Product Variety: The Intensive vs. Extensive Margins of Efficiency

by

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Abstract

The paper creates and analyzes a circular location model where a social planner chooses the addresses on the circular product spaces where a firm can enter. The paper identifies two different regimes that a social planner can use in order to determine exactly how many firms enter at the different locations. At each different address, firms compete via Cournot to serve the consumer base. In symmetric equilibrium there will be the same number of firms and output produced at each location. When the social planner adds more locations, he increases product variety and competition among the locations, but also decreases the number of firms at each location, which in turn increases inefficiency.

1. Introduction

Location models have given economists the ability to incorporate geographic proximity and product variety into the analysis of consumer preferences. One of the most significant advances in location models was the idea of the Circular City. Salop (1979) introduced the idea that firms are evenly spread along a space with no endpoints, and they create differentiated products; each firm acts as a local monopolist on its space within the circle. Consumers take into account their preferences and location, and purchase goods from the firm that maximizes their utility. The circular city has individual producers spread out evenly along the product space. This model, however, is not consistent with what happens in reality. We observe that firms tend to consolidate at different locations.

Economic theory tells us that unregulated monopolists will set marginal costs equal to marginal revenue, and produce less than the socially optimum quantity. We expect that there should be some inefficiency when firms have control over demand. In Salop’s model, however, the city is not underserved because the entire city is covered and every customer is served exactly one unit; this type of inefficiency is not
observed in the circular city. However, although inefficiency is not encountered in
the model, it does not meet our expectations about what happens when a firm has
market power.

Related to the idea of efficiency are the issues about how a consumer responds
to changes in firm behavior. We expect that when a producer changes its price, the
attractiveness of the firm increases and more customers will begin to shop at that
location. We also expect that customers will purchase more of the good at that
location, because the good becomes more appealing and customers are able to afford
more of it. Economists refer to this as the extensive margin and intensive margin of
demand. Changes in the extensive margin are changes in demand due to changes in
the number of consumers, while the changes in the intensive margin reflect how much
of a product that a consumer will buy due to a price shift. The issue of intensive
demand, however, is sidestepped in Salop’s model because of the nature of quantity
in the model. The only way for demand of a firm’s good to change in the circular
model is to have new users come to the location, but once a user is there, he will only
buy one unit of a good.

In this paper, we create a three-stage model that extends Salop’s Circle model
to include multiple firms at every location on the circle. At the first stage, a socially
optimum number of locations are created around the circle. The second stage
involves firms entering at each location. In the model we demonstrate two ways
whereby firms can enter the market. A social planner can allot the socially optimum
number of firms to each location or firms can enter at free entry. Consistent with the
results of Mankiw-Whinston (1986), more firms enter in a free entry regime than
what is socially optimal. The last stage of the model has firms at each location compete among each other over the quantity served per consumer.

In order to solve the model, we use backwards induction, a practice common in solving multi-stage games. When we are solving for the optimal quantity of a firm, it becomes apparent that it is algebraically difficult to calculate the output for each individual firm. In section four we assume that there is some symmetric equilibrium quantity. In section five, we use a reasonable approximation to find a concise, close-formed solution for the individual output. After we have a specific solution for the equilibrium output, we can continue to solve for explicit solutions. In section six, we do not approximate for the output for each firm, and illustrate new comparative static results.

By setting up our model in this manner, we address the previous issues that are outlined above. Firms colocate at a given location and then compete with each other. The output of a location is inversely related with the number of firms. When a social planner adds a location, there are fewer firms at each location. Few firms at a location produce socially low output. Thus, there is a tradeoff between having more locations on the product space and having more concentrated, inefficient markets. Lastly, unlike many traditional location models, we are able to show the effects of changes in intensive demand. In section six we discover one last effect; firms not only compete among themselves at a location, but compete with the neighboring locations as well, which in turn compels each location to increase their quantity.
2. Literature Review

Economists have long used location models in the study of industrial organization in order to incorporate geographic proximity into differentiated product models. Location models are models of monopolistic competition where geographic address is included in consumer preferences. A location can either be a physical address or a proxy “that measures how close the brands actually produced are to the consumer’s ideal brands (Shy 1995).” Hotelling’s (1929) “Location Model” is perhaps the most famous model that incorporates geographic locations into the analysis of firm behavior. Hotelling analyzed the relationship between a firm’s address and its price. In his model, he posited a linear street with a length greater than 0 where two firms are identical except in geographic address, and consumers are spread evenly throughout the street. In a game where firms choose their locations, the sub-game perfect equilibrium result is that the firms locate next to each other at the center of the street. This result is commonly referred to as the Principle of Minimum Differentiation.

Salop extended Hotelling’s Model to include firms around a unit-circle as opposed to firms on a linear street. In Salop’s Model, consumers can consider an outside, undifferentiated good. Firms maximize the distance between themselves and other firms, and therefore locate equidistant from each other. Consumers take into account price and the transportation cost and then buy from the firm that maximizes their surplus. If firms are sufficiently close, then they must take into account the pricing decisions of other firms. Salop called the area where firms compete over a consumer the “competitive region.” If there are only a few firms, however, they do
not compete for the same consumers, and each firm is a local monopolist. If free entry is allowed, there are \( n \) firms in the market, and the distance between each firm is \( 1/n \).

The Salop Circle Model is a very versatile model and has been used in many applications for a variety of different subfields in economics. Dornbusch (1987) reviews the direction of macroeconomic research and claims that applying industrial organization models to open economy macroeconomics has provided many significant economic insights. He uses the Circular City to establish how economic shocks affect goods pricing. In the Dornbusch model, there are four firms in two different countries that produce differentiated products. Firms one and three are from the home country and firms two and four are from the foreign country. The equilibrium has been achieved, so each firm has an equal market share. Dornbusch analyzes what happens when the home countries currency depreciates, and determines that the price of foreign firms increase and the range of consumers expand to the home countries. The author then utilizes the same circular product space, but imposes fixed costs and capacity constraints to make the model more realistic. In this setting, both the home and foreign firms raise prices when the value of the home currency depreciates.

Dell’Ariccia (2001) evaluates how the market informational structure can affect the banking industry. In particular, he is concerned about how informational asymmetries can change industry competition and interest rate. He uses a circular product space in order to model the location of the banks. The space can have two representations. First, the location on the circle can represent a specialization in a
particular type of loan. Secondly, the length of the circumference along the circle that
a firm has control over can also illustrate the market power of a specific bank. The
author derives the short-run equilibrium and the steady-state equilibrium if all
borrowers are treated equally, and then determines the results when a bank can
account for different risks in borrowers. In this instance the primary conclusion is
that there is an off-setting effect. When consumers are trustworthy, more banks will
enter the market and thus lower the equilibrium interest rate; however, with many
low-risk borrowers, there is less incentive to compete for a greater market share,
which has the effect of raising equilibrium interest rates. Therefore, the direction of
the interest rate is ambiguous when a new bank enters the market.

Gal-Or (1999) analyzes the propensity to merge of both hospitals and
insurance companies depending on market share. Furthermore, she determines
whether insurance companies gain from restricting the choice of hospitals for
consumers. In the model, the insurance companies have different locations on the
circle. The market is in equilibrium such that the companies are equidistant from the
neighboring firms. There is another circular product space that similarly describes the
hospital. The model is a single-stage game where the insurance company sets
insurance rates for consumers and concurrently bargains with hospitals for
reimbursement rates. In the analysis of the game, the author concludes that mergers
are more likely to happen between hospitals when the market for hospitals is not
competitive. The same logic applies to the insurance companies as well. Hospitals
and insurance providers, however, act differently to market power in the opposing
industry. When the market for hospitals is competitive, the incentive for insurance
companies to merge increases; when the insurance companies are competitive, hospitals have are averse to mergers. Furthermore, insurance providers are more likely to restrict access to hospitals when their “bargaining position vis-à-vis hospitals is weak (348).”

The Circle Model can be found in many more applications. Pietz and Waelbroeck (2006) use the Circular City to show that music labels do not necessarily suffer from free downloading through peer-to-peer networks. Cairns and Liston-Heyes (1999) use the model to demonstrate that deregulation of taxi fares is less socially optimal than regulated fares. Jonard and Schenk (2001) employ a circular product space to determine when differentiated but highly substitutable products should merge networks. Malla and Gray (2007) use Salop’s Model to determine that it may sometimes be socially beneficial to charge a price or restrict access to intellectual property.

Many economists have extended the theoretical structure of location models to analyze different economic paradigms. Of particular interest to this paper, is how agglomeration is incorporated into location models. D’Aspremont, et al. (1979) claimed that the Hotelling Result was invalid if firms engaged in Bertrand Competition. The equilibrium result would be that firms would disperse on a linear street in order to reduce price competition. Andersen and Neven (1991) extended Hotelling’s model to show that if firms engage in Cournot competition on a two-dimensional street, there are certain parameter values where equidistant dispersion is the equilibrium and certain values where agglomeration is the equilibrium. Under Bertrand competition, the equilibrium result always favored dispersion.
Pal (1998) demonstrated that on a two-dimensional circle a duopoly will locate equidistant from each other no matter whether they engage in Cournot competition or Bertrand price competition. He believed the equilibrium would extend to the n-firm case. This is in contrast to a one-dimensional linear street: Cournot competition promotes agglomeration but Bertrand competition yields dispersion. Matsushima (2001) expanded Pals model to incorporate an n-firm oligopoly. He discovered that another equilibrium point exists when firms compete via Cournot Competition on a circle: half the firms agglomerate at one point, and the other firms agglomerate at the point half-way across the circle.

Matsumura et al. (2005) evaluated the different findings and tried to extend the models to include non-linear transport costs. The authors concluded that “Pal-type equilibrium is more robust (225)” because that type of equilibrium appeared in all circumstances, whereas the alternate equilibrium was very much dependant on the transport cost being neither too concave nor too convex. If the transport cost was linear then the profit to firms, the consumer surplus, and the total social welfare was exactly equal in the two equilibriums. Nonetheless if the transport cost was nonlinear then the equidistant equilibrium yielded greater profits for producers and higher total welfare and the agglomerated equilibrium generated greater consumer surplus. Matsushima and Matsumura (2003) create a model where a welfare-maximizing public firm competes with private firms on circular product space. The equilibrium result is that all the private firms agglomerate at the location that is on the opposite side of the circle as the public firm.
The presence of market failure is generally an accepted conclusion for when firms have market power. Mankiw and Whinston (1986) demonstrate that oftentimes a socially inefficient number of firms enter the market under free entry in imperfect competition. Indeed, in a homogeneous market when fixed costs are present for firms, there will unambiguously be a socially excessive number of firms that enter the market. When a new firm enters a market, there are two primary effects to social welfare. The first effect is the increase in consumer surplus due to the decrease in price. The second effect is the “business-stealing effect,” the effect that happens when a firm takes away the business of existing firms. When business stealing takes place, the output of each firm declines, and the producer surplus decreases. In a market with no market variety and fixed costs, there is a point where a new firm’s contribution to the reduction of consumer surplus is not fully compensated compared to the costs of entry. Thus, the profit of a firm due to business-stealing “makes entry more attractive than is socially warranted (49).” Adding product variety or when fixed costs are zero can greatly reduce or eliminate this instance of market failure.

The different effects of changes in intensive margin and extensive margin are currently being explored in labor economics and equilibrium macroeconomics. It is a topic, however, that has been discussed for quite some time. Generally, the extensive margin refers to the amount or the quantity of an input that is being utilized. Intensive margin is to what degree or the extent that input is being exploited. Black (1929) discusses in a note the difference between the extensive margin and the intensive margin. He states that many new students to economics have trouble discerning between the two concepts. They are not opposing notions; a producer does
not have to decide whether to focus on one or the other. Instead, for every extensive margin, there will be an associated intensive margin. Heckman (1993) believed that incorporating extensive and intensive margin was one of the greatest advances in how to study labor during the previous twenty years: “a crucial theoretical distinction with important empirical payoff is that between labor supply choices at the extensive margin (i.e., labor-force participation and employment choices) and choices at the intensive margin (i.e., choices about hours of work or weeks of work for workers (116).”

3. Salop’s Circular City

We will determine equilibrium conditions and welfare outcomes for Salop’s original model. The results of the model influence us to develop a new model that addresses the criticisms we have with the Circular City.

\( F \) is the fixed cost, \( C \) is the cost function, \( q_i \) is the output level of the firm-producing brand \( i \), \( \pi_i(q_i) \) is the profit level of the firm producing brand \( i \). The following assumptions about the profit function are made

\[
\pi_i(q_i) = (p_i - C)q_i; \text{ if } q_i > 0
\]
\[
\pi_i(q_i) = 0; \text{ if } q_i = 0
\]

We will assume the cost function is based on function that has constant marginal costs, \( c \). Consumers are uniformly distributed on the unit circle. We denote \( \tau \) as the consumers’ transportation cost per unit of distance. Each consumer will minimize their cost by minimizing the sum of the price and the transportation cost
We assume that each firm is located equidistant from each other. Thus the distance between firms is 1/L.

Suppose there is a firm 1 between a firm 2 and a firm L. Now we will assume that firms 2 and L charge a uniform price $p$. The consumer who is indifferent to whether he buys from firm 1 or firm 2, or similarly firm L is located at $\hat{\theta}$. The indifferent consumer’s utility functions on the two sides of firm 1 are can be used to find the total amount of demand that firm 1 has.

The utility that an indifferent consumer will receive from purchasing using location one is

$$U_1 = v - p_1 - \tau\hat{\theta}$$

The utility that the same indifferent consumer will receive from purchasing using location L is

$$U_2 = v - p - \tau\left(\frac{1}{L} - \hat{\theta}\right)$$

We can set the two equations equal to each other and solve for $\hat{\theta}$ to find the location of the indifferent consumer

$$p_1 + \tau\hat{\theta} = p + \tau\left(\frac{1}{L} - \hat{\theta}\right)$$

$$\hat{\theta} = \frac{p - p_1}{2\tau} + \frac{1}{2L}$$

The last customer that firm one will serve is the indifferent customer. Therefore, firm one will serve all the customers that are closer to the address. There is an indifferent customer on both sides of the firm. Firm one will serve the entire distance between the two indifferent customers. This is firm one’s demand function and it is given by $2\hat{\theta}$.
\[
2\theta = q_1(p_1, p) = \frac{p - p_1}{\tau} + \frac{1}{L}
\]

Now we are trying to find the profit function facing firm one. The formula for economic profit is as follows

\[
\pi(q_1, p_1, p) = p_1 q_1 - c(q_1)
\]

By substitution we know

\[
\pi(p_1) = p_1 \left( \frac{p - p_1}{\tau} + \frac{1}{L} \right) - c\left( \frac{p - p_1}{\tau} + \frac{1}{L} \right)
\]

We can solve for the optimum level for the price of firm one.

\[
\frac{\partial \pi(p_1)}{\partial p_1} = \frac{-p_1}{\tau} + \left( \frac{p - p_1}{\tau} + \frac{1}{L} \right) + \frac{c}{\tau} = 0
\]

Now we can find the reaction function facing firm one

\[
p_1(p) = \frac{p}{2} + \frac{\tau}{2L} + \frac{c}{2}
\]

Thus, as we would expect, their pricing strategies are strategic complements. When the outside firm raises its price, firm one reacts by raising its price.

Now we will invoke the condition of symmetric equilibrium, the condition where all firms use the same strategy

\[
p_1^* = p.
\]

By using substitution of \(p_1^*\) for \(p\) we can find the equilibrium price.

\[
p_1^* = \frac{p_1^*}{2} + \frac{\tau}{2L} + \frac{c}{2}
\]

\[
p_1^*(p_1^*) = \frac{\tau}{L} + c
\]

From previously we know what the quantity is depending on the prices of firm one and the outside firm. In equilibrium the quantity is equal to

\[
q_1^*(p_1^*, p_1^*) = \frac{1}{L}
\]
We have derived the equilibrium price and quantity of firms in the circular city. We can find perform welfare analysis on the city to see what the surplus is for the producers and the consumers in equilibrium. The producer surplus is

\[ PS = p_1 q_1 - c(q_1) \]
\[ = \left( \frac{\tau}{L} + c \right) \left( \frac{1}{L} \right) - c \left( \frac{1}{L} \right) \]
\[ = \frac{\tau}{L^2} \]

This above result is the amount of producer surplus for one firm in the city. The amount of producer surplus for the entire city is

\[ \sum PS = \frac{\tau}{L^2} \cdot L \]
\[ \sum PS = \frac{\tau}{L} \]

The Consumer Surplus net of transportation costs is

\[ 2 \int_0^{1/2L} (v - p_1^* - \tau \hat{\theta}) d\hat{\theta} \]
\[ = 2(v - p_1^*) \theta - 2(1/2) \tau \theta^2 \bigg|_0^{1/2L} \]
\[ = \left( \frac{v - p_1^*}{L} - \frac{\tau}{4L^2} \right)\cdot L \]

This above result is the amount of consumer surplus generated by the sales of one firm. It does not show how consumers also gain because of lowering transportation costs. The amount consumer surplus net of transportation costs for the entire city is

\[ = \left( \frac{v - p_1^*}{L} - \frac{\tau}{4L^2} \right) \cdot L \]
\[ = v - p_1^* - \frac{\tau}{4L} \]
The total welfare of the city is equal to the sum of the producer surplus and the consumer surplus. The welfare of the city is

\[
\tau \frac{L}{L} + v - p_1^* - \frac{\tau}{4L}
\]

With defined welfare results, we can derive meaningful comparative static results that reveal how different variables and equations move if we change a given parameter. We can show that the price that a firm offers is inversely related to the amount of firms in the market.

\[
\frac{\partial p_1^*(p_1^*)}{\partial L} = \frac{-\tau}{L^2}
\]

We can see that the term is always negative, so as the number firms increase, the price always unambiguously decreases, and as the number of firms decrease, the price unambiguously increases. Intuitively this makes sense because as more firms enter a competitive market, the price will trend downwards to marginal cost.

We can find the comparative static comparing the producer surplus and the number of locations

\[
\frac{\partial PS}{\partial L} = \frac{-\tau}{L^2}
\]

This term is unambiguously negative. Again, this makes sense because profit will decrease as the prices a firm can offer decreases. Above, we showed that the number of firms and price are inversely related. It naturally follows that the number of firms and the profit has the same relationship.

The comparative static comparing consumer surplus to the number of locations is

\[
\frac{\partial CS}{\partial L} = \frac{\tau}{L^2} + \frac{\tau}{4L^2}
\]
\[ \frac{\partial CS}{\partial L} = \frac{5\tau}{4L^2} \]

Consumer surplus unambiguously increases as firms enter the market. People will have to pay a lower price and move less distance as indicated by the term, \( \frac{\tau}{L^2} \), to get to their desired product, and that will thus keep consumer costs lower.

The comparative static result showing how total welfare changes as the number of firms is modified is

\[ \frac{\partial TW}{\partial L} = \frac{-\tau}{L^2} + \frac{5\tau}{4L^2} \]

\[ \frac{\partial TW}{\partial L} = \frac{\tau}{4L^2} \]

The total welfare clearly increases as we add locations. The increase in total welfare is due solely to the fact that consumers will have to pay a lower transport cost. When there is a change in the number of firms, the profit change to producers is exactly offset by the change in price. In this model, adding locations is unambiguously a net positive for society.

We have derived the Circular City and shown a few welfare and comparative static results. Salop developed a very adaptable model, but, it does not demonstrate some effects that we know to be true. Part of this is because we assume that each user buys exactly one unit per time period, so efficiency does not change as we add additional firms. However, intuitively we would expect, since this is an oligopoly setting, that there is deadweight loss due to oligopoly competition, and increasing the number of firms would reduce the overall inefficiency. So we must somehow introduce the fact that imperfect competition adversely affects welfare. Furthermore, there is the issue of whether the intensive margin or extensive margin for demand
increases as total demand increases. In general, models dealing with space (i.e. the
circular city), tend to sidestep the issue of the intensive margin because every
customer buys exactly one unit. We will try to model the intensive margin in the new
model.

Furthermore, there is strong reasoning to believe that firms can collocate at
the same address, even in a product space that is marked by differentiated products.
An example of this is a farmer’s market. There are multiple vendors selling exactly
the same product, but there are differentiated products throughout market. So the
different items sold at the farmer’s market are represented by different addresses on
the product space. In this example, red peppers may be close to green peppers on the
product space, but very far from cakes. There are different vendors selling red
peppers, green peppers, and cakes. This example serves as a representation of
multiple firms at a location in the Circular City.

4. The Model

We will create a three stage model on a circular product space in order to
incorporate the idea that the intensive margin of demand affects the decision of firms.
The first stage of the model is that locations must be created around the circle. There
are varying methods in which locations can be created. First, each location on the
circle could be free, and any firm that wanted to could use the infrastructure of the
city at each location. Another method in which locations can be created is that a
social planner may create and allow access at certain points on the circle. This is how
we will be creating locations in the model.
The next stage of the model is that n firms enter each location at symmetric equilibrium around the circle. In the instance where entry is free, firms would enter at each location until profits were equal to zero. If a social planner were to be in charge of creating each location, we could imagine that his goal would be to maximize the benefits to society, so the number of firms allowed at each location would maximize total welfare. We will demonstrate the effects of both of these regimes in our model.

The last stage of the model is that firms compete among each other at each location. In this model we will assume firms compete over quantity per consumer at a particular location on the circle, and then set their price accordingly. We are not concerned with the total amount of goods that a firm provides. Instead we are concerned with the intensive level of service that each firm provides for the consumer. For example, if a firm sells CDs, they have an unlimited warehouse of CDs to sell to a consumer, but the amount of features is fixed. In this simple example, the quantity that firms will compete over is the number of features, as opposed to the total number of CDs that a firm sells. We can also think of q as quality instead of quantity.

We will use sub-game perfect Nash equilibrium and derive equilibrium conditions using backwards induction.

The total quantity produced for a single consumer at location zero on the circular product space is given by \( Q_0 \). The quantity produced by a representative firm at location zero is given by \( q_{0i} \). The total output of a location is therefore equal to the sum of the individual firms at a location.
Like Salop’s Model, consumers are spread evenly along the product space, and are represented by the location $\theta$. Each consumer has a downward sloping, linear demand function given as

$$p = a - bQ$$

The price at location zero is given by

$$p_0 = a - bQ_0$$

We can now find the consumer surplus that a person would obtain from buying from the firms at point zero.

$$CS = \frac{1}{2} (a - (a - bQ))Q$$

$$CS_0 = \left(\frac{1}{2}\right) bQ_0^2$$

A person’s utility is equal to the net consumer surplus they get from purchasing a product minus the transport costs they incur. Thus, the utility a person gets from buying at location zero is equal to

$$U_0 = \left(\frac{1}{2}\right) bQ_0^2 - \tau \theta$$

We can use similar reasoning to derive the consumer surplus and the utility of a person that shops at either location one or location L.

$$CS_1 = \left(\frac{1}{2}\right) bQ_1^2$$

$$U_1 = \left(\frac{1}{2}\right) bQ_1^2 - \tau \left(\frac{1}{L} - \theta\right)$$

An indifferent consumer achieves the same utility from buying at location zero and at location one. If we assume that the locations are at an equal distance from
one another and that firms at the two locations on both sides of location zero offer a uniform price, we can set the utility function of a customer buying at location zero equal to the utility function of a customer buying at location one to find where the consumer with the indifferent utility lies.

\[
\left( \frac{1}{2} \right) bQ_0^2 - \tau \theta = \left( \frac{1}{2} \right) bQ_1^2 - \tau \left( \frac{1}{L} - \theta \right)
\]

\[
\hat{\theta} = \left( \frac{1}{4\tau} \right) (bQ_0^2 - bQ_1^2) + \frac{1}{2L}
\]

Like Salop’s Circular City, the firms at location zero will attract customers from two sides. Therefore, the number of consumers who purchase quantity at location zero is twice \( \hat{\theta} \). The demand of a firm that is positioned at location zero is

\[
2\hat{\theta} = \left( \frac{1}{2\tau} \right) (bQ_0^2 - bQ_1^2) + \frac{1}{L}
\]

\[
2\hat{\theta} = \left( \frac{b}{2\tau} \right) (Q_0^2 - Q_1^2) + \frac{1}{L}
\]

The profit that a representative firm at point zero receives from one consumer is equal to \((p_0 - c)q_{0i}\). As we described earlier, \(q_{0i}\), represents a per consumer quantity. In order to calculate the profit from all customers we must multiply the above expression by the demand function. In this model, we also assume that firms must pay a fixed cost, \(F\), in order to begin production.

\[
\pi_{0i} = (p_0 - c)q_{0i}2\hat{\theta} - F
\]

\[
\pi_{0i} = (a - bQ_0 - c)q_{0i}\left( \left( \frac{1}{2\tau} \right) (bQ_0^2) - b(Q_1^2) \right) + \frac{1}{L} - F
\]

We can now solve for the profit-maximization condition for a representative firm

\[
\frac{\partial \pi}{\partial q_{0i}} = -bq_{0i}2\hat{\theta} + (p_0 - c)2\hat{\theta} + (p_0 - c)q_{0i}\frac{\partial 2\hat{\theta}}{\partial q_{0i}} = 0
\]
The intensive margin of demand, the demand that takes into account consumption per consumer, is represented by the first two parts of the above expression. The extensive margin of demand, the demand changes from bringing in a new user, is $\frac{\partial^2 \tilde{\theta}}{\partial q_{0i}}$.

We now invoke the condition of symmetric equilibrium. In symmetric equilibrium, all firms will produce the same quantity, and all locations will have the same number of firms.

$$2\tilde{\theta} = \frac{1}{L}$$

$$Q_0 = Q_1 = nq_{0i}$$

$$\frac{\partial \pi}{\partial q_{0i}} = -bq_{0i}2\tilde{\theta} + (p_0 - c)2\tilde{\theta} + (p_0 - c)q_{0i}\left(\frac{1}{\tau}bnq_{0i}\right) = 0$$

Unfortunately, the derivative is a cubic, and will it therefore be difficult to find a concise, close-formed solution for $q_{0i}$. For now, we will assume there exists a unique solution, $q_{0i}$, in symmetric equilibrium that a social planner uses to make his decisions. In section 5, we will use an approximation to find concise, close-formed solutions.

The net welfare gains to the producer for each representative firm is equal to the economic profit, $\pi(p_0)$, at equilibrium.

$$\pi(p_0) = (p_0 - c)q_{0i}2\tilde{\theta} - F$$

$$\pi(p_0) = (a - bnq_{0i} - c)\left(\frac{q_{0i}}{L}\right) - F$$

This is the amount of producer surplus for one firm at one location. The amount of producer surplus for the entire city is

$$= \left((a - bnq_{0i} - c)\left(\frac{q_{0i}}{L}\right) - F\right)nL$$

$$= ((a - bnq_{0i} - c)(nq_{0i}) - FnL)$$
We previously derived the consumer surplus of a person who buys from location 0. The total consumer surplus that is generated from buying from location 0 when in symmetric equilibrium is

\[ CS = 2 \int_0^{2\pi} \left( \frac{1}{2} bQ_0^2 - \tau \theta \right) d\theta \]

\[ = 2(1/2)(bn^2q_0^2\theta) - 2(1/2)\tau\theta^2 \right|_0^{1/2\pi} \]

\[ = \frac{bn^2q_0^2}{2L} - \frac{\tau}{4L^2} \]

This is the amount of consumer surplus for one location. The consumer surplus for the entire city is

\[ = \left( \frac{bn^2q_0^2}{2L} - \frac{\tau}{4L^2} \right)L \]

\[ = \left( \frac{1}{2} \right) bn^2q_0^2 - \frac{\tau}{4L} \]

The total welfare of the city is equal to the sum of the consumer surplus and the net welfare to producers.

\[ TW = (a - bnq_0i - c)(nq_0i) + \left( \frac{1}{2} \right) bn^2q_0^2 - \frac{\tau}{4L} - FnL \]

We have determined the welfare results of the city. We can see that there are four components of total welfare: producer surplus, consumer surplus net of the transport costs, transport costs, and fixed costs. We are now interested in the first stage of the model, the stage where locations around the city are created. We assume that the locations are created by a social planner that is concerned with maximizing the total welfare of the city. He would, therefore, create the amount of locations that would be the socially optimal.
We explore two different regimes that the social planner could employ. One possibility is a welfare-maximizing regime where the social planner would create the locations and then choose how many firms may enter at each location; another possibility is that the social planner would be consigned to setting up locations, but then once the locations are created, he must allow for free entry at each location. If the first regime is employed he would solve the welfare-maximizing result for each location, and then create the amount of locations that are necessary to maximize the welfare of the city. If the second regime is utilized, he must anticipate the number of firms that would enter at each location, and then subsequently must create the number of locations that would maximize welfare of the entire city.

In order to find how many firms would enter at each location if the social planner was allowed to choose the socially optimum number of firms per location we take the derivative of total welfare with respect to \( n \), set it equal to 0, and then solve for the number of firms per location.\(^1\)

\[
\frac{\partial TW}{\partial n} = (a - bnq_{0i} - c) \left( n \frac{\partial q_{0i}}{\partial n} + q_{0i} \right) - (nq_{0i}) \left( bn \frac{\partial q_{0i}}{\partial n} + b q_{0i} \right) + bn^2 q_{0i} \frac{\partial q_{0i}}{\partial n} + bnq_{0i}^2 + F L = 0
\]

We denote the solution as \( n^{WM} \).

Instead if the regime that is employed is the one that must allow for free entry, the total number of firms that will enter at a given location is determined by setting the profit of a firm equal to 0, and solving for \( n \).

\[
(a - bnq_{0i} - c) \left( \frac{q_{0i}}{L} \right) - F = 0
\]

\(^1\) See appendix for details
We can denote this solution as \( n^{FE} \).

By knowing how many firms will enter at each location, a social planner could derive the socially optimal number of locations.

\[
TW = bnq_{0i}^2 + \frac{1}{2}bn^2q_{0i}^2 - \frac{\tau}{4L} - FnL - GL
\]

\[
\frac{\partial TW}{\partial L} = 0
\]

The first thing of note is that there is now a fixed cost, \( G \), to society of creating a location. In this model, we can assume that the social planner incurs the cost of creating a location. As shown above, the social planner can determine the socially optimal number of locations by setting the derivative of total welfare with respect to \( L \) equal to zero and then solve for \( L \).

5. Simplifying the Model

In order to find an explicit expression for \( q_{0i} \), we can use an approximation whereby each individual firm ignores the changes in the extensive margin.

\[
\frac{\partial \pi}{\partial q_{0i}} = -bq_{0i}2\hat{\theta} + (p_0 - c)2\hat{\theta}
\]

\[
q_{0i} = \frac{a - c}{b(1 + n)}
\]

This assumption is reasonable and actually quite standard. The change in quantity that one firm makes is small, and as the number of firms per location increases, the change in profit becomes increasingly trivial. Nevertheless, in the next section, we later explicitly prove a few results using the complete profit maximization condition. We can see that with the approximation, the sub-game Nash equilibrium quantity is the standard Cournot quantity.
LEMMA 1. As the number of firms per location increases, the amount of additional profit that a firm obtains by accounting for the external margin of demand becomes increasingly trivial.

PROOF.

\[(p_0 - c) \frac{\partial^2 \theta}{\partial q_{0i}} = (p_0 - c) q_{0i} \left( \left( \frac{1}{\tau} \right) b n q_{0i} \right) \]

\[= \left( a \left( \frac{1}{1 + n} \right) + c \frac{n}{1 + n} - c \right) q_{0i} \left( \left( \frac{1}{\tau} \right) b n q_{0i} \right) \]

\[= \left( \frac{a - c}{1 + n} \right) q_{0i} \left( \left( \frac{1}{\tau} \right) b n q_{0i} \right) \]

\[= \left( \frac{1}{\tau} \right) b^2 n q_{0i}^3 \]

\[= \left( \frac{(a - c)^3}{b^3 \tau} \right) \left( \frac{n}{(n + 1)^3} \right) \]

As \( n \) increases, the external demand approaches 0 on the order of \( n^{-2} \).

This is important because it justifies the approximation of ignoring the external demand term in order to determine the equilibrium quantity.

Now that we have the equilibrium quantity, we can determine the price that a representative firm will set.

\[p_0 = a - b n q_{0i}\]

\[p_0 = a - b \left( \frac{a - c}{b(1 + n)} \right) n\]

\[p_0 = a \left( \frac{1}{1 + n} \right) + c \frac{n}{1 + n}\]

We observe that this is the standard Cournot price.
We have determined the equilibrium quantity and price and can now determine explicit welfare results. The total welfare of the city is equal to the sum of the consumer surplus and the net welfare to producers.$^2$

$$TW = bna_0^2 + \left(\frac{1}{2}\right) bn^2 q_0^2 - \frac{\tau}{4L} - FnL$$

$$TW = bna_0^2 \left(1 + \frac{n}{2}\right) - \frac{\tau}{4L} - FnL$$

We can now find an explicit solution for if the social planner was allowed to choose the socially optimum number of locations.$^3$

$$n^{WM} = \left(\frac{a - c}{\sqrt{Fb}}\right)^{3/2} - 1$$

We are able to do likewise for the free-entry regime.$^4$

$$n^{FE} = \frac{a - c}{\sqrt{Fb}} - 1$$

We are now able to derive comparative static expressions for each different type of regime. The primary concern is how different variables and expressions change with respect to the number of locations. In the last step we will determine the socially optimal number of locations for each regime.

We will first prove comparative static results for the welfare-maximizing regime.

**Lemma 2.1.** *As the number of locations in the welfare-maximizing regime increases, the number of firms per location decreases.*

**Proof.**

$$n^{WM} = (a - c)^{2/3} (Fb)^{-1/3}(L)^{-1/3} - 1$$

$^2$ See appendix for details

$^3$ See appendix for details

$^4$ See appendix for details
\[
\frac{dn_{WM}^W}{dL} = \left( -\frac{1}{3} \right) (a - c)^{2/3} (Fb)^{-1/3} (L)^{-4/3}
\]

\[
\frac{dn_{WM}^W}{dL} = \left( -\frac{1}{3} \right) (n_{WM}^W + 1) (L)^{-1}
\]

\[
\frac{dn_{WM}^W}{dL} < 0
\]

The above expression is unambiguously negative. When a location is added to the product space, the market structure of the circle readjusts, and each location serves a smaller market size. There is less demand for each location, so the equilibrium number of firms per location decreases.

**LEMMA 2.2.** As the number of locations in the welfare-maximizing regime increases, the quantity increases.

\[
\frac{dq_{0i}}{dL} > 0
\]

The expression is unambiguously positive.

**LEMMA 2.3.** As the number of locations in the welfare-maximizing regime increases, the total quantity per location decreases.

\[
\frac{dQ_0}{dL} < 0
\]

The expression is unambiguously positive. The logic of Cournot competition gives us the intuition that the total quantity of a location will be greater if there are more firms engaged in competition. Therefore, as the number of locations in the city increases, the number of firms at each location decreases, which in turn will decrease
the total quantity produced at a location, though each individual firm increases its quantity.

**LEMMA 2.4.** If there are multiple firms at each location, as the number of locations in the welfare-maximizing regime increases, the producer surplus increases. If there is a monopolist at each location, then producer surplus doesn’t change as another location is added.

\[ \frac{\partial PS}{\partial L} > 0 \]

Producers will make a higher profit if there are more locations because that means there will be more possible locations for firms to enter at, and thus there will be fewer firms at each location. There will be fewer firms engaged in Cournot competition at each location, and firms will charge a higher price. This increase in profits for the producer is negated somewhat by the fact that each location will have smaller amounts of total profits. Nonetheless, the first effect is stronger than the secondary effect. In the special case that there is a monopolist at each location, then the total producer surplus does not change if you add another location.

**LEMMA 2.5.** As the number of locations in the welfare-maximizing regime increases, the consumer surplus net of transport costs decreases.

**PROOF.**

\[ CS = \frac{1}{2} b(Q_0)^2 \]

\(^7^\) See appendix for the complete proof
\[
\frac{\partial CS}{\partial L} = b(Q_0) \frac{dn_{WM}q_0}{dL}
\]

\[
\frac{\partial CS}{\partial L} = b(Q_0) \left(- \frac{1}{3} (L)^{-1}(q_{0i})\right)
\]

\[
\frac{\partial CS}{\partial L} = - \frac{1}{3} b n_{WM} (q_{0i})^2 (L)^{-1}
\]

\[
\frac{\partial CS}{\partial L} < 0
\]

We have previously determined that the total quantity at a location decreases as a social planner increases the number of locations. As there is less quantity being served by a location, the consumer will have to pay more for the product at a specific address. We must remember that this is the consumer surplus without transport costs, so we are not saying that the welfare for consumers necessarily declines when a new location is added. Nonetheless, consumers will have to face a higher price at whichever location they shop at.

**Lemma 2.6.** As the number of locations in the welfare-maximizing regime increases, the producer surplus added to the consumer surplus net of transport costs decreases.

**Proof.**

\[
\frac{\partial PS}{\partial L} + \frac{\partial CS}{\partial L} = \frac{bn_{WM}q_{0i}^2 - b q_{0i}^2}{3L} - \frac{bn_{WM} (q_{0i})^2}{3L}
\]

\[
\frac{\partial PS}{\partial L} + \frac{\partial CS}{\partial L} = - \frac{b q_{0i}^2}{3L}
\]

\[
\frac{\partial PS}{\partial L} + \frac{\partial CS}{\partial L} < 0
\]

When we add locations, the decrease in consumer surplus net of transport costs is greater than the producer surplus. The gain to producers having a stronger market share at each location is not fully offset by the fact that consumers will have to
pay a higher price for the different goods. As locations become more proliferated, each location itself becomes a stronger oligopoly, because fewer firms will be at each address. The results of Cournot competition tell us that stronger oligopolies produce less quantity and charge higher prices than instances where there is more robust competition. The welfare gains to producers as we add more locations do not make up for welfare lost to consumers net of increased transport costs. Thus, deadweight loss increases as we increase the number of locations in the city.

**Lemma 2.7.** As the number of locations in the welfare-maximizing regime increases, the total transport costs decrease

**Proof.**

\[
\text{Transport Cost} = \frac{\tau}{4L}
\]

\[
\frac{\partial TRC}{\partial L} = \frac{\tau}{4}(-L)^{-2}
\]

\[
\frac{\partial TRC}{\partial L} = -\frac{\tau}{4L^2}
\]

\[
\frac{\partial TRC}{\partial L} < 0
\]

As a social planner adds more locations on the product space, consumers will have to pay a lower cost of transportation. The distance between firms is less, and thus, customers will not have to travel as far to reach their desired location.

**Lemma 2.8.** As the number of locations in the welfare-maximizing regime increases, the total fixed costs increase.

---

8 See appendix for the complete proof
\[
\frac{\partial FC}{\partial L} > 0
\]

The total fixed costs to society increases as we add more locations. When we add a location, we know fixed costs are going to rise by the cost of creating a location added to the number of firms at that location multiplied by the fixed cost of one firm. The total effect of the change in fixed costs as we add locations is somewhat offset by the fact that as we increase the number of locations there are fewer firms per location. Nonetheless, the increase in costs outweighs the reduction of costs when a location is added.

**PROPOSITION 1.** As the number of locations in a welfare-maximizing regime increases, it is ambiguous whether total welfare increases or decreases. There are an optimal number of locations, and before that number is reached, it is beneficial to keep adding locations. After the city reaches the optimal number of locations, however, it harms society to add more locations.

\[
\frac{\partial TW}{\partial L} = -Fn^{WM} - G + \frac{\tau}{4L^2} = 0
\]

\[
\sqrt{\frac{\tau}{4Fn^{WM} + 4G}} = L^*
\]

We needed to show that the total welfare function is concave with respect to L at the optimal point, in order to make sure \( L^* \) is at maximum point.

\[
\frac{\partial^2 TW}{\partial L^2} < 0
\]

When a social planner adds a location, the decrease in total welfare due to increased inefficiency and fixed costs are counterbalanced by the smaller transport

---

\(^{9}\) See appendix for the complete proof
costs. Thus we see one of the core results of this paper: there is a tradeoff between increased inefficiency and increased product variety. The optimal number of locations in a city is determined by society’s preference for product variety.

With the exception of differing magnitudes, the comparative static results for a free entry regime parallel that of a welfare maximizing regime. We will leave the proofs of the results in the appendix. For this regime, there are two results that we will use for later proofs.

LEMMA 3.6.\textsuperscript{10} As the number of locations in a free entry regime increases, the producer surplus added to the consumer surplus net of transport costs decreases.

\[
\frac{\partial PS}{\partial L} + \frac{\partial CS}{\partial L} = \frac{-b q^2_{0i}}{2L}
\]

PROPOSITION 2.\textsuperscript{11} As the number of locations in a welfare-maximizing regime increases, it is ambiguous whether total welfare increases or decreases. There are an optimal number of locations, and before that number is reached, it is beneficial to keep adding locations. After the city reaches the optimal number of locations, however, it harms society to add more locations.

\[
\frac{\partial TW}{\partial L} = \frac{\tau}{4L^2} - \frac{F n^{FE}}{2} - G = 0
\]

\[
\sqrt{\frac{\tau}{2F n + 4G}} = L^*
\]

Total welfare of our model depends on four factors. As we increase the number of locations the producer surplus increases, the consumer surplus net of

\textsuperscript{10} See appendix for the complete proof
\textsuperscript{11} See appendix for the complete proof
transport costs decreases, the transport costs decrease, and the fixed costs increase.

As we have previously shown, the totaling of producer surplus and consumer surplus with respect to L in both models is a negative term. This is due to increased inefficiency as the market at each location becomes more concentrated. Increasing the number of locations also causes fixed costs to increase which decrease total welfare. In fact, the only reason total welfare increases as we add locations is because of the reduction in transport costs. In other words, there is a tradeoff to society between product variety (modeled by the number of locations) and increased deadweight loss and fixed costs.

Increased product variety comes at the cost of lost efficiency. Thus we see that it is beneficial to society to add a location only if the decrease in transport costs is greater than other negative changes. If the transport cost is high or if consumers greatly values the differentiated products, then it makes sense to have more locations. Conversely, if transport costs are low, or if consumers care very little among the differentiated products, then there will be less locations. Indeed if there is no cost to travel, or if society is indifferent between differentiated products, then it is most beneficial for society to have all the firms aggregate at one location.

We have also established two different regimes that the social planner could make use of to govern the city. The social planner may be able to choose how many firms can enter at each location (we assume that he chooses the welfare-maximizing amount), and then based on the number of firms per location, he would create the socially optimal amount of locations in the city. Conversely, the social planner may have to allow free-entry at every location he creates, so he would have to anticipate
the number of firms that would enter locations, and then create the socially optimal number of locations.

Even though we previously proved that the two regimes responded in similar ways when a social planner changes the number of locations, we can make a few comparisons because of the differing scales between the two regimes.

PROPOSITION 3.12. If the location has enough customers that a monopolist could make a quarter of the fixed costs in profit, then there would be more firms per location in a free entry regime than a welfare-maximized regime

\[ n^{FE} = \frac{a-c}{\sqrt{FLb}} - 1, \quad n^{WM} = \frac{3(a-c)^2}{\sqrt[3]{FLb}} - 1 \]

\[ \frac{a-c}{\sqrt{FLb}} - 1 > \frac{3(a-c)^2}{\sqrt[3]{FLb}} - 1 \]

As long as fixed costs are not so great that they do not overwhelm the profits at a location, under free entry there will be more firms at a location than what is socially optimal. Some possible ways this result would not hold would be if there were too many locations, and thus profit per location was very low or if the fixed costs were so high that the profit of a monopolist would be a quarter of the fixed costs. Therefore, it is a reasonable postulation to say that under most conditions, the free entry number of firms for each location is greater than the welfare-maximized number of firms for each location. This confirms the Mankiw-Whinston result that in a homogenous market in an unregulated oligopoly setting, more than the socially optimal number of firms enter the market under free-entry. The circular product

\[12 \text{ See appendix for the complete proof} \]
space clearly draws out the idea that there is no product differentiation among firms competing at the same location.

LEMMA 5.13 When we sum the producer surplus and the consumer surplus net of transport costs, adding a location under a welfare-maximizing regime will contribute to a greater loss of welfare than under free entry.

\[
-\frac{b (q_{WM}^2)}{2L} > -\frac{b (q_{FE}^2)}{3L}
\]

In both cases adding an additional location causes the sum of producer surplus and consumer surplus net of transport costs to be negative. We proved, however, that for the welfare-maximizing regime it is more negative. This is due to the fact that the firms per location in the welfare-maximizing regime is already in a more concentrated oligopoly than a free entry regime, so adding another location will allow the firms at each location to have even stronger market power. This suggests that a free-entry regime may be better able to withstand a social excess amount of locations better than a welfare-maximizing regime.

PROPOSITION 4. \( \forall L, \text{ if } 2n_{WM} > n_{FE}, \text{ then } L \text{ in the welfare maximizing regime will be smaller. If } 2n_{WM} < n_{FE}, \text{ then } L \text{ in the welfare maximizing regime will be bigger. If } 2n_{WM} = n_{FE}, \text{ then } L \text{ in the two regimes will be identical} \)

PROOF.

By previously proving the concavity in both regimes of Total Welfare with respect to \( L \), we have proved that the Total Welfare with respect to \( L \) is downward sloping at

\[13 \text{ See appendix for the complete proof}\]
the optimal point. Now that we have shown that both first derivates are downward sloping we can compare the two and show which regime would have a larger L at the social optimum

\[ \frac{\partial T W_{FE}}{\partial L} = \frac{\tau}{4L^2} - \frac{F n_{FE}}{2} - G = 0 \]

\[ \frac{\tau}{4L^2} = \frac{F n_{FE}}{2} = G \]

\[ \frac{\partial T W_{WM}}{\partial L} = \frac{\tau}{4L^2} - F n_{WM} - G = 0 \]

\[ \frac{\tau}{4L^2} - F n_{WM} = G \]

We have determined two possible scenarios to determine the number of firms for each location; there can be free-entry at each location, or a socially optimizing agent can choose the welfare maximizing number of firms for each location. The number of firms at a location under a free-entry regime will always be larger than when a welfare-maximizer gets to choose the number of firms. Thus, since the
number of firms per location under each regime will be different, the socially optimum number of locations under the two different regimes will also be different.

If the number of firms per location under free-entry is less than twice as big as the number of firms under a welfare-maximizing regime, then the socially optimal number of locations will be greater under free entry. Intuitively this makes sense because the relative cost of adding one more location when there are a large number of firms per location is small compared to adding a location when there are many firms at each location. The loss in efficiency is great when there are a few firms per location and then we add another location; we are further concentrating the market power of an already concentrated oligopoly. However, when the number of firms per location is already large, adding another location does not change the efficiency with nearly the same magnitude. In other words, the tradeoff between lowering transport costs and adding deadweight loss when adding a location will be more pronounced under a welfare-maximizing regime than under a free-entry regime.

If the number of firms per location under free-entry is more than twice as big as the number of firms in a welfare-maximizing regime, then the socially optimal number of locations will be greater under the latter regime. Under any regime, when a location is added, there are fixed costs of the individual firms, in addition to the fixed cost of creating a location. If the number of firms per location is extremely high, then fixed costs start to become a very important factor. Under the circumstance where the welfare-maximized number of firms per location is less than half as big as the free-entry number of firms per location, the fixed costs outweigh the relative gain in efficiency that will be made when a firm is added. Thus, in this situation a free-
entry system will have comparatively fewer locations than a welfare-maximizing system.

6. The model without assumptions

We have solved all three stages of the model and determined welfare results for the assumption that a firm ignores the extensive margin for demand. This is a sensible assumption and one that allows for concise, close-formed solutions. We can, however, solve for the symmetric equilibrium quantity, where a firm takes into account the extensive margin when they make their profit-maximizing condition.

\[
\frac{\partial \pi}{\partial q_{0i}} = (a - b Q_0^* - c) 2 \theta - b q_{0i} 2 \theta + (a - b Q_0^* - c) q_{0i} \left( \frac{1}{\theta} \right) b Q_0^* = 0
\]

As stated previously, since this equation is a cubic, the solutions are not nearly as concise as the approximated equations

In order to solve for the symmetric equilibrium quantity of the welfare-maximizing regime, we must modify the profit maximizing condition and solve for the quantity of an entire location.\(^{14}\)

\[
(a - b Q_0^* - c) \frac{\partial Q_0^*}{\partial L} - (Q_0^*) \left( b \frac{\partial Q_0^*}{\partial L} \right) + \left( \frac{1}{2} \right) b Q_0^* \frac{\partial Q_0^*}{\partial L} + \frac{\tau}{4L^2} - F_n = 0
\]

The equilibrium quantity is a dependant on the number of locations, and in order to find \(\frac{\partial Q_0^*}{\partial L}\), we must use the implicit function theory.

\(^{14}\) See appendix for complete proof
LEMMA 6.\textsuperscript{15} \textit{If a firm takes account of the profit due to the extensive margin, as the number of locations in the welfare-maximizing regime increases, the total quantity per consumer per location increases.}

\[ \frac{\partial Q_0}{\partial L} > 0 \]

We have proved that a unique quantity exists for the welfare-maximizing regime in symmetric equilibrium. Right away, we notice that the direction of the sign is not the same, as with the approximated quantity. In this version of the model, the quantity per location increases as we add more locations. This does not mean, however, In other words, the total quantity increases around the city when the social planner adds locations. In this version of the model, each address is pressured from the two adjacent locations. When a social planner increases the number of addresses where firms can enter, each individual location covers less demand, and the competition among the locations becomes fiercer. This increased inter-location competition causes firms to increase output.

If the number of firms per location decreases, every firm has stronger market power on the intensive margin of demand. The model with the approximation explicitly draws out this effect. It, however, ignores the countering effect that occurs because of the increased inter-location competition. The model without the approximation clearly exposes the second effect and shows it outweighs the first effect.

We get similar results for the free-entry regime. For this regime, we can determine explicit solutions.

The number of firms per location is\textsuperscript{16}

\[ n^{FE} = \left( a - bQ_0 - c \right)Q_0 \]

\textsuperscript{15} See appendix for complete proof
\textsuperscript{16} See appendix for details
The number of firms in the free entry regime is equal to the profit per customer divided by the fixed costs and number of locations. The explicit result clearly demonstrates that there will be many firms per location if profits are high and fixed costs and the number of locations are low.

The total quantity of each location is\(^{17}\)

\[
Q_0^* = \frac{-L^2 F(a - c)b + 2\tau(a - c)b - \sqrt{L^4 F^2(a - c)^2b^2 + 4\tau^2 b^3 LF - 4\tau b^3 L^3 F^2}}{2(\tau b^2 - b^2 L^2 F)}
\]

The total quantity per location is directly affected by the total number of locations. If a location is added, the number of firms per location decreases, but the competition among the locations increases.

**LEMMA 7.**\(^{18}\)

*If a firm takes account of the profit due to the extensive margin, the total quantity per location will usually increase*

If \(L^3 F(a - c)^2 + \tau^2 > 3\tau^2 b L^2 F\), then \(\frac{\partial Q_0^*}{\partial L} < 0\)

If \(L^3 F(a - c)^2 + \tau^2 < 3\tau^2 b L^2 F\), then \(\frac{\partial Q_0^*}{\partial L}\) is dependent on the parameters

We know the first term is unambiguously positive, and we cannot determine the sign of the second term. If \(L^3 F(a - c)^2 + \tau^2 > 3\tau^2 b L^2 F\), then the derivative is unequivocally positive. If the opposite is true, then the derivative can be negative dependent on the other parameters. As long as the number of locations is somewhat high, then the comparative static expression will be positive. This result is generally

\(^{17}\) See appendix for details
\(^{18}\) See appendix for complete proof
consistent with the welfare-maximizing regime that increasing the number of locations will increase the total quantity.

**LEMMA 8.** As fixed costs approach 0 we can show that the quantity produced if firms took into account the changes in the extensive margin nearly approach the approximation derived previously.

**PROOF.**

\[
\lim_{F \to 0} (q_{0i}) = \frac{2\tau(a - c)b}{2n\tau b^2}
\]

\[
\lim_{F \to 0} (q_{0i}) = \frac{a - c}{nb}
\]

This is similar to the approximated quantity

\[
q_{\bar{a}} = \frac{a - c}{b(n + 1)}
\]

The approximated equilibrium quantity is slightly higher than the full equilibrium quantity where there are no fixed costs. This makes sense because the approximation assumes that a firm foregoes a small amount of marginal profit. In the full case, however, the producer accounts for the entire profit, so it makes sense that he would want to produce a slightly higher quantity. For the unique case that there are no fixed costs, the price in equilibrium will equal marginal cost, and we will be at perfect competition at each location.

In this section we have proven results if a firm takes into account the entire potential profit. While some of the results remain consistent with the previous section, other outcomes are different than what the approximation revealed. In this section we illustrate that firms can actually increase total output when a social planner adds a location. Firms at each address are not only in competition with themselves, but also,
in competition with firms from adjoining locations. As the number of locations increase, the second type of competition intensifies. While this model does an excellent job of describing the inter-location competition, it may not fully demonstrate the effect that adding locations can increase market power. This occurs, because we allow continuous changes in the number of firms per location when we change the total number of locations. In reality though, we think the structure of the city as such that the social planner chooses the total number of locations based upon how many firms will enter at each location. When the number of locations is changed, there will be an integer change to the number of firms at each location. Therefore, there is some numerical reduction in the number of firms per location where the market power effect actually outstrips the competition effect. We may need to use numerical methods in order to accurately compare the two offsetting effects.

There are several advantages to the method in this section that we used to derive equilibrium quantity. Most importantly, we have found that there exists a unique equilibrium quantity for the full model. Furthermore, the equilibrium output we derive does not ignore any part of the demand. The primary disadvantage of this method is that we take away the independent effect that the number of firms per location has on the equilibrium output. We expect that the number of firms per location helps determine what the quantity will be in equilibrium. Instead, in this model, all the equilibrium results are determined in the same stage of the game.

7. Conclusion
We have developed Salop’s Circular City and have determined a few issues that the model does not adequately explain. The model assumes that each consumer purchases exactly one unit per time period. Therefore, there are no efficiency concerns in his model, despite the fact that the model is in an oligopoly setting. Nonetheless, we expect there to be issues of deadweight loss when there are unregulated monopolists in a market.

In response to these problems, we created a three-stage location model using a circular product-space. The first stage involves a social planner creating a socially optimum number of locations. In the second stage, firms enter the market at symmetric equilibrium. We have discussed two different regimes that the social planner may allow firms to enter the market. One possibility is that the he must allow free entry at every location; the other possibility is that he would permit the number of firms that would optimize social welfare. In the last stage, firms would compete a la Cournot among themselves at each location.

We have found results for two versions of the model. To solve the model completely, the first version of the model uses a simplifying assumption that firms ignore the extensive margin in order to find concise, close-formed solutions. With this approximation, firms produce the standard Cournot quantity. For a variety of reasons, we believe this is a reasonable assumption. Nevertheless, we have found the equilibrium quantity for the second version, the full model, and shown some comparative static results. The two different methods used to find the equilibrium quantities have their own advantages and disadvantages.
The first version of the model shows that the free-entry number of firms per location is greater than what is socially most desirable. This result is consistent with Mankiw-Whinston that too many firms enter in a homogenous oligopoly market. Since there is no product variety at each location, the profit that each firm sees from “business-stealing” allows for a socially undesirable result. Interestingly, consumers prefer a free-entry regime, because more firms per location means that consumers face a lower price. A social planner that uses the concept of welfare-maximization actually ends up protecting producers from themselves. This is an interesting benefit of regulation; usually, social planners are thought of as protecting consumers when they face a market of imperfect competition. In the scenario of oligopolistic competition in a homogenous market, however, a social planner increases the amount of welfare for each individual firm.

We cannot say with certainty that there will be more locations in one regime than another. When a location is added, there is reduction in social welfare for two primary reasons: increases in deadweight loss and more fixed costs. The negative efficiency effect is relatively bigger for the welfare-maximizing regime than it is for a free-entry regime. The fixed cost effect is relatively bigger for the free-entry regime. If the free entry number of firms per locations greatly exceeds the socially optimum number of firms per location than there will be more locations in the welfare-maximized regime. However, if the difference between the two regimes is not that great, then there will be more locations under free entry than there will be under a welfare-maximizing regime.
The socially optimum number of locations is dependent on a variety of different factors. However, what the first version of the model especially specifies is the exchange between lower transport costs for increased inefficiency. Transport costs unambiguously decrease as more locations are added to the product space. Inefficiency, however, unambiguously increases as more locations are added to the product space. Therefore there is a crucial tradeoff between product variety, as modeled by the number of locations, and deadweight loss. If consumers greatly value product variety (i.e. transport costs are high) then they will not mind to suffer the costs of inefficiency. However, if they are generally indifferent between the differentiated products (i.e. transport costs are low) then having many locations will not be beneficial for society.

The second version of the model explores what happens when a firm decides to account for the change in the extensive margin. In this version, each location is in competition with the adjacent addresses. When a social planner adds another location to the city, the addresses become closer together, and each location controls less of the total demand of the city. The increased competition among the addresses causes the output for each location to increase. This is the opposite of what happens in the previous section; in the first version of the model, firms do not take account of the neighboring locations and the negative efficiency effect dominates. Nonetheless, we think that this version of the model may exaggerate the inter-location competition effect because out setup allows for smooth, continues changes in the parameters. In reality, we think the changes that occur will be integer changes.
The primary advantage of the method with the approximation is that it allows for concise, close-formed solutions. We can easily solve this model to derive results for all three stages of the game. Moreover, we can see how the equilibrium output is derived from the parameters it is dependent on. It does, however, not account of the entire profit maximizing condition. The second method to finding equilibrium conditions where there is no approximation technique allows the firm to completely maximize its profit. The primary disadvantage with this method to finding quantity is that the parameters which we expect it to be dependent on, such as the number of firms per location, and the number of locations, are now determined in the same stage of the game as the equilibrium quantity. In reality, however, we expect that the equilibrium quantity is determined after we know the number of locations and the number of firms per location.
Appendix

If we assumed the existence of $q_{0i}$ then the details for the determining how many firms will enter in a regime where the social planner chooses the socially optimum number of firms are given by

$$TW = (a - bnq_{0i}^* - c)(nq_{0i}^*) + \left(\frac{1}{2}\right) bn^2 q_{0i}^* - \frac{\tau}{4L} - FnL$$

$$\frac{\partial TW}{\partial n} = (a - bnq_{0i}^* - c) \left( n \frac{\partial q_{0i}^*}{\partial n} + q_{0i}^* \right) - (nq_{0i}^*) \left( bn \frac{\partial q_{0i}^*}{\partial n} + bq_{0i}^* \right) + bn^2 q_{0i}^* \frac{\partial q_{0i}^*}{\partial n}$$

$$+ bnq_{0i}^* - FL = 0$$

If we have $q_{0i}$ at equilibrium then the net welfare to producers is given by

$$PS = (p_0 - c)q_{0i} 2\theta - F$$

$$PS = \left( a \left( \frac{1}{1 + n} \right) + c \frac{n}{1 + n} - c \right) q_{0i} \left( \frac{1}{L} \right) - F$$

$$PS = \left( \frac{a - c}{1 + n} \right) q_{0i} * \left( \frac{1}{L} \right) - F$$

$$PS = \frac{bq_{0i}^2}{L} - F$$

This is the amount of producer surplus for one firm at one location. The amount of producer surplus for the entire city is

$$= \left( \frac{bq_{0i}^2}{L} \right) nL - FnL$$

$$= bq_{0i}^2 n - FnL$$

If we have $q_{0i}$ at equilibrium, then the consumer surplus is given by

$$CS = 2 \int_0^{1/2L} \left( \frac{1}{2} bQ_0^2 - \tau \theta \right) d\theta$$
\[
\begin{align*}
\theta &= 2(1/2)(bn^2 q_{0i}^2 \theta) - 2(1/2) \tau \theta^2 \\
\theta &= \frac{bn^2 q_{0i}^2}{2L} - \frac{\tau}{4L^2}
\end{align*}
\]

This is the amount of consumer surplus for one location. The consumer surplus for the entire city is
\[
\begin{align*}
&= \left(\frac{bn^2 q_{0i}^2}{2L} - \frac{\tau}{4L^2}\right) L \\
&= \left(\frac{1}{2}\right) bn^2 q_{0i}^2 - \frac{\tau}{4L}
\end{align*}
\]

If we have \( q_{0i} \) at equilibrium then the details for the determining how many firms will enter in a regime where the social planner chooses the socially optimum number of firms are given by

**PROOF.**

\[
\begin{align*}
TW &= bnq_{0i}^2 \left(1 + \frac{n}{2}\right) - \frac{\tau}{4L} - FnL - GL \\
TW &= bn \left(\frac{(a-c)^2}{b^2(n+1)^2}\right) \left(1 + \frac{n}{2}\right) - \frac{\tau}{4L} - FnL - GL \\
TW &= n \left(\frac{(a-c)^2}{b(n+1)^2}\right) \left(1 + \frac{n}{2}\right) - \frac{\tau}{4L} - FnL - GL \\
TW &= \frac{(a-c)^2}{b} n \left(\frac{1}{(n+1)^2}\right) \left(1 + \frac{n}{2}\right) - \frac{\tau}{4L} - FnL - GL \\
\frac{\partial TW}{\partial n} &= 0 \\
\frac{\partial TW}{\partial n} &= \frac{(a-c)^2}{b} \left(\frac{1}{(n+1)^2}\right) \left(1 + \frac{n}{2}\right) - \frac{(a-c)^2}{b} 2n \left(\frac{1}{(n+1)^3}\right) \left(1 + \frac{n}{2}\right) \\
&\quad + \frac{(a-c)^2}{b} n \left(\frac{1}{(n+1)^2}\right) \left(\frac{1}{2}\right) - FL = 0
\end{align*}
\]
\[
\begin{align*}
&= (n + 1)\left(1 + \frac{n}{2}\right) - 2n(1)\left(1 + \frac{n}{2}\right) + n(n + 1)\left(\frac{1}{2}\right) = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= \left(1 + \frac{n}{2}\right)(n + 1 - 2n) + n(n + 1)\left(\frac{1}{2}\right) = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= \left(1 + \frac{n}{2}\right)(1 - n) + n(n + 1)\left(\frac{1}{2}\right) = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= \left(1 + \frac{n}{2}\right)(1 - n) + \frac{n}{2}(n + 1) = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= (1)(1 - n) + \left(\frac{n}{2}\right)(1 - n) + \frac{n}{2}(n + 1) = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= (1 - n) + \left(\frac{n}{2}\right)(1 - n + n + 1) = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= (1 - n) + \left(\frac{n}{2}\right)(1 + 1) = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= (1 - n) + \left(\frac{n}{2}\right)(2) = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= (1 - n) + n = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= 1 = \frac{FLb(n + 1)^3}{(a - c)^2} \\
&= (n + 1)^3 = \frac{(a - c)^2}{FLb} \\
&= n + 1 = \sqrt[3]{\frac{(a - c)^2}{FLb}} \\
&= n^{WM} = \sqrt[3]{\frac{(a - c)^2}{FLb}} - 1
\end{align*}
\]
If we have an explicit functional form of $q_0i$ at equilibrium then the proof for the determining how many firms will enter in a regime where the social planner must allow for free entry is given by

**PROOF.**

\[
\frac{q_0^2 b}{L} - F = 0
\]

\[
\frac{q_0^2 b}{L} = F
\]

\[
q_0^2 = FL/b
\]

\[
\left(\frac{a - c}{b(n + 1)}\right)^2 = FL/b
\]

\[
\left(\frac{(a - c)^2}{b^2(n + 1)^2}\right) = FL/b
\]

\[
\frac{(a - c)^2}{(n + 1)^2} = FLb
\]

\[
\frac{(a - c)^2}{FLb} = (n + 1)^2
\]

\[
a - c = \sqrt{FLb}(n + 1)
\]

\[
n^{FE} = \frac{a - c}{\sqrt{FLb}} - 1
\]

**PROOF OF LEMMA 2.2**

\[
q_{0i} = \frac{a - c}{b(1 + n^{BWM})}
\]

\[
q_{0i} = \frac{a - c}{b \left(1 + \left(\frac{3(a - c)^2}{FLb} - 1\right)\right)}
\]
\[ q_{0i} = \frac{a - c}{b \sqrt{\frac{(a - c)^2}{F L b}}} \]

\[ q_{0i} = \frac{(a - c)(FLb)^{1/3}}{b(a - c)^{2/3}} \]

\[ q_{0i} = \frac{(a - c)^{1/3}(FLb)^{1/3}}{b} \]

\[ q_{0i} = \frac{(a - c)^{1/3}(FLb)^{1/3}}{b} \]

\[ q_{0i} = (a - c)^{2/3}(FL)^{1/3}(b)^{1/3} \]

\[ q_{0i} = (a - c)^{2/3}(FL)^{1/3}(b)^{-2/3} \]

\[ q_{0i} = (a - c)^{1/3}(F)^{1/3}(b)^{-2/3}(L)^{1/3} \]

\[ \frac{dq_{0i}}{dL} = \left( \frac{1}{3} \right) (a - c)^{1/3}(F)^{1/3}(b)^{-2/3}(L)^{-2/3} \]

\[ \frac{dq_{0i}}{dL} = \left( \frac{1}{3} \right) (q_{0i})(L)^{-1} \]

\[ \frac{dq_{0i}}{dL} > 0 \]

**PROOF OF LEMMA 2.4**

\[ PS = bn^{WM} q_{0i}^2 \]

\[ \frac{\partial PS}{\partial L} = b \frac{dn^{WM}}{dL} q_{0i}^2 + 2bn^{WM} q_{0i} \frac{dq_{0i}}{dL} \]

\[ \frac{\partial PS}{\partial L} = b \left( \left( -\frac{1}{3} \right) (n^{WM} + 1)(L)^{-1} \right) q_{0i}^2 + 2bn^{WM} q_{0i} \left( \frac{1}{3} \right) (q_{0i})(L)^{-1} \]

\[ \frac{\partial PS}{\partial L} = b \left( \left( -\frac{1}{3} \right) (n^{WM} + 1)(L)^{-1} \right) q_{0i}^2 + 2bn^{WM} q_{0i} \left( \frac{1}{3} \right) (q_{0i})(L)^{-1} \]

\[ \frac{\partial PS}{\partial L} = -\frac{b(n^{WM} + 1)q_{0i}^2}{3L} + \frac{2bn^{WM} q_{0i}^2}{3L} \]

\[ \frac{\partial PS}{\partial L} = -\frac{bn^{WM} q_{0i}^2}{3L} - \frac{bq_{0i}^2}{3L} + \frac{2bn^{WM} q_{0i}^2}{3L} \]
\[
\frac{\partial PS}{\partial L} = b n^M q_0^2 - bq_0^2 \frac{3L}{3L} \\
\frac{\partial PS}{\partial L} = 0 \text{ when } n \text{ is } 1, \text{ and is greater than } 0 \text{ when } n > 1
\]

PROOF OF LEMMA 2.8

Fixed Costs = \( F n^W L + GL \)

\[
\frac{\partial FC}{\partial L} = F n^W + FL \frac{dn^W}{dL} + G
\]

\[
\frac{dn}{dL} = \left( -\frac{1}{3} \right) (n^W + 1)(L)^{-1}
\]

\[
\frac{\partial FC}{\partial L} = F n^W + FL \left( \left( -\frac{1}{3} \right) (n^W + 1)(L)^{-1} \right) + G
\]

\[
\frac{\partial FC}{\partial L} = F n^W + \left( \left( \frac{-FL(n^W + 1)}{3L} \right) \right) + G
\]

\[
\frac{\partial FC}{\partial L} = F n - \left( \frac{F(n^W + 1)}{3} \right) + G
\]

\[
\frac{\partial FC}{\partial L} = \frac{3F n^W}{3} - \left( \frac{F(n^W + F)}{3} \right) + G
\]

\[
\frac{\partial FC}{\partial L} = \frac{2F n^W - F}{3} + G
\]

\[
\frac{\partial FC}{\partial L} > 0
\]

PROOF OF PROPOSITION 1

\[
TW = b n^W q_0^2 \ast \left( 1 + \frac{n^W}{2} \right) - \frac{\tau}{4L} - F n^W L
\]

\[
\frac{\partial TW}{\partial L} = \frac{\partial PS}{\partial L} + \frac{\partial CS}{\partial L} - \frac{\partial TRC}{\partial L} - \frac{\partial FC}{\partial L} = 0
\]
\[
\frac{\partial TW}{\partial L} = -bq_0^2 + \frac{\tau}{3L} + \frac{2Fn_{WM}}{3L^2} + \frac{F}{3} - G = 0
\]

\[
\frac{\partial TW}{\partial L} = -\frac{F(n_{WM} + 1)}{3} + \frac{\tau}{3L} - \frac{2Fn_{WM}}{3L^2} + \frac{F}{3} - G = 0
\]

\[
\frac{\partial TW}{\partial L} = -\frac{Fn_{WM}}{3} - \frac{F}{3} + \frac{\tau}{3L^2} - \frac{2Fn_{WM}}{3} + \frac{F}{3} - G = 0
\]

\[
\frac{\partial TW}{\partial L} = \frac{\tau}{4L^2} = Fn_{WM} + G
\]

\[
\tau = 4F L^2 n_{WM} + 4G L^2
\]

\[
\tau = (4F n_{WM} + 4G) L^2
\]

\[
\frac{\tau}{4F n_{WM} + 4G} = L^2
\]

\[
\sqrt{\frac{\tau}{4F n_{WM} + 4G}} = L
\]

\[
\frac{\partial^2 TW}{\partial L^2} = -\frac{\tau}{4L^3} - F \frac{\partial n_{WM}}{\partial L} < 0
\]

\[
\frac{\partial^2 TW}{\partial L^2} = -\frac{\tau}{4L^3} - F \left( -\frac{1}{3} \right) (n_{WM} + 1) (L)^{-1} < 0
\]

\[
\frac{\partial^2 TW}{\partial L^2} = -\frac{\tau}{4L^3} + F \left( \frac{1}{3} \right) (n_{WM} + 1) (L)^{-1} < 0
\]

\[
\frac{\partial^2 TW}{\partial L^2} = \frac{\tau}{4L^3} > F \left( \frac{1}{3} \right) (n_{WM} + 1) (L)^{-1}
\]

\[
\frac{\partial^2 TW}{\partial L^2} = \tau > \left( \frac{4}{3} \right) F (n_{WM} + 1) (L)^2
\]

\[
\frac{\tau}{4L^2} = Fn_{WM} - G = 0
\]

\[
\frac{\tau}{4L^2} > \frac{F n_{WM}}{2}
\]

\[
\tau > 4F n_{WM} L^2
\]
We can substitute $4F n^{WM} L^2$ for $\tau$ because the former term is strictly greater than the second term

$$4F n^{WM} L^2 > \left(\frac{4}{3}\right) F((n^{WM} + 1)(L)^2)$$

$$4n^{WM} > \left(\frac{4}{3}\right)((n + 1))$$

$$4n^{WM} > \left(\frac{4}{3}\right)((n^{WM} + 1))$$

$$n^{WM} > \left(\frac{1}{3}\right)((n^{WM} + 1))$$

LEMMA 3.1. *As the number of locations in a free entry regime increases, the number of firms per location decreases.*

PROOF.

\[
n^{FE} = (a - c)(Fb)^{-1/2}(L)^{-1/2} - 1
\]

\[
\frac{dn^{FE}}{dL} = \left(-\frac{1}{2}\right)(a - c)(Fb)^{-1/2}(L)^{-3/2}
\]

\[
\frac{dn^{FE}}{dL} = \left(-\frac{1}{2}\right)(n^{FE} + 1)(L)^{-1}
\]

LEMMA 3.2. *As the number of locations in a free entry regime increases, the quantity that each firm produces increases.*

PROOF.

\[
q_{0E} = \frac{a - c}{b(n^{FE} + 1)}
\]

\[
q_{0E} = \frac{a - c}{b \left(1 + \frac{a - c}{\sqrt{F L b}} - 1\right)}
\]
\[ q_0i = \frac{a - c}{b} \left( \frac{a - c}{\sqrt{FLb}} \right) \]

\[ q_0i = \frac{(a - c)(FLb)^{1/2}}{b(a - c)} \]

\[ q_0i = \frac{(FLb)^{1/2}}{b} \]

\[ q_0i = (F)^{1/2} (b)^{-1/2} (L)^{1/2} \]

\[ \frac{dq_0i}{dL} = \left( \frac{1}{2} \right) (F)^{1/2} b^{-1/2} (L)^{-1/2} \]

\[ \frac{dq_0i}{dL} = \left( \frac{1}{2} \right) (q_0i)(L)^{-1} \]

**Lemma 3.3.** As the number of locations in a free entry regime increases, the total quantity per location decreases.

**Proof.**

\[ Q_0 = n^{FE} \left( \frac{a - c}{b(n^{FE} + 1)} \right) \]

\[ Q_0 = \left( \frac{a - c}{b} \right) \left( \frac{n^{FE}}{n^{FE} + 1} \right) \]

\[ Q_0 = \left( \frac{a - c}{b} \right) (n^{FE})(n^{FE} + 1)^{-1} \]

\[ \frac{dQ_0}{dL} = -\left( \frac{a - c}{b(n^{FE} + 1)^{-2}} \right) \frac{dn^{FE}}{dL} + (n^{FE} + 1)^{-1} \left( \frac{a - c}{b} \right) \frac{dn^{FE}}{dL} \]

\[ \frac{dQ_0}{dL} = \left( (n^{FE} + 1)^{-1} - (n^{FE})(n^{FE} + 1)^{-2} \right) \frac{dn^{FE}}{dL} \left( \frac{a - c}{b} \right) \]

\[ \frac{dQ_0}{dL} = \left( \frac{1}{n^{FE} + 1} - \frac{n^{FE}}{(n + 1)^2} \right) \frac{dn^{FE}}{dL} \left( \frac{a - c}{b} \right) \]

\[ \frac{dQ_0}{dL} = \left( \frac{n + 1}{(n + 1)^2} - \frac{n^{FE}}{(n^{FE} + 1)^2} \right) \frac{dn^{FE}}{dL} \left( \frac{a - c}{b} \right) \]
\[
\frac{dQ_0}{dL} = \left(\frac{1}{(n^{FE} + 1)^2}\right) \frac{dn^{FE}}{dL}\left(\frac{a - c}{b}\right)
\]

\[
\frac{dQ_0}{dL} = (n + 1)^{-2} \frac{dn^{FE}}{dL}\left(\frac{a - c}{b}\right)
\]

\[
\frac{dQ_0}{dL} = (n^{FE} + 1)^{-2}\left(\frac{-1}{2}(n^{FE} + 1)(L)^{-1}\right)\left(\frac{a - c}{b}\right)
\]

\[
\frac{dQ_0}{dL} = -\frac{1}{2}(n^{FE} + 1)^{-1}(L)^{-1}(q_{0i}(n^{FE} + 1))
\]

\[
\frac{dQ_0}{dL} = -\frac{1}{2}(L)^{-1}(q_{0i})
\]

\[
\frac{dQ_0}{dL} < 0
\]

**Lemma 3.4.** As the number in a free entry regime increases, the producer surplus increases as long as there are multiple firms per location. When there is a monopolist at each location, then the producer surplus does not change when locations are added.

**Proof.**

\[PS = bn^{FE}q_{0i}^2\]

\[
\frac{\partial PS}{\partial L} = b \frac{dn^{FE}}{dL}q_{0i}^2 + 2bn^{FE}q_{0i} \frac{dq_{0i}}{dL}
\]

\[
\frac{\partial PS}{\partial L} = b \left(\frac{-1}{2}(n^{FE} + 1)(L)^{-1}\right)q_{0i}^2 + 2bn^{FE}q_{0i}\left(\frac{1}{2}(q_{0i})(L)^{-1}\right)
\]

\[
\frac{\partial PS}{\partial L} = \left(\frac{-1}{2}(n^{FE} + 1)(L)^{-1}\right)q_{0i}^2 + 2bn^{FE}q_{0i}\left(\frac{1}{2}(q_{0i})(L)^{-1}\right)
\]

\[
\frac{\partial PS}{\partial L} = -\frac{1}{2}b(n^{FE} + 1)(q_{0i})^2(L)^{-1} + bn^{FE}q_{0i}^2L^{-1}
\]

\[
\frac{\partial PS}{\partial L} = -\frac{b(n^{FE} + 1)q_{0i}^2}{2L} + \frac{bn^{FE}q_{0i}^2}{L}
\]
∂PS
∂L = 0 when n^{FE} is 1, and is greater than 0 when n > 1

LEMA 3.5. As the number of locations in a free entry regime increases, the consumer surplus net of transport costs decreases.

PROOF.

CS = \frac{1}{2} bn^{FE} q_0^2

CS = \frac{1}{2} b(Q_0)^2

\frac{∂CS}{∂L} = b(Q_0) \frac{dnq_0}{dL}

\frac{∂CS}{∂L} = b(Q_0) \left(- \frac{1}{2} (L)^{-1} (q_0) \right)

\frac{∂CS}{∂L} = - \frac{1}{2} b n^{FE} (q_0)^2 (L)^{-1}

\frac{∂CS}{∂L} < 0

LEMA 3.6. As the number of locations in a free entry regime increases, the producer surplus added to the consumer surplus net of transport costs decreases.

PROOF.

\frac{∂PS}{∂L} + \frac{∂CS}{∂L} = \frac{bn^{FE} q_0^2 - b q_0^2 - bn^{FE} (q_0)^2}{2L}

\frac{∂PS}{∂L} + \frac{∂CS}{∂L} = - \frac{b q_0^2}{2L}

\frac{∂PS}{∂L} + \frac{∂CS}{∂L} < 0
LEMMA 3.7. As the number of locations in a free entry regime increases, the transport costs decrease.

PROOF.

Transport Costs = $\frac{\tau}{4L}$

\[
\frac{\partial TRC}{\partial L} = \left(\frac{\tau}{4}\right)(-L)^{-2}
\]
\[
\frac{\partial TRC}{\partial L} = -\frac{\tau}{4L^2}
\]
\[
\frac{\partial TRC}{\partial L} < 0
\]

LEMMA 3.8. As the number of locations in a free entry regime increases, the fixed costs increase.

PROOF.

Transport Cost = $F_{n FE} L + GL$

\[
\frac{\partial FC}{\partial L} = F_{n FE} + FL \frac{dn_{FE}}{dL} + GL
\]
\[
\frac{\partial FC}{\partial L} = F_{n FE} + FL \left(\left(-\frac{1}{2}\right)(n_{FE} + 1)(L)^{-1}\right) + G
\]
\[
\frac{\partial FC}{\partial L} = F_{n FE} + \left(\left(-\frac{FL(n_{FE} + 1)}{2L}\right)\right) + G
\]
\[
\frac{\partial FC}{\partial L} = F_{n FE} - \left(\frac{F(n_{FE} + 1)}{2}\right) + G
\]
\[
\frac{\partial FC}{\partial L} = \frac{2F_{n FE}}{2} - \left(\frac{(F_{n FE} + F)}{2}\right) + G_v g_g
\]
\[
\frac{\partial FC}{\partial L} = \frac{F_{n FE} - F}{2} + G
\]
\[
\frac{\partial FC}{\partial L} = \frac{F_{n FE}}{2} - \frac{F}{2} + G
\]
\[ \frac{\partial FC}{\partial L} = F n^{FE} - \frac{F (n^{FE} + 1)}{2} + G \]

\[ \frac{\partial FC}{\partial L} > 0 \]

**PROOF OF PROPOSITION 2**

\[ TW = b n^{FE} q_0^2 \left( 1 + \frac{n^{FE}}{2} \right) - \frac{\tau}{4L} - FnL \]

\[ \frac{\partial TW}{\partial L} = \frac{\partial PS}{\partial L} + \frac{\partial CS}{\partial L} - \frac{\partial TRC}{\partial L} - \frac{\partial FC}{\partial L} = 0 \]

\[ \frac{\partial TW}{\partial L} = -\frac{b q_0^2}{2L} + \frac{\tau}{4L^2} - \frac{F n^{FE}}{2} + \frac{F}{2} - G = 0 \]

\[ \frac{\partial TW}{\partial L} = -\frac{F}{2} + \frac{\tau}{4L^2} - \frac{F n^{FE}}{2} + \frac{F}{2} - G = 0 \]

\[ \frac{\partial TW}{\partial L} = \frac{\tau}{4L^2} - \frac{F n^{FE}}{2} - G = 0 \]

\[ \frac{\tau}{4L^2} = \frac{F n^{FE}}{2} + G \]

\[ \tau = 2FN^{FE} + 4G L^2 \]

\[ \tau = (2 FN^{FE} + 4G) L^2 \]

\[ \frac{\tau}{2FN^{FE} + 4G} = L^2 \]

\[ \sqrt{\frac{\tau}{2FN^{FE} + 4G}} = L^* \]

\[ \frac{\tau}{4L^2} = \frac{F n^{FE}}{2} - G = 0 \]

\[ = \frac{\tau}{4L^2} > \frac{F n^{FE}}{2} \]

\[ = \tau > 2FN^{FE} L^2 \]

We can substitute \(2FN^{FE} L^2\) for \(\tau\) because the former term is strictly greater than the second term
\[2Fn^{FE}L^2 > F((n^{FE} + 1)(L)^2)\]
\[2n^{FE} \geq (n^{FE} + 1)\]

**PROOF OF PROPOSITION 3**

\[n^{FE} = \frac{a - c}{\sqrt{FLb}} - 1\]
\[n^{WM} = \frac{3((a - c)^2)}{\sqrt{FLb}} - 1\]
\[\frac{a - c}{\sqrt{FLb}} - 1 > \frac{3((a - c)^2)}{\sqrt{FLb}} - 1\]
\[\frac{a - c}{\sqrt{FLb}} > \frac{3((a - c)^2)}{\sqrt{FLb}}\]
\[\frac{a - c}{(FLb)^{\frac{1}{2}}} > (FLb)^{\frac{1}{3}}\]
\[(a - c)^{\frac{1}{3}} > (FLb)^{\frac{1}{3}}\]
\[(a - c)^2 > FLb\]
\[\frac{(a - c)^2}{Lb} > F\]

Profit per location = \(q_0^2bn/L\)

\[\pi^* = \frac{bn(a - c)^2}{Lb^2(n + 1)^2}\]
\[\pi^* = \frac{n(a - c)^2}{Lb(n + 1)^2}\]
\[\pi^* = \frac{(a - c)^2}{Lb} \ast \frac{n}{(n + 1)^2}\]

If there were a monopolist or n=1
\[ \pi_{\text{Mon}} = \frac{(a - c)^2}{Lb} \times \frac{1}{4} \]

\[ 4\pi_{\text{Mon}} > F \]

\[ \pi_{\text{Mon}} > \frac{F}{4} \]

The Details for solving for the optimal quantity when a firm takes into account the additional profit a firm would receive from new customers

\[ TW = (a - bnq_{0i}^* - c)(nq_{0i}^*) + \left(\frac{1}{2}\right) bn^2 q_{0i}^* - \frac{\tau}{4L} - FnL \]

\[ TW = (a - bQ_0^* - c)(Q_0^*) + \left(\frac{1}{2}\right) bQ_0^* - \frac{\tau}{4L} - FnL \]

\[ TW = (a - bQ_0^* - c)(Q_0^*) + \left(\frac{1}{2}\right) bQ_0^* - \frac{\tau}{4L} - FnL \]

\[ \frac{\partial TW}{\partial L} = (a - bQ_0^* - c) \frac{\partial Q_0^*}{\partial L} + (Q_0^*) \left( -b \frac{\partial Q_0^*}{\partial L} \right) + \left(\frac{1}{2}\right) bQ_0^* \frac{\partial Q_0^*}{\partial L} + \frac{\tau}{4L^2} - Fn = 0 \]

Substitute for \( \frac{\partial Q_0^*}{\partial L} \) and solve for \( Q_0^* \)

PROOF for LEMMA 6

In this regime the regulator has the option to choose the optimal number of firms per location and the optimal number of locations. We can use the implicit function theorem in order to derive the relevant comparative statics expressions. We must manipulate the functions in order to find a useful comparative static expression. For this exercise, \( Q_0 \) is the choice variable that we are looking to maximize, and \( n \) and \( L \) are the state variables of interest
\[
\frac{\partial \pi}{\partial q_0} = -\frac{bq_0}{L} + \frac{(a - bQ_0 - c)}{L} + (a - bQ_0 - c)q_0 \left(\frac{1}{\tau} b\eta q_0\right) = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = \Pi = -\frac{bQ_0}{L} + \frac{(a - bQ_0 - c)n}{L} + (a - bQ_0 - c)Q_0^2 \left(\frac{b}{\tau}\right) = 0
\]

In order to invoke the IFT we must show that the previous function does not equal 0 when we take its derivative with respect to the choice variable.

\[
\Pi_{Q_0} = -\frac{b}{L} - \frac{bn}{L} + \left(\frac{b}{\tau}\right) (2Q_0(a - bQ_0 - c) - b(Q_0^2))
\]

\[
\Pi_{Q_0} = -\frac{b}{L} - \frac{bn}{L} + \left(\frac{b}{\tau}\right) (2Q_0(a - c) - 2bQ_0^2 - b(Q_0^2))
\]

\[
\Pi_{Q_0} = -\frac{b}{L} - \frac{bn}{L} + \left(\frac{b}{\tau}\right) (2Q_0(a - c) - 3bQ_0^2)
\]

\[
\Pi_{Q_0} = -\frac{b}{L}(n + 1) + \left(\frac{b}{\tau}\right) (2Q_0(a - c) - 3bQ_0^2)
\]

\[
\Pi_{Q_0} = -\frac{b}{L}(n + 1) + \left(\frac{2bQ_0}{\tau}\right) ((a - bQ_0 - c) - Q_0)
\]

\[
(a - bQ_0 - c) \left(\frac{n}{L} + \frac{bQ_0^2}{\tau}\right) = bQ_0
\]

\[
(a - bQ_0 - c) \left(\frac{\tau n}{L} + \frac{bLQ_0^2}{\tau}\right) = bQ_0
\]

\[
(a - bQ_0 - c) = \frac{b\tau Q_0}{\tau n + LbQ_0^2}
\]

\[
\Pi_{Q_0} = -\frac{b}{L}(n + 1) + \left(\frac{2bQ_0}{\tau}\right) \left(\frac{b\tau Q_0}{\tau n + LbQ_0^2} - Q_0\right)
\]

\[
\Pi_{Q_0} = -\frac{b}{L}(n + 1) + (2b^2Q_0^2) \left(\frac{\tau}{\tau n + LbQ_0^2} - 1\right)
\]

\[
\Pi_{Q_0} = -\frac{b}{L}(n + 1) + (2b^2Q_0^2) \left(\frac{\tau}{\tau n + LbQ_0^2} - \frac{\tau n + LbQ_0^2}{\tau n + LbQ_0^2}\right)
\]
\[ \Pi_{Q_0} = -\frac{b}{L}(n + 1) - (2b^2Q_0^2) \left( \frac{\tau n + LbQ_0^2 - \tau}{\tau n + LbQ_0^2} \right) \]

\[ \Pi_{Q_0} < 0 \]

Now we can use the IFT to show that there exists a unique \( C^1 \) implicit function

\[ Q_0^* = \xi(n, L) \]

such that \( \Pi = (\xi(n, L), n, L) \equiv 0 \) for some neighborhood of \( n \) and \( L \)

\[ \Pi = -\frac{b \xi(n, L)}{L} + \frac{(a - b \xi(n, L) - c)n}{L} + (a - b \xi(n, L) - c)\xi(n, L)^2 \left( \frac{b}{\tau} \right) \equiv 0 \]

\[ \Pi_n = -\frac{b}{L} \frac{\partial Q_0}{\partial n} + \frac{(a - c)}{L} - \frac{nb}{L} \frac{\partial Q_0}{\partial n} - \frac{bQ_0}{L} - \left( \frac{b}{\tau} \right) Q_0^2 \left( b \frac{\partial Q_0}{\partial n} \right) + (a - bQ_0 - c) \left( \frac{2b}{\tau} Q_0 \frac{\partial Q_0}{\partial n} \right) = 0 \]

\[ = \frac{(a - c)}{L} - \frac{bQ_0}{L} \]

\[ = \frac{b}{L} \frac{\partial Q_0}{\partial n} + \frac{nb}{L} \frac{\partial Q_0}{\partial n} + \left( \frac{b}{\tau} \right) Q_0^2 \left( b \frac{\partial Q_0}{\partial n} \right) - (a - bQ_0 - c) \left( \frac{2b}{\tau} Q_0 \frac{\partial Q_0}{\partial n} \right) \]

\[ = \frac{(a - bQ_0 - c)}{L} = \left( \frac{\partial Q_0}{\partial n} \right) \left( \frac{b}{L}(1 + n) + \left( \frac{b}{\tau} \right) Q_0^2 \right) \]

\[ = \frac{(a - bQ_0 - c)}{L} = \left( \frac{\partial Q_0}{\partial n} \right) \left( \frac{b}{\tau n} (1 + n) + \left( \frac{Lb^2}{\tau L} \right) Q_0^2 \right) - (a - bQ_0 - c) \left( \frac{2Lb}{\tau L} Q_0 \right) \]

\[ = \frac{(a - bQ_0 - c)}{L} = \left( \frac{\partial Q_0}{\partial n} \right) \left( \frac{b}{\tau L} (1 + n) + (Lb)Q_0^2 \right) - (a - bQ_0 - c)(2LQ_0) \]

\[ = \frac{(a - bQ_0 - c)}{\tau(1 + n) + (Lb)Q_0^2} = \left( \frac{\partial Q_0}{\partial n} \right) \left( \frac{b}{\tau L} \right) \]
\[\frac{\partial Q_0}{\partial n} = \frac{(a - bQ_0 - c)\tau L}{b(\tau(1 + n) + (Lb)Q_0^2 - (a - bQ_0 - c)(2LQ_0))} = \frac{\partial Q_0}{\partial n}\]

\[\frac{\partial Q_0}{\partial n} = \frac{(a - bQ_0 - c)\tau L}{b(\tau(1 + n) + (Lb)Q_0^2 - \left(\frac{b\tau Q_0}{\tau n + LbQ_0^2}\right)(2LQ_0))} = \frac{\partial Q_0}{\partial n}\]

\[\frac{\partial Q_0}{\partial n} = \frac{(a - bQ_0 - c)\tau L}{b\left(\tau(1 + n) + (Lb)Q_0^2 - \left(\frac{2Lb\tau Q_0^2}{\tau n + LbQ_0^2}\right)\right)} = \frac{\partial Q_0}{\partial n}\]

\[\frac{\partial Q_0}{\partial n} = \frac{(a - bQ_0 - c)\tau L}{b\left(\frac{(\tau + \tau n) + (Lb)Q_0^2}{\tau n + LbQ_0^2} - \left(\frac{2Lb\tau Q_0^2}{\tau n + LbQ_0^2}\right)\right)} = \frac{\partial Q_0}{\partial n}\]

\[\frac{\partial Q_0}{\partial n} > 0\]

\[\Pi_L = bQ_0L^{-2} - bL^{-1} \frac{\partial Q_0}{\partial L} - n(a - bQ_0 - c)L^{-2} - nbL^{-1} \frac{\partial Q_0}{\partial L}\]

\[+ (a - bQ_0 - c)2Q_0 \left(\frac{b}{\tau}\right) \frac{\partial Q_0}{\partial L} - Q_0^2 \left(\frac{b}{\tau}\right) \left(b \frac{\partial Q_0}{\partial L}\right) = 0\]

\[= bQ_0L^{-2} - n(a - bQ_0 - c)L^{-2}\]

\[= bL^{-1} \frac{\partial Q_0}{\partial L} + nbL^{-1} \frac{\partial Q_0}{\partial L} - (a - bQ_0 - c)2Q_0 \left(\frac{b}{\tau}\right) \frac{\partial Q_0}{\partial L}\]

\[+ Q_0^2 \left(\frac{b}{\tau}\right) \left(b \frac{\partial Q_0}{\partial L}\right)\]
\[= bQ_0L^{-2} - n(a - bQ_0 - c)L^{-2}\]

\[= bL^{-1} \frac{\partial Q_0}{\partial L} + nbL^{-1} \frac{\partial Q_0}{\partial L} - 2(a - bQ_0 - c)Q_0 \left( \frac{b}{\tau} \right) \frac{\partial Q_0}{\partial L}\]

\[+ Q_0^2 \left( \frac{b^2}{\tau} \right) \left( \frac{\partial Q_0}{\partial L} \right)\]

\[= bQ_0L^{-2} - n(a - bQ_0 - c)L^{-2}\]

\[= \left( bL^{-1} + nbL^{-1} - 2(a - bQ_0 - c)Q_0 \left( \frac{b}{\tau} \right) + Q_0^2 \left( \frac{b^2}{\tau} \right) \right) \left( \frac{\partial Q_0}{\partial L} \right)\]

\[= bQ_0L^{-2} - n(a - bQ_0 - c)L^{-2}\]

\[= \left( bL^{-1} + nbL^{-1} - 2 \left( \frac{b^2Q_0^2}{\tau n + LbQ_0^2} \right) \right) \frac{\partial Q_0}{\partial L}\]

\[+ \left( \frac{Q_0^2 b^2}{\tau} \right) \left( \frac{\tau n + LbQ_0^2}{\tau n + LbQ_0^2} \right) \left( \frac{\partial Q_0}{\partial L} \right)\]

\[= bQ_0L^{-2} - n(a - bQ_0 - c)L^{-2}\]

\[= \left( bL^{-1} + nbL^{-1} - 2 \left( \frac{b^2Q_0^2}{\tau n + LbQ_0^2} \right) \right) \frac{\partial Q_0}{\partial L}\]

\[+ \left( \frac{Q_0^2 b^2 \tau n + Q_0^2 b^2 LbQ_0^2}{\tau (\tau n + LbQ_0^2)} \right) \left( \frac{\partial Q_0}{\partial L} \right)\]

\[= bQ_0L^{-2} - n(a - bQ_0 - c)L^{-2}\]

\[= \left( bL^{-1} + nbL^{-1} - 2 \left( \frac{b^2Q_0^2}{\tau n + LbQ_0^2} \right) + \frac{Q_0^2 b^2 \tau n}{\tau (\tau n + LbQ_0^2)} \right) \left( \frac{\partial Q_0}{\partial L} \right)\]

\[+ \frac{Q_0^2 b^2 LbQ_0^2}{\tau (\tau n + LbQ_0^2)} \left( \frac{\partial Q_0}{\partial L} \right)\]
\[ = bQ_0L^{-2} - n(a - bQ_0 - c)L^{-2} \]
\[ = \left( bL^{-1}(1 + n) + \frac{b^2Q_0^2}{\tau n + LbQ_0^2} (n - 2) + \frac{Q_0^2b^2LbQ_0^2}{\tau(\tau n + LbQ_0^2)} \right) \left( \frac{\partial Q_0}{\partial L} \right) \]
\[ = bQ_0L^{-2} - n(a - bQ_0 - c)L^{-2} \]
\[ = \left( bL^{-1}(1 + n) + \frac{b^2Q_0^2}{\tau n + LbQ_0^2} (n - 2) + \frac{Q_0^2b^2LbQ_0^2}{\tau(\tau n + LbQ_0^2)} \right) \left( \frac{\partial Q_0}{\partial L} \right) \]
\[ = bQ_0L^{-2} - n(a - c)L^{-2} + nbQ_0L^{-2} \]
\[ = \left( bL^{-1}(1 + n) + \frac{b^2Q_0^2}{\tau n + LbQ_0^2} (n - 2) + \frac{Q_0^2b^2LbQ_0^2}{\tau(\tau n + LbQ_0^2)} \right) \left( \frac{\partial Q_0}{\partial L} \right) \]
\[ = bQ_0L^{-2}(1 + n) - n \left( \frac{b\tau Q_0}{\tau n + LbQ_0^2} + bQ_0 \right) L^{-2} \]
\[ = \left( bL^{-1}(1 + n) + \frac{b^2Q_0^2}{\tau n + LbQ_0^2} (n - 2) + \frac{Q_0^2b^2LbQ_0^2}{\tau(\tau n + LbQ_0^2)} \right) \left( \frac{\partial Q_0}{\partial L} \right) \]
\[ = bQ_0L^{-2} - bQ_0L^{-2}n - n \left( \frac{b\tau Q_0}{\tau n + LbQ_0^2} + bQ_0 \right) L^{-2} \]
\[ = \left( bL^{-1}(1 + n) + \frac{b^2Q_0^2}{\tau n + LbQ_0^2} (n - 2) + \frac{Q_0^2b^2LbQ_0^2}{\tau(\tau n + LbQ_0^2)} \right) \left( \frac{\partial Q_0}{\partial L} \right) \]
\[ = bQ_0L^{-2} + bQ_0L^{-2}n - \frac{bn\tau Q_0L^{-2}}{\tau n + LbQ_0^2} - bnQ_0L^{-2} \]
\[ = \left( bL^{-1}(1 + n) + \frac{b^2Q_0^2}{\tau n + LbQ_0^2} (n - 2) + \frac{Q_0^2b^2LbQ_0^2}{\tau(\tau n + LbQ_0^2)} \right) \left( \frac{\partial Q_0}{\partial L} \right) \]
\[
\begin{align*}
\pi_{0i} &= (p_0 - c) q_{0i} 2\hat{\theta} - F = 0 \\
\pi_{0i} &= (a - bQ_0 - c) q_{0i} \left( \left( \frac{1}{2L} \right) (b(Q_0 * Q_0) - b(Q_1 * Q_1)) + \frac{1}{L} \right) - F = 0 \\
\pi_{0i} &= \left( a - b \sum_{i=1}^{n} q_{0i} - c \right) q_{0i} \left( \frac{b}{2\tau} ((Q_0^2) - (Q_1^2)) + \frac{1}{L} \right) - F = 0
\end{align*}
\]

The details for finding the equilibrium quantity and number of firms per location under a free-entry regime where a firm can account for the entire profit
Now, we invoke the conditions of symmetric equilibrium

\[ \pi_{0i} = (a - bnq_0 - c)q_0 \left( \frac{b}{2\pi} (nq_0 - nq_0) + \frac{1}{L} \right) - F = 0 \]

\[ \pi_{0i} = (a - bQ_0 - c)q_0 \left( \frac{1}{L} \right) - F = 0 \]

We can sum over all the firms at a location to find the free entry profit for all firms at a location.

\[ \sum_{i=1}^{n} \pi_{0i} = (a - bQ_0 - c)nq_0 \left( \frac{1}{L} \right) - nF = 0 \]

\[ \sum_{i=1}^{n} \pi_{0i} = (a - bQ_0 - c)Q_0 \left( \frac{1}{L} \right) - nF = 0 \]

\[ \sum_{i=1}^{n} \pi_{0i} = ((a - c)Q_0 - bQ_0^2) \left( \frac{1}{L} \right) - nF = 0 \]

\[ \sum_{i=1}^{n} \pi_{0i} = ((a - c)Q_0 - bQ_0^2) \left( \frac{1}{L} \right) = nF \]

\[ \sum_{i=1}^{n} \pi_{0i} = \frac{(a - c)Q_0 - bQ_0^2}{LF} = n^* \]

Now we take the partial derivative of the profit function with respect to \( q_0 \) to find the profit maximization condition for an individual firm

\[ \frac{\partial \pi}{\partial q_0} = -bq_0 2 \hat{\theta} + (p_0 - c) 2 \hat{\theta} + (p_0 - c)q_{0i} \frac{\partial 2 \hat{\theta}}{\partial q_0} = 0 \]

Now, we invoke the conditions of symmetric equilibrium
\[
\frac{\partial \pi}{\partial q_0} = -bq_02\hat{\theta} + (p_0 - c)2\hat{\theta} + (p_0 - c)q_0 \left(\frac{1}{\tau}\right) bnq_0 = 0
\]

\[
\frac{\partial \pi}{\partial q_0} = -\frac{bq_0}{L} + \frac{(a - bQ_0 - c)}{L} + (a - bQ_0 - c)q_0 \left(\frac{1}{\tau}\right) bnq_0 = 0
\]

Now we will sum the marginal profits over all the firms for a location

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bnq_0}{L} + \frac{(a - bQ_0 - c)n}{L} + (a - bQ_0 - c)nq_0 \left(\frac{1}{\tau}\right) bnq_0 = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bQ_0}{L} + \frac{(a - bQ_0 - c)n}{L} + (a - bQ_0 - c)Q_0 \left(\frac{1}{\tau}\right) bQ_0 = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bQ_0}{L} + \frac{(a - bQ_0 - c)n}{L} + (a - bQ_0 - c)Q_0^2 \left(\frac{b}{\tau}\right) = 0
\]

If we substitute \( n^* \) from the summed free-entry profits, we can find get the optimal level of profits for an entire location by solving for \( Q_0 \).

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bQ_0}{L} + \frac{(a - bQ_0 - c)}{L} \left(\frac{(a - c)Q_0 - bQ_0^2}{LF}\right) + (a - bQ_0 - c)Q_0^2 \left(\frac{b}{\tau}\right) = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bQ_0}{L} + \frac{(a - c - bQ_0)}{L} \left(\frac{(a - c) - bQ_0}{LF}\right) Q_0 + (a - bQ_0 - c)Q_0^2 \left(\frac{b}{\tau}\right) = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bQ_0}{L} + \frac{(p - c)}{L} \left(\frac{p - c}{LF}\right) Q_0 + (a - bQ_0 - c)Q_0^2 \left(\frac{b}{\tau}\right) = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bQ_0}{L} + \frac{(p - c)^2}{L^2 F} Q_0 + (a - bQ_0 - c)Q_0^2 \left(\frac{b}{\tau}\right) = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bQ_0}{L} + \frac{((a - c) - bQ_0)^2}{L^2 F} Q_0 + (a - c - bQ_0)Q_0^2 \left(\frac{b}{\tau}\right) = 0
\]
\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{bQ_0}{L} + \frac{(a-c)^2 - 2(a-c)bQ_0 + b^2Q_0^2}{L^2F}Q_0 + (a - c - bQ_0)Q_0 \left( \frac{b}{\tau} \right)
\]

\[
= 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{b}{L} + \frac{(a-c)^2 - 2(a-c)bQ_0 + b^2Q_0^2}{L^2F} + (a - c - bQ_0)Q_0 \left( \frac{b}{\tau} \right) = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{b}{L} + \frac{(a-c)^2 - 2(a-c)bQ_0 + b^2Q_0^2}{L^2F} + ((a-c)Q_0 - bQ_0^2) \left( \frac{b}{\tau} \right) = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{b}{L} + \frac{(a-c)^2 - 2(a-c)bQ_0 + b^2Q_0^2}{L^2F} + \left( \frac{(a-c)bQ_0}{\tau} - \frac{b^2Q_0^2}{\tau} \right) = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -\frac{b^2Q_0^2}{\tau} + \frac{b^2Q_0^2}{L^2F} + \frac{(a-c)bQ_0}{\tau} - \frac{2(a-c)bQ_0}{L^2F} - \frac{b}{L} + \frac{(a-c)^2}{L^2F} = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = -b^2Q_0^2L^2F + \tau b^2Q_0^2 + L^2F(a-c)bQ_0 - 2\tau(a-c)bQ_0 - brLF
\]

\[
+ \tau(a-c)^2 = 0
\]

\[
\sum_{i=1}^{n} \frac{\partial \pi}{\partial q_0} = (\tau b^2 - b^2L^2F)Q_0^2 + (L^2F(a-c)b - 2\tau(a-c)b)Q_0
\]

\[
+ (\tau(a-c)^2 - b\tauLF) = 0
\]

We utilize the quadratic formula to find the optimal level of quantity for a given location

\[
Q_0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

where \(a = (\tau b^2 - b^2L^2F)\),

\[
b = (L^2F(a-c)b - 2\tau(a-c)b), \quad c = (\tau(a-c)^2 - b\tauLF)
\]

\[
b^2 = ((L^2F(a-c)b - 2\tau(a-c)b))(L^2F(a-c)b - 2\tau(a-c)b))
\]
\[ b^2 = L^4 F^2 (a - c)^2 b^2 - 4 \tau L^2 F b^2 (a - c)^2 + 4 \tau^2 b^2 (a - c)^2 \]

\[ 4ac = 4(\tau b^2 - b^2 L^2 F)(\tau(a - c)^2 - b\tau LF) \]

\[ 4ac = 4(\tau b^2 \tau(a - c)^2 - b^2 L^2 F \tau(a - c)^2 - \tau b^2 b\tau LF + b^2 L^2 F b\tau LF) \]

\[ 4ac = 4(\tau^2 b^2 (a - c)^2 - \tau^2 L^2 F b^2 (a - c)^2 - \tau^2 b^3 LF + \tau b^3 L^3 F^2) \]

\[ 4ac = 4\tau^2 b^2 (a - c)^2 - 4\tau L^2 F b^2 (a - c)^2 - 4\tau^2 b^3 LF + 4\tau b^3 L^3 F^2 \]

\[ b^2 - 4ac = L^4 F^2 (a - c)^2 b^2 - 4\tau L^2 F b^2 (a - c)^2 + 4\tau^2 b^2 (a - c)^2 \]

\[ - 4\tau^2 b^2 (a - c)^2 + 4\tau L^2 F b^2 (a - c)^2 + 4\tau^2 b^3 LF - 4\tau b^3 L^3 F^2 \]

\[ b^2 - 4ac = L^4 F^2 (a - c)^2 b^2 + 4\tau^2 b^3 LF - 4\tau b^3 L^3 F^2 \]

\[ Q_0 = \frac{-L^2 F (a - c) b + 2\tau (a - c) b \pm \sqrt{L^4 F^2 (a - c)^2 b^2 + 4\tau^2 b^3 LF - 4\tau b^3 L^3 F^2}}{2(\tau b^2 - b^2 L^2 F)} \]

\[ Q_0^* = \frac{-L^2 F (a - c) b + 2\tau (a - c) b \pm \sqrt{L^4 F^2 (a - c)^2 b^2 + 4\tau^2 b^3 LF - 4\tau b^3 L^3 F^2}}{2(\tau b^2 - b^2 L^2 F)} \]

If we assume that \( L^2 F > \tau \), then the equilibrium quantity will be

\[ Q_0^* = \frac{-L^2 F (a - c) b + 2\tau (a - c) b - \sqrt{L^4 F^2 (a - c)^2 b^2 + 4\tau^2 b^3 LF - 4\tau b^3 L^3 F^2}}{2(b^2 (\tau - L^2 F))} \]

\[ Q_0^* = \frac{-L^2 F (a - c) b + 2\tau (a - c) b - b\sqrt{L^4 F^2 (a - c)^2 b^2 + 4\tau^2 b^3 LF - 4\tau b^3 L^3 F^2}}{2b^2 (\tau - L^2 F)} \]

\[ Q_0^* = \frac{(a - c) (-L^2 F + 2\tau)}{2b (\tau - L^2 F)} - \frac{\sqrt{L^4 F^2 (a - c)^2 + 4\tau^2 - 4\tau^2 b L^2 F}}{2b (\tau - L^2 F)} \]

\[ Q_0^* = \frac{(a - c) (L^2 F + 2\tau)}{2b (\tau - L^2 F)} - \frac{\sqrt{L^4 F^2 (a - c)^2 + 4\tau^2 - 4\tau^2 b L^2 F}}{2b (\tau - L^2 F)} \]

\[ Q_0^* = \frac{(a - c) (L^2 F + \tau + \tau)}{2b (\tau - L^2 F)} - \frac{\sqrt{L^4 F^2 (a - c)^2 + 4\tau^2 - 4\tau^2 b L^2 F}}{2b (\tau - L^2 F)} \]
\[ Q_0^* = \frac{(a-c)}{2b} \left( \frac{\tau}{(\tau - L^2F)} + 1 \right) - \frac{\sqrt{LF} \sqrt{L^3F(a-c)^2 + 4\tau^2 - 4\tau^2 bL^2F}}{2b(\tau - L^2F)} \]

\[ Q_0^* = \frac{(a-c)}{2b} \left( 1 - \frac{\tau}{L^2F - \tau} \right) + \frac{\sqrt{LF} \sqrt{L^3F(a-c)^2 + 4\tau^2 - 4\tau^2 bL^2F}}{2b(L^2F - \tau)} \]

\[ Q_0^* = \frac{(a-c)}{2b} \left( 1 - \frac{\tau}{L^2F - \tau} \right) + \frac{\sqrt{LF} \sqrt{L^3F(a-c)^2 + 4\tau^2 - 4\tau^2 bL^2F}}{2b(L^2F - \tau)} \]

\[ Q_0^* = \frac{(a-c)}{2b} \left( 1 - \frac{\tau}{L^2F - \tau} \right) + \frac{\sqrt{LF} \sqrt{L^3F(a-c)^2 + 4\tau^2 - 4\tau^2 bL^2F}}{2b(L^2F - \tau)} \]

**PROOF for LEMMA 7**

\[ \frac{\partial Q_0^*}{\partial L} = \frac{(a-c)\tau 2L}{2b} (L^2F - \tau)^{-2} \]

\[ + \left( 2b(L^2F - \tau) \right)^{-1} \left( \frac{1}{2} \right) (L^4F^2(a-c)^2 + 4\tau^2 LF \]

\[ - 4\tau^2 bL^3F^2)^{-1} (4L^3F^2(a-c)^2 + 4\tau^2 F - 12\tau^2 bL^2F^2) \]

\[ \frac{\partial Q_0^*}{\partial L} = \frac{(a-c)\tau L}{b} (L^2F - \tau)^{-2} \]

\[ + \left( 2b(L^2F - \tau) \right)^{-1} \left( \frac{1}{2} \right) (L^4F^2(a-c)^2 + 4\tau^2 LF \]

\[ - 4\tau^2 bL^3F^2)^{-1} \left( 4F(L^3F(a-c)^2 + \tau^2 - 3\tau^2 bL^2F) \right) \]

\[ \frac{\partial Q_0^*}{\partial L} = \frac{(a-c)\tau L}{b(L^2F - \tau)^2} + \frac{2F(L^3F(a-c)^2 + \tau^2 - 3\tau^2 bL^2F)}{(2b(L^2F - \tau))(L^4F^2(a-c)^2 + 4\tau^2 LF - 4\tau^2 bL^3F^2)^{1/2}} = 0 \]

The first term is clearly positive and the denominator of the second term is also clearly positive. Thus the sign of the derivative is dependent on the numerator.
References


