes the intuition that properties come in incompatible pairs whereas objects do not. For the predicate 'is brown' there is 'is not brown' and for an object to be brown requires that it not by the case that the same object be not brown, and vice versa. Objects are not subject to this pattern. For any object A there is not an anti-A object that fails to have all properties that A has and has every property that A lacks, and even if there were an attempt to define an anti-object for every object, it would be subject to contradictions. To see this, consider a triangular object, T. Well then T is neither square, nor round. Anti-T would then have to be round and square, and it would be hard to believe that any such object exists. The basic working of AC is to examine propositions of the form C(t), where 't' is a linguistic expression, and C( ) is the incomplete proposition structure resulting from the removal of 't'. Then AC says that 't' functions as a singular term if and only if it is not the case that for all substitutions of C( ), $\hat{C}$( ), and any substitution for 't', a, that $\hat{C}(a) \leftrightarrow - \hat{C}(t)$. Wright dismisses the approach, as is, because quantifiers, which are not thought to be objects, can be concatenated with predicates to make the conditions fail proper linguistic assignment.[19] The proposition 'everything is brown if and only if it is not the case that everything is not brown' in invalid, showing that predicates cannot split the world in half with regard to the quantifier 'everything' and that 'is brown' can pass as a singular term. We might try to restrict the variables to first order predicates to preclude quantifiers that cloud the results of the test, but there is no way to enact such a restriction without

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19 Ibid

reference to singular terms. We would have to define a first order predicate by saying that it is an incomplete expression that is completed by a singular term, but with no clear way of determining singular terms, this would be impossible.

Hale provides an argument that an elimination procedure can be carried out to restrict the possible C( ) substitutions to first order predicates. To see his suggestion, it is necessary to formalize AC.[20] Let α substitute for 't' in any expression C(t) as we have been regarding it. β will be allowed to substitute for C( ). (α ,β) represents a sentence of the form C(t).

AC: C(t) is a sentence. As is (α, β). Let $\Sigma\alpha$ and $\Pi\beta$ quantify over the set of variables which can replace α and β respectively.

We say t functions as a singular term in $C(t) \leftrightarrow - \Sigma\alpha\ \Pi\beta((\alpha, \beta) \leftrightarrow -(t, \beta))$

We noted that when quantifiers take the place of variables above, predicates can pass the test as singular terms. Hale's suggestion is to rewrite AC as:

t functions as φ in C(t) iff $\Sigma\alpha\ \Pi\beta((\alpha, \beta) \leftrightarrow -(t, \beta))$[21]

Now , φ is a quantifier if t can pass AC (reformulated). We saw how the quantifier 'everything' above could pass this test. We use this fact to make the restriction on $\Pi\beta$ to $\Pi\beta \setminus \{ \beta: \beta \text{ is a quantifier}\}$. Running the test again we get that φ is a first order predicate if t it passes the test. Rectifying the quantifier flaw and the inability to define first order predicates of the original formulation, we can now say that φ is a singular term if t fails the test when we restrict $\Pi\beta$ to first order predicates. Through the eliminative procedure the desired result is obtained without reference to singular terms.

Perhaps an example would be helpful. Let us start with the sentence C(t): 'Plato is a philosopher.' We might construe C( ) and t two different ways. The first would be to

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[20] Hale (2001a: 41-45)

[21] Note that there is no longer a negation before the quantifiers

say that C( ) is 'Plato…' and 't' is the expression 'is a philosopher'. Now 'is a philosopher' will function as φ in C(t), where φ can be a singular term, a first order predicate, or a quantifier. If we are to determine the validity of $\Sigma\alpha\ \Pi\beta((\alpha, \beta)\leftrightarrow -(\text{is a philosopher}, \beta))$, we substitute the only logical choice for α, 'is not a philosopher, and we try substitutions for β. We might try 'everyone' and end up with the invalid statement 'everyone is not a philosopher iff it is not the case that everyone is a philosopher' implying that 'is a philosopher' is a singular term. We suspect that there is something amiss, and can recognize that 'everyone' can pass the test if it is substituted for 't' in the proposition, 'everyone is a philosopher' (substitute 'not everyone' for α). Now, 'everyone' cannot be substituted for β as it has been marked as a quantifier. We can substitute 'is not a philosopher' again for α, to obtain the biconditional (is not a philosopher, β)↔ -(is a philosopher, β). Now if we cannot substitute quantifiers for β, any substitution by an expression that would yield a complete statement makes a valid AC schema, e.g. 'Obama is not a philosopher iff it is not the case that Obama is a philosopher'. This would imply that 'is a philosopher' is a first order predicate and that 'Obama' is a singular term. The second way we could construe 'Plato is a philosopher' is to take 'Plato' to be 't' and '…is a philosopher' to be C( ). Now, it should be clear that no α can be thought of such that for any substitution for a first order predicate substitution of β, we get (α, β) ↔ −(Plato, β). We cannot find an α such that α is a philosopher if and only if Plato is not, *and* α likes wine if and only if Plato does not, and so on. Nothing has every trait that Plato does not. There is not an 'anti-Plato'. In fact, above, I showed why such a non-person would be a contradiction.

Unfortunately the matter is not settled, as I have skipped over some confounding factors. Notice that the first step of the elimination procedure can fail to recognize quantifiers in the presence of higher order generalities or quantifiers. An attempt to substitute in an opposite quantifier to yield a valid formula in the presence of a second order generality replacing β might fail. Consider 'something somehow' which Hale fortunately interprets for us. Unfortunately, he gives two interpretations that both accord with 'something somehow' (ss).

ss_1 : Some property is instantiated by something: (∃F)(∃x) Fx

ss_2: Something has some property: (∃x)(∃F)Fx

Luckily the two are logically equivalent despite the uncertainty of which quantifier has wide scope; this is expected as existential quantifiers can be rearranged conservatively. The associated propositions where 'nothing' is substituted which should be incompatible with the 'something' propositions are where there test breaks. Because of the negation of the existential quantifier to form 'nothing', the issue of scope presents itself, for 'some property is instantiated by nothing' and 'nothing is such that it instantiates some property' are not equivalent. The formulation with first order quantifiers having wide scope –'something has some property if and only if it is not the case that nothing has some property' – is valid[22], and the quantifiers 'something' and 'nothing' are flagged. When the second order generalities are given wide scope as in, 'some property is instantiated by something if and only if it is not the case that some property is instantiated by nothing', the biconditional is invalid and the criterion fails to flag the quantifiers. In the second case 'something' would remain in the domain of Πβ. The eliminative procedure breaks down and AC fails to properly pick out singular terms. Any further eliminative procedure to purge the β-set of second-order quantifiers would be obscured by even higher order quantifiers.

[22] Valid only when some property refers to the same property in both instances.

It seems that in the face of these failures, the hope of establishing syntactic criteria for objecthood is diminished. Even worse, it seems possible that as the conditions are refined, we could reach a point where we have something which precludes things that are definitely not objects and accepts those things which are definitely objects, leaving us in no position to judge the remaining fringe cases. How do we know when the conditions treat the margins properly or that the object-sifting screen is fine enough? These problems must be addressed if we are to believe the plausibility of establishing a usable framework for determining objects syntactically.

Rather than keeping on with crafting criteria for isolating singular terms and thereby objects, we might keep in mind the structure of the argument; singular terms are supposed to have some logical function in true propositions. This thought motivates Dummett, who proposes certain inferential patterns which can recognize this function of singular terms. It also recognizes the logical function of quantifiers and seeks to distinguish their function from that of singular terms. His procedure aims to pick out only singular terms. His formulation runs thus: 'a' is a singular term only if:

> (i) For any sentence A(a), the inference from A(a) to there is something such that A(it) is valid.
>
> (ii) For any sentences A(a) and B(a), the inference from them to there is something such that A(it)and B(it) is valid.
>
> (iii) For any sentences A(a) and B(a), the inference from 'it is true of a that A(it) or B(it)'to 'A(a) or B(a)' is valid.

At stage (i), the quantifiers 'nothing', 'no one', etc are ruled out. At (ii) 'something', 'somebody', etc. And at (iii), 'everything', 'everybody', etc.[23] Dummett's inferential patterns work to pick out singular terms much like AC and is likewise plagued by higher order quantification or generality. To fix this, Dummett supplements the inferences with a further test to recognize the presence of higher order quantification. When we use first order 'something', a request for a specification of which thing it is can be made that is both grammatically correct and that is meaningful in the sense that it displays an understanding of 'something' as used in the proposition. Take two distinct objects Crispin and Bob. The proposition 'Bob and Crispin are something' can be met with the request to specify which thing they are, to which 'a philosopher' would be an appropriate response. But, inquiring after which philosophy they are would be considered a misunderstanding, for Bob and Crispin are distinct and the 'a' in 'a philosopher' is a general article, not a replacement for 'one'. If at any point in a line of specifying questions, the request for specification of a quantifier phrase appears misplaced or senseless, the quantifier is of higher order and the inference invalid. The advantage is that the line of questions must stop with a misunderstanding or an appropriate response. There can be no infinite regress as there was in trying to carry out an elimination procedure in AC, where every time an n^{th} order quantifier could be tied down, an $n+1^{st}$ order quantifier could disrupt things.

The specification test proposed by Dummett seems to be effective. Hale adopts the Dummettian framework as an elimination procedure for quantifiers and attempts to explain any residual difficulty. The first difficulty is that the sequence of questions cannot be ensured to play out in the expected way. The question regarding what Bob and

[23] Wright (1983: 57). Dummett's original proposal in Dummett (1981)

Crispin are might have ended with the reply that the thing they are is 'old'. Because there is no prior established way to recognize predicates, one would not know that under this interpretation 'something' would be second order because it has a predicate in its range. Using the specification test in conjunction with the inferential tests would avail this problem. The reverse may happen, in that a respondent is ready to make a specification where there was not supposed to be any further specification. Hale clarifies the specification test to state that so long as there is a point that can be reached where a request for specification could be dismissed as a misunderstanding, the inference is invalid. Take the example provided by Wetzel, using the proposition 'a sheep is ruminant'. To see whether the inference from that to 'there is something such that it is ruminant' is valid and so one may ask for specification of this 'something' to which a replay may be 'a sheep'. The next question would ask after which sheep it is that is ruminant, and the inquiry may end with the reply 'any sheep'. Being able to reject this answer is a dismissal of further specification, and an invalidation of the inference. In other terms, we are free to challenge the devil's advocate who will craft responses to get around the test. The answers need not always be accepted.[24] The moral is that these tests are hypothetical, not actual, and that there may be ways to exploit loopholes, but there is always commonsense which can be provided as protection against this. We are working within the framework of an established language where the meanings of most propositions are understood, rather than constructing a more logical English in which there may room to question meaning on technical grounds. This is not to say that the tests are infallible or that they fully determine the set of objects. In fact, it may not be possible to achieve such precision. This seems to be the attitude of Wright, who sees the

[24] Hale(2001b: 56), originally from Wetzel (1990: 251)

tests more as just illustrating of the possibility for capturing the sense of object syntactically.

To take stock of the situation: a handful of attempts at laying down syntactic criteria for objecthood have been described. In order to adopt the linguistic account of objects, there needs to be such syntactic grounds for identifying singular terms corresponding to objects and predicates corresponding to concepts. The results of the attempts so far are inconclusive. It seems as though the tests are strong enough to conform to basic belief about the range of objects and that the issues raise have been mostly as hoc. Yet, the conditions are still prone to yield problems. While we can recognize the obvious ones, there could be more subtle ones. Also, fine-tuning the criteria only captures our vague intuition. It seems as though at this point two sets of criteria could be proposed that differ in their object extension. If we reflect upon the procedure of refining the criteria for objecthood, we notice that the legitimate attempts all capture the intuition of objects well. On the margins, technicalities of a recognizable kinds are able to generate problems, but it is not the case that under one set of criteria trees count as objects whereas bushes do not, nor is it the case that refining the criteria slightly results in things changing object status. On the margins, vagueness might have to be expected. There is no reason to think that the universe splits nicely into objects and non-objects. There is empirical evidence to the contrary. Photons, under certain circumstances display particle behavior, and under other circumstances display wave properties. There is not a presumption that the neo-Fregean program will provide a complete ontological taxonomy. That is not the aim. The promise was to provide syntactic criteria for picking out genuine instances of singular terms which refer to objects in certain true contexts. The hope was that numbers would be seen as objects under this sort of procedure. Abnormal behavior on the margins of the criteria can be seen as reflecting reality of things just as much as it can be seen as a gap in explanation. The upshot of the trials above is that the criteria seem to be displaying asymptotic behavior getting closer and closer to an acceptable account of object. The marginal problems seem to be an analog of residual area under the tail of the asymptote and so the absence of a precise demarcation of the set of objects is, thus, not a serious worry. The notion of object given by the admittedly imperfect criteria settled with above does accord well with intuition of reality and has no immediate counterexamples. The neo-Fregean is committed to the general form of the syntactic criteria for objecthood and to the fact that further refinements to the criteria might avoid technicalities, but might also never completely determine the set of objects and non-objects. For the case of numbers, there is a lucid notion of the sortal concept which steers clear of the margins qualifies as an object so long as the conditions are not radically altered.

C. Justifying the Program

The discussion has been focused on the technical possibility of accounting for numbers as objects through syntactic criteria. Although the criteria are not yet settled, they are sufficient for the purposes of believing that the approach can work and that numbers should be considered objects. Given the claim that numbers are objects, one would hope to establish a theory of reference to and knowledge of them. We know that there is the standing challenge from Benacerraf[25] which Field generalizes. It asks for an account explaining how knowledge of abstract objects is possible. The challenge is framed as a dilemma in which acceptance of numbers as abstract leaves them causally inert and cannot be known by the predominant causal theory of knowledge, and opting

[25] Benacerraf & Putnam (1983)

for knowledge seems to make the abstract claim untenable. There is also a reductionist challenge, that reads HP as revealing the fact that numbers can be reduced to or exist only in virtue of physical phenomena occurring on the right hand side of HP (the 1-1 relationship between concepts).

i. Achieving an Account of Abstract Reference

The archetype of the neo-Fregean claim of reference to abstract objects is seen in the direction establishing principle:

> $D^=$: The direction of **a** is the same as the direction of **b** if and only if **a** is parallel to **b**.

The neo-Fregean asserts the priority of directions $D^=$. The reductionist claim is that it provides a way to reduce all occurrences of apparent singular terms (directions) to genuine singular terms (lines) for which there is an unproblematic account of reference and knowledge because of their concrete and causal nature. The neo-Fregean argument is that reference to lines in disingenuous as it hides the real reference to directions. The left hand side makes this reference explicit. Further, the left side provides for the intelligibility of the right hand side. The dispute is over which side of $D^=$ holds priority. Neither side seems to have the default advantage.

The reductionist position which sits comfortably on the causal theory of knowledge and the concreteness of lines can be brought into question. Knowledge and reference to everyday objects is taken for granted and probably for the best. The means, however may be questioned. The typical story is that reference is achieved through some sort of ostension. When asking for the reference of a singular term, one may point ostensively to the object intended, or believe that his reference is achieved through some historical chain to somebody who has referred to the intended object in this way. This is obviously not how reference would be achieved for abstract numbers. Ostensions are limited to the spatio-temporal realm. To challenge the ostensive method of reference, it can be pointed out that the gap between linguistic terms and actual objects cannot be overcome with the point of a finger or some other ostensive action. Grasp of a singular term requires more than this. If I sit in French class and am asked "Que-ce que c'est un livre," the professor would be right to question my understanding if I were to just pick up my textbook. Such an ostension would fulfill myriad requests (e.g. text, object, confusing) if it were to succeed. Ostensive actions cannot even differentiate among linguistic classes of singular terms and predicates. No non-linguistic effort can demonstrate an understanding. One can only be attributed with an understanding of a term if her is able to use the term in context, and use it correctly. This is not a reiteration of the context principle but a justification of it. For reasons like this Frege thought that mental images do not suffice for true understanding and hence for the foundation of mathematics.

Correct use and understanding of a term requires knowledge of its linguistic class as determined by the syntactic criteria. To correctly use 'Plato' requires recognizing that it takes the place of a singular term and has the capacity to complete incomplete sentence structures, say '…is a philosopher'. The language user ought to recognize that the incomplete sentence requires a singular term or a quantifier (indexicals would be acceptable as well and are almost like a linguistic ostension, but they do not import the incomplete structure with a constant meaning). Not every substitution for a singular term into the incomplete expression will yield a true statement, but any such substitution will yield a statement that can potentially be semantically evaluated. To say 'Plato can be read' is to be simple, but to say that 'shiny can be read' is to linguistically fail. We can observe the various occurrences of 'Plato' in propositions and see that it almost always functions as a singular term. Of course recognition of the acceptable linguistic class is only a necessary, but not sufficient condition for understanding a term. If recognition of the acceptable linguistic category is to be the way of understanding, rather than ostension, then abstract objects are not different from their concrete counterparts in this manner. Abstract objects cannot be rejected because they cannot be pointed to, or referred to ostensively. The reductionist position underestimates the gap between language and knowledge and reality in thinking that reference of their speech can be attained through literally reaching out into the world and grasping the intended reference of their words.

Dummett, as a reductionist, leverages the existence of such a gap to draw a strong line between what the syntactical criteria determine to be singular terms and what are genuinely singular terms. The reductionist claims that there is a gap between what is considered an object under a linguistic model and what the real world actually consists of.[26] He does not accept that linguistic use or the function of a term is sufficient to determine objecthood. He places the proposed ontological commitments of syntactical conditions on trial and seeks justification for doing ontological analysis on a linguistic level. Taking $D^=$ as illustrative of directions being synonymous with pairs of lines is mistaken. $D^=$ should not be read as a rephrasing; it is an equivalence where each side has a claim to legitimacy and a face value construal. To see the left side as derivative of the right is to reject any intrinsic linguistic value to directions, and to see them merely as convenient constructions that are artificially injected into speech. Having questioned the traditional view of reference by ostension, it becomes unclear which side holds priority,

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[26] Dummett (1991 and others)

if either does at all. Reading $D^{=}$ through reductionist lenses is a misguided and austere reading of the situation which goes against linguistic practice and belief.

There is a more constrained reductionist reading that accords abstract singular terms some kind of reference, but maintains the denial of real reference, or that they refer to objects. The view of reference remains steadfast in requiring that object reference can only be succeed when in the object being referred to is located in the real world, i.e. the spatio-temporal world. Allowing a term to refer does not create or determine an object on this reading and the furnishings of the world are how they are irrespective of linguistic methods or discoveries. This is a complete rejection of the syntactic priority thesis. The neo-Fregeans unwaveringly hold to the position that there is to be no understanding of a term or its reference without internalizing the process. This involves conceptual analysis of the kind prescribed. If ostension is not what sets apart the accepted concrete from the unaccepted abstract on an epistemological level, it is not clear what does. It cannot be that they are visible, for certain concrete objects are beyond microscopic and still considered less problematic than abstract ones. The best option for the reductionist would be to maintain that existence of concrete objects are potentially verifiable through at least indirect evidence. This is the exact point of the neo-Fregean account thought. It is to show that despite being objects, numbers and the facts about them are known analytically, whereas the concrete are known only synthetically. It might be the case that a speaker is confident that there are referents to his concrete singular terms because he can point to them, but the linguistic history of some of those terms might be wrapped up with abstract identity conditions.

An example might help. To use a mathematical one: when explaining addition a teacher might say "if I have 2 cookies and then you give me 2 cookies, then I would have

4 cookies." The foundation of addition, however, does not rest in cookies or any other concrete substitute. It is based in arithmetic and logic. To understand the teacher requires a cognitive capacity to maintain the distinction between the four cookies which could then be either appropriately added or counted. If the example was changed to having 2 molecules of this and 2 molecules of that, then it may not be the case that I have 4 molecules in the end, and yet the foundations of addition are not questioned. Those who appeal to surface justifications cannot be accorded an understanding of addition if their explanation cannot be furthered to mathematical objects. Addition, then, can only seemingly appear to be explained in concrete terms, but actual understanding addition requires the grasp of abstract theory. Otherwise we would be stuck with "Cookie Addition" and "Molecule Addition" and other addition theories that may not be isomorphic. Their equivalences to each other would be an empirical matter. To a degree this presupposes what I am trying to show. If we think of addition as having some underlying theory, the abstract theory has all the means of providing the proofs for commutativity and associativity of addition. It is difficult to see how the cookie theory can provide the same proofs. The cookie theory seems to be grounded in inductive logic whereas the abstract theory is based in some kind of deductive logic. There would then be a gap in inferential power between the two theories. To take the example further, consider the question of whether a series converges. Looking at $1/n$ and $1/n^2$ it would appear that either both converge or both diverge. To know the truth, that $1/n$ diverges whereas $1/n^2$ converges requires formal theory. There is no practical way of coming to know the fact otherwise. Correct mathematics requires grasp of abstracts, a fact reductionists recognize; typically the reductionist does not want to be a fully-fledged empiricist of this kind. Most believe in abstracts as far as logic goes. The predominant

view it seems, is that there is no problem with abstracta; their objecthood is what is challenged. Further below, I will discuss how the conservativeness of HP makes adding numbers to the list of objects is unproblematic if the reductionist holds to this view.

The force of the example of sequences in real analysis exemplifies a step up in the argument against reductionists. The debate has focused on whether lines have some priority over directions. The real debate ought to be over whether parallelism is prior to directions. It would seem that parallelism is nothing that can be pointed to in the way that lines can. Parallelism seems to have no special epistemological foothold over directions. If parallelism is to be recognized in the world, it seems incredible to deny this status to directions. Yet the reductionist seems to be trying to slip in reference to a property of parallelism on the back of seemingly unproblematic reference to lines. It would be one thing to maintain lines as physically reduced are objects. It is another to lump in parallelism. The reductionist surely cannot opt to give up on the parallel relation in the world, as to give up on all properties is to greatly impoverish one's ability to speak about the world. It does not seem as though maintaining that properties are mental constructs goes much further either, as no statement would be true of the world. This would amount to a weak ontology.

Reductionism unnecessarily restricts the set of objects to the concrete, thinking that anything beyond the physical is not real. Just because concrete terms enjoy more currency is speech does not mean that there is not something more at the base of this speech which allows for the success of language. There is a correspondence gap between thought and language and the real world. The only bridge available is through abstraction. The abstract is that which we have direct access to. Any use of the concrete makes latent use of some abstraction. If the problem began as what we have access to, it would seem the tables have turned and abstract seem to have a better cognitive footing. Some of the concepts we have the best understanding of are abstract. Take mathematics, which can only be understood (as pure mathematics) in the abstract. There is no concrete interpretation of convergence or of the limit. The absolute clarity of these concepts is unparalleled in the concrete realm and perhaps it is this that causes confusion, that we are not used to such vivid understanding. It is difficult to tell whether something is a person or not as evidenced by the volume of philosophical and political discussion on the matter. It is this clarity of mathematics that the neo-Fregeans can account for, which, in part, makes it the better realist philosophy considered in this paper. The methodological naturalist does not have an account of the seeming clarity of mathematics.

Of course, any difference that the neo-Fregean opens up between the abstract and concrete is a potential point to be exploited as much as it can explain the superiority of the account. There is simply a different attitude taken to this difference by the neo-Fregeans and the reductionists. The neo-Fregean reads the difference only as having epistemological implications whereas the reductionist reads the difference as having ontological implications. Even if mathematics can be done without reference to the abstract (as in Field's *Science Without Numbers*[27]) there is still no reason to reject numbers as objects. There is abstraction required in the sense of HP or $D^{=}$ in order to form a bridge of understanding from language to reality, even if no reference is being made. While the two sides of the abstraction principles may have differences in epistemological status, they do not have different ontological statuses, as this would allow for gaps in the truth preservation of the biconditional to open up. Because the two sides share truth-

[27] Field (1980)

values, it would be difficult to see how one could be a realist on the right and an antirealist on the left. If the abstract are held to be just as accessible as the concrete and are truth functionally related, then much of the oppositional force is eroded.

Referring to abstract objects is, then, unproblematic. The arguments for the possibility of mathematical knowledge have been started in this discussion as well with the discussion of analytic knowledge of the abstract. The concern is that causal theories of knowledge require interaction, but if there is a separate theory of knowledge appropriate for the abstract, the dilemma could be resolved.

ii. Knowledge of the Abstract

The concern regarding the antirealist challenges against theories of abstract knowledge might be met with the logicist claim of the neo-Fregean program. The laws and inferences of logic seem to be known in an unproblematic manner, so if mathematics can be put on a par with logic, there would be little left to doubt of mathematics. The claim will be that the causal view of knowledge is not appropriate for mathematics. Matters are not so simple as it is one thing to show that our knowledge of mathematical facts is unproblematic, and another thing to interpret the content of these facts as being unproblematic.

The challenge is that knowledge of a statement cannot be found but by some causal relation with the truth-makers of a statement and that the purported abstractness of the truth-makers of mathematics (for us, numbers) makes for the impossibility of mathematical knowledge. It does not matter what the exact theory of knowledge is that is used in the challenge. Any theory basing knowledge in causal relations to the facts is sufficient to maintain the force of the challenge. Above, the difference between the abstract and the concrete was claimed to be epistemological. This difference is recognized throughout the history of philosophy in the distinctions made between the a priori and a posteriori as well as between the analytic and synthetic. The exact meanings of the terms are disputed, and sometimes the terms are rejected outright, but they (very) roughly convey distinctions that can be read into the difference between concrete and abstract objects. If anything is said to be known a priori, it means that there does not have to be something that causes the knowledge of that thing in the sense of causal used in the antirealist challenge. Analytic knowledge is derived from conceptual analysis or is definitional. Again, it seems as though there is no direct cause for analytic knowledge. The neo-Fregean works off a belief in the a priori status of mathematics in claiming that the demand for causal links to mathematical knowledge is inappropriate. HP gives the sense that the grasp of number can be prior to the grasp of the 1-1 relation corresponding to a number. Further, the 1-1 relation is not necessarily among physical concepts. The concepts used to construct the natural numbers are purely logical. Logic and other abstract concepts have this a priori status as well.

With regard to number, it is not clear what causes the stipulation of HP, however. Is it the case that someone could come up with HP prior to experience or did they witness phenomena that needed to be counted and proceeded to lay down the definition. This question misses the crux of the neo-Fregean program guided by context principle. Neither option captures the situation correctly. It is not legitimate to consider a term in isolation, but only in context. What this establishes, is that the question, "what is number," is invalid. There needs to be a contextual construal of how such object-referring singular terms feature syntactically in true statements. It should not be asked what knowledge there is of books or of numbers, but what knowledge there is of the propositions in which their associated singular terms occur. The strategy embraced is to see what function 'number' plays with regard to the meaning of the statements in which it occurs. By using the context principle, knowledge of number is just the same knowledge of the statements which feature numbers essentially.

Knowledge of mathematical statements rests in their deducibility from certain axioms. The means for evaluating the truth of the statements is logic and analysis of the axioms. If HP can be accepted as a non-logical axiom, then mathematical knowledge is unproblematic. Because HP is non-logical, the neo-Fregean cannot be a pure logicist as Frege was. Instead mathematics is said to be analytic, where the truths of the statements are seen as deductions from the definition of number, HP, and second-order logic. It is because the truths of mathematics can be traced back in this way that there is no causal element required, so long as HP is an acceptable axiom. Mathematical knowledge, then, really can be asserted to be justified true belief. It is not clear what exactly the justification is for mathematical statements. There is computation which can be used to verify specific problems, and there is proof used to verify general propositions. The two are inferentially related, but not the same. Computation seems to be as reliable a process we can have. Proof causes some difficulties however, which Gödel's incompleteness theorems bring to light. They show that not the case that every arithmetical truth can be traced back to the axioms, that the truths are not recursively axiomatizable.

Gödel showed that arithmetic is incomplete at its essence, meaning that the incompleteness does not depend on a particular axiomatization of arithmetic. He showed that there exist true but unprovable statements in any reasonable arithmetic; really, he showed that no set of axioms that describe arithmetic can be both complete and consistent. The incompleteness theorems are aimed at the formalization of mathematics as undertaken by Hilbert. To Gödel, the result meant that in order to know the truths of mathematics, we must have a mathematical intuition which gives insight into the facts otherwise underdetermined by the formalized system.

Logicism, especially the near logicism of the neo-Fregeans, is not to be equated with Hilbert's formalism. Hilbert's system is an uninterpreted and otherwise meaningless mathematics. Logicism, by invoking HP, does not leave mathematics uninterpreted. HP may be seen as a logicist analog to Gödel's mathematical intuition. Whereas Gödel thinks that certain mathematical results require the supplement of intuition, the neo-Fregean claims that each result is analytic and therefore potentially knowable without invoking some vague faculty for recognizing mathematical truth. The neo-Fregean stops just short of claiming that each result is a logical consequence. Hilbert sought an absolute finitary consistency proof of arithmetic, but the neo-Fregeans admit just relative consistency.[28] We can only say that HP plus second-order logic is probably consistent and that the problems that Gödel's theorems present are just on a par with concerns regarding the use of second-order logic. The force of Gödel's theorem comes from the fact that first-order logic is complete and arithmetic is not, which implicates arithmetical axiomatization as the culprit. If logic is extended to higher orders, then it is no longer clear that arithmetic is the problem, and instead the problem might be that logical truth is as difficult to account for as arithmetical truth. This can be taken as a reason why higher order logic is only tenuously referred to as logic. This is not a *good* situation either, but it highlights the differences between the consequences of Gödel's incompleteness theorems for logicism and formalism, in that incompleteness is not an especial problem for logicism. The problem shifts to justifying the use of second-order logic with HP to derive arithmetic. The problems with second-order logic crop up again in the "Bad

[28] HP is consistent with second order logic, that is.

Company" objections to using HP, and hopefully there, it will be successfully argued that second-order logic is not especially problematic. To give a quick response here, it might be said that even though arithmetic is incomplete, the addition of HP and the realist attitude implicit in it have the ability to resolve any of the questions which are mathematically significant.

Hale and Wright are unconcerned by incompleteness. In the introduction to *The Reason's Proper Study*, they claim that by merely seeking to obtain the foundations of arithmetic rather than attempting to show that every mathematical truth is derivable as a theorem they avoid "an obvious clash."[29]Their claim is even more basic than the one above. All they hope for is to show that PA or an equally good set of arithmetical axioms can be derived from HP and second-order logic. That this is proved by Frege's theorem means that as long as HP can be accepted, all is well. If PA is incomplete, this is a mathematical problem, not a philosophical one. I am not sure that it is so simple. If mathematical truth is to be displayed through proof, and there exist true but unprovable statements, a gap appears. There is the option of saying that the truth of the statements, while unprovable, are subject to conceptual analysis to a degree that they can be rendered analytic. It seems this is the best option to take for the neo-Fregean. The program already relies on a view of mathematic being analytic. There may be room to engage in definitional manipulation outside of logic that would make the truths knowable. Also, there is the possibility of laying down new definitions by a process like HP to uncover further truth.

D. Further Considerations

[29] Hale and Wright (2001: 4, fn. 5)

The first order of business that must be handled is the Caesar problem. Recall that the inability to determine the truth or falsity of propositions such as '2=Caesar' pushed Frege to make the ill-fated move of stipulating the inconsistent BLV. It is necessary that if we are said to have knowledge of numbers then we should be able to evaluate whether 2 and Caesar are the same thing or not. HP does not provide insight into the matter. It provides identity criteria among numbers, but does not provide general identity criteria for numbers among the universe of objects.

To resolve the Caesar problem, Wright introduces a supplement to HP.[30]

> N^d: Gx is a sortal concept under which numbers fall only if there are, or could be, singular terms 'a' and 'b' purporting to denote instances of Gx such that the truth-conditions of 'a=b' could adequately be explained as those of some statement to the effect that a 1-1 correlation obtains between a pair of concepts.

The supplementary principle makes explicit the somewhat obvious claim that identity can only hold among things which can be situated under the same identity criteria. Numbers cannot have one set of identity criteria in one context, and different set in the next. If a user understands Caesar, he must grasp that Caesar is a person, and thus subject to person identity conditions, of which we have not defined, and will not define, but that surely do not include 1-1 relations. By adopting the innocuous principle we can deal with the (absurd) Caesar proposition. Any attempt to identify Caesar with 2 through bijection is incoherent, and yet it is intrinsic to 2 that it can coherently be analyzed under such relations. It is a necessary property of any natural number, and so Caesar's failing to be relatable through the 1-1 relation precludes him from identifying with 2.

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[30] Wright (1983: 116)

Up to this point, we have more or less accepted HP as true in virtue of an adequate explanation of the concept number, and its use justified. Considering the body of work regarding HP, this is not the case. The acceptability of HP is disputed and must be defended. The primary concern is that the method of defining numbers contextually might be seen as misguided. Frege's original program did not work out and it might be blamed on the underlying problems with implicit definition. Even if the problem can be blamed on the inconsistency of Basic Law V, we only know that HP is relatively consistent and that its truth is uncertain. To make matters worse, it might not be analytically true which would have obvious repercussions for the discussions above that required that every theorem of mathematics be a logical truth or analytic. Finally, if HP is analytically true, then there are other similar definitions that seem to do the same thing and that are just as acceptable as HP, yet taken collectively are not mutually consistent. This problem entails that not all of the definitions that individually have good standing, can have good standing at once. There needs to be further justification for why HP would be selected to remain in good standing, reasoning that would differentiate HP from the bad company.

HP and the general use of abstraction principles is justified on the ground that they are mere reconstruals or recarvings of an already accepted thought or content featured on the other side of the biconditional. The two sides of an abstraction principle may seem to have differing ontological commitment, which would lead one to question the truth of any of them. The point is that, despite appearances, the two sides share sameness in content. Speaking of two concepts standing in one to one relation to each other is to be referring, perhaps unknowingly, to numbers. Speaking of parallel lines is to be speaking of direction. The ontological commitments are the same. The successful principles do just reconstruct the content of one side on the other side in new or different terms. The two sides of HP share truth conditions and so the veracity of it seems certain. Some principles appear to have this property but fail in their recarving role. Others fail because they are obviously untrue or inconsistent. If care is taken when laying down an abstraction principle the difficulties can be avoided. The structure of the principles is such that they give a contextual definition of the definiendum by fixing the meaning of the term with recognized constraints. In HP, the numbers are associated with two sortals are constrained such that they will only be equal if there exists a 1-1 relation between the object instances of the sortals, meaning they can be paired off such that neither sortal concept extension contained any unpaired elements. At first this reflects a connection between numbers and 1-1 relations, but there is a deeper interpretation of HP that sees the principle as showing a connection between numbers and second-order logic (as 1-1 can be defined in second-order logic). This is interesting because it shows that the logicist program does not just happen to work out on account of Frege's theorem, but that the logicism is deeply ingrained in HP. A connection to second-order logic seems less arbitrary than a connection merely to 1-1 relations. This goes a way to displaying the naturalness of HP, which is difficult to express in a technical restriction sought for by opponents.

The defense of abstraction principles and HP in particular might revolve around finding distinctions between those which are acceptable and those that are not. Consistency sets HP apart from BLV, for instance. HP is not vulnerable to Russell's paradox. Some have claimed that the source of error in BLV was not its inconsistency, but the fact that the right side quantifies over an unspecified domain that contains the very things the principle intends to establish, and that this circularity has bad consequences.[31] This is the Dummettian complaint against the enterprise of generating objects through abstraction principles.[32] He argues that the quantificational statements featured in the abstraction principles only have determinate content when the domain of quantification is fixed. The truth of a statement asserting the infinity of primes vacillates depending on, among other things, whether the domain of quantification for the implicit existential quantifier is infinite or not. If the domain is to be fixed prior to assertion, or the assertion only interpreted after the domain is fixed, then there is a problem in that the platonist must assert the existence of an infinite domain prior to knowing whether there are infinitely many objects. The requirement of a fixed domain prior to interpretation can and should be challenged. Wright points out that the statement has a meaning before a domain is fixed and that understanding HP comes down to recognizing what kind of domain is needed in order for the principle to hold true. The truth of the principle is assumed in order to see the conditions that would support that truth, rather than fixing sets of conditions and determining the meaning of the statement. The difference in method is based in each camp's realist and antirealist attitudes respectively. The realist has a notion of unconstructed truth that he can use, whereas the antirealist does not. If a fixed domain is required, it must be obtained through a higher level of quantification over a set of domains. It is not difficult to see that this would lead to an infinite regress as quantifying over domains required that those domains be fixed in some manner, which requires further quantification.[33] The other option is to allow that further domain specification would stop at some level of unrestricted generality, in which case that type of domain would be available for use

[31] The quantification is embedded in the 1-1 relation
[32] Dummett (1998)
[33] Wright (1998a: 352)

from the start. By momentarily accepting a domain of unrestricted generality, HP can be described as giving the truth conditions for accepting itself. Wright says, "what [HP] is supposed to bestow, in the first instance, is only, for each particular F and G, a knowledge of how things have to be in a given population if 'Nx:Fx=Nx:Gx' is to be true of it."[34] The explanatory power of HP remains, even in the face of indeterminate domains of quantification. If there is not a problem with the consistency or coherency of HP, its acceptance is well founded. The impredicativity of HP does not seem to cause the systemic failures feared. One distinction, then, between HP and bad abstractions has been described. Wright is even able to give a formal restriction applicable to all abstraction principles that is sufficient for distinguishing the principles vulnerable to Russell's paradox from those which are not.[35]He states that if it is the case that any object established by an abstraction principle falling under some concept that it is associated with through the term forming operator on the left hand side entails that it falls under every concept it is associated with in this way, then the relation on the right hand side is called Russellian. The Russellian relations make a principle fall victim to Russell's paradox. Coextensiveness, as featured in BLV is Russellian whereas 1-1 is not. This is because 3, for example, does not fall under all of the concepts which can be used to construct it, whereas the value ranges established by BLV do. Unfortunately, Wright's restriction does not go far enough, as Dummett points out that such a restriction does not rule out all contradictory principles.[36]

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[34] Ibid. 355
[35] Ibid. 347
[36] Dummett (1998: 377)

It is fortunate that the dialectic between Wright and Dummett can be found in one source with each responding directly to the other.[37] In these chapters, Wright claims the entitlement to stipulate HP, which Dummett rebuts. Dummett does agree on slightly different terms that abstraction principles are acceptable. He admits that there are good abstraction principles, but because there are no means by which to distinguish them from the bad ones, one cannot be entitled to stipulate any particular one. There is no reason to believe that the stipulated principle is in good standing. Dummett further criticizes the neo-Fregean enterprise for employing abstraction principles in an uninteresting and un-Fregean manner; for not demonstrating how we come to the domain that is big enough to contain numbers. A truly Fregean program would seek to simultaneously specify the domain to which would-be numbers would fall while giving reference to those numbers. The neo-Fregean leaves the domain unspecified, instead only offering the fact that an unrestricted domain can accommodate numbers once they are picked out. Frege's method would provide an account of numbers as indefinitely extensible and "unsuited to delineate a determinate totality."[38] Without a prior conception of an indefinitely extensible domain, the account of numbers is lacking, for assuming an unrestricted domain does not allow for the generation of an indefinitely extensible domain; it only allows for anything that can be established by an acceptable abstraction principle to fall into it. The fact that the natural numbers are generated by something like a successor function should be enough to conceive of an indefinitely extensible domain. The domain is obtained later in the procedure, but it is still well described.

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[37] Schirn (1998) Chapters 13-15.

[38] Dummett (1998: 380)

The use of an unrestricted domain, required by the neo-Fregean, is questionable. The intelligibility and hence legitimacy of such a domain can be doubted. It is a difficult concept to understand. It would allow for the existence of anything generated by abstraction. In a way it can be thought of as a vacuum. It is not empty, but it is defined by the empty space within it. Still it is hard to see how a vacuum-like structure could serve as a domain. Maybe we should alter the analogy so that the vacancies in the domain are not vacuum-like but more like dark matter, waiting to be uncovered. I am tempted to claim that the two disputants have reached a point at which there is no resolution. The realist can faithfully ascribe to this sort of analogy, thinking that once cannot place restrictions on a domain because one does not know what abstracta there are to be discovered. The realist is predisposed to hold that there is darkness which can be uncovered, and that the door must be left open for this, if we are to ever grasp reality as it is. The antirealist is predisposed to accepting the sentiment that there is no sense to be made of analogies to vacuum-like structures or dark matter within a domain. The domains must be constructed and as such cannot contain anything that was not put into it. It is possible that the use of unrestricted domains differs on a case by case basis depending on what an abstraction principle demands. $D^{=}$ is agreed by most to be a good principle. That may be because the domain is limited by the various orientations of a line. The line's rotation is limited by the nature of space. For each direction obtained via $D^{=}$, there is an easy construction via rotation. The hope is that if further criteria can be given for recognizing the acceptable abstraction principles, the use of unrestricted domains will be seen as unproblematic with regard to those principles, and their abstracta can be accepted.

Kit Fine gives two necessary conditions for the truth of a principle – that the relation on the right hand side be an equivalence relation (i.e. reflexive, symmetric, and transitive) and that the identity criterion must not be inflationary, meaning the equivalence classes must not be greater than the number of objects.[39] Given the non-circularity of an identity criterion, there is not impediment to its truth. Stopping here would be to say that everything that can exist does, as the restrictions really only serve to rule out objects which would be contradictory. The two conditions are not enough then, unless we are willing to prescribe to a maximalist ontology.[40] Fine goes on to offer some very technical restrictions. I will not discuss them except to say that some of them make use of set theoretic models, which for our purposes, with mathematics in question, cannot be appealed to. Giving further conditions is not easy. Ad hoc conditions may be crafted but cannot really hold sway.

It is not the neo-Fregean belief that numbers exist because they can, rather it is the other way around. Because they do exist, they can, which seems trivial, but the statement should be interpreted as saying that they do exist by dint of the nature of reality, and it is up to us to discover and make possible understanding of this existence. This belief implies that there is a higher standard for acceptance of abstraction principles than merely fulfilling the two conditions at the top of the paragraph. The principle has to genuine in some respect but it is difficult to convey this through a technical restriction. This condition is meant to convey the belief that there should be reason to expect the abstract thing to exist, and preclude the attempts by others to craft abstracts just to obfuscate matters by overpopulating the set of abstractions to obscure the line between good and bad. Numbers, for example, give us expressive power in mathematics, but also

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[39] Fine (1998, §1)

[40] Hale and Wright (2009) explicitly reject maximalism

reflects the idea of the simply infinite, which is not strictly mathematical. There is a fine line between this condition and a condition which would have it that the abstractions have to be experience based. I believe the line holds, however thin it is. The view is that HP, and other abstractions, are innovative and novel, uncovering new ways to speak about reality.

Other restrictions might be framed to answer the standing problems with the current set of abstraction principles. The current set (resulting from the restrictions given by Fine) is unacceptable because they collectively hyperinflate the universe of objects; i.e. taken together they provide for more equivalence classes than objects. Among this set there still remains pairwise inconsistent principles. The first restriction that we might take on is conservativeness – the conjunction of a principle to a theory does not affect the ontology of the theory.[41] Others suggest conditions based off this which Eklund refers to as ultraconservative conditions. Among these are stability – for some cardinality K, the principle holds in all models of cardinality ≥K, irenicity – conservative and compatible with all other conservative principles, and modesty – the addition of a principle to a mutually consistent theory yields no consequences for the ontology of the combined theory.[42]

HP is conservative. As Wright points out, the fact that HP requires an infinite domain is misleading. It requires an infinite model for its satisfiability, but only with regard to numbers.[43] It does not ask of a believer that he change his views with regard to non-numerical ontology to accept HP. It does not require that there be concepts beside the logical ones created to generate the numbers that have larger cardinality than can be

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[41] Hale and Wright (2001: 19)
[42] Eklund (2009: 395-6)
[43] Wright(1997: 295)

imagined in the world. If someone accepts HP and prior to the acceptance believed that the world is finitely populated, he can retain his belief that the non-numerical world is finite. Given its conservativeness, HP can easily be seen to be stable and modest. I am less sure of its irenic status. Wright and Hale believe that conservativeness is an important condition to place on the acceptability of abstraction principles, and it is not hard to see why. Conservative principles provide for their own acceptance. In addition to not entailing their own contradiction, they carve out a place in reality for their acceptance without disturbing the nature of things prior to their acceptance. It is a condition for sustainable belief.

While I believe that conservativeness is a good addition to the conditions required of an abstraction principle, it does not seem that adding more and more restrictions is the way to go. It is not believed that conservativeness is sufficient to erase all of the difficulties mentioned. There are many technical conditions that can be advanced to render the set down further, and there is a lot of literature on this. Opponents highlight a problem showing that a particular set of restrictions on abstraction principles is not sufficient, and demand further conditions. Sometimes the proponent can offer a condition, and typically the opponent can find a new problem to show that the restrictions are still not strict enough. It seems as if their goal is not to be antagonistic pests but rather to show that it would be impossible to ever know which principles are true of reality and which are artificial constructions. The restriction game loads the deck in favor of the opponents. As above, when there did not seem to be a stopping point for imposing conditions on the syntactic criteria to make the filter fine enough, there does not seem to be a set of non ad hoc restrictions that would yield only the acceptable abstraction principles. By discussing restrictions there is only the opportunity to show that HP is not bad and places the neo-Fregean in a perpetually defensive role. It does not allow for discussion of why HP should be believed. I hope to leave behind the framework of proposing further conditions on principle acceptability in search of a fairer playing field on which to evaluate HP. The hope is that analysis of existential quantification will provide an even ground on which to assert the acceptability of HP.

The meaning of existential quantification and existential beliefs has been treated as well understood. The fact is that there is not a universal interpretation of existence. To see the different interpretations I will use a framework developed by Sider.[44] He examines three possible interpretations of the existential quantifier. On one interpretation, tables exist, and on the other tables do not exist. There is nothing unique to tables. The third position is a mediation which claims that the two opposing interpretations talk past each other. The difference in opinion of the existence of tables might be blamed on differing understanding of tables, but this is dismissed, as there is a general agreement as to what defines a table. Even if it were the case that there is equivocation over the predicate 'is a table' it seems less believable that there is anything to equivocate upon with the predicate 'is a number'. Sider claims that both parties can agree that table can be defined as a collection of simples arranged table-wise. The debate is over the question of whether there is such a thing as a collection. Numbers can be defined within the framework as the content associated with sortal concepts. Sider claims "that "the real issue is whether any of these [quantifier] interpretations is *metaphysically* distinguished, whether any of them uniquely matches the structure of the world, whether

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[44] Sider (2009)

any *carves at the joints* better than the others," (emphasis his).[45] The claim is that disagreements over the interpretation of the existential quantifiers reflect different carvings of the world. There are many ways to carve up reality, and many would not be considered at getting at objects. There is a battle over what sort of carvings are natural and would reflect the real structure of the world, and hence reveal the actual objects. If realism is to be tenable, there must be an interpretation of existence that corresponds to a better carving at the joints.

If the debate hinges on the naturalness of the joints chosen, the questions have only been delayed. The discussion is now at a point where the neo-Fregean must either embrace a religious sentiment of believing he is carving the world correctly or justify his carving as being a good explanation of the world. The latter choice would either place mathematical entities in reality that gets experienced or construct the entities to explain reality. Neither would be characteristic of the neo-Fregean. No doubt certain joints are more natural than others. To carve the world up by those things which are F's or G's for two unrelated concepts, F and G, and those which are neither cannot be believed to get at reality or any objects. When attempting to describe those joints which would yield acceptable objects, we run into a generalized version of the "Bad Company" argument. The concern is simpler – how to justify a particular choice of competing realities. There is a sense in which all carvings can be viewed as artificial. Before, the entitlement to stipulate HP was defended against criticism claiming that the presence of bad abstractions obscure which are good. Now, the claim that the joints that yield numbers as objects are natural must be defended against a constructivist view. HP says that number is formed along joint cut out by 1-1 relations. And it goes the other way as well

[45] Ibid. 392

– that the 1-1 relations are formed along the joints cut by numbers. In fact, there is even more to it than this. HP shows a connection between numbers and second-order logic. If the joints carved out by 1-1 relations are questioned as artificial, then appeal to second-order logic can be seen as overlaying a joint which we believe cuts deeper. On this note, number can be justified over competing theories. The other abstractions cannot be ruled out on technical restrictions, but some of them are hard to believe. If each were put to the philosophical test as HP is here, many could not be so thoroughly defended. This goes to show something that appeal to formal restrictions cannot: that number has a special place in belief. It is difficult to go beyond this, which is why it is difficult to reject the view that numbers are not objects without seeming dogmatic. At least on this framework, it can be seen that the opposing views are engaged in a stalemate to be won on appeal to something deeper rather than having the realist always on the defense. When the point is reached where the realist maintains that the number joint is a natural one and the antirealists dispute it, there is genuine disagreement. The two sides meet, understand what the other is saying, and disagree over a deep-seated belief.

The neo-Fregean cannot appeal to anything in the world to show that numbers are part of the furniture of reality. He cannot say that the physical world seems to act according to numerical norms. The opponent denies the claim and he likewise cannot say much more. He can say that we do not need numbers because physical reduction is enough, but has no justification for the claim that reality adheres to ontological economy even if it is the case that reductionism works out. Both sides are in a position of expressing something akin to religious belief, having faith that the world *just is* that way. I hope I have shown that the antirealist is in an equally vulnerable position – that of the skeptic who can only stubbornly deny that the world *just is not* that way. It is nice to find a point of genuine disagreement in which the opposing sides are in mutually vulnerable positions. If philosophy is at all about our experience and beliefs as humans, we ought to be able to discover such points of disagreement which reduce the opponents to their individual experiences and deep beliefs. The upshot is that there is no shame in believing in the objecthood of mathematics.

It should be noted that Hale and Wright do not endorse the view of quantifier variants.[46] They demand a single existential quantifier that is understood, in spite of the difficulties presented here. They justify HP by demanding outright entitlement to employ all abstraction principles. An abstraction principle must be shown to be in bad standing before it is rejected. It is difficult to see how this would not lead to a sort of maximalist ontology, and one that has inconsistencies. They believe they distance themselves from such metaontologies. They claim that their entitlement derives from the fact that there cannot be any metaphysical collateral to block the way of accepting abstraction principles, as they are innovative and cannot answer to prior ontological expectations. Perhaps they are not worried by inconsistencies in the abstract realm, which is not completely incredible. But, their claim to outright entitlement of abstractions, a sort of "Good Company" objection, in order to include HP seems radical. I am not sure that it is a fully realist claim. There is an aspect to realism in which there are unknowable parts of reality, but there is also the implicit claim of a determinate reality. At the very least, if they expand the universe as much as it would appear by granting entitlement to all seemingly good abstractions, it would seem uninteresting that numbers exist. It can be conceived that number competitors could also be allowed to exist, and much of the

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46 Hale and Wright (2009)

force of realism is lost. I see the claimed entitlement to all abstraction principles as too radical a view to adopt, which is why I diverge from Hale and Wright and sympathize with others who attempt to fit acceptance of HP (and other select abstractions) in by other means.

There is another option proposed by Eklund. He entertains the idea that the neo-Fregeans need not be committed to HP or abstraction principles at all in order to establish platonism.[47] There are other implicit definitions of number, PA for example, which would be suitable for conditioning the truth of statements with occurrences of number, or even general mathematical use. It does not have to be an either/or situation. The fact that there are independent implicit restrictions is something fairly unique to HP and number. The problematic principles do not have independent restrictions. HP does not invent numbers but discovers them and so it is unsurprising that there is a different set of conditions that describe number. It seems that if we endorse the proposal of using implicit definitions restricting the use of number or other abstracta in general, then there should be no impediment to using the abstraction principle as the principal method of knowing about the objects other than the questioned status of the principles as a priori or conceptual truths. It is necessary then to explore in what sense these sentences involving numbers are said to be true.

The priority thesis does seem a little hard to accept, or at least its simplicity is. The commitment to what a true statement refers to seems to require some other premise. It seems as though there is a latent commitment to a correspondence theory of truth, as this would require that truth be expressed in terms of the ontological reality. Now when number theoretic statements are said to be true, it is because the ontology of

[47] Eklund (2009: 406)

reality is being touched upon by the number-objects which feature in this truth. This is not the correct estimate of the neo-Fregean view. Correspondence, coherence, or any other account of truth which would have us presuppose some structure to the world are unacceptable. It is the things which are taken to be true that determine the ontology of the world and not the other way around. Truth is, as asserted in response to the Benacerrafian challenge, not subject to further verification. Or rather, the truths of certain subjects are unquestionable. This is how Hale and Wright should be interpreted when they speak of the metaontology required for their view. Above, I said that we should not accept the radical view of accepting all abstractions. But, in the present context I think we can make sense of how they should be interpreted when they state that, "the kind of justification which we acknowledge *is* called for is precisely justification for the thought that no such [ontological] collateral assistance is necessary. There is no hostage to redeem."[48] We should not believe that all abstraction principles are assumed to be good until they are shown to be false, but merely believe that the ones that are identified as being good for some reason other than presumption need not answer to anything further. There is, as they say "no gap for metaphysics to plug."[49] The good principles supply the ontology needed, and make for their own truth. This much is agreed upon; I agree that at the level of ontology nothing more is needed for the truth of the principles. The disagreement comes at the level of what suffices for acceptance of an abstraction. I believe that widespread entitlement is not the appropriate attitude to base acceptance on, that this is too radical to believe.

The oddity of the situation now, is that the truth spoken of appears to be of an antirealist flavor. This antirealist truth still imposes realist conditions upon reality. There

[48] Hale and Wright (2009: 193)
[49] ibid

is no problem accepting the picture of truth requiring no presupposition as to the complete order, structure, or characteristic of reality. This does not mean that the concrete cannot be presupposed, only that the concrete cannot be assumed to be all there is to reality or that all statements and terms can be reduced to the physical. In this regard, it becomes quite uncontroversial to allow for the existence of abstract objects. One question that I have ignored to this point is where HP comes from. HP must somehow be known, but it might be asked if it is somehow installed through experience or known a priori. It is claimed to be analytic, but this has gone unsupported.

One framework that we might look at to analyze the neo-Fregean truth is a Quinean one suggested by Ebbs.[50] The framework places truth among conventions (similar to the abstraction stipulations) as attempted by Quine and Carnap. Carnap thinks that as far as metaphysical questions are concerned either they are trivial and decidable analytically, or they come down to a pragmatic choice. The choice is laying down a statement as an axiom for a particular system which would have the appearance of a particular ontological commitment. Of course, Carnap has an antimetaphysical attitude and the choices among metaphysical claims are not to be invested in seriously. The neo-Fregean fights to stipulate HP much in the same way that a Carnapian would lay down a convention that would go towards determining a system, but the neo-Fregean hopes to avoid the Carnapian conclusion that any number of stipulations could have been made in place of HP, which would have resulted in just as acceptable a system. Carnap upholds the distinction between the analytic and synthetic to rid scientific debate of metaphysical issues. The analytic statements are determined by the stipulations laid down to determine the language and allow for the truth of the synthetic statements of science to be

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[50] Ebbs (forthcoming)

established empirically and interpreted in light of a fixed language. Quine rejects any sharp or absolute distinction between the analytic and synthetic, but does maintain along with Carnap that certain stipulations are made irrespective of reality. This is Ebbs's view. Whereas some Quineans have taken Quine to be saying that there are not the conventional or primitive truths that Carnap thinks there are, the real difference is merely the implications of the conventions.[51] Carnap does not endow the conventions with any sense of hypothesizing the nature of reality as this would be read as an external proposition, which he sees as misguided. For Quine, the conventions laid down (even the freely postulated ones) are on a par with hypotheses about reality, which are tested by experience. The conventions laid down are motivated by experience. He does, however, acknowledge that there are such propositions which differ from the ordinary propositions, even if they are subject to the same verification procedures.

The neo-Fregean would probably want to argue along slightly different lines than Quine to maintain the analytic status of mathematics. Burgess points out that analyticity is not the same as self-evidence, and it is the latter which provides the foundation of mathematics that philosophers seek.[52] They might argue that mathematical method with its apparent existential commitment to numbers, entitles the stipulation. These methods present the neo-Fregean with good justification for accepting HP. HP can be justified over similar stipulations, like "'numbers are objects' is true," as HP reveals more about the objective nature of numbers than the similar stipulation. Obtaining entitlement to stipulation by this strategy may come at a high cost. The ability to lay down the truth of HP on pragmatic consideration is based in a belief that pragmatic decisions are all there is. This does not present HP as the firmly established truth, but like Quine's

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[51] Burgess (2004:41) similarly maintains that Quine "*needs* to recognize a notion of analyticity."
[52] Burgess (2004: 39)

conventions, it would be merely true until further notice. The platonist goal can be achieved at the cost of the logicist one. This could be avoided if one take the view that logic is likewise conventionally true. This may be tempting, but I think it erodes away the realist agenda of the neo-Fregean. The point of the program is to show that mathematics and logic both have some foundational place in reality and that they are not experience based. Victory on Quinean grounds would be vacuous.

I think the neo-Fregean must hold that HP is special in some regard, in that it allows for us to see an essential carving of reality. Acceptance of HP should be based on a view of reality independent of empirical observation and instead determined by logic. HP does entail ontological commitment in reality which some are uncomfortable with. I am not sure that I understand the fears of these commitments. A common claim is that there is nothing to ensure that the domain of objects is infinite. For starters, HP provides for an infinite domain. Also, finitude seems as serious a commitment as infinitude. A presumption of finitude cannot be seen as the safe or default ontology. In fact, infinity seems to be the safer presumption. Take time as an example. Time can be thought of as either infinitely divisible or quantized. If it is said to be infinitely divisible, there is an intuitive sense of infinity, which those committed only to the reality of the spatio-temporal realm can accept. If time is quantized, it cannot be broken down into infinite divisions. But, time is still thought of as a continuous dimension that extends infinitely into past and into the future. If one tries to think of some first moment in time, it would seem that the question of what happened just prior to that moment makes sense. Likewise, if one thinks about any moment in the future, it seems possible to ask what happens next. In this way, time still provides an intuitive sense of the infinite. The acceptance of the natural numbers only requires a belief in the simply infinite which parallels the time example. There is only commitment to the successor of a number which we already have. This idea of infinity, as the always-possible successor is far less ontologically laden than the idea of infinity as a far-off, unreachable bound. It is the former notion that is being pushed.

There have been advances in providing for the acceptance of HP. The "Bad Company" objection is still on the table, but its force reduced. It can remain quarantined for now, as it does not directly entail the rejection of HP, but instead says that we cannot tell that it is good. To be a realist about reality sometimes entails its unknowability or the lack of an identifiable reason to explain why reality is the way it is. Certain abstraction principles can be identified which would reveal parts of reality that are otherwise unknowable, and should not be rejected because of this.

III. Methodological Naturalism

A. Background

Though Frege believes that ultimately it was the use of mathematics and numbers that founds truth and realist attitude, he attempted to give an account of the nature of mathematics through philosophical inflection, ultimately arriving at a linguistic explanation. The Fregean and neo-Fregean accounts of mathematics can be labeled as philosophy-first. Maddy subverts philosophical analysis, taking a Quinean approach to mathematics. She claims the answers to philosophical questions regarding mathematics are contained within mathematics and science.[53] She calls herself a methodological naturalist and takes on a philosophical modesty that makes philosophy secondary in the context of discussion; mathematics and science are primary. In might be tempting to see her view as derivative of Carnap, in claiming that metaphysical questions are to be settled

[53] To be clear, Quine says that the answers are only within science.

within the languages of mathematics and science, but it is important to see her as part of the Quinean movement away from the antimetaphysical Carnap. She also moves with Quine away from the positivists' strict demarcation of what counts as analytic and what does not, and allows for a more holistic picture, where sentences are not verified or justified on an individual level. But, in the spirit of all parties under consideration, Maddy dismisses the doctrine of philosophy first in internalizing the metaphysical to science and mathematics. She, however, intends to preserve the meaningfulness of purely metaphysical questions. Questions about the metaphysical nature of higher mathematics that are seen as meaningless by Quine, remain meaningful here. As a naturalist, Maddy is not merely describing the methods of mathematics, but continuing in the field by clarifying residual, sometimes philosophical, problems.

Maddy breaks from Quine by providing mathematics a stronger foothold in the structure of reality than the contingent nature it seems to have as a tool of science. To capture the general attitude that mathematics is necessary or a priori, Maddy supplements the naturalized view with something like a Gödelian faculty for intuiting the objects of mathematics. This faculty provides for the obviousness and intuitiveness of some mathematical facts. Gödel believes in some mystical interaction with the mathematical realm that "force themselves upon us as being true."[54] Gödel's intuitive faculty is similar to the perception that gives us macroscopic physical entities. His intuition gives the objects and structure of the mathematical realm, at least to a point. Maddy wishes to demystify Gödel's intuition. She hopes to establish for the perception of sets. Set theory has long been synonymous with foundational mathematics and so if a

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[54] Gödel (1947/83: 484)

perception of sets can be accepted, at least certain mathematical facts can be placed on a par with other evident facts of the world.

Those who lack Gödel mathematical vision might question the results he urges based on it. The results of higher un-intuitive mathematics as well as his insistence on mathematical realism are difficult to grasp without sharing his intuition. For these issues, Maddy turns to the indispensability arguments offered by Quine and Putnam, which demands a commitment to the entities featured indispensably in the current best theory of the world. If science is the best theory of the world, and mathematics is indispensable to it, then there is a commitment to the entities postulated by mathematics. Maddy does not require that the entities actually be indispensable to the theory, but only demands that they serve some important purpose (which Quine agrees with, given certain qualifications of what the important purpose is). The immediate oddity of the indispensability argument – that the best theory accords with reality – is reflective of the abandonment of the doctrine of philosophy first. Whatever the current best theory dictates of reality is to be accepted. No further philosophical questions may be posed to ask whether reality actually is that way. This is typically defended by reasoning along the lines of: it is unacceptable to say that science is the best theory of the world, and proceed to maintain that it is still not enough and never can be. Maddy hopes to capture mathematics and science as they are practiced. She departs from the traditional indispensability argument in promoting the status of mathematics to that of science, and argues for the validity of mathematical justification outside of science.

B. Mathematical Intuition

Our talk about mathematics and numbers often mirrors the speech about ordinary objects in the world. When pushed to answer what these mathematical things are exactly, the mathematician encounters difficulty. He can cite the fact that he has two hands, that 2+2=4, and myriad other facts about 2. Some are obvious, others are not. None of these number theoretic statements yield the requisite knowledge for evaluating an ontological claim about numbers or mathematical entities. It is not clear that any mathematician would even attend to such questions. Mathematics has progressed without stopping to respond to philosophical question. If is a fruitful and powerful discipline without doing so. Its success suggests that mathematicians have a good sense of what numbers are and how they work in all of the relevant ways, even if they cannot account for their ontological status.

The Benacerrafian challenge was examined and responded to by the neo-Fregean program. The naturalized response will be slightly different. In light of the current account of mathematics it is not so much the abstract nature of mathematics that causes problems, but the fact that it is mind independent. If Maddy is going to be a realist and claim that mathematics is mind independent, she can provide for some causal link to numbers or risk embracing a compromised view of mathematics as either mind dependent or unknowable. The challenge provides for some unease about the epistemological status of a naturalized mathematics that does not just accept its secondary role in science. Maddy cannot employ the typical response of saying that there is nothing more that can be said, and any request for more is ungrounded, because she splits verification of mathematics from verification of science, cleaving mathematics from the safety of the web of belief. The typical response to nominalists, that they are unacceptably being alienated epistemologists, cannot be given, because Maddy herself alienates the epistemology of mathematics. Therefore, Maddy owes us an account both for the source of her realism as well as for the knowability of mathematics.

The challenge is to show how a naturalist of Maddy's kind can put herself into an interaction with the content of mathematical knowledge. The knowledge must be more than justified true belief; it requires some reliable and causal link to which the states of affairs which make the truth of the known statements true. The naturalist focuses on sets and so must supply a picture that fulfills the following criteria: that there are sets, and that we form beliefs about them in a reliable way. She can focus on sets alone because all of mathematics can be grounded in set theory. The picture of mathematical knowledge is split in two. There is a base intuition provided by a Gödel-like perception of sets, while any further mathematical knowledge is provided on appeal to indispensability.

While the naturalist dispenses with abstractedness of mathematics, in turning to Gödel she provides for a view of mathematics at its base level as quasi-physical. Invoking modern neuroscience and psychology she employs the same mechanisms responsible for our beliefs in everyday objects to describe the near perception of sets. She hopes to locate sets in reality, and claims that they exist side-by-side, or perhaps behind each object. Where there is an object, there is a set. There is a set of ten (fingers) before me, and concurrently a set of two (hands). Whether the physical heap in front of me is participating in two-ness (or-ten-ness) where two (and ten) is an abstract form, or whether two is located everywhere there is a pair – the distinction is irrelevant unless the mathematician has some reason for choosing the "universal" two over a "local" two (although it is hard to see what mathematical reason there would be). For now, the picture of spatio-temporal derived sets suffices. The sets are quasi-concrete. We cannot pick up or point to sets. We can only perform such actions on the members of a set. The naturalist is not constructing a philosophical account, but crafting one around mathematical and scientific belief which underdetermines these types of questions and she therefore does not need to take a definitive stance about whether there is an abstract form which each instance of two participates in or whether two is just in every pair. There is no need to declare the existence of an abstract form in addition to all the physical instances.

Establishing the quasi-concreteness of sets puts the naturalist in good position to explain the path of sets to the mind. The view must be clarified to develop the perception of sets and the formation of beliefs based on those perceptions. Experiments in neuroscience have established that at a young age humans (and for that matter many animals) develop the ability to see objects in a field of vision. Whereas the field of vision may, at first, appear as an abstract painting with no discernible foreground, the mind quickly develops the ability to identify objects which present themselves clearly. The basic explanation is that through our experiences with objects we come to associate certain visual features with certain behavior. So perhaps a visual stimulus with clear boundary that moves about within a visual frame is learned to be one thing and not a series of different visual stimuli appearing and disappearing from the frame near continuously. The concept of identity is learned through observed behavior of similar looking visual stimuli being recalled and recognized. As the behavior of objects is observed further, neural connections are formed, conditioned, and restructured to account for a wider range of object behavior until objects can be readily picked out.[55] Humans do not have a super perception capable of seeing everything as it really is and are still prone to being duped by optical illusions and camouflage, for example, but in general the process is reliable. Cognition of macroscopic objects develops through constant observation and neural conditioning that eventually sets up a complex neural

[55] The summary of neuroscience comes from Maddy (1990: 50-67)

cell-assembly that becomes efficacious at recognizing objects. It is the macroscopic objects that play the central role in the development of beliefs about identity and objecthood.

Maddy does not go as far as Gödel in providing for a direct intuition or perception of sets. Only the physical members of the quasi-concrete set are accessible. The picture of sets as quasi-concrete is not easy to grasp. I think the picture is that sets are almost hiding behind the curtains of their physical members. The same Fregean argument applies as to why sets or numbers cannot be synonymous with any physical heap, in that physical heaps do not by themselves divide into countable units. Frege formalizes the identity conditions of the concepts to which numbers can attach. Maddy provides the scientific picture of how the mind grasps an object by certain visual stimuli. The sets are not physical, but they are asserted to be intimately tied to the physical. This may be the case, but that does not mean that there is a perceptual faculty of sets. The neo-Fregeans likewise believe number to be tied to sortal concepts, many of which are of physical objects. They allow for the appropriate distance by claiming that numbers are abstract. If Maddy's neural assembly is another name for the process by which the neo-Fregean grasps number by way of sortal, the account would be far clearer. As it stands, proximity to the physical does not entail that sethood can be perceptual. The grasp of sets or numbers is far clearer on the neo-Fregean account, by which the conceptual process is described, rather than a conditioned response to the visual being described.

Regardless of which account is clearer, we can describe how the intuition generates beliefs about sets. When three eggs are perceived, three concrete entities are perceived, but also the threeness of the eggs collectively can be sensed by the Gödelian faculty. The latent three-set controls what can be perceived. If the eggs are rearranged (transparently), the belief continues to be that there are three eggs and requires no inference. The three-set ensure that if an egg is taken away a belief that there is now just two eggs will be formed. There is no need to stop and reevaluate the contents of the perceptual field. Similarly, a beginning mathematician does not need to think about the equality of 1+2 and 2+1 (though an advanced mathematician might). While interaction is only with the concrete members of a set, the underlying set controls the perceptions and beliefs formed about the concrete members, and so it can be said that there is a natural intuition of sets. These set intuitions motivate the beginning of set theory.

The skeptic may demand evidence for the intuition of sets, thinking that it is not some underlying intuition that is to be attributed with the belief that taking away one egg from three results in two but instead the reevaluation of the current perception. He might claim that an intuition of sets is being confused with a new count of objects in the visual field. Scientific research provides evidence to support that the perceptual beliefs are noninferential. Experiments on young children show that, like the case for objects, they do not have a concept of sets of objects at first. When objects are moved around they form a new, different belief about the size of the set of objects. Three objects placed close together are viewed as different in number from three objects placed far apart. At some point, it seems as though the child develops the concept of set, and no transparent permutation or rearrangement of a fixed number of objects will sway the child's belief in the numerical equality throughout. The range of perceptual beliefs is very small, and it is difficult to form beliefs about particularly large sets. Even with a set of three objects, the number of beliefs I can form about the set is very small. I believe that permutation does not change the cardinality of the set but I have no noninferential belief about the size of the power set of three. Power sets are a basic construction of the

Zermelo-Fraenkel axioms of set theory (hereafter ZF)[56] and I am doubtful that the basic intuition formed can account for the facts about power sets. We can intuit a base notion of set, but not much more.

Up to this point, it has been claimed that it is sets that are the content of the mathematical intuition. However, nothing about the behavior of observed collections of objects determines that the collection corresponds to a set as opposed to a class or even an aggregate. The differences are only seen at a higher level. The difference between a set and a class is that a set can always be a subset; there is an iterative order that does not hold of classes. It probably does not hold of aggregates either, although they are not rigorously defined. Sets are defined extensionally, whereas classes are defined intensionally. For this reason, classes are prone to set theoretic paradoxes and are not widely used in mathematics. The naturalist, at this point, has no intuitive knowledge that this is the case and therefore has no reason to believe that she is dealing with one kind of collection over another. The naturalist does not have to definitively state which of the options is perceived or intuited, so long as sets are among the options, which seems to be the case. With regard to the small numbers and small amount of beliefs generated, the distinction is mostly stylistic. It is only on a grand scale that intuition of classes and aggregates reveal their flaws. The naturalist is working within the framework of an established mathematics and has reasons beyond basic perceptual beliefs to hold the primacy of sets. While it is true that the perceived collections do not lend themselves to a particular interpretation, i.e. set, class, etc, the perceptions do not fail to be able to be interpreted as sets. Science and mathematics provide a formal account of the unsuitability of the other options even if perception cannot discern as much. While this

[56] The Zermelo-Fraenkel Axioms are the accepted and widely used axioms for set theory

is all true, it once again reflects the opaque nature of the supposed intuition we supposedly have about sets. The neo-Fregean lays down the numeric properties from the start. HP is the result of deep reflection about what could provide a foundation for mathematical theory. There is no such foresight in this account, and therefore no shored up foundation. The pieces can be made to fit, but if it is the case that things are ambiguous prior to formal consideration, it should be asked whether the Gödelian faculty plays any role in the actual acquisition of knowledge. While the perceptual beliefs can be fitted to accord with sets, the perception can also turn out to be of a class. The second part of the program still provides for knowledge of mathematics, but mathematics is no longer as intuitive as hoped for. Only where sets can be interpreted as classes could it plausibly be maintained that the perceptual beliefs formed have anything to do with mathematical theory, and only by way of analogy.

Emily Carson finds Maddy's account unsatisfying in a similar manner, claiming that perceiving something as having a number property that includes combination and that has sub-collections is not enough to assert that a set is perceived or that a set is hiding behind that which is perceived. Carson pinpoints the essential difference between sets and other possible ways of defining a collection. It is that sets can be members of other sets, and that unless this basic property can be perceived, it cannot be said that a set is perceived or intuited or sensed.[57] This much has already been stated more or less. Maddy has her defense in maintain that what is perceived can be retroactively determined to be a set. The perceived object turns out to be a set in the end, but the perception lacks the basic property of sets, iterative ordering. Maddy cannot account for the ability to perceive the basic set structure. It is not the case that I simultaneously see

[57] Carson (1996: 7)

the many sets: my hands, the singleton[58] containing the set of my hands, and so on. Carson goes further than above in saying that without picking up on this structure, we cannot be said to be seeing sets. Above it was possible, but not determinate. Here, it is questioned whether it is even a possibility. Some of the criticism can be attributed to the fact that she does not admit that we see collections as one object. She insists that any attempt to view a collection as a single object, which is necessary to be seeing a set, is conscious. This seems true for large sets (larger than the visual field at least) such as a crowd of people. Three eggs however can probably be apprehended as a collection, 'three'. It seems equally probably that the perception is of three eggs as it is of egg egg egg. Depending on the viewing conditions, number can just as easily be perceived as the individual objects can. Without establishing that the collection is viewed as a set it is difficult to say that it can be exploited for the development of set theory, for the reasons above.

The details discovered by mathematicians can be imposed on the collection to consider it as a set, but the collection itself cannot generate beliefs that can properly be called mathematical in nature. They may share mathematical behavior with real sets but should not be seen as intrinsically mathematical. This is for all of the same reasons that "Cookie Theory" does not count as mathematics. There is a formalization, which, while it can be applied to the world, is separate. This is implicit in the naturalized position that when observation refutes our cluster of beliefs, mathematics is the last belief we revoke.[59] That Maddy thinks that the latter details can be provided to believe that there is a set perceived is backwards. How are the perceptual beliefs supposed to undergird basic

[58] A set with one member

[59] Quine (1991/2004: 59) Something along the lines of mathematical necessity is reflected in his maxim of minimum mutilation. Maddy has to hold to at least this much.

mathematical intuition, if it takes mathematical theorization to reconstruct an intuition from what is perceived? I am not sure that mathematics has the strength to impose a reading of the collection as a set. Mathematics shows that sets are the suitable foundation for a rigorous theory. This does not mean that sets are seen behind a collection of objects. Maddy's claim for some faculty for supposedly perceiving sets would be appropriate if the later theory could provide some reason why classes are not what is seen behind a collection, even if the reason is not immediately perceptible. The fact is that it is not until the size of the collection becomes impossibly large that the distinction between sets, classes, families, or some other grouping. The everyday objects of perception cannot be perceived in groups of a large enough size to differentiate on theory what the group could be.

While it is certainly believable that the number property of a collection can be intuited, vague intuition is not the foundation of mathematics. Mathematics adheres to rigor and this sort of intuition does not do much to explain the understanding of sets and the theory built upon them. It is believable that cell assemblies are formed to internalize behavior of collections of objects. It can even be said that this number-holding thing which lies behind a collection of objects motivates the discipline of mathematics. It is doubtful that what is seen or intuited *is* the content of mathematics. It may be coextensive to a degree, but it is not well enough defined to believably found mathematical intuition or even support the claim that mathematics is obvious. The claim should be that the high rate of coincidence between mathematics and the real world provides an excellent analogy, but cannot overcome the essential gaps pointed out. The set may be motivated by the collections which account for the high rate of coincidence, but once the set object is thoroughly defined it remains only a likeness to its real world inspiration. The set has certain properties such as {1,2,3,4} not being the same as {1,2,3,4,5} but being the same as {2,1,3,4}. The set also diverges from its real world primitive self in having an exact definition which provides for certain qualities not perceptible or recognizable of a collection.

I do not mean to say that this is Maddy's view of things. I do not think she believes or is committed to a view of sets as artificial constructions inspired by perception and intuition. She really does believe that what is seen can be known to be sets and that sets are mind independent. Further, she seems to believe that sets are something described, not constructed, by mathematics. Without providing direct access to the sets, which she tries to do through intuition, the justification for believing in the reality of sets is weak. It takes a degree of faith to believe that the perceptual beliefs that are actually formed correspond to mathematical reality at all. The neo-Fregeans are clear about the strictly separate mathematical reality. It is difficult to see where mathematical entities live on Maddy's view. Even if the intuition could provide insight, we have the idea that sets are not concrete but are chained to the concrete. With its failure, not only is the location in question, but also the independent existence. This might explain why Maddy gives far less attention to an intuitive faculty in *Naturalism in Mathematics*.[60] If I have things right, this is a change for the better. Without something to take its place, however, the obviousness and immediacy of some of the truths of mathematics cannot be explained. If her program is a response to this lacuna in the Quinean approach, then it fails in this respect.

The failure of this part of the program should not be surprising. The picture of set intuition is parallel to that of intuition of macroscopic objects. Both are said to

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[60] Maddy (1997)

generate concepts via neural conditioning to visual stimuli. The analogy to knowledge of physical objects can only be drawn out to the stage of scientific hypothesis. The phenomena processed about everyday objects are subject to verification through experiment. The hypotheses regarding the nature of physical objects can be tested and observed either directly or indirectly. If I only ever observe straws in glasses of water, I would have reason to form the belief that it is in the nature of the straw to be angled. Should this belief be questioned I can potentially pull the straw out of the glass of water and verify the exact opposite, that straws are in fact straight by nature. Purported knowledge of most objects is really just a hypothesis with some level of justification (that may be very strong). This hypothesis is subject to a verification process, which does in theory exist for physically based hypotheses, but which does not exist for purely mathematical hypotheses. It is unsurprising that the standard of knowledge involves a verification process. The inability to go through a verification process entails the unknowability, or instability of knowledge of mathematics. The supposed perceptual beliefs of sets generated by perception fit with the real world, but cannot be verified, especially without having definite content.

Taking a naturalized position may allow for viewing mathematics as a fruitful fiction. The fiction referred to is not metaphysically loaded but is instead to be interpreted in the context of science. Just as atoms were given an inferior ontological level until there was enough verification, mathematics might be stuck in a compromised ontological state and be regarded more in a fictionalist light. The naturalized view is committed to what is used in science, but it holds certain commitments stronger than others. Commitment is a matter of degree not kind, as it must be given that science is always changing and our beliefs needs to be versatile to respond to those changes. The goodness of mathematics is a fairly stable belief, but that may only be because mathematics is not itself stable, and gets reworked along with science, albeit at a much slower rate. Maddy cannot even make appeal to the applicability of mathematics in the world to establish realism, because she does not base her general realist attitude of mathematics in the real world. The realist attitude is based in intuition and the demands of mathematics. The intuition is not clear, and would have only revealed a tiny portion of mathematics. Appealing to the classic indispensability arguments does not allow for a realist attitude about higher mathematics. Maddy seems to be aware of these facts, because the realist agenda is also less present in *Naturalism in Mathematics*. She is actually accepting of fictionalist and formalist attitudes, seeing refined versions of either as possibly more consistent with mathematical methodology than staunch realism.[61]

C. Mathematic Naturalized

The failure to develop a mathematical intuition by appeal to perception of everyday objects means that Maddy must begin again with a new approach. There is not a case for realism yet, and the presumption that the obviousness of basic mathematics can be accounted for is likely gone. Even if the first part of the program succeeds, Carson's criticism of the program shows how little of mathematics can be accounted for by intuition. The failure of the first part to develop a realist construal of any part of mathematics implies that the entirety of the account must be given now. Under the naturalized view, direct access to mathematical reality cannot be hoped for and instead the claim will be more that it is acceptable to be a realist given the considerations of the program. I saw the potential for direct access to mathematical reality as an advantage of Maddy's program over neo-Fregeanism, and so with that hope dashed, it will be

[61]Maddy (1997: 202)

necessary to see if a realist position can be maintained with her methodological naturalism.

Maddy's naturalism is amended from Quine's. Maddy embraces the general attitude of the indispensability argument, in claiming that because mathematics is needed in science, which is the best explanation of experience, it is to be accepted in whole as good. She sees acceptance on these grounds as justifying the grant of autonomy of mathematicians to develop the field according to their own principles and methods, without direct consideration of science, and certainly without much consideration for philosophy. By looking to the history of mathematics we can see many instances where disregard for science in moving forward with mathematics has turned out to benefit science in the end. Because of this, mathematics has earned the allowance to develop according to its own methods. To think that mathematics is subject to the advances in science is to be mistaken about the history of its relationship with science. This is also the case for those who believe that scientific needs fuel mathematical innovation. Take the development of non-Euclidean geometry. Although its application was uncertain at the time of development, it turned out to better approximate the physical world once physics caught up with mathematics to realize this. Lobachevsky, one of the founders of non-Euclidean geometry, actually said that, "There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world."[62] The break from traditional naturalism is legitimizing mathematics on its own merit rather than keeping mathematics answerable to science. The view has the nice consequence that unapplied mathematics is not viewed as false in virtue of being unapplied. The non-Euclidean geometry would have been seen as a meaningless fiction

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[62] Quoted in Rose(1988)

prior to the insight that it better accorded to physical theory. We might think of the new position as granting mathematicians the role of arbiter of truth, a promotion to what scientists have had all along. Restricting mathematics to have application in mind would be an undue restriction on the eventual applicability of mathematics. This consideration sways the naturalist to accept mathematics on its own terms, as its own science.

Because mathematics is split from science in this way, Maddy makes explicit what considerations ought to drive mathematics, if not use in science. If mathematics is on its own, it could seemingly proceed according to any maxims it chooses. Maddy tries to keep mathematics focused, or rather tries to fit the maxims of mathematics into a focused framework, by providing an account of mathematical methodology. This methodology also seeks to properly place philosophy in mathematics, if it is meant to be in the subject at all. She hopes to identify the goals of mathematics and ask whether certain methodological choices have historically played a role in furthering the goals. It is not always clear what the goals are. In seeking out axioms about higher cardinalities and continuums, both Cantor and the French analysts arrived at a position of saying that the goal is to find a complete account of sets of real numbers despite having different mathematical interests.[63] If we look at the Continuum Hypothesis (CH), we see that questions surrounding it seem to be both philosophical and mathematical. These questions aim at the possible *existence* of a set whose cardinality is between that of the natural numbers and the real numbers, but the existence or nonexistence of such a set plays an *important role in mathematical theory*. Maddy's methodological naturalism seeks to answer the undecideable questions by first eliminating the considerations that have not been historically helpful at adjudicating similar concerns. It is largely the case that the

[63] Maddy (1997: 194)

philosophical considerations, in this case whether there is such a thing as the continuum, have not been helpful, at least on Maddy's view. Eliminating the bad considerations serves to amplify the good considerations, which she believes will elucidate which methods are appropriate for the problem and hence provide direction with which to move forward in resolving the issue. Once a methodology is proposed for a particular problem, it comes up for review. Arguments can be formulated within the proposed methodology and tested for plausibility. The potential consequences can be examined. The rationality of the arguments can be examined for soundness. This work largely falls to the philosopher. The philosopher is the one who helps draw out the conclusions that would come from a proposed result.

The history of the Axiom of Choice (or just Choice) us a good example for Maddy to use. Choice states that from a collection of nonempty sets, there is a function whose domain is the collection of sets, and this function will map each set to a member of itself. Choice may not sound particularly special when formulated like this, but some of its equivalents convey significance immediately. They include well ordering - that every set, infinite ones included, can be ordered by some relation; trichotomy – that any two sets will either have the same cardinality, or one will have greater cardinality than the other; and that every vector space has a basis. Trichotomy, in particular, seems both highly plausible and important for a rich theory, which lends to acceptance of Choice. On the other hand, Choice has some absurd seeming consequences. The Banach-Tarski Paradox follows from Choice and proves that a sphere of a given radius can be decomposed into a finite number of pieces which can be reassembled to form two spheres with the same given radius, i.e. same volume. It is easy to see that an infinite number of spheres all copies of a given sphere could be produced from just one. Given this consequence, Choice seems obviously false.

In the development of set theory, there was much discussion about whether Choice ought to be accepted or not. The set of axioms consisting of ZF and Choice (ZF+C or ZFC) and the set of axioms consisting of ZF and the negation of Choice (ZF+(-C)) are both consistent and so it is not a question of whether Choice is a theorem of the base axioms. Some of the discussion surrounding the choice to be made regarding Choice was philosophical, asking whether it Choice is plausible or further, true. No definitive philosophical argument could be advanced either way. Today, though widely accepted by mathematicians, the philosophical status of Choice is still debated. Choice won acceptability on the arguments of the mathematicians who studied the consequences of accepting or denying Choice. They argued that Choice has been featured latently in mathematics for quite some time. This reflects the plausibility of the axiom, but it also shows the central feature it plays in many proofs and how reliant mathematics is on it. When enough research had been done to determine that certain important results of mathematics, which enriched the theory, would not be available without something equivalent to Choice, the consensus was that Choice had to be accepted as an axiom of standard set theory. Nowhere in this story does the applicability of Choice or its consequence in science play a role.

The moral taken from the indispensability argument is that mathematics is incredibly useful in the sciences and must be accepted on that ground. Most questions about the philosophical status of mathematics can be settled on that front, meaning that mathematics applicability reflects its truth. Despite the reliance on science for initial justification, mathematics is an insular practice. Mathematicians are allowed to disregard science, and should disregard philosophy in doing mathematics. Instead, they should follow their own mathematically significant criteria when answering unsettled questions, such as the existence of choice functions. The sentiment of indispensability remains that these answers cannot be challenged outside of mathematics. The criteria considered in answering questions are important. For Gödel, the criteria include usefulness in drawing consequences, and simplifying proofs. In particular, he notes the ability to get results we already have from more elementary methods. These types of consideration, and not Gödel's philosophical arguments, urged acceptance of Choice in set theory. The role of philosophy, then, is to work within mathematics to develop and clarify the subject.

This kind of naturalism is a potentially better for our purposes than Quine's, which is unequipped to account for higher mathematics with no foreseeable prospect of application in natural science. The first thing difference is that mathematics is accepted *in whole*[64] on account of its application in science. Whereas the first part of the program tried (and failed) to establish for a small number of mathematical facts, this approach legitimizes at least mainstream mathematics. On the intuitive account of mathematics, it could be argued that there exists an upper limit to the natural numbers. It is hard to believe that anyone could grasp the number one billion or even one million from a visual field, or even as a belief generated by perception. Strictly limited to intuition, we may be like the child who asserts that his parents are ten years old, because to him, ten is the biggest number there can be. Science commits to far larger numbers that are difficult or impossible to grasp intuitively. I know some of the mathematical properties of Avogadro's number but I cannot conceive of the number like I can conceive of two or

[64] Of course, it is difficult to determine what all of mathematics is. There are certainly areas being developed which may not be considered part of the body of mathematics yet. For the current purposes, the whole of mathematics can be said to be that which most mathematicians would recognize as established mathematics

three. Allowing just the addition function, which I maintain is different than inductive combination, commits us to the set of the natural numbers. Analysis and calculus are needed (or very valuable) in science, which furthers our commitment to the real numbers, and even the complex, hence imaginary, numbers. Most of familiar mathematics and more is committed to by science. While this provides justification for believing certain mathematical statements to be true, it comes up short in capturing mathematical methodology, and hence does not justify certain statements which have yet to be asserted. There is no guarantee that applicable mathematics can be ensured by ruling out inapplicable mathematics. Even Quine allows for justification to be extended to some unapplied mathematics to make sure that the applied mathematics is not compromised. The two are not insular and so it would seem that an account which can capture mathematics in whole would be more correct. There is not the worry that mathematicians will change their ways because unapplied mathematics is not accepted as true, but it would seem disingenuous to tell a story that maintains a distinction of applicability. The realist who does not base mathematics in the spatio-temporal realm should allow for the possibility that true mathematics may never be applied. Unfortunately, we have little understanding of the content of true mathematics because we have no access to the entities, and base wholesale justification on partial applicability. Although Maddy is certainly not an empiricist, it is difficult to see what she bases mathematical reality in.

It is for this reason that unapplied mathematics should be acceptable, that Maddy goes beyond the original sense of the indispensability argument to supplement it with mathematical autonomy. This mathematical autonomy should be accepted on account of actual mathematical practice. When we examine mathematics as an activity, as dynamic, rather than a static body of axioms and proofs, we see that mathematicians work in much the same way as scientists. They explore and experiment with different sets of assumptions to explore certain properties or entities. They develop systems to explore counterfactual systems to learn about the factual, much like the scientist who simplifies his assumptions to focus on one concept. For instance, scientists momentarily assume the Earth is flat, and that gravity is constant with regard to distance, in order to study certain properties of gravity. They know the Earth is round, and that gravity has an inverse distance squared relation between objects, but acknowledging these facts may make a simple problem very hard. If we want to know how long it will take a ball to drop to the floor, it is best to assume constant gravitational force rather than trying to capture the fact that as the ball gets closer to Earth, the force of gravity is stronger. Some of the unapplied theories of mathematics might be seen as exploring the counterfactual and odd sets of axioms to see what can be proven. These considerations bring to light the fact that mathematicians do not necessarily regard all of mathematics as true. With the help of philosophers, they can be selective in what they regard as true, and so mathematical autonomy should not be seen as skewing the truth as feared.

Maddy grants mathematical autonomy so that acceptance of mathematical statements is to be protected from extramathematical consideration. The philosophical questions regarding the general ontology of mathematics are answered by science. We must accept mathematics because of its place in the best theory of experience. The epistemological questions are on more of a case-by-case basis. It was hoped that providing a mathematical intuition could begin to answer the questions about how knowledge of the mathematical realm is obtained, and of what that knowledge consists. That seems gone. Maddy can resort to the naturalized position that knowledge is gained through what best accounts for experience. This is not to say knowledge is empirical. It is only to say that experience regulates knowledge to a point. Quine asserts that experience underdetermines the account of reality. It is not entirely clear what degree of realism can be maintained on experience. First, however, it should be questioned to what degree the distinction can be maintained between philosophical and mathematical questions.

Choice seems to be a philosophical statement, or at least has philosophical consequences. Philosophical questions regarding the ontological status of numbers seem to have mathematical significance. The point is less that there needs to be a distinction and more that mathematical autonomy dictates that mathematical answers, even if not framed in philosophy, sometime must be accepted as philosophical statements. Acceptance of Choice was a mathematical decision. Mathematicians probably did not intend for it to be read philosophically, but the choices made in mathematics have to be accepted as they are the best theory we have of mathematical entities.

The primary philosophical question that must be answered right now is what kind of realism is it that Maddy claims and what kind of realism can be claimed on the view. She shares the view with Quine that philosophical questions are answered within the framework of the best theory. For Quine that means science, while for Maddy that means science as well as mathematics. The paragraph above attempts to show that the line between philosophical and mathematical questions is not clearly defined. If mathematical propositions intended to be purely mathematical determine the ontology of mathematics, it seems as though we have an antimetaphysical account of a Carnapian kind. In discussing the status of HP above, I made use of a framework to see the difference between Quine and Carnap. Quine is able to preserve metaphysics and realism in opposition to Carnap. The fact that certain existential commitments follow from the stipulation of particular mathematical axioms would imply either that the axioms hypothesize about reality, just like scientific posits, or that the metaphysics is somehow trivial and that we preclude any genuine metaphysical discussion. It seems as though the break from science pushes Maddy towards the Carnapian side of the coin. While scientific method is clearly aimed at reality it is not even clear what mathematical methodology is aimed at.

Maddy has at best a shallow commitment to realism. Even so, she admits that subtle versions of formalism and fictionalism "may well coincide with that of our mathematical naturalist."[65] The subtleness she admits concedes more ground than a traditionally Quinean account of things would. Consider the "subtle formalism" she does not rule out. She defines it as mathematics being "the study of which conclusions follow from which hypotheses, and that all consistent axiom systems *are on a metaphysical par*, but also that there are rational reasons springing from the goals of mathematics itself, that justify the choice of one axiom system over another for *extensive study*" (emphasis mine).[66] This claim would certainly fall in line with Carnap.

One might try to read Quine's pragmatism into the rational reasons, but it is important to note that Maddy holds that the reasons should follow mathematical methodology, not scientific. A mathematical goal, such as providing a rich theory of real numbers has no claim to being about reality. For Quine and Carnap, metaphysical matters are settled by the choice of axiom system, and cannot be settled otherwise. They attach very different levels of significance to this, but both deny that further questions can be asked. Maddy seems to think that there is still a metaphysical question after. I

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[65] Maddy (1997: 202)
[66] Ibid

would offer that this might be because of the nature with which axioms are accepted that she wavers. Choice is almost reluctantly accepted as a compromise between the traditional justificatory methods of mathematics and the pragmatism of its practice. It seems that acceptance of the *actual* existence of a choice function should not be made on such shaky ground. The reason the ground is shaky is because of the break from Quine in the first place. The shakiness is symptomatic of mathematical goals that Maddy invests in, that point orthogonal to reality. I fear that if allow exploration of the strength of the justification of acceptance, then all metaphysical results that could be drawn on these lines would be further questioned, which is especially problematic because we have removed the grounds which could be appealed to in resolving the further questions. In this way, the program would do itself in. There is always risk of widespread misunderstanding in believing that that best current theory is normative, but if one takes on the view anyway, one would do best not to doubt it later on.

D. The Problems with Methodology

The original motivations for defending mathematical realism were to give a clear picture to the content of mathematics and to preserve mathematical reasoning. The former seems completely lost here. The latter more or less defines methodological naturalism, and should be seen as justified still. I think this is right, but there is some doubt to raise. It may be the maxim of the methodological naturalist to heed to the way mathematics is practiced, but surely we have not reached an end point of a complete and stable methodology. Part of the job of the philosopher is to attempt to bring the theory in line with the methods and draw out the consequences of commitments to certain methods or axioms. If we could open the mathematicians' eyes to his ignorance, and show him that his theory is antirealist and that antirealism is more consistent with constructive logic, he might believe that a change of methodology is in order. This hypothetical is hard to believe, but cannot be ruled out.

I think that I am not being petty; the hypothetical is a bit farfetched, but it shows that in the shift from Quinean naturalism, we lose more than it appears. By presupposing the goodness of mathematics, we disregard the important verification procedure that ties science and mathematics to reality, without replacing a procedure to orient mathematics within reality. When it is not clear whether to regard a theory in realist or antirealist light, there are means of potentially uncovering the facts that would speak to this uncertainty. There is no such guarantee of potentially uncovering the truth or falsity of a theory when adopting philosophical predilections with regard to mathematics. Wrongs in science can be corrected. Being wrong with respect to the axiom of choice cannot be corrected by further evidence to the contrary. For Quinean naturalism, it does not matter what the philosophical stance is with regard to a specific theory because there are better methods that allow science to proceed. There is no reason to believe that atomic theory is false; it explains experience better than competing theories, and so realism is justified when philosophical commitments are entertained. When Quine lays down axioms, they are true by dint of their explanatory power. It is with the aim of explaining experience and the presentable world that the conventions are laid down.

This speaks to the above concern that a methodological approach within mathematics is misguided and potentially leads the subject astray. The step to promote mathematical methodology cannot be taken as easily as imagined because the beliefs generated by mathematical theory no longer explain any world that can be experienced. The axioms are laid down according to a methodology that has no interest in getting things right. The naturalist is committed to explaining the presentable world (i.e.

concrete or quasi-concrete) as she relies on some link to science for legitimizing mathematics, but then realigns mathematics with no compass, and no guarantee that anything is being said about reality.

It is further problematic to have so much faith in mathematical methodology. Mathematics has been stuck on certain problems. Is it the case that methodology is not enough to answer these questions? It is one thing to say that all philosophical questions (about mathematics) are to be answered within mathematics and proceed to answer those questions or say why they are not yet answered. It is another to say that the questions can only be answered within mathematics, only to find out that mathematics is inadequate for handling these questions. If progress is a goal of mathematics, which has to be the case, then allowing for philosophical approaches to these problems might further that goal. Philosophy is secondary when we do physics because there is experimentation and verification. When mathematics is seen as a tool of science, its justification is easy to see (and likewise the lack of justification for unapplied mathematics). If Maddy is going to break mathematics from science, that justification is lost with nothing to take its place.

If we consider the problem of CH that has gone unanswered, we can see that methodological naturalism is ill equipped to handle the problem. It must be asked why there is indecision on the matter. There are two options: accept or reject. Maddy might say that the reason why there is no definitive answer is because the problem has not yet been thoroughly explored. This is not the case. The question has been on the table and seriously explored for at least one hundred years. For a problem which is known to lack proof or disproof in ZFC, this is a long time for a problem to remain open. Maddy seems to think that leaving its status open is a viable option. I do not understand how this can be a good option. If the subject does have clear goals which are suitable for progress, then why are they not given and why are they unhelpful in this instance?

It would seem to be the case that philosophical discussion is needed to identify what the goals or mathematics are and what they should be. Maddy installs an artificially purposive nature into mathematics that I do not see in the subject. The dilemma is this: philosophy must be accepted into the mathematical domain as a method by which to explore solutions to problems that cannot be resolved under the current methods and aims of the discipline, or philosophy is needed in order to identify the methods and aims of mathematics. In the latter case, it is not the case that the philosophical views would be seen as trying to accord with mathematical practice. Instead, philosophical views would *define* mathematical practice. Philosophy, then, is not the afterthought that Maddy thinks it is in a view reliant on mathematical practice. It must play a central role due to the methodological insularity from the self-regulating sciences.

It is important to remember that most of the criticism here about Maddy's view is attributable to my attempting to use her account to push a realist agenda. I thought that her view might provide for mathematical realism, but when we try to maintain realism and methodological naturalism at once, both become unstable. Most of the criticism is not generated from bad philosophy, but merely an intentional choice to attempt to interpret the account with realism in mind. She seems aware of this as she, herself, has shifted away from attempting to defend realism, at least as a main aim. The hope was that a shift from Quine could reinforce realism by adding direct access to mathematical reality. The positive result, is that it does seem as though philosophy does have a larger role in her view, though, than she admits when she claims to be subduing philosophy. By positive I mean both a matter of fact, as well as good. I intend the latter, because even if I fail to establish mathematical realism, there is a sub-claim implicit in this paper that philosophy does contribute something to mathematics whether it is acknowledged or not.

IV. Evaluation

A. Pragmatism and Amenability

One of the criteria laid down at the outset was that a good philosophical account ought to be amenable to mathematics. I see both philosophies largely as being conservative and preserving the status quo. I would think that conservatism is embedded in the methodology of mathematics, as it builds on itself. By defending realism, it was hoped that the classical logic favored in mathematics could be justified.

The methodological naturalist defines her view by its amenability to mathematics. By asserting that the methods which have proven fruitful should remain, the mathematician who has been trained in the ways that have been successful will be pleased. There is a sense in which the account might be seen as too conservative. In the history of mathematics, there have been moments where mathematicians have employed a progressive approach to a problem which was initially criticized, but proved effective. To justify the progressive approach, the mathematician may have had to step out of his role as mathematician-qua-mathematician and step into mathematician-qua-philosopher, or mathematician-qua-scientist relying on empirical evidence for a theory. It is hard to say whether or not a methodological naturalist transplanted back to one of these moments in time would have approved of the unorthodox methods.

It seems arbitrary to take mathematics as it stands in 2011 and say that everything that has been done before today that has worked is acceptable, and anything that has not worked or was irrelevant will continue to be so and hence will be unacceptable. Holding this view in the midst of unresolved foundational issues is difficult to accept. The position allows for rational reconsideration to be attached to the historical analysis of contemporary problems, which may temper the conservativeness. Still, it might be the case that either she is too conservative in restricting mathematical methods to those of the past, or she says too little and does not adequately provide boundaries for what is or what is not acceptable mathematical method. When Maddy gives an outline for how problems are to be addressed along a methodological agenda, it is hard to see whether the view boils down to anything beyond slightly scrutinized mathematical practice. There is a presumption that mathematics is generally good, which is an acceptable but optimistic attitude. If it were not the case that BLV was not inconsistent, Frege would have introduced a radical reconceptualization of mathematics that would have greatly affected the discipline. It is difficult to believe that the methodological naturalist would have been open to accepting a language based account of arithmetic.

The neo-Fregean account of mathematics allows for certain mathematical amenities. It does defend the nonconstructive reasoning favored in mathematics. It also seems an impossible account of the subject, given the seer difficulty of establishing just arithmetic. It should be interpreted less as a how-to guide to mathematics and more as a foundational reinforcement. If it is not read as mathematically suggestive in this sense, then it becomes more amenable. If we think of language as central to human thinking and mathematics among the intellectual activities, then it seems correct to think that mathematical knowledge is based in language. The view should be interpreted as an aside to the actual practice of mathematics, both because it does not capture the active component of mathematics as a human discipline and because it places an enormous obstacle in the way of doing higher mathematics. The attempt by Hale to establish the real numbers by abstraction does not inspire much confidence.[67] I have neither the time nor expertise to analyze the attempt, but at the very least it lacks the elegance of the neo-Fregean account of the natural numbers. It requires several abstractions, each of which does not have the apparent ease of understanding on the level of HP. Where the methodological naturalist is able to offer a strategy for attacking undecideable questions, the neo-Fregean approach seems impotent. It can offer a realist outlook, but little else. Further abstraction principles would have to be offered and analyzed for each addition axiom or set of axioms that could settle the undecideable questions of mathematics. This is a serious impediment towards providing a foundation for each level of mathematics. It is nice to know that the theory gets at the reality of the situation, but it is very difficult to show that it can do so for more than arithmetic. The achievement of providing a theory of arithmetic is no less monumental because of this. The neo-Fregean is not committed to the truth of (almost) all mathematical practice in the way that Maddy is. As far as the undecideable questions, they should have determinate answers on a realist account of mathematics, although it is possible to imagine the mathematical realm underdetermining the truth of certain questions. Either way, the prospect of knowing the truth value of something like Choice may not be great, but there is no problem with developing both axiom systems, ZFC and ZF(-C), as there is no value judgment about the goals of mathematics that would preclude this. The neo-Fregean program runs parallel with practice, rather than being part of that practice. Having certain results in mind when developing axiomatic systems is anti-Fregean. It should not matter that the status of further axioms is open. That is to be expected.

[67] Hale (2000)

The two accounts approach mathematics with different ideas about what the subject is. Maddy sees mathematics as authoritative and philosophy as secondary. She recognizes goals in the subject and aligns them to instill a purposive property almost to a teleological extent. The naturalized position declares the supremacy of science and mathematics, hailing them as the best theory of reality. It might be challenged that this is a self reinforcing view. There is little verification that can be conducted to show that it does indeed represent the best theory of the world. It allows no means to see the falsity of itself. It is easy to see why it is difficult to supplement the theory with philosophy as the theory largely explains itself. This feature of naturalism makes it uninteresting in a way, as there is little explanation of how things are. There is an oddity in naturalism leaving open the possibility for formalism and fictionalism. It would seem that either contradict the claim that science and mathematics represent the best theory of the world, formalism contradicting that the theory is about the world at all, and fictionalism contradicting the approximation of the world.

The neo-Fregean account has the complete opposite approach to the subject. Wright claims that "mathematics is not a monolith of certainty"[68]in raising the merit of philosophy in a mathematical context, especially a foundational one. I think that the neo-Fregeans mirror the actual attitude of the mathematical community better than the naturalists. The only sense in which mathematics can be seen as purposive is in the sense that certain results are considered interesting either because they generate many other results or because they seem intuitively true. It does not seem to be the case that mathematic is practiced with an agenda as the methodologist imagines. The current state of mathematics reflects this curiosity for discovering the world, rather than attempting to

[68] Wright (1988: 432)

bring something to the world. Mathematics is sometimes said to be so spread out that the different branches have difficulty communicating and that even intra-branch communication has become strained by extreme specialization. This is because mathematicians have set out to study potentially interesting models by stipulating or removing axioms from generally accepted theories. The neo-Fregean spirit encourages the explorative nature of mathematics more than the methodological naturalist who is more results driven.

B. Ontological Criterion

One of the criteria a good philosophy of mathematics must fulfill is providing a good philosophical account of the content of mathematics. It would be good to know whether numbers are objects or not, and also whether numbers live in the spatio-temporal world or elsewhere. Here, the neo-Fregean philosophy is much clearer than methodological naturalism. It says of numbers that they are objects and abstract at that. These objects are independent of any human conception of them. Whether or not one believes these claims about numbers and mathematical entities is a different question, but as far as giving a clear picture the neo-Fregean program does this far better. All there is to numerical existence is that number terms feature syntactically in a certain way in certain true sentences.

On the naturalized view, there are two views of existence. There is the Gödelian faculty that is something like intuition driven perceptual beliefs, which provide for the quasi-concrete nature of number-holding properties present among collections of objects. This was an unclear view ontologically speaking, as it proves to be no less mystical than Gödel's account. The existential claims based off the indispensability argument cannot provide a link to experience once mathematics is deemed ontologically authoritative. This also does not provide a clear view of the nature of mathematical existence, which is unsurprising because metaphysics is not taken too seriously on this account. There might be commitment to the things occurring in mathematical statements but this has the effect of making existence seem contingent on the way a theory is formulated as well as leaving mathematical existence indeterminately placed along the realism spectrum.

It would be instructive to place Maddy and the neo-Fregeans along the realist spectrum to see what the views come out to. The spectrum can be defined in terms of other philosophers. So, on the far end, espousing the strictest realism is Field. If we take both Field and Maddy to be reactionary to Quine, which makes sense as both pursue programs which attempt to reconcile philosophical attitudes regarding mathematics with philosophical attitudes regarding science. Maddy take the centrality of mathematics in science to encourage a realist attitude, taking Quine's naturalistic commitments in a liberal sense. She thinks there is commitment to those theories which might serve no purpose other than mere simplification. There are actually two degrees of Maddy. In her earlier version, she carries a stricter realist criteria, needing to establish for realism on appeal to science and hence the physical world. Later Maddy (if she is still a realist) regards mathematics with light realism but sways in doing so, admitting that there can be room for refined antirealism. Field takes the criterion for ontological commitment strictly to be absolute indispensability. He takes mathematics to be one of the posits Quine mentions at the end of *Two Dogmas*, and takes a negative view of these posits. He focuses on Quine's remark that the posits are "comparable, epistemologically, to the

God's of Homer."[69] He takes it that showing that mathematics is not truly indispensable means that it is not actually true, and attempts to demonstrate that part of physics can be done without commitment to mathematical entities, showing the dispensability of mathematics there. He does not show that mathematics can be dispensed with in all sciences but takes his progress as showing that it might be the case, which motivates his fictionalism. Instead of regarding mathematics as true, he regards it as conservative placing it on a lower ontological level. Because it is truth preserving, its use is still acceptable, but its existential statements are not, on a literal level.

Wright has been discussed here as possibly laying down HP as a Quinean posit. He would be seen as focusing on the Quinean remark that there have to be "irreducible posits" which legitimizes ontological seriousness towards them. I do not mean to say that Wright is secretly a Quinean who lays down HP to explain experience. I only mean to say that he has a similar commitment to posits. Wright does see HP as a posit "different in kind" from others, contrary to Quine's belief that all posits differ "only in degree."[70] The neo-Fregean criterion for ontological commitment is strict in its own right but significantly different in that it takes the human conceptual apparatus and the syntactic reflections of it to correctly depict the nature of reality. In this way, the neo-Fregeans have a reverse view. The others take mathematics' association with experience through science as garnering it into a tenuous realism. The neo-Fregeans, however, see mathematical entities as objects by dint of their syntactic featuring in true statements. Some of the statements are regarded as true prior to experience. The mathematical entities, then, have an epistemological status different from the scientific entities. It is

[69] Quine (1951: 41)
[70] Quine (1951: 41)

science and experience that get the contingency label and are not part of reality by necessity.

If abstract realism is to be maintained from either of the considered views, it must come from the neo-Fregean one. Coming to believe in the realism of objects which cannot be experienced by the senses would seem to require an almost religious encounter with the purportedly existing objects. Otherwise it is easy to default to an antirealist attitude, holding that anything which cannot be proven to exist is made up or constructed. Here the proof referred to is the colloquial meaning that matches up with scientific proof – something like showing something to be true. Both views attempt to stand back from religious appeal to realism and support the realism with something firmer, but both views are charged with being realist in an antirealist manner[71], being antirealist on a metametaphysical level. Had the Gödelian route been more coherent, Maddy could have provided direct access to a non-concrete mathematical reality and could have been realist through and through. As it is, that account is mystical and religious in nature. It does not appear as if there is a way to uphold realism in a metametaphysically realist manner. The neo-Fregean view, which on most counts is the better view for maintaining mathematical realism does so on a linguistic turn. It takes a linguistic construction to meet reality halfway. The neo-Fregean presupposes a prior understanding of truth to hold that certain number theoretic statements which true in everyday speech actually are true. Calling the sentences true is a seeming sleight at trying to pass off the sentences as true in a robust sense of truth, or a realist conception of truth. The featured truth is best understood as deflationary as opposed to a correspondence based truth. The hope is that by calling the truths analytic they will

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[71] Perhaps the neo-Fregeans could be called lightweight ontological realists instead

remain true irrespective of reality. This just shifts the burden to the definitions for approximating reality correctly, which the neo-Fregean believes to be the case. This neo-Fregean belief is more dogmatic than philosophical. In light of this, it is possible that an existentially committing statement can only be analytic if the existence spoken of is held to be of a light ontological type. It is difficult to give gravity to realism that is not achieved via direct access. Although, nothing says that maintaining a view of light ontological existence is bad or wrong, or even that it does not better characterize the prephilosophical attitude regarding mathematics.

C. The Epistemological Criterion

Much of the epistemological concern was common to both philosophies, and was in response the Benacerraf's challenge. The neo-Fregean defense rests in the claim that because we have analytic knowledge of mathematics the typical causal interaction is not needed for reliable knowledge. HP has both the mechanism to lay down what numbers are as well as provide the way by which we can understand the numbers established. The naturalist gets her epistemological account seemingly for free. Knowledge of the entities is gained by examining the existential commitments of the theories deemed best on other grounds. Mathematical knowledge is grounded in the proven methods of mathematics and science. The naturalized truths are either laid down by convention according to scientific and mathematical norms or derived from logic (which may itself be a matter of convention). The naturalized account of knowledge is simpler for sure, but because of the failure to maintain realism in a suitable manner, it is only necessary to see if the neo-Fregean epistemological account is satisfactory.

Certainly there is controversy with abstraction principles, which was brought up by the "Bad Company" objections and Dummettian concerns for the ability of the principles to simultaneously provide for the existence of objects while providing a noncircular understanding. Despite the controversy, abstraction principles are not inherently flawed and HP should be acceptable. Simply, knowledge of numbers is established along the same grounds as knowledge of 1-1 relations. If HP can be accepted as analytic then the true statements of mathematics can be known through logic up to the point where incompleteness might prove problematic, which was left somewhat open. Something akin to logicism will place all of mathematics on a firmer epistemological base. Again, knowledge of higher mathematics is less grounded without corresponding abstraction principles.

V. Can Implies Ought?

At this stage, it is clear that I endorse the neo-Fregean defense of mathematical realism. What I want to do now is see if a stronger claim can be made. I would like to see whether I can put the neo-Fregean view on offense to establish, rather than merely defend, mathematical realism. One of the advantages of the view, is that it was able to argue for realism without invoking a mystical faculty for perceiving mathematics of the kind Gödel relies on. I hope it is clear that there are too many problems tied to such a notion, and that my dismissal was in good order. But at the same time, without direct access to numbers, it is hard to see what positive evidence can be put forth to compel mathematical realism. To be sure, I do find the neo-Fregeans persuasive, but I believe they fall just short of compelling realism. This can be seen in examining the dialectic with fictionalism. The dialectic leads to both parties agreeing that the fact about mathematics is a contingent one, and that there seem to be no definitive grounds for thinking the contingency goes in a particular direction.

The fictionalist opposition is advanced almost exclusively by Field. Just as nobody believes that the characters of literature are part of reality, nobody should believe that numbers are real. Numbers are just the characters in the story of mathematics. As mentioned, Field has a high stand for ontological commitment, namely pure indispensability. He provides the start of an argument that mathematics is dispensable by providing a prima facie plausible nominalist theory of physics. The discussion here can assume that his dispensability program works out, because the neo-Fregeans do not attempt to base their realism on that claim. Of course, the dispensability of mathematics does not actually render it down to the status of literature, say. Mathematics *can* be used to talk about real world facts, whereas it is hard to believe that any fact about Oliver Twist has any bearing on the real world. The asymmetry presented is explained away by the distinction of mathematics in being conservative. Conservative, here, is the property of a theory that it can be added to another theory without generating new truths could not have been inferred from the original theory to begin with, and that all inferences using the new theory preserve truth. In the present circumstances this means that given a nominalistic theory (body of statements) and logic, the set of nominalistic truths generated from that alone is the same as the set of nominalistic truths generated by the nominalistic theory, logic, and mathematics.

The fictionalist view follows from the assumption that reality is synonymous with the spatio-temporal. There is a gap between mathematical statements and truth, which is apparently not overcome by the neo-Fregean account. The neo-Fregean provides justification for believing the statements but not for the truth of the statements. Their only justification for the latter is the assertion that what is true “by ordinary criteria” really is true. According to Field, they fail to give any reason for thinking that the ordinary criteria are sufficient for bridging the gap to truth. The lacking nature of the criteria still supports mathematical belief, but fails to establish mathematical truth.

Field is convincing in attacking the irregularity of the neo-Fregean account which has it that if mathematicians believe 'p', then p. This schema can be explained to an extent. We can say, because all agree that the inferences in mathematics are good, that if mathematicians accept 'p' as an axiom, then p is true. Further, because mathematicians are good at crafting consistent systems and recognizing inconsistencies, taking 'p' to be an axiom would mean that 'p' is logically consistent with any other axioms. But consistency still falls short of truth.

Consistency cannot rely on a model-theoretic construal (as that would make Field committed to mathematical entities) and so in this context it accords with possibility in a primitive sense. This is the stage at which Field diverges from the neo-Fregeans. Accepting 'p' means it is possible that p. The neo-Fregeans assert that the type of statement that 'p' is, a mathematical one, means that it is necessary. It is necessarily true, or necessarily false, and so the mere possibility of it being true would entail that it is true and necessarily so.[72] Field denies any sense of necessity which would lend itself to that arguments. It cannot be thought that the statements are logically necessary, as logic statements cannot entail the existence of objects. They are content neutral, and the content here, if the neo-Fregean is to make use of the questionable 'p' statements cannot be neutral. Field concludes that it is possible that mathematics is true, but it is contingently false. To be a little more precise, the conclusion is only that we ought to believe mathematics is false; the conclusion is about the commitment that should be held regarding mathematics rather than about mathematics itself. The jump from possibility

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[72] This argument is owing to Hale (1987: 109-110)

to falsity is explained by the presupposition that there is a strictly physical criterion for ontological commitment. As long as mathematics is dispensable in the theories which are committed to, then there is reason not to believe in the truth of mathematics, but instead merely in its conservative instrumentality.

It is this restriction to the physical realm which grounds Field's contentions. He concedes that the neo-Fregean has the proper account of singular terms, and that their account wins over the reductionist. Whereas the neo-Fregeans invoke syntactic priority to explain the semantic role of singular terms (objects), Field invokes a strict physical realism that denies any semantic value to genuine singular terms (on linguistic grounds alone). His skepticism is well grounded. There are plenty of concepts that can be explained that have no instances. The concept of a unicorn is perfectly clear, and not much different from the concept of a horse. If unicorns exist, if they exist, would be concrete however, and subject to empirical verification, but this is not to be expected of numbers. If that is not the right analogy, Field tries again with analogies to God. If existence follows from the explanation of a concept, we can explain the concept of God, and become believers, except for the fact that in light of explanations of what God *would* be, we are a secular society. This example is equally lacking. It would seem to be the case that while God may not be physical, we conceive of potentially verifiable physical consequences of God's existence. This may not be correct, but then it would seem the concept of God is not all that clear, as we do not know what to expect of his potential existence. This is not to say that God does not exist, but only to say that there is a distinction between the clarity of explanations of God and explanations of number.

Even if the examples can be distinguished from numbers in important ways, it is still not the case that we ought to believe in that numbers exist or that the arithmetical statements are true. Field denies the belief on the grounds that while it is a matter of contingency, the idea that mathematics can be completely dispensed "undercuts all reasons for believing it and hence raises the plausibility of the no-truth view."[73] A proponent of neo-Fregeanism will not share the belief that there is a physical criterion of commitment.

The neo-Fregeans rely on a picture of conceptual truth for mathematics. Justifying the beliefs of mathematics as well as providing for their truth requires having grounds for believing them to be true. The proposition '2+2=4' is grounded in computation, for example. Field denies the truth on a claim that there is no such thing as 2 or 4, and that the computational ground only provides justification within an accepted mathematical theory. The contention on the table is based on the condition of priority which Field refers to as (S).[74]

> (S): What is true according to ordinary criteria really is true, and any doubts that this is so are vacuous.

The intended meaning of true "by ordinary criteria" is actually that

> "whenever a statement is associated with certain *canonical* grounds, i.e. grounds such that to suppose that the statement is true is a commitment to the availability of such grounds for believing it to be true."[75]

This means that when the neo-Fregean says of arithmetical statements that they are true, it is not by the criteria which associate lightning with "Zeus's ballistic extravagances," but rather by the criteria which fix the concept number, HP. The neo-Fregean is modest in the assertion of truth, admitting that defeasibility of the grounds for accepting such

[73] Field(1993: 290)
[74] Field (1984: 646)
[75] Wright (1988: 449), based on Wright (1982: 211)

statements. This modesty is that important as it extends to the concrete as well; there are very few things accepted that are indefeasible, if any. In turn, defeasibility of justification for a belief is no reason to think the belief is not true.

It is important to see that Field does not challenge the legitimacy of HP, and that it is available as canonical grounds for defeasible verification of the statements in question. The challenge being made is how to read or how to use HP and other abstraction principles. Field precludes commitments based in them by interpreting any ontological use of the principles as conditioned on the existence of the objects whose content is fixed by the principles.

$D^{=}_{Field}$: If directions exist, then $D^{=}$

HP_{Field}: If numbers exist, then HP

The situation is this: the neo-Fregeans condition the existence of numbers on HP. HP simultaneously fixes the content and provides reference, and so the antecedents of Field's construals are unintelligible. Field, they claim, allows that we know what it would take for numbers to exist, while taking away the only seeming condition for knowing as much. This is only the case if one adheres to the thought that there can be nothing more to numerical existence than conceptual establishment. Of course, Field sees the antecedent as being determined by physical considerations, namely whether mathematics is dispensable in science. It is not the case, then, that Field takes much interest in the principles above as far as a mathematical account. He does, however, admit that HP conveys the correct sense upon number. On this note, Field is still committed to the explanation of number by HP, as number must have the meaning conferred to it by HP in order for it to be conservative.

The rejection of HP as truth is the rejection of truth preservation across the biconditional. It is the denial that a 1-1 relation among concepts suffices for the truth of corresponding identity of numbers associated with those concepts, and only because it is denied that any function takes concepts to numbers (as objects). If the numbers could be antecedently determined to exist, there is no obstacle for the success of such a function. In light of this, what does the neo-Fregean have to say about the presumed success of the number forming function? They make a few remarks. All parties admit that HP gives a sense to number, but the controversy is whether it fails to refer. Reference can be verified by verification of a sentence in which the term holds a reference-demanding position. The idea is that one would want to find an identical term which does refer. Hale and Wright offer the example of how one would go about verifying the existence of some term 'Bin Laden'. One would want to verify the identity 'q= Bin Laden'. 'q' might achieve reference in the traditional ways, e.g. definite description, or ostention.[76] To employ this method presumes that 'Bin Laden' is not a fundamental term – one which does not stand for some new kind of object which has no prior means of reference. But HP, if successful, would establish a fundamental object.

There is a question of whether there are any fundamental terms that cannot be related to anything else. Someone, like the neo-Fregeans, who attempts to place mathematics on an analytic base are prone to see an ontological hierarchy in which terms can reduce down to a base set of fundamental terms. The challenge posed, however, is not a dogmatic refusal to accept anything nonphysical. It can be seen merely as a response to such a hierarchical view, proposing that instead there is a web-like

[76] Hale and Wright (2009: 202)

ontological structure in which there are no irreducible terms. Here we witness the Quinean influence.

It is difficult to make progress on this problem as Field and the neo-Fregeans are looking for objects in, as they accuse of each other, the wrong places. Having a physical standard requires the conceptual insufficiency of HP for establishing numbers, and searching for numbers in true contexts requires that certain of the true contexts be undoubtedly accepted as true. If the neo-Fregeans cannot argue directly against Field's presumption by offering a better set of conclusions, perhaps they can show that Field's presumptions lead to absurd or troublesome conclusions.

It appears as though contingency is inevitable of this dialectic. The contingency we are dealing with, conceptual contingency does not lend itself to being resolved easily. The neo-Fregeans try to say that it is not actually conceivable that HP could not be true, but in the end, this presupposes a richer reality than some are willing to grant, and so they cannot deflate the contingency problem. Field attempts to resolve the contingency on appeal to ontological economy, but there is no reason to suppose this to be true of the world. I do not think that the contingency can be overcome. The deciding factor for me is the thought that experience underdetermines reality, and that if we would like to have a satisfactory ontology, one that can explain reality to a fuller extent, then we need to supplement ourselves with stipulations such as HP. There is no reason to impoverish one's ontology in fear of being incorrect. As far as allowing HP in favor stipulations of God or little green men, there are the considerations of clarity and plausibility. Accepting HP refreshingly demystifies part of the experienced reality, ordering the bizarre. It does so without being overbearing.

VI. Conclusion

Defending mathematical realism is not easy. There are many problems standing in the way of such a philosophical stance. I chose to defend mathematical realism because I see it as an interesting philosophical view, because it has importance in mathematics, and because I believe it. It is not easy to convince the reader that the sum of this paper is anything more than the listing of beliefs which each person must ultimately decide for his or herself. This is both correct and incorrect. The conclusion is largely that realism comes down to a belief or maybe even a faith in the existence of mathematics apart from humans. I hope I have provided for the possibility that such a belief is completely surd as Hale and Wright would say. Yet I hope I have shown something more; that philosophy has more import than conveyed in its conception as belief examination. Quine regards the "old opposition of realism and conceptualism…[as] no mere quibble; it makes an essential difference in the amount of classical mathematics to which one is willing to subscribe."[77] Even if philosophy does come down to disagreement over belief, that same philosophy is able to show the high stakes of those beliefs. This, in turn, gives gravity to the "mere" beliefs we ascribe to. So, when I say that I believe in mathematical realism, I do not intend for this to be taken lightly. I think it has just as much force as saying "I know mathematical realism is right."

I took on the question of whether numbers exist. Quine once again expresses the sentiment which I have taken on: "It is no wonder, then, that ontological controversy should tend into controversy over language."[78] The neo-Fregean view understands this connection. Their evaluation of the question of number existence is at once clear, and I believe, correct. Most people are inclined to believe that numbers exist, citing "something special" about mathematics. It is not hard to believe in numerical existence.

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[77] Quine (1948: 34)
[78] Ibid, 35

The difficult part comes in moving past the psychological picture of numerical existence, and providing a clear view of what exactly such existence both means and entails. The neo-Fregeans take this further step which is glossed over in a naturalized view of mathematics. It is one thing to know that I am committed to the variables ranging over bound quantifiers in a theory that explains so much. It is quite another to elaborate further on this commitment.

It is philosophically unsatisfying to be left ignorant of the content of a commitment. It makes believing difficult. Given a naturalized account of mathematics, I have no problem accepting a commitment to numerical existence. I do have a problem with being unable to talk about the nature of the commitment any further. The naturalized position seems to cut off the ability to deeply entrench a belief about mathematics. This is the part that I find to be unsatisfactory. Above I mentioned that philosophy in part is expected to come down to belief, and that this should not cause worry. But there is a frustration in not being able to go beyond commitment to a belief. It does not allow for the stakes to be known. The question of numerical existence can be answered within the framework, but at a loss of significance. I am open to the suggestion that anything further is illusory, and that a desire to know more is misplaced. When presented with the clarity of the neo-Fregean account of mathematics, I am swayed to disagree with that sentiment.

There are three aspects of the neo-Fregean program which I find to be at once elegant, but also demanding of respect. Hume's principle, at first, seemed trivial. It does not, at first glance, reveal the depth and complexity that it gives rise to. The more I studied the principle, the more I got back. The analogy that describes it as carving the world at its joints, seems the best interpretation, and illuminated the situation. The analogy put me in a position to evaluate my beliefs about the nature of things. But I would not have been in a good situation to do this evaluation without two earlier aspects of the program; the first being an insistence on a shift from the psychological to the logical. I take this in part to mean that an idea which cannot be expressed cannot actually be believed. Without the expression of numbers as abstract objects with well-defined properties, I was never sure what to believe about them. This situation is reflected in the noncommittal sentiment of prephilosophical beliefs about numbers. The situation is mirrored by some of the classic philosophical questions – the questions about universals and questions about deities. These situations seem to reflect my attitude about numerical existence. The more a supposed belief is undefined, the more faith it takes to believe it. The neo-Fregeans minimize the faith required for believing in numerical existence, reducing it to a level that I am comfortable with.

More importantly to the specific question of this essay though, the neo-Fregeans provide the arguments for increasing the amount of faith it takes to deny numerical existence. My conclusion is that it would take the same amount to hold either belief. The third aspect of the program is one of the arguments to this effect. Wright's simple argument that goes to show the inadequacy of ostension in giving the objects of the world sets the stage for the neo-Fregean to give a redefinition of object. It shows that there is something more to being able to express a belief, a conceptual component. Once this door is opened, the opportunity to reevaluate beliefs about a reality previously taken for granted is opened. These three aspects are the lasting considerations that lead me to believe in the neo-Fregean program, and ultimately in mathematical realism.

As far as defending mathematical realism within the guidelines set forth at the beginning, I consider neo-Fregeanism the best option. Certainly not every issue is fully resolved, but I find the account to be approximately correct. I do not expect that any theory that is worth believing is without difficulty. The difficulties have been sufficiently answered to allow for belief in mathematical realism. All things considered, I believe neo-Fregeanism provides the best answer, of all the views I have considered, to the question of numerical existence and so it does more than defend realism; it makes realism the appealing philosophical option. I had hoped to end this paper to the tune of, "Mathematics is real. Numbers are objects. QED." (Instead) Mathematics is real. Numbers are objects. QEC.[79]

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[79] Quod erat credendum. As opposed to QED, what was to be demonstrated, we have what was to be believed.

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