Calculating the Degree of Higher Order Alexander Polynomials

by

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Abstract

Given two knots, $K_1$ and $K_2$, a fundamental problem in low dimensional topology is determining if $K_1$ and $K_2$ are equivalent. Knot invariants are a key tool in determining this equivalence. We define an infinite sequence of integer invariants, $\delta_n$ for $(n \geq 0)$, based on the derived series of fundamental groups of knot complements. While these $\delta_n$ are useful, calculating them is a non-trivial task, usually requiring manipulations of modules over non-commutative, non-principal ideal domains. We detail the process of evaluating $\delta_1$, and then discuss an implementation of a computer program that calculates $\delta_1$. 
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1 Background

A knot is defined to be an embedding $K : S^1 \to S^3$. Given two knots $K_1$ and $K_2$, we call $K_1$ and $K_2$ equivalent if there exists an orientation preserving homeomorphism $h : S^3 \to S^3$ such that $h \circ K_1 = K_2$. A fundamental problem in low-dimensional topology is to classify all knots into equivalence classes. Knot invariants are a powerful tool in aiding this classification. By knot invariant we mean an object defined for each knot that is the same for all knots in the same equivalence class. We will often refer to the image in $S^3$ of a knot $K$ as $K$ itself. Given a knot $K$, the complement of the image of $K$, which we will call $S^3 - K$, is a topological space. Given two equivalent knots $K_1$ and $K_2$, the complement $S^3 - K_1$ is homeomorphic to $S^3 - K_2$. Therefore, any invariant of the homeomorphism type of $S^3 - K$ is also an invariant of $K$.

Recall that the fundamental group of a space $X$, written $\pi_1(X, x_0)$, is the group of path homotopy classes of loops in $X$ based at $x_0$ ([11], Section 52). Since the fundamental group of a topological space is an invariant of its homeomorphism type, it follows that the fundamental group of the complement of a knot, $\pi_1(S^3 - K, x_0)$, is a knot invariant of $K$. Using the Seifert-Van Kampen Theorem, it is easy to write a presentation (called the Wirtinger presentation) for $\pi_1(S^3 - K, x_0)$. However, the fundamental groups of knot complements are generally infinite, non-abelian, and complicated, and determining when two presentations of such groups represent isomorphic groups is difficult. Mathematicians have thus defined many powerful, yet computable, invariants of groups based on their presentations.

In this paper, we will define an infinite sequence of integer invariants, $\delta_n$ for $n \geq 0$, that arises when examining the first homology with local coefficients in quotients of the fundamental group by elements of the derived series. These invariants were first defined by Cochran ([2]) and Harvey ([10]). As an invariant, $\delta_n$ reflects some of the structure of $G^{(n+1)}/G^{(n+2)}$. While these $\delta_n$ are useful, they are not all easily computable. For $\delta_0$,
which is closely related to the classical Alexander Polynomial, the calculation is simple and well documented. For $\delta_n$ with $n \geq 1$, the computation usually requires manipulations of modules over non-commutative, non-principal ideal domains, a necessarily difficult task. After defining the $\delta_n$, we will detail how theoretically to compute $\delta_1$, and then give a partial implementation of a computer program that calculates $\delta_1$. 
2 Definition of the Invariant

In this section, we provide the definition and a brief overview of higher order Alexander polynomials for knots. For a more complete description, we direct the reader to Cochran ([2]) and Harvey ([10]). We begin by considering a module structure on successive quotients of the derived series of a group, then briefly describe the classical Alexander module and polynomial, then the higher-order Alexander modules and polynomials, and finally define the $\delta_n$.

2.1 Module Structure on $G^{(n)}/G^{(n+1)}$

Recall that given an unital ring $R$, a left $R$-module is an abelian group $(M,+)$ together with an action, $*$, of $R$ on $M$ that satisfies the following properties for all $r,s \in R$ and $x,y \in M$:

1. $r \ast (x + y) = r \ast x + r \ast y$
2. $(r + s) \ast x = r \ast x + s \ast x$
3. $(rs) \ast x = r \ast (s \ast x)$
4. $1_R \ast x = x$.

Let $G$ be a group, we will show that the quotient of the derived series $G^{(n)}/G^{(n+1)}$ can be considered as a left $\mathbb{Z}[G/G^{(n)}]$-module. Note that the elements of $\mathbb{Z}[G/G^{(n)}]$ are of the form:

$$\sum_{g \in G} n_g [g]_{G^{(n)}}$$

where $n_g \in \mathbb{Z}$ with all but finitely many equal to 0, and $[g]_{G^{(n)}}$ is the coset of $G^{(n)}$ containing $g$. For $h \in G^{(n)}$, denote the coset of $G^{(n+1)}$ containing $h$ by: $[h]_{G^{(n+1)}}$. 

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Lemma 2.1. The ring $\mathbb{Z}[G/G^{(n)}]$ has a well-defined action on $G^{(n)}/G^{(n+1)}$ which induces a left $\mathbb{Z}[G/G^{(n)}]$-module structure on $G^{(n)}/G^{(n+1)}$.

Proof. Let $g \in G, h \in G^{(n)}$, then $G/G^{(n)}$ acts on $G^{(n)}/G^{(n+1)}$ by:

$$[g]_{G^{(n)}} \ast [h]_{G^{(n+1)}} = [ghg^{-1}]_{G^{(n+1)}}.$$ 

Since $G^{(n)}$ is normal in $G$, we have that $ghg^{-1} \in G^{(n)}$, thus $[ghg^{-1}]_{G^{(n+1)}} \in G^{(n)}/G^{(n+1)}$.

We must also check that this action is well-defined. First, let $h_1, h_2 \in G^{(n)}$ with $[h_1]_{G^{(n+1)}} = [h_2]_{G^{(n+1)}}$, then $h_1h_2^{-1} \in G^{(n+1)}$. Because $G^{(n+1)}$ is normal in $G$, for any $g \in G$ we have $gh_1h_2^{-1}g^{-1} \in G^{(n+1)}$. So:

$$gh_1g^{-1}gh_2^{-1}g^{-1} = (gh_1g^{-1})(gh_2g^{-1})^{-1} \in G^{(n+1)}.$$ 

Therefore we have:

$$[gh_1g^{-1}]_{G^{(n+1)}} = [gh_2g^{-1}]_{G^{(n+1)}}.$$ 

Next, we let $g_1, g_2 \in G$ with $[g_1]_{G^{(n)}} = [g_2]_{G^{(n)}}$, so $g_1^{-1}g_2 \in G^{(n)}$. So for any $h \in G^{(n)}$, we have the commutator $[g_1^{-1}g_2, h] \in G^{(n+1)}$. So we have:

$$(g_1^{-1}g_2)h(g_1^{-1}g_2)^{-1}h^{-1} = g_1^{-1}g_2h_2^{-1}g_2^{-1}g_1h^{-1} \in G^{(n+1)}.$$ 

Since $G^{(n+1)}$ is normal in $G$:

$$g_1(g_1^{-1}g_2h_2^{-1}g_1h^{-1})g_1^{-1} = g_2h_2^{-1}g_1h^{-1}g_1^{-1}$$

$$= (g_2h_2^{-1})(g_1^{-1}g_1h^{-1})^{-1} \in G^{(n+1)}.$$ 

Hence:

$$[g_1h_1g_1^{-1}]_{G^{(n+1)}} = [g_2h_2g_2^{-1}]_{G^{(n+1)}}.$$
Thus the action is well-defined. We extend this to an action of $\mathbb{Z}[G/G^{(n)}]$ on $G^{(n)}/G^{(n+1)}$ by:

$$\sum_{g \in G} n_g [g]_{G^{(n)}} \ast [h]_{G^{(n+1)}} = \prod_{g \in G} [ghg^{-1}]^{n_g}.$$ 

Note that this product is actually finite since all but finitely many $n_g = 0$. From here it is routine to check that $G^{(n)}/G^{(n+1)}$ is indeed a $\mathbb{Z}[G/G^{(n)}]$-module.

Thus each quotient in the derived series, $G^{(n)}/G^{(n+1)}$, can be considered as a left $\mathbb{Z}[G/G^{(n)}]$-module with the action defined as above.

### 2.2 The Classical Alexander Module and Polynomial

Let $K$ be a knot, $X = S^3 - K$ be its complement, $x_0 \in X$, and $G = \pi_1(X, x_0)$ be the fundamental group. Let $p : \tilde{X} \to X$ be the universal abelian cover of $S^3 - K$, with $p(\tilde{x}_0) = x_0$. That is, $\tilde{X}$ is the cover induced by the standard homomorphism from $G$ onto $G/G^{(1)}$, the abelianization of $G$. Since $\pi_1(\tilde{X}) \cong G^{(1)}$, $H_1(\tilde{X}) \cong G^{(1)}/G^{(2)}$, the abelianization of $G^{(1)}$. As defined in Lemma 2.1, this homology group can be endowed with a $\mathbb{Z}[G/G^{(1)}]$-module structure. This is the classical Alexander module, denoted $A_K$. Since $S^3 - K$ is a knot complement, $G/G^{(1)}$ is infinite cyclic ([5], Section 8.1), thus $\mathbb{Z}[G/G^{(1)}] \cong \mathbb{Z}[\mu, \mu^{-1}]$, the Laurent polynomial ring with integer coefficients. We can present (using methods described in Section 3.2) $A_K$ by an $(n - 1) \times (n - 1)$ matrix, $P$, where $n$ is the number of crossings in the knot, $K$. We define the zeroth elementary ideal, $E_0 \subseteq \mathbb{Z}[\mu, \mu^{-1}]$, of $A_K$ as the ideal generated by the determinant of $P$.

It can be shown that any two presentations of $G$ will produce the same zeroth elementary ideal ([5], Section 7.4), and since it is generated by the determinant, $E_0$ is always principal. We define the Alexander polynomial, written $\Delta(K)$, to be the determinant of $P$. It follows that $\Delta(K) \in \mathbb{Z}[\mu, \mu^{-1}]$ is an invariant of $K$ ([1], Page 281).

The Alexander polynomial, while useful, is limited to reflecting the structure of
$G^{(1)}/G^{(2)}$ as a module over $\mathbb{Z}[G/G^{(1)}]$. Any information associated with other members of the derived series of $G$ is invisible to $\Delta(K)$. We turn our attention to invariants that arise when considering solvable covering spaces of a knot complement.

2.3 Higher Order Alexander Modules, Polynomials, and their Degrees

Given a knot $K$, consider the complement $X = S^3 - K$. We let $G = \pi_1(X)$ and $\Gamma_n = G/G^{(n+1)}$. Then for each $n \geq 0$, there is a covering space $X_n$ of $X$ that corresponds to the epimorphism $G \rightarrow \Gamma_n$. Therefore, $\pi_1(X_n) \cong G^{(n+1)}$. Then we have $H_1(X_n) \cong G^{(n+1)}/G^{(n+2)}$, the abelianization of $\pi_1(X_n)$. By Lemma 2.1, we can endow $G^{(n+1)}/G^{(n+2)}$ with a $\mathbb{Z}\Gamma_n = \mathbb{Z}[G/G^{(n+1)}]$-module structure. In other words, $H_1(X_n)$ can be considered as a $\mathbb{Z}\Gamma_n$-module. We denote this module as $H_1(X;\mathbb{Z}\Gamma_n)$. This is the homology of $X$ with local coefficients induced by $\Gamma_n$, as described in ([6], Chapter 5). Note that this module is an invariant of $K$, and that $H_1(X;\mathbb{Z}\Gamma_0)$, the homology with local coefficients in $\mathbb{Z}\Gamma_0$, is exactly the classical Alexander module. It is possible, using methods described in Section 3.1, to find a presentation matrix for $H_1(X;\mathbb{Z}\Gamma_n)$ given a presentation of $G$. However, since $\mathbb{Z}\Gamma_n$ is not usually a principal ideal domain (henceforth written P.I.D.), there is not a canonical presentation for such modules. Thus it is not algorithmically clear how to determine if two such presentation matrices present isomorphic modules.

We will show how to construct a non-commutative P.I.D. associated to each $\mathbb{Z}[G/G^{(n)}]$.

We first provide the definition of a skew Laurent polynomial ring.

**Definition 2.2.** Let $K$ be a skew field (a non-commutative division ring), let $\alpha$ be an endomorphism on $K$, and let $t$ be an indeterminate. Then the skew Laurent polynomial
ring in \( t \) over \( K \) associated with \( \alpha \) is the set of all expressions of the form

\[
t^n a_n + ... + t a_1 + a_0 + t^{-1} a_{-1} + ... + t^{-m} a_{-m}
\]

where each \( a_i \in K \). Addition is term-wise, multiplication is defined in the usual way, with the added condition that \( at = t\alpha(a) \) ([3], Page 54).

The endomorphism \( \alpha \) will be referred to as the twisting function on the polynomial ring. It can be shown that a skew Laurent polynomial ring has a division algorithm, and is thus a P.I.D. ([4], Section 2.1). Our goal is to construct a skew Laurent polynomial ring related to a knot \( K \). We first define Ore localization, a method of constructing a ring of fractions from a non-commutative ring.

Recall that given a commutative domain \( R \) and a multiplicatively closed set \( S \subset R \), one can construct the localization of \( R \) at \( S \), written \( RS^{-1} \), where there is a homomorphism from \( R \) into \( RS^{-1} \) with the image of \( S \) being a set of units ([8], Section 7.3). When \( R \) is non-commutative, the construction of such a ring is possible only when additional constraints hold.

**Definition 2.3.** Let \( R \) be a ring, let \( S \subset R \). Then \( S \) is an Ore set of \( R \) if:

\( i. \) \( 1 \in S \) and \( 0 \notin S \),

\( ii. \) \( S \) is closed under multiplication,

\( iii. \) for each \( r \in R \) and each \( s \in S \), there exists \( \bar{r} \in R \) and \( \bar{s} \in S \) such that \( r\bar{s} = s\bar{r} \).

Given a ring \( R \) and an Ore set \( S \subset R \), we can construct the right fraction ring, \( RS^{-1} \), where each element in \( RS^{-1} \) can be written as \( rs^{-1} \) for some \( r \in R \), \( s \in S \), and every element of \( S \) is invertible in \( RS^{-1} \). The product of two elements \( r_1s_1^{-1}, r_2s_2^{-1} \in RS^{-1} \) is defined to be:

\[
(r_1s_1^{-1})(r_2s_2^{-1}) = (r_1\bar{r})(s_2\bar{s})^{-1}
\]
where $\bar{r} \in R$, $\bar{s} \in S$ such that $r_2 \bar{s} = s_1 \bar{r}$, as guaranteed by condition (iii) above. Addition is defined as:

$$(r_1 s_1^{-1}) + (r_2 s_2^{-1}) = (r_1 \bar{r} + r_2 \bar{s})(s_2 \bar{s})^{-1}$$

where $\bar{r} \in R$, $\bar{s} \in S$ and $s_2 \bar{s} = s_1 \bar{r}$. This construction is further detailed in Ore ([12]) and Passman ([13], Page 427).

**Definition 2.4.** A ring $R$ is called an Ore domain if $R - \{0\}$ is an Ore set.

Then given an Ore domain, $R$, we can construct the right ring of fractions, $R(R - \{0\})^{-1}$, where every non-zero element of $R$ has a right inverse in $R(R - \{0\})^{-1}$. We will show that $\mathbb{Z}\Gamma_n$ is an Ore domain.

**Definition 2.5.** A group $G$ is called poly-(torsion free abelian) (henceforth written P.T.F.A.) if there exists a normal series of subgroups

$$G = G_0 \triangleright G_1 \triangleright ... \triangleright G_n = \{1\}$$

where each $G_i/G_{i+1}$ is torsion free abelian.

We observe that if $G$ is a P.T.F.A. group, then any subgroup $H \subset G$ is also P.T.F.A. Recall that $\Gamma_n = G/G^{(n+1)}$. Consider the normal series:

$$\Gamma_n = \frac{G}{G^{(n+1)}} \triangleright \frac{G^{(1)}}{G^{(n+1)}} \triangleright ... \triangleright \frac{G^{(n+1)}}{G^{(n+1)}} = \{1\}.$$

By the third isomorphism theorem, the successive quotients are:

$$\frac{\left( \frac{G^{(m)}}{G^{(n+1)}} \right)}{\left( \frac{G^{(m+1)}}{G^{(n+1)}} \right)} \cong \frac{G^{(m)}}{G^{(m+1)}}.$$
So the quotients are abelian. Additionally, if \( G \) is a knot group, Strebel ([14]) showed that \( G^{(m)}/G^{(n+1)} \) is torsion-free abelian. Thus, \( \Gamma_n \) is P.T.F.A.. We have the following result of Cochran linking P.T.F.A. groups and Ore domains.

**Proposition 2.6.** ([2], Proposition 3.2) If a group \( \Gamma \) is P.T.F.A., then the group ring \( \mathbb{Z}\Gamma \) is an Ore domain, and embeds in its classical right ring of quotients \( \mathcal{K} \), a skew field.

Then since \( \Gamma_n \), as defined above, is P.T.F.A., we have that \( \mathbb{Z}\Gamma_n \) is an Ore domain, and therefore \( \mathbb{Z}\Gamma_n \) embeds in its classical right ring of quotients, \( \mathbb{Z}\Gamma_n (\mathbb{Z}\Gamma_n - \{0\})^{-1} \), written \( \mathcal{K}_n \) ([13], Section 2.4). Let \( \psi : \Gamma_n \to G/G^{(1)} \cong \mathbb{Z} \) be the natural epimorphism and \( \Gamma_n = \ker(\psi) \). Note \( \Gamma_n = G^{(1)}/G^{(n+1)} \). Then since \( \Gamma_n \) is a subgroup of the P.T.F.A. group \( \Gamma_n, \Gamma_n \) is P.T.F.A., and is therefore an Ore domain. Let \( \mathbb{K}_n = \mathbb{Z}\Gamma_n (\mathbb{Z}\Gamma_n - \{0\})^{-1} \) be its classical right ring of fractions. We have the following result.

**Proposition 2.7.** ([10], Proposition 4.4) The quotient ring \( \mathbb{Z}\Gamma_n (\mathbb{Z}\Gamma_n - \{0\})^{-1} \) is isomorphic to the skew Laurent polynomial ring \( \mathbb{K}_n [t, t^{-1}] \).

While the proof of this proposition is not included here, some of the relevant details are given in Section 3.2.

**Definition 2.8.** Given a knot \( K \), let \( X = S^3 - K \) and group \( G = \pi_1(X) \), and \( \mathbb{K}_n \) be as defined above, then the \( n^{th} \) Alexander module of \( K \) is:

\[
\mathcal{A}_n(K) = \mathbb{K}_n[t, t^{-1}] \otimes_{\mathbb{Z}\Gamma_n} H_1(X; \mathbb{Z}\Gamma_n).
\]

This is equivalent to \( H_1(X; \mathbb{K}_n[t, t^{-1}]) \), the homology of \( X \) with local coefficients in \( \mathbb{K}_n[t, t^{-1}] \) ([10], Chapter 5).

Cochran showed that given any knot \( K \), the \( n^{th} \) Alexander module will always be a finitely-generated torsion module over the P.I.D. \( \mathbb{K}_n[t, t^{-1}] \) ([2], Proposition 4.2). We have the structure theorem for finitely presented modules over P.I.D.s.
Theorem 2.9. ([3], Page 292) Given a P.I.D. \( R \), any finitely generated torsion \( R \)-module \( M \) can be decomposed into a direct sum of cyclic modules

\[
M \cong R/e_1 R \oplus ... \oplus R/e_r R.
\]

Definition 2.10. Let \( \mathcal{A}_n(K) \) be the \( n \)th Alexander module of the knot \( K \), and let \( \{e_1, ..., e_r\} \) with \( e_i \in \mathbb{K}_n[t, t^{-1}] \) be the \( e_i \)'s as given in the decomposition of \( \mathcal{A}_n(K) \) in the preceding theorem. Then

\[
\Delta_n(K) = \prod_{i=1}^{r} e_i
\]

is the \( n \)th Alexander polynomial of \( K \), viewed as an element of \( \mathbb{K}_n[t, t^{-1}] \).

It is a fact that the isomorphism between \( \mathbb{Z}\Gamma_n(\mathbb{Z}\Gamma_n - \{0\})^{-1} \) and \( \mathbb{K}_n[t, t^{-1}] \) from Proposition 2.7 is dependent on the choice of a splitting function from \( \mathbb{Z} \) to \( \Gamma_n \) ([10], Chapter 4). Therefore, so is the \( n \)th Alexander polynomial. However, the degrees of each higher order Alexander polynomial are, in fact, independent of this choice.

Theorem 2.11. ([2], Section 5) Let \( K \) be a knot, \( n \geq 0 \), then the degree of the \( n \)th order Alexander polynomial, written \( \delta_n(K) \), is an invariant of \( K \).

This defines the sequence of invariants, \( \delta_n(K) \) for \( n \geq 0 \). We now turn our attention to algorithmically computing \( \delta_1 \) for an arbitrary knot.
3 Computing $\delta_1$

In this section, we address the issue of practically computing the value of $\delta_1$. From the previous section, we have $\delta_1$ is the degree of the 1st order Alexander polynomial. Calculating $\delta_n$ for $n > 1$ will not be discussed in this paper. Let $K$ be a knot, $X = S^3 - K$, and $G = \pi_1(X, x_0)$ be the fundamental group relative the basepoint $x_0$ (note for a knot the choice of basepoint is irrelevant up to isomorphism). We first discuss Fox Calculus, the primary method for obtaining a presentation matrix of $H_1(X, x_0; \mathbb{Z}G)$.

3.1 Fox Calculus

Fox Calculus, sometimes called Free Differential Calculus, is a useful tool in constructing algebraic invariants of knots. We begin by defining a derivative map on the group ring $\mathbb{Z}G$.

**Definition 3.1.** A derivative on the group ring $\mathbb{Z}G$ is a map $D : \mathbb{Z}G \to \mathbb{Z}G$ which satisfies the following for any $v_1, v_2 \in \mathbb{Z}G$:

\[
D(v_1 + v_2) = D(v_1) + D(v_2)
\]

\[
D(v_1 v_2) = D(v_1)\epsilon(v_2) + v_1 D(v_2).
\]

where $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is defined by: $\epsilon(\sum n_i g_i) = \sum n_i$.

From the definition of the derivative, we observe several important facts. First, since $D$ is an additive homomorphism, we have that:

\[
D(\sum n_i g_i) = \sum n_i D(g_i).
\]

Next, since $D(1) = D(1 \cdot 1) = D(1) + D(1)$, we have that $D(1) = 0$, and therefore for
any \( n \in \mathbb{Z} \):

\[ D(n) = 0. \]

Lastly, given any \( g \in \mathbb{Z} \) we have \( 0 = D(g^{-1}g) = D(g^{-1}) + g^{-1}D(g) \), and thus it follows that:

\[ D(g^{-1}) = -g^{-1}D(g). \]

While we have defined a derivative for a general group ring \( \mathbb{Z}G \), we are only interested when \( G \) is the fundamental group of a knot complement. We shall restrict ourselves to cases when \( G \) is finitely presented. Suppose \( G = \langle x_1, x_2, ..., x_n | r_1, r_2, ..., r_s \rangle \), then the elements of \( \mathbb{Z}G \) are all linear combinations of elements in the group \( G \). For each generator, \( x_j \), we can assign a unique derivative map \( D_j : \mathbb{Z}G \to \mathbb{Z}G \), written \( D_j = \frac{\partial}{\partial x_j} \)

defined on an element of \( G \) recursively by the length of a word, \( w_k \), by the following rules:

\[
\frac{\partial x_i}{\partial x_j} = \begin{cases} 
1 & \text{if } x_i = x_j \\
0 & \text{if } x_i \neq x_j.
\end{cases}
\]

And for any \( u, v \in G \):

\[
\frac{\partial uv}{\partial x} = \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x}.
\]

Note from the equations above, we have:

\[
\frac{\partial x_i^{-1}}{\partial x_j} = x_i^{-1} \left( \frac{\partial x_i}{\partial x_j} \right) = \begin{cases} 
-x_i^{-1} & \text{if } x_i = x_j \\
0 & \text{if } x_i \neq x_j.
\end{cases}
\]

It is easy to check that this mapping indeed satisfies the definition of a derivative on \( \mathbb{Z}G \).
Fox ([5], Chapter 7) showed that using these derivatives we can obtain a presentation matrix for $H_1(X, x_0; ZG)$ as:

$$\Lambda = \left( \frac{\partial r_i}{\partial x_j} \right).$$

### 3.2 The 1st Higher Order Alexander Module

Recall from Section 2.3 that $\Gamma_1 = G/G^{(2)}$. By considering elements of the group $G$ as representing cosets in $G/G^{(2)}$, the matrix, $\Lambda$, produced by Fox Calculus is also a presentation matrix for $H_1(X, x_0; Z\Gamma_1)$. We have the split short exact sequence:

$$1 \to G^{(1)} / G^{(2)} \to G / G^{(2)} \to G / G^{(1)} \to 1.$$

It follows from a version of the splitting lemma for short exact sequences that $G/G^{(2)} \cong G^{(1)} \ltimes \tau G^{(1)}/G^{(2)}$, where $\tau : G/G^{(1)} \to \text{Aut}(G^{(1)}/G^{(2)})$ is given by conjugation as in Lemma 2.1. Since $G/G^{(1)} \cong \mathbb{Z}$, a splitting $\phi : G/G^{(1)} \to G/G^{(2)}$ is determined by $\phi(1) = a$, where $a \in G$. Using the semidirect product structure, it follows that any element in $G/G^{(2)}$ can be written uniquely in the form $a^m x$ where $x \in G^{(1)}/G^{(2)}$.

Therefore, we can write the entries in $\Lambda$ as elements of $\mathbb{Z}[G^{(1)} \ltimes G^{(1)}/G^{(2)}]$. Recall that $\mathbb{K}_1[t, t^{-1}] \cong \mathbb{Z}\Gamma_1(\mathbb{Z}\Gamma_1 - \{0\})^{-1}$. So there is a monomorphism $\mathbb{Z}\Gamma_1 \to \mathbb{K}_1[t, t^{-1}]$. Under this monomorphism, $a \mapsto t$ and $x$ maps to a unit. Applying this monomorphism to the entries of $\Lambda$, we obtain a presentation matrix, $\bar{\Lambda}$ for $H_1(X, x_0; \mathbb{K}_1[t, t^{-1}])$.

Since $\mathbb{K}_1[t, t^{-1}]$ is a P.I.D., we can diagonalize the matrix $\bar{\Lambda}$. However, the standard diagonalization algorithm requires division of elements of $\mathbb{K}_1[t, t^{-1}]$. In order to perform the long division of these polynomials, it suffices to understand division with remainders of elements in $\mathbb{K}_1$. Since $\mathbb{K}_1 \cong \mathbb{Z}\Gamma_1(\mathbb{Z}\Gamma_1 - \{0\})^{-1}$, all non-zero elements of $\mathbb{Z}\Gamma_1$ have formal inverses in $\mathbb{K}_1$. However, it is not immediately clear when an element of $\mathbb{Z}\Gamma_1$, written as a linear combination of cosets $[g]_{G^{(2)}}$, is 0. Therefore, we would like to have a canonical representation of the elements in $\mathbb{K}_1 \cong G^{(1)}/G^{(2)}$. 

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We will again use Lemma 2.1 to obtain information about $G^{(1)}/G^{(2)}$ by endowing it with a $\mathbb{Z}[G/G^{(1)}]$-module structure. We are able easily to obtain a presentation matrix for this module from the matrix, $\Lambda$, produced by Fox Calculus by a similar process (details in the next section). In this case, let $G/G^{(1)} \cong \langle \mu \rangle$, then $\mathbb{Z}[G/G^{(1)}] \cong \mathbb{Z}[\mu, \mu^{-1}]$, and we have a module over the Laurent polynomial ring with integer coefficients. Note that this is exactly a presentation matrix for the Alexander module. That is, $H_1(X; \mathbb{Z}[\mu, \mu^{-1}])$ is isomorphic as an abelian group to $G^{(1)}/G^{(2)}$. Therefore:

$$H_1(X; \mathbb{Q}[\mu, \mu^{-1}]) \cong \mathbb{Q} \otimes \frac{G^{(1)}}{G^{(2)}}.$$ 

Since $\mathbb{Q}[\mu, \mu^{-1}]$ is a P.I.D., we can reduce the matrix to a rational canonical form, which corresponds to writing a module as the sum of a finite number of cyclic modules, as defined in Theorem 2.9 ([9]). In particular, we have a canonical representation of elements of $G^{(1)}/G^{(2)} \otimes \mathbb{Q}$. From this, we can extrapolate information about the elements of $G^{(1)}/G^{(2)}$ since $G^{(1)}/G^{(2)}$ is $\mathbb{Z}$-torsion free (since $G$ is a knot group ([14])). Using this information, we can easily determine when elements of $\Gamma_1 \cong G^{(1)}/G^{(2)}$ are invertible, giving us the means to perform division on $\mathbb{K}_1$, and therefore on $\mathbb{K}_1[t, t^{-1}]$.

We now have a presentation matrix $\bar{\Lambda} = H_1(X, x_0; \mathbb{K}_1[t, t^{-1}])$, and have shown a way to do division of elements in $\mathbb{K}_1$. We recall that the 1st order Alexander module is defined as:

$$\mathbb{K}_1[t, t^{-1}] \otimes_{\mathbb{Z}_1} H_1(X; \mathbb{Z}_1).$$

From the long exact sequence of a pair we have that:

$$H_1(X, x_0; \mathbb{Z}_1) \cong H_1(X; \mathbb{Z}_1) \oplus \mathbb{Z}_1.$$ 

Cochran ([2], Proposition 3.7) showed that $H_1(X; \mathbb{Z}_1)$ is a torsion $\mathbb{Z}_1$-module. Addi-
tionally, the fact that $H_1(X, x_0; \mathbb{Z} \Gamma_1)$ has a rank 1 free summand implies one column of $\bar{\Lambda}$ will always be a linear combination of the others. Removing this column gives exactly a matrix presentation for $H_1(X; \mathbb{K}_1[t, t^{-1}])$, the 1st higher order Alexander module.

3.3 Step by Step Computation

We will give a step by step description of the process of computing $\delta_1$.

1. Wirtinger Presentation

Given a diagram of an $n$ crossing knot $K$, one can easily produce the Wirtinger presentation of $G = \pi_1(S^3 - K)$, where each arc corresponds to a generator and the crossings describe the relations between them. One relation is always superfluous, so we will have $n$ generators and $n - 1$ relations. This process is described in detail by Crowell and Fox ([5], Chapter 6). We can write the presentation for $G$ as:

$$G = \langle a, x_1, ..., x_{n-1} | r_1, r_2, ..., r_{n-1} \rangle$$

where each relation, $r_i$, is of the form $r_i = x_{k_1}x_{k_2}x_{k_3}^{-1}x_{k_4}^{-1}$ with $k_1 = k_3$. Each of the generators in this presentation maps to a generator of $G/G^{(1)} \cong \mathbb{Z}$ under abelianization. We therefore define our splitting homomorphism by $\phi(1) = a$, the first generator listed in the presentation.

2. Change Generating Set:

We add new generators, $y_i$, and relations, $y_i = x_i a^{-1}$, to $G$, giving us:

$$G = \langle a, x_1, ..., x_{n-1}, y_1, ..., y_{n-1} | r_1, ..., r_{n-1}, x_1 a^{-1}y_1^{-1}, ..., x_{n-1} a^{-1}y_{n-1}^{-1} \rangle.$$ 

Using Tietze transformations we are able to simplify $G$ to

$$G = \langle a, y_1, ..., y_{n-1} | r'_1, ..., r'_{n-1} \rangle.$$
by the substitutions $x_i \mapsto y_i a$ for $(1 \leq i \leq (n-1))$ into each $r_j$ for $(1 \leq j \leq (n-1))$. Since $y_i = x_i a^{-1}$, it is easy to show that $y_i \in G^{(1)}$. When we wish to rewrite $G/G^{(2)}$ in terms of $G/G^{(1)} \times G^{(1)}/G^{(2)}$, we have $a$ as the generator of $G/G^{(1)}$, and each $y_i \in G^{(1)}/G^{(2)}$.

3. Fox Calculus:

As described in Section 3.1, we next use Fox calculus on the modified presentation of $G$ to produce an $(n-1) \times n$ presentation matrix $\Lambda$ for $H_1(X, x_0; \mathbb{Z} G)$. As described in Section 3.2, this matrix also represents $H_1(X, x_0; \mathbb{Z} \Gamma_0)$. We wish to modify this matrix to present $H_1(X, x_0; \mathbb{K}_1[t, t^{-1}])$ by applying the monomorphism from $\mathbb{Z} \Gamma_0 \rightarrow \mathbb{K}_1[t, t^{-1}]$ to each entry in $\Lambda$, as described in Section 3.2. Notice that under the monomorphism, $a$ maps to $t$ and the $y_i$ map to units in $\mathbb{K}_1$. We are left with the presentation matrix $\bar{\Lambda} = H_1(X, x_0; \mathbb{K}_1[t, t^{-1}])$.

4. Extracting Information from $G^{(1)}/G^{(2)}$:

We wish to determine the decomposition of $G^{(1)}/G^{(2)}$ considered as a $\mathbb{Z}[\mu, \mu^{-1}]$-module into a direct sum of cyclic modules. Recall the matrix $\Lambda$ presents $H_1(X, x_0; \mathbb{Z} G)$, therefore, by considering the entries as they occur in $\mathbb{Z}[G/G^{(1)}]$, it also represents $H_1(X, x_0; \mathbb{Z} \Gamma_0)$. Since $\Gamma_0 = G/G^{(1)} \cong \langle \mu \rangle$, we can modify the matrix by applying the epimorphism from $G$ onto $G/G^{(1)}$ to each entry $\Lambda$. Since the chosen splitting is given by $\phi(1) = a$, $a$ is sent to $\mu$, and as each $y_i \in G^{(1)}$, everything else is trivialized. We have a presentation matrix for $H_1(X, x_0; \mathbb{Z}[\mu, \mu^{-1}])$. For the same reasons as described in Section 3.2, we have:

$$H_1(X, x_0; \mathbb{Z}[\mu, \mu^{-1}]) \cong H_1(X; \mathbb{Z}[\mu, \mu^{-1}]) \otimes \mathbb{Z}[\mu, \mu^{-1}].$$

We can simply remove the 1st column of the matrix, leaving us with a presentation of $H_1(X; \mathbb{Z}[\mu, \mu^{-1}])$. 

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We are left with a matrix with entries in \( \mathbb{Z}[\mu, \mu^{-1}] \). Since this is not a P.I.D., we can consider the entries as they occur in \( \mathbb{Q}[\mu, \mu^{-1}] \), as discussed in Section 3.2. We reduce the matrix to a rational canonical form. Once reduced, we can use the matrix to extrapolate information about elements in \( G^{(1)}/G^{(2)} \) when performing operations on \( K_1 \), allowing us to check for non-invertible elements when performing long division of these polynomials.

5. Reducing:

Lastly, we wish to diagonalize \( \bar{\Lambda} \) using a standard row and column operation algorithm, which corresponds to finding a decomposition of the module as defined in Theorem 2.9. This process is further detailed in the next section. When performing division, we can refer to the information acquired in step 4 as necessary. Finally, \( \delta_1 \) is the sum of the degrees of the entries along the diagonal of \( \bar{\Lambda} \).
4 Implementation and Example

In this section we discuss our implementation of an algorithm to evaluate $\delta_1$, as well as present an example calculation performed by the program for a simple 4-crossing knot.

4.1 Algorithm Implementation

We chose to implement our algorithm using Java, as an object oriented programming language seems conducive to constructing algebraic structures. We designed classes for each algebraic object and operation involved in computing $\delta_1$. We address each step in the process as described in Section 3.3. A complete API and documentation for the methods and class structure is available, though is not included in this paper.

Suppose we have been given the $n$-crossing knot $K$, then we have the following.

1. Wirtinger Presentation

   It is left to the user to determine $\pi_1(S^3 - K)$ in the form described in Section 3.3. The program takes a String representing the group presentation as input. It is expected that the generators of the group will be entered as single characters, such as $a, b, c, d$, etc. For now, we will call the first generator listed $a$, and the remaining generators $g_i$ for $(1 \leq i \leq (n - 1))$.

2. Change Generating Set:

   Given a String representation of a group, the program constructs a group class in which generators are stored in a variable array and relations are stored in an array as ordered lists of variables and exponents. By default, we choose $a$, the generator at index 0 in the stored array, to be the image of the generator under the splitting homomorphism, $\phi$. The class provides methods for changing the generators of the group, calculating inverses, multiplying words, and Fox Calculus.
The change generators method will replace the $g_i$ with $x_1, x_2, ..., x_{n-1}$, as well as perform the substitutions into the group relations. That is, for each entry in the array of relations, we replace each occurrence of $g_i$ with $x_i a^{-1}$ for ($1 \leq i \leq (n-1)$).

3. Fox Calculus:

The Fox Calculus method in the group class produces an $(n-1) \times n$ matrix with entries as formal sums of group elements with coefficients in $\mathbb{Z}$, as described in Section 3.1. Additionally, the program uses the relations in the presentation of the group to simplify each term if it is convenient. For example, given the relation $r = x_1 ax_1^{-1} a^{-1} x_2^{-1}$, we take the Fox Derivative of $r$ with respect to $x_1$ and get:

$$\frac{\partial r}{\partial x_1} = 1 - x_1 ax_1^{-1}.$$

We observe that, since $x_1 ax_1^{-1} a^{-1} x_2^{-1}$ is trivial in the group, then $x_1 ax_1^{-1} = x_2 a$. Thus:

$$\frac{\partial r}{\partial x_1} = 1 - x_2 a.$$

We then wish to write this term as an element of $\mathbb{K}_1[t, t^{-1}]$. We algorithmically apply the monomorphism from $\mathbb{Z} \Gamma_1$ into $\mathbb{K}_1[t, t^{-1}]$ by checking each term in turn for instances of $a$. Since the first term 1 contains no occurrences, we need not modify it. For $x_2 a$, we rewrite it as $x_2(t)$. However, as we wish to write the $t$'s on the left, we apply our twisting function, given by conjugation, and the entry becomes:

$$1 - t \ast (a^{-1} x_2 a).$$

Performing these operations for each entry the matrix gives exactly the matrix, $\tilde{\Lambda}$, as defined in Section 3.2.
4. Extracting Information from $G^{(1)}/G^{(2)}$:

We are able to obtain a matrix presentation with entries in $\mathbb{Z}[\mu, \mu^{-1}]$ by eliminating the first column of the matrix obtained in part 2, and by mapping $a$ to $\mu$ and all other generators to 1, resulting in an $(n - 1) \times (n - 1)$ matrix with Laurent polynomial entries. As we are considering the matrix as a presentation of a module over $\mathbb{Q}[\mu, \mu^{-1}]$, the polynomials have rational coefficients. We reduce the matrix to a rational canonical form, an algorithm is given in Devitt and Mollin ([7]). Since column operations change the basis of the module, whenever a column operation is necessary we store the operations on the generating set in a separate array as sums of generators, allowing us to keep track of the new basis. We note that, given two columns $C_j$ and $C_k$ with corresponding generators $g_j$ and $g_k$, multiplying $C_j$ by some $m \in \mathbb{Q}[\mu, \mu^{-1}]$ and adding it to column $C_k$ replaces the generator corresponding to $C_j$ to $g_j - (m \times g_k)$.

5. Rewriting elements of $\mathbb{Z}[G/G^{(1)} \times G^{(1)}/G^{(2)}]$:

This step is not included above because it is not absolutely necessary for calculating $\delta_1$. Since we have decomposed $G^{(1)}/G^{(2)}$ into a direct sum of cyclic modules, we can rewrite each entry in $\tilde{\Lambda}$ in terms of the generators of these cyclic modules. Doing so would give a canonical representation of each element in $\mathbb{K}_1$, which gives the advantage of not having to check if an element is invertible, as anything that represents 0 will actually be 0! While we have not completed this section of the program, we would prefer to include this step rather than have to check for non-invertible elements every time we wish to divide elements in $\mathbb{K}_1$. 
6. Reducing:

Finally, since \( \mathbb{K}[t, t^{-1}] \) is a P.I.D., and we are only interested in the rank of the matrix, we can diagonalize using a standard matrix reduction algorithm. Unlike in Step 4, we are not concerned with changing the basis of the module, therefore we need not keep track of the column operations. We first scan the matrix for the entry of lowest degree, then use that entry to reduce the degree of the highest degree element in its row and column. Repeating this process will eventually diagonalize the matrix.

We recall the definition of the 1st order Alexander module from Section 2.3 as:

\[
\mathcal{A}_1(K) = \mathbb{K}_1[t, t^{-1}] \otimes_{\mathbb{Z}_1} H_1(X; \mathbb{Z}_1).
\]

Note that \( \mathbb{K}_1[t, t^{-1}] \) is a bi-module, then since \( H_1(X; \mathbb{Z}_1) \) is a left-module, we consider \( \mathcal{A}_1(K) \) as a left-module. Therefore, any column operations are done on the right, and row operations are always done on the left. We also recall that \( \mathbb{K}_1 \) is a skew field. By definition, we must always write inverses on the right. Finally, since \( \mathbb{K}_1[t, t^{-1}] \) is a twisted polynomial ring, after multiplying, we must apply the twisting function \( \alpha \), given by conjugation by \( a \), to move any \( t \)'s to the left of an argument.

For example, multiplying \( t * (ax_1 a^{-1}) \) on the right by \( t^2 * (a^{-3} x_1^2 a^3) \) gives:

\[
t * (ax_1 a^{-1}) \times t^2 * (a^{-3} x_1^2 a^3).
\]

In order to “commute” \( ax_1 a^{-1} \) past the \( t^2 \) we apply our twisting function \( \alpha \) twice to \( ax_1 a^{-1} \), giving:

\[
t^3 * ((a^{-1} x_1 a)(a^{-3} x_1^2 a^3)).
\]
When the matrix has been fully diagonalized, we sum the degrees along the diagonal, giving us $\delta_1$.

### 4.2 Example: The Figure 8 Knot

We shall demonstrate a running of the algorithm for the 4-crossing Figure 8 knot.

![Figure 8 Knot](image)

Calculating the fundamental group gives the Wirtinger presentation, in the form described in Section 3.3, as:

$$G = \pi_1(X) = \langle a, b, c, d | ac^{-1}d^{-1}, cab^{-1}, bab^{-1}d^{-1} \rangle.$$ 

We then pick our splitting homomorphism; the algorithm by default chooses the first generator listed in the group presentation, namely $a$. We let $\phi : G/G^{(1)} \to G/G^{(2)}$ given by $\phi(\mu) = a$, and define $x_1, x_2, x_3$ to be:

$$x_1 = ba^{-1}, \ x_2 = ca^{-1}, \ x_3 = da^{-1}.$$ 

So $G$ becomes:
\[ G = \langle a, b, c, d, x_1, x_2, x_3 | aca^{-1}d^{-1}, bab^{-1}d^{-1}, x_1^{-1}ba^{-1}, x_2^{-1}ca^{-1}, x_3^{-1}da^{-1} \rangle. \]

Using Tietze transformations we can simplify \( G \) to:

\[ G = \langle a, x_1, x_2, x_3 | ax_2a^{-1}x_3^{-1}, x_2ax_2^{-1}a^{-1}x_1^{-1}, x_1ax_1^{-1}a^{-1}x_3^{-1} \rangle. \]

Fox Calculus is now used to obtain a presentation matrix for \( H_1(X, x_0; \mathbb{Z}G) \) as:

\[
\begin{pmatrix}
1 - ax_2a^{-1} & 0 & a & -ax_2a^{-1}x_3^{-1} \\
-x_2ax_2^{-1}a^{-1}x_1^{-1} & 1 - x_2ax_2^{-1} & 0 & \\
x_1 - x_1ax_1^{-1}a^{-1} & 1 - x_1ax_1^{-1} & 0 & x_1ax_1^{-1}a^{-1}x_3^{-1}
\end{pmatrix}.
\]

However, using the relations from the presentation of \( G \) we can simplify the presentation to:

\[
\begin{pmatrix}
1 - x_3 & 0 & a & -1 \\
x_2 - x_1 & -1 & 1 - x_1a & 0 \\
x_1 - x_3 & 1 - x_3a & 0 & -1
\end{pmatrix}.
\]

We now use this presentation to obtain a presentation matrix for \( G^{(1)}/G^{(2)} \), considered as a \( \mathbb{Q}[\mu, \mu^{-1}] \) module, by removing the first column, sending \( a \) to \( \mu \), the generator of \( G/G^{(1)} \), and trivializing everything else.

\[
\begin{pmatrix}
0 & \mu & -1 \\
-1 & 1 - \mu & 0 \\
1 - \mu & 0 & -1
\end{pmatrix}.
\]
We triangularize, using only row operations, and are left with:

\[
\begin{pmatrix}
-1 & -\mu + 1 & 0 \\
0 & \mu & -1 \\
0 & 0 & \mu^2 - 3\mu + 1
\end{pmatrix}.
\]

We then get the matrix into a rational canonical form with the following column operations:

1. Multiply 1st column by \((-\mu + 1)\) and add it to the 2nd column.

2. Multiply 3rd column by \(\mu\) and add it to the 2nd column.

3. Multiply 2nd column by \(-\mu^{-1}\) and add it to the 3rd column.

Leaving us with:

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & \mu^2 - 3\mu^2 + \mu & 0
\end{pmatrix}.
\]

And the columns correspond to the new generators:

\[x_1 + (\mu - 1) \ast x_2, \quad \mu^{-1} \ast x_3, \quad x_3 - \mu \ast x_2.\]

From here, we can extrapolate information about each generator, \(x_i\), in terms of \(x_3\), since the new generators and relations imply:

\[
0 = -x_1 + (-\mu + 1) \ast x_2
\]
\[
0 = (\mu^2 - 3\mu + 1) \ast x_3
\]
\[
0 = (\mu) \ast x_2 - x_3.
\]
Therefore:

\[ x_1 = (-1 + \mu^{-1}) \ast x_3 \]
\[ x_2 = (\mu^{-1}) \ast x_3. \]

Applying the module action gives the group relations in \( G^{(1)}/G^{(2)} \):

\[ x_1 = x_3^{-1}(a^{-1}x_3a) \]
\[ x_2 = a^{-1}x_3a. \]

We return to the presentation matrix for \( H_1(X, x_0; \mathbb{Z}G) \). Writing each element in terms of an element of \( G/G^{(1)} \) acting on an element of \( G^{(1)}/G^{(2)} \) is an easy task. As described previously, we send \( a \) to \( t \), and “commute”, using \( \alpha \), until all of the \( t \)'s are on the left. Giving the presentation of \( H_1(X; \mathbb{K}[t, t^{-1}]) \):

\[
\begin{pmatrix}
-x_3 + 1 & 0 & t & -1 \\
x_2 - x_1 & -1 & 1 - t \ast (a^{-1}x_1a) & 0 \\
x_1 - x_3 & 1 - t \ast (a^{-1}x_3a) & 0 & -1
\end{pmatrix}
\]

Now, using the equations obtained above, we substitute for \( x_1 \) and \( x_2 \) into the presentation matrix for \( H_1(X, x_0; \mathbb{K}[t, t^{-1}]) \):

\[
\begin{pmatrix}
-x_3 + 1 & 0 & t & -1 \\
(a^{-1}x_3a) - x_3^{-1}(a^{-1}x_3a) & -1 & -t \ast ((a^{-1}x_3^{-1}a)(a^{-2}x_3a^2)) + 1 & 0 \\
x_3^{-1}(a^{-1}x_3a) - x_3 & -t \ast (a^{-1}x_3a) + 1 & 0 & -1
\end{pmatrix}
\]

Since \( \mathbb{K}[t, t^{-1}] \) is a P.I.D. we are able to reduce this matrix to:
As discussed in Section 3.2, one column was a multiple of the others. We remove the 3rd column of zeroes, leaving the presentation for $H_1(X; \mathbb{K}_1[t, t^{-1}])$. Checking the degree of the product of the diagonal reveals that $\delta_1 = 1$ for the Figure 8 knot.

5 A Word on Higher Order $\delta_n$

One of the primary challenges encountered in designing an algorithm to evaluate $\delta_1$ lies in writing elements of $G^{(1)}/G^{(2)}$ canonically. However, we showed that considering $G^{(1)}/G^{(2)}$ as a $\mathbb{Z}[G/G^{(1)}]$-module makes the calculation possible. For $\delta_n$ with $n > 1$, $\Gamma_n \cong G^{(1)}/G^{(n+1)}$ cannot be endowed with the same module structure nor is it necessarily an abelian group. Determining when elements of $\Gamma_n$ are invertible must be done by hand. Finding an algorithmic approach to check for containment in $G^{(n+1)}$ would be helpful. For the time being, finding $\delta_n$ for $n \geq 2$ will require many pages of hand-written, tedious calculations.
References


