Wesleyan University The Honors College

# Molly gone,
# Polygons

by

Duane Lê
Class of 2010

A thesis submitted to the
faculty of Wesleyan University
in partial fulfillment of the requirements for the
Degree of Bachelor of Arts
with Departmental Honors from the College of Letters

Middletown, Connecticut April, 2010

Molly gone,

Polygons

## *Acknowledgements*

Professor Needler, *for your patient assistance, creative suggestions, and enthusiasm.*

My friends and family, *for your constant encouragement and support.*

Odel,

I have found the solution to the Garden of Arenne problem. If you recall, the problem is presented in the following way:

There is a young boy who finds himself at the edge of a certain garden in Arenne. There is a ditch around the garden, and so the boy climbs a nearby tree and lowers himself from one of its limbs into the garden. The boy walks along a path lined with flowers and brush, and comes to a small clearing. There is a circular pool in the center of the clearing, along with four large rocks that are spread out evenly around its edge. The path leads down to the water, as do three others, one between each pair of rocks. The boy bends over the pool to drink, but as he does so, he slips into the water.

When he surfaces and climbs out of the pool, the boy cannot remember from which direction he has come. He chooses two of the rocks, and takes the path between them leading from the pool. Beyond the clearing, the path is littered with leaves. The boy finds that the path ends at a wall, and so he takes the path back to the pool. The boy then tries another path, but again finds that it leads him to a wall. The boy tries the remaining paths, and tries all of the paths repeatedly, but the paths are

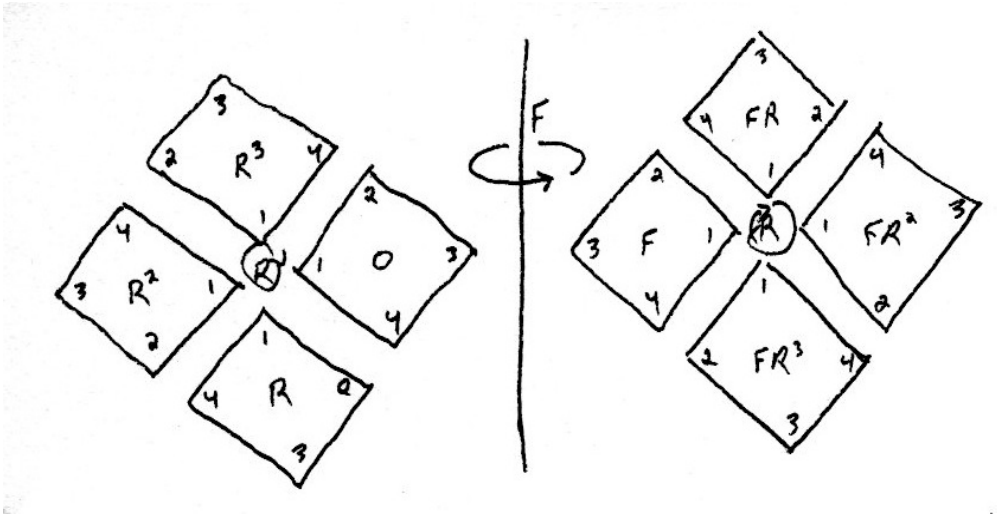
never like the one he remembers, and they always end in a high wall that he cannot surmount. The boy cannot escape, and inevitably dies in the garden.

The solution is the Dihedral 8 group, the set of symmetries on a square. Recall that from a starting position, a square can be maneuvered in 8 different ways so that it will still look the same. We can give the square zero, one, two, or three turns, so that the four corners of the square rotate places. But, we can also give the square a flip, so that the top face of the square becomes the bottom face, and the bottom the top. Combining these, we get all eight possible positions, and thus the eight symmetries on the square. This set forms a mathematical group, and so no matter what combination of the eight maneuvers we perform on the square, it will be equivalent to having performed just one of these maneuvers.

From here, that this is the solution to the Garden of Arenne problem should be obvious. The four rocks in the garden must represent the four corners of the square. The symmetry of the paths represents the ordinary symmetry of a square. Can you guess what the falling into the pool represents? It must be a flip of the square! Then, from the boy's perspective, the entire garden must be upside-down. Thus, the ditch is inverted,

and becomes a wall. Likewise, the robust and colorful flowers are replaced by dead, brown leaves: an inversion of seasons, and thus, of location. Since the boy never again enters the pool in order to flip the square, he is stuck on the wrong face of the square. He only performs turns, and so he can never leave in the direction from which he came. QED!

Ludo



Odel,

Is there any doubt that my solution is correct? It isn't inconsistent with the problem. The effects of symmetry and reversal are clearly present in the story. Do you think that there is some aspect of the story that the dihedral group doesn't account for? Or could there be a simpler solution? It seems to me that once we consider the dihedral group's relevance to the problem, the analogy is obvious. This renders some of the minor details of the story superfluous, but this is more likely a failure in the posing of the problem than evidence of a better solution.

I intend to verify that my solution is correct. The problem was undoubtedly posed by a mathematician who was from or who stayed in Arenne. I will go to Arenne to inquire after this mathematician, to determine the original manner in which the problem was posed, and to confirm the validity of my solution. If I meet the author of the problem, I will suggest that the statement of the problem be pared down in the future, so as to avoid the excess of detail that makes it somewhat less elegant. I'll let you know when I'm finished.

Ludo

Odel,

I stopped at the first house that I came to in Arenne, and spoke with a villager there. I first asked him whether there are any well-known mathematicians from the village. He replied that he didn't know of any, except, perhaps, for Milla, who teaches mathematics to the children in Arenne. I then asked if he was familiar with the problem of the Garden in Arenne. He said that he probably knew which garden I was referring to, and asked me to tell him the problem.

I recounted the problem, and when I finished, the villager said that he was indeed familiar with the story. He told me that there are not four rocks around the pool in the garden, though, but three. This, of course, hardly affects the solution of the problem--we would merely consider the group of symmetries on the equilateral triangle rather than on the square. He added, however, that the boy is not trapped in the garden by a wall. He said that the boy simply gets lost in the garden, and that it was not until the boy died that the wall was built.

I pointed out that this version of the story was at odds with the nature of the problem, and that the detail of the wall would serve no purpose in the problem if it

were added after the boy disappears. I suggested that the villager had, perhaps, not heard the problem correctly told. He said that the story refers to a real boy that disappeared into a real garden in the vicinity of Arenne.

I found this hard to believe. How could anyone have known that the boy had gotten lost in the garden if he disappeared? The villager could not explain this, but said that the story was well known in Arenne, and that it was repeated in order to dissuade the children of the village from climbing over the wall into the garden.

This turned out to be true. I stopped at every house that I passed as I entered the village, and all of the villagers I spoke to were familiar with a story of a boy that gets lost in the garden (of which the villagers are apparently frightened). There was some disagreement, however, as to whether the wall was built before or after the boy's disappearance. And, none of the villagers could account for how it might be known that the boy had disappeared into the garden. The average villager knows a story about this boy's disappearance that is both incomplete and inconsistent.

In the center of the village, there is a well shared by the villagers. When I reached this area, some children that had been playing around the well after

lunch were beginning to make their way back to the school. A few of them appeared to be looking for someone that was missing, but they eventually gave up and followed the other children. I decided to go speak with the teacher Milla to see what she might know about the story of the boy, so I followed them.

I waited until the school day had ended to speak to Milla. She is familiar with the story of the boy that gets lost in the garden. Milla told me she has heard various versions of the story. She agrees with me that the version of the story in which the boy disappears, and in which this is inexplicably known, is unsatisfactory. She said that some of the villagers believe that the boy was eventually able to escape from the garden, and to tell what had happened to him before dying. Others say that the body of the boy must have been found inside of the garden, and others maintain that the boy simply disappeared.

Milla also said that some of the villagers believe that the wall was built before the boy disappeared, and others that it was built afterwards, because he got lost in the garden. I asked how the boy could have been trapped by the wall in the garden if the wall had been built after his disappearance. She said that one could simply get lost in the garden without being trapped by

the wall. She said that the garden is supposed to be difficult to navigate because its symmetries make it difficult to determine one's location and the direction in which one is traveling.

I told Milla the problem of the lost boy as I knew it. She laughed when I told her that I had found the solution to the problem, though she thought that the set of symmetries on the triangle is an interesting analogy to the garden. Milla pointed out that if the boy were not careful in choosing the paths, then the boy could simply have tried the wrong paths repeatedly, and missed the right one. I mentioned to Milla that I would be interested to see this garden, and asked her where it is. Milla said that the garden is in the woods at the edge of the village, and that of the houses in the village, she lives in the one closest to the garden. She said that there is a trail near her house that leads to the wall of the garden.

Milla agreed to show me the trail, but we stopped at her house, and before we could set out again, Molly’s parents arrived. Molly is one of Milla's students, a young girl of the village. She did not return to school after lunch with the other students today. Milla had assumed that the girl returned home for some reason, but, according to the girl's parents, she had not. The girl's

parents had believed her to be in school, and became worried when she did not return home later in the day. They came to ask Milla where the girl is.

Milla and the girl's parents were upset when they realized that she was unaccounted for. They decided to leave immediately to search for the girl. Milla has agreed to let me stay in an empty room in her house.

Ludo

Odel,

I went into the woods today on a trail that Milla showed me. She refused to accompany me to the garden, saying that she needed to continue helping with the search for the missing girl.

I followed the trail, which is rough and winding. I don't know how far it was that I walked before I came to the wall, nor do I know how far I was from the start of the trail at this point. The woods are extremely regular in that the trees are evenly distributed throughout. Yet, the directions from a particular tree to its nearest neighbors are unpredictable. So, on the one hand, even if we move a large distance along the path, the distribution of the trees appears generally the same. We would have no way of knowing that we are in a different part of the woods. On the other hand, even if we move only slightly along the path, our perspective of all of the trees changes entirely. Since it would be impossible to remember the precise placement of each tree along the path, there would be no way to tell just from looking at the trees that we hadn't moved a great distance. I gave up trying to keep track of my progress along the trail and my direction after a while, and I had nearly forgotten my purpose when the trail came to the wall.

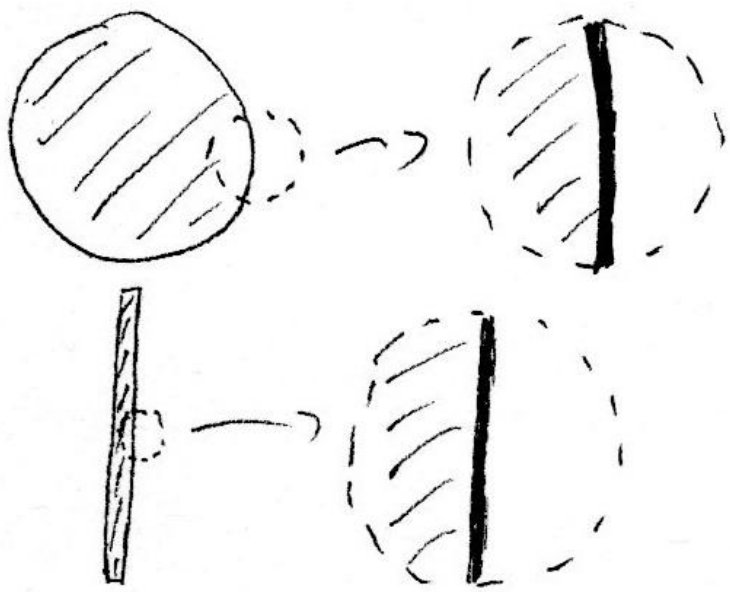
The wall was just low enough for me to reach the top when I jumped, and I was able to pull myself on top of it. The wall seemed to extend straight out to either side; that is, it had no apparent curvature. This, of course, must mean one of two things. Either the wall has similar curvature all the way around, like a circle, in which case the area inside of the wall must be very large, or the wall turns more quickly in certain places than in others, in which case the area inside of the wall need not be very large; for example, the wall could simply form a narrow rectangle, in which case I would have been standing on a longer side. This does not mean, however, that if the wall turns quickly in some places, then the area within the wall must be small.

I say the area within the wall, and not the garden, because there was no way to tell from where I stood on the wall that there is any garden at all. From what I could see, the trail that I had followed to the wall continued on the other side of the wall. And, the type and distribution of the trees on either side of the wall was identical.

It looked so similar on either side that, I admit, I must have gotten confused when I was standing on the wall, and gotten down on the wrong side. I climbed down on what I believed to be the inside of the wall, in order

to continue making my way to the garden.  As I followed the trail, no part seemed particularly familiar or unfamiliar.  I must have been eager to reach the garden, because I was disappointed when I walked out of the woods and found myself once again at the start of the trail, near Milla's house.

Ludo



Odel,

Milla appeared exhausted this morning. Apparently, the girl, Molly, has still not been found, though the villagers are continuing to search for her. Milla said that yesterday she and Molly's parents spoke to the children that had been playing near Molly before the children returned to school. The children told them that Molly had been playing near the well, as she usually did, and that none of them saw where Molly had gone. Milla and Molly's parents ran to the well, fearing that they would discover Molly's body inside. They were relieved not to find it there. The children could not provide any more clues to where the girl might be.

I told Milla about my trip to the wall of the garden. She seemed disappointed that I had not been able to get into the garden, though she was not surprised to hear about my mistake. Milla said that she has occasionally tried to find the way into the garden in the past, but has never been successful. She said that the villagers all avoid the wall and the garden, because it is so easy to get lost in the area around the garden.

On the other hand, if the villagers all avoid the garden, then they will never learn to find their way around it. The problem must eventually be dealt with.

Ludo

Ludo finds his way into the garden, arriving along one of three paths that lead into a place of the garden. Ludo recognizes the place from a problem that Ludo has heard about a young boy being trapped in a garden. In the center of the place is a circular pool of water. Three rocks evenly spaced around the water form an equilateral triangle. The three identical paths leading straight into the place each meet the pool of water at a point between a pair of rocks.

Before, Ludo imagined that the rocks in the place were just at the edge of the water, so that the triangle they formed would be circumscribed by the circular pool. Ludo sees now, however, that the rocks are at some distance from the water, and that the triangle they form seems to circumscribe the pool. Ludo walks slowly, heel-to-toe between the rocks, from each rock to the water, around the water, and from the water to the edge of the place, measuring all of these distances. The place is precisely symmetrical.

Ludo sits for a while on one of the three rocks, looking at the water. Ludo sees the image of a young girl in the water, and, believing it to be a reflection, looks up, expecting to see a young girl standing in the place. There is no girl, but Ludo sees that Lemma is standing on the path opposite the rock that Ludo is

sitting on, watching Ludo. Ludo stands up, and Lemma takes a step back. When Ludo begins approaching Lemma, Lemma turns and walks along the path towards another place of the garden.

Ludo follows the path that Lemma has taken, but loses sight of Lemma. Ludo arrives in another place, which looks identical to the one that Ludo has previously been in, and so Ludo does not know whether Ludo has walked in a loop, and returned to the same place of the garden, or if Ludo has arrived in a new place that looks just like the first. That the path that Ludo followed was apparently straight leads Ludo to suspect that this is, indeed, a new place. That it looks so similar to the previous one, though, leads Ludo to reconsider.

Ludo doesn't see Lemma anywhere in this place, though Lemma has only started off down another path, and is watching Ludo from the edge of that path. Ludo realizes that it will get dark soon, and worries that Ludo will have a hard time finding the way back through the woods if Ludo does not go back immediately. Ludo marks an "X" on the path that Ludo has taken to this place, so that Ludo will know after retracing the previous path whether Ludo has been in two identical places or in the same place twice.

Lemma follows from a distance as Ludo returns on the path. When Ludo arrives back at the place in which Ludo started, Ludo immediately marks the path that Ludo has just taken with a "Y." Since Ludo does not see the "X" that Ludo marked in the other place, Ludo concludes that there must be at least two places in the garden that are identical. It is now dark, and Ludo cannot remember from which path Ludo entered the garden. Ludo chooses one of the remaining two paths that Ludo did not mark with a "Y." Ludo looks around before marking this path with a "Z." Ludo realizes that it would have been enough for Ludo to have left the "Y," and to have remembered that Ludo had walked clockwise around the place to reach the path Ludo has now chosen, but Ludo worries that Ludo will forget.

Ludo follows this path out of the place. Because it is so dark, Ludo has a difficult time determining whether Ludo is staying on the path, or straying into the trees. Lemma waits before following the path that Ludo took out of this place. Lemma cannot see Ludo, and before reaching the next place, Lemma realizes that Ludo's scent has faded.

Odel,

Milla admitted that she had not thought that I would find my way into the garden, and was interested to learn that I had determined that there are probably multiple clearings in the garden. According to Milla, there are supposed to be three trails that lead towards the garden from the wall of the garden. The villagers formerly believed that there was exactly one clearing in the garden, and that each of these three trails becomes a path leading to the pool at the center of this clearing. Since it now seems that there are at least two clearings in the garden, the garden must have a more complex structure than Milla and the villagers believed. So, Milla asked what sort of structure the garden might have.

I certainly haven't followed all of the paths in the garden. And, unless the three paths of the first clearing that I visited are the same three paths of the second clearing that I entered, it is reasonable to suspect that there are other clearings that I haven't seen. So, I cannot, of course, claim to know what the structure of the garden is. On the other hand, we can make some general remarks about what the entire garden might look like if all of the paths and clearings are like the paths and clearings that I saw. That is, let's

suppose that all of the paths of the garden share some characteristics, and that all of the clearings in the garden share some characteristics. Then, if we choose good characteristics based on what I saw in the garden, we can see what sorts of shapes the garden might have.

For convenience, let’s assume that there are no dead ends in the garden. That is, every path in the garden ends at points that are also in the garden. If there were a stray path that didn’t do this, and ended somewhere outside of the garden, then we would really know nothing about it unless we followed it to the end. Since we’re only imagining, we might as well ignore these paths.

Another thing which we might assume is that the paths of the garden don't intersect one another. Assuming this is true isn't a very bad thing, since it's true that none of the paths that I've seen so far have intersected other paths of the garden. This I would have noticed. And, notice that making this claim does a great deal to limit what the garden could look like. For example, we no longer need to worry about paths that end at other paths. That is, we know that there is some collection of pools in the garden, and the paths can only end at these. (Of course, if we had any arrangement of paths in the garden, we could force these assumptions to

be true just by naming the places where paths intersect one another or end the pools of the garden, but let’s insist, for now, that only pools are pools.)

This condition also puts some good limitations on the general shape of the garden. For example, if we take five pools in the garden, there cannot be a path between every pair of them. And, if we have two sets of three pools, then there cannot be a path from each pool in one set to each pool in the other set.

I'm sure you've already realized the last thing which we ought to take to be true about the paths, which is that the paths of the garden are all perfectly straight. I admit that I don't like making this assumption, because, though the paths appeared to be straight, they were rather long. So, the paths might actually curve without my having noticed. As long as the paths of the garden don't curve too much, though, the layout of the garden could still be similar to one with perfectly straight paths.

But, this assumption helps a lot! Using it, for example, we can say that every path is determined by the two pools that it connects. If we already know which two pools a path connects, then since the path that connects them is straight, it must be the one path leading directly from one pool to the other. After all, any

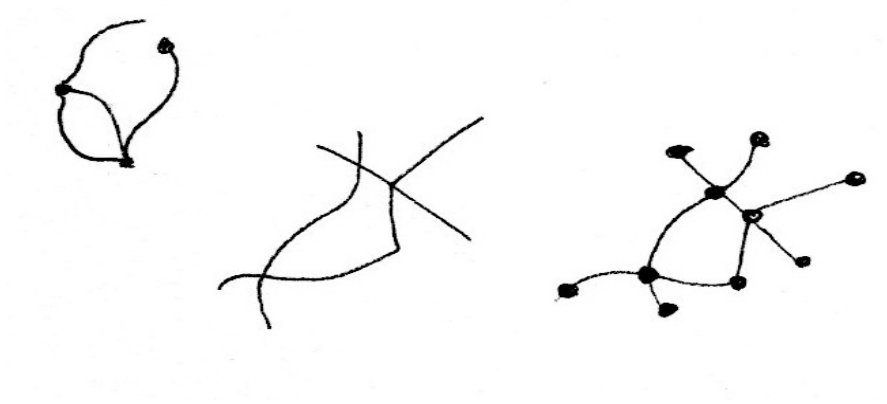
other path connecting these two would have to sit directly on top of this one in order to be straight, and thus, they would really be the same path. (I have assumed that every path connects two pools. What if the path connects a single pool to itself? Surely, this cannot happen if the path is straight. For the path to leave from a pool in a particular clearing, and return to the same pool, it must change direction somewhere. If the path simply left a clearing in a straight line, stopped somewhere, and returned along the line to the same clearing, this stop would effectively be an end of the path that isn't in the garden. We've already decided that we want to ignore such paths, and so for us, this is impossible. So, if the path changes direction somewhere, then it must curve away from the straight line. In other words, it isn't straight.)

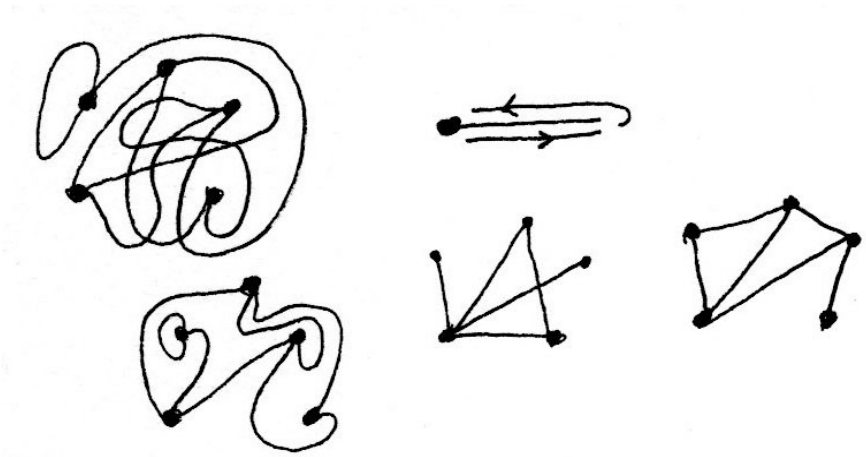
Each of these assumptions makes it much easier to imagine what the garden could look like. If we assume both of these things together, then the garden becomes even more manageable. For example, suppose there is a route in the garden that we could follow so that we would never repeat a path, and would never return to a clearing, except that the last clearing would be the same as the first. Then, this route must outline a polygon, whose sides are the paths of the route we've taken, and

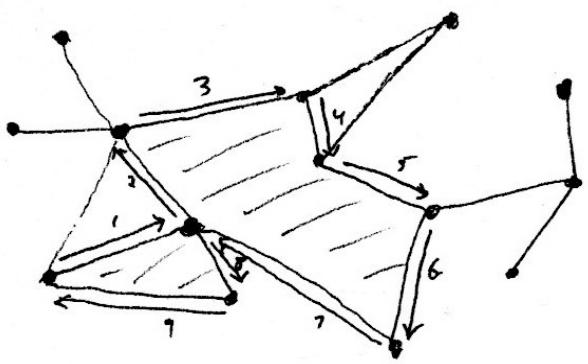
whose vertices are the clearings of the garden that we've passed through. I suppose this is rather obvious. But, we could also say that even if a route passes through some of the clearings multiple times, as long as it never repeats any of the paths in the garden, then it must outline some number of polygons, with no extra paths!

Milla thought these major assumptions to make after my having visited only two clearings of the garden, though she admitted that without such assumptions I couldn't have said anything interesting about the garden. I mentioned to Milla the false assumption that I had made in expecting that the circumference of the pool circumscribed the triangle formed by the rocks. Milla responded that she had imagined that the triangle circumscribed the circle, and so she must be more intelligent than I am, because she assumed what was really the case.

Ludo






3
4
5
2
1
6
7
7

Odel,

I've had trouble in trying to return to the garden. I followed the trail near Milla's house to the wall of the garden, as I did the last time that I went to the garden. What I meant to do was to climb over the wall, and to walk clockwise around the inside of the wall. I wanted to see the other two trails that lead in from the wall. I realize now that I ought to have marked the trails as I passed them.

I climbed up the wall and down on what I believed was the other side, and walked along the wall. The wall seemed entirely straight, and I felt that I was making my way very far away from the trail back to Milla's house. I passed the first trail, and expected that I would soon notice the wall turn, but I did not. When I reached the second trail, I decided to follow it into the garden, rather than continue walking around the wall. I eventually found myself not in the garden, but once more, somehow, back at the start of the trail, near Milla's house.

I must have been extremely inattentive in trying to enter the garden, because I cannot imagine a single mistake that would account for what happened. It seems to me that I must have made at least two mistakes in

trying to get into the garden. Firstly, I must have walked in a full circle around the wall, without even noticing that I was turning, and so returned to the trail from which I had started. Secondly, I must have become confused, yet again, and gotten down on the wrong side of the wall. So, rather than walking clockwise around the inside of the wall, I ended up walking counter-clockwise around the outside.

These are certainly foolish mistakes. On the other hand, this could only have happened because the wall of the garden is so poorly placed. We should keep in mind that the wall should be a signal of the boundary of the garden. It's convenient to think of the boundary of the garden as being an actual place, but this isn't the case. The boundary is what we necessarily cross when we enter and exit the garden. We could never say at a particular moment, however, that we are precisely located at the boundary of the garden.

The wall ought to have been placed over the true boundary of the garden, so that one might look down on one side, and see the woods, and look down on the other, and see the garden. You might argue that because we cannot know where exactly the boundary of the garden is, we could not place the wall on the boundary. Recall, however, that it would be enough for us to know when we

are in the garden, and when we are outside of the garden. In that case, consider the following process:

We begin outside of the garden, and walk towards the garden with measured steps, until we indeed find ourselves inside of the garden. We stop, and then with smaller steps, walk in the reverse direction, until we find ourselves outside of the garden. Though we will never end up at the boundary of the garden, we can, by repeating this process until we only move back and forth by very small distances, determine that we are very close to the boundary. Because the wall of the garden has some depth to it, we can get close enough to the boundary that the wall must lie on top of it. This will ensure that there is only garden on one side of the wall, and only woods on the other.

I mentioned this to Milla, who agreed that the wall ought to have been placed differently, so that it would not be so difficult to determine where the garden is. Milla said that it is too late to change the placement of the wall, however, because it has already been built. She commented that it's odd that the wall is so poorly placed, because she has heard that the wall is, in fact, supposed to have been designed by Milo, one of the earliest residents of Arenne, who was a clever mathematician and architect. Milla said that Milo is

believed to have known the boy that disappeared in the garden, and that as a result, Milo personally took on the responsibility of building the wall. When I asked more about this, Milla said that there are some sketches of Milo's at the school, and she has agreed to show them to me.

Milla told me that my having made at least two mistakes reminded her of trouble that she ran into with Molly one day while she was trying to do an exercise in addition with the class. Milla told the students to each pick up 10 pebbles from the ground, and then to sit down with the 10 pebbles in a pile in front of them. Milla told the students all to take 2 pebbles from the pile and set them to the side. She told them to take another 3 pebbles from the pile, and to set them apart as well. She had the children combine the pile of 2 pebbles and the pile of 3 pebbles, and asked the children to count the total number of pebbles. She told them that since they now had 2 pebbles and 3 pebbles together, the number of pebbles in this new pile was the sum of 2 and 3. Milla asked the children how many pebbles they had counted. All of the children said that they had 5 pebbles, except for Molly, who said that she had 4.

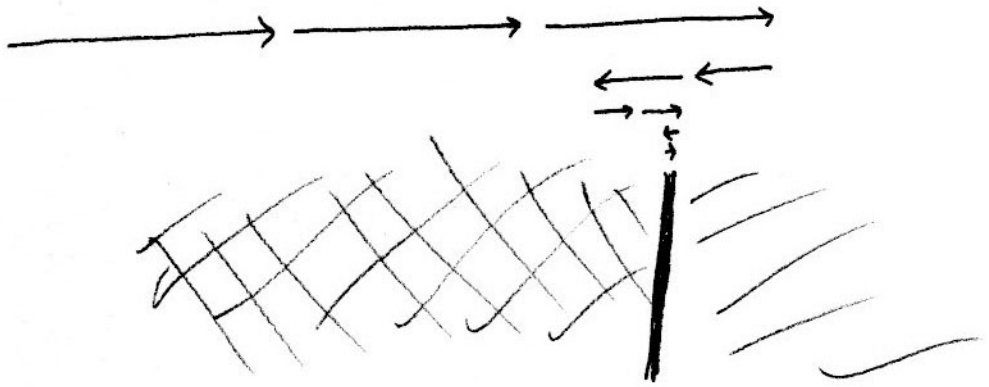
Milla told Molly to count them again out loud, and she did - 1, 2, 3, 4. Milla asked Molly to put these

pebbles back into the large pile of 10, and to start over, counting each batch of pebbles as she took them. She did this for the batch of 2, and then for the batch of 3, but, again, counted only 4 when she put them together. Milla got up, and went to see what she had done with her pebbles. The new pile of pebbles that she had just made indeed had only 4 pebbles. Moreover, the pile of pebbles which she had taken from, which was supposed to have started with 10, now had 5 left--the amount it ought to have had. Somehow, Molly had made a mistake in taking the first 10 pebbles, and then another in counting out the groups of 2 and 3 pebbles, so that the original pile had the correct number of pebbles left in it. Milla would have guessed that Molly was playing some sort of game, except that the girl's face bore no trace of mischief or pleasure.

Milla then put all of Molly's pebbles back into one pile, and told the girl to do the experiment again. This time, Molly counted out 2, and then 3, and put the pebbles together while Milla watched. Finally, the girl counted 5 pebbles. She didn't seem surprised to have counted 5 pebbles this time, rather than 4. Milla asked Molly whether she understood at this point that 2+3=5. Molly asked what that had to do with the pebbles. Milla told her that 2+3=5 means that every time we have 2 of

something, and we take 3 more of the same thing, and put them together, we get 5 of that thing. Molly replied that before she had gotten four pebbles. When Milla told her that she had probably made a mistake, the girl was depressed.

Ludo



Ludo is relieved to arrive once more in the garden. Ludo arrives in the place that Ludo arrived in the first time that Ludo came to the garden, though the "Y" and "Z" that Ludo made in the paths in this place are gone. While Ludo is crawling around on the paths, closely examining each of the three paths that leads to the pool in the center of the place, looking for traces of the "X," "Y," or "Z," Lemma arrives along one of the paths and sits watching Ludo. Ludo notices Lemma.

"For someone who is supposed to be a mathematician," says Lemma, "you look ridiculous."

Ludo immediately stands up and faces Lemma.

"How did you know that I'm a mathematician?" asks Ludo.

"I didn't," says Lemma.

"How did you know that I'm supposed to be a mathematician?" asks Ludo.

"I didn't," says Lemma.

"Then how could you claim that I'm supposed to be a mathematician?" asks Ludo.

"I didn't," says Lemma.

"Then, what did you claim?" asks Ludo.

"I claimed," says Lemma, "that in the case that you were supposed to be a mathematician (which I now gather

you are), you would have seemed particularly ridiculous. Though, to be honest, you looked ridiculous anyway."

"I was trying to determine where I am," says Ludo,

"You're here," says Lemma.

"I mean I was trying to determine which clearing I'm in," says Ludo, "I was looking for some marks that I've made in the paths--an 'X,' a 'Y,' and a ‘Z.'"

"I prefer," says Lemma, "to call them 'places' rather than 'clearings.' Of course, we could call them anything we like. To avoid confusion, though, I think it'll be best if we both refer to them as 'places.'"

This doesn’t seem fair to Ludo.

"Luckily for you," says Lemma, "I can tell you that you're in the place in which you marked the paths with a 'Y' and a 'Z.' In fact, that path is the path that you marked with a 'Y.' And, you're not in the place in which you marked a path with an 'X.'"

"So, they aren't the same place," says Ludo.

"No," says Lemma, “Otherwise you'd have to be here and not here at the same time, and so, you couldn't be at all."

"Didn't I see you the last time that I was in the garden?" asks Ludo. "You ran away from me along the path."

"I certainly didn’t," says Lemma. "I was merely interested to see how you would find your way around the paths and places. It amazes me how awful the people of the village are at it. They can never remember which paths or places they've been to, and always end up wandering off this way or that. I'm surprised that you haven't."

Lemma stretches and stands again.

"Let's go for a walk?" asks Lemma. Lemma begins walking out of the place along one of the paths. Ludo catches up, and the two continue along the path. Ludo tries to walk in as straight of a line as possible, though Lemma occasionally switches from walking on one side of Ludo to walking on the other.

"You've seen the villagers?" asks Ludo.

"Some," says Lemma. "They occasionally wander into the place closest to the village."

"The place closest to the village?" asks Ludo.

"The one you arrived in," says Lemma. "I've never seen a villager in another place whose scent I couldn't trace back to the place in which you arrived (which, for convenience, I call the 'First Place.'). So, for a villager to get to any place, the villager must first arrive in the First Place. In that sense, the First Place must be the place 'closest' to the village. Of

course, the ordinary distance isn't necessarily the shortest distance between a place and the village."

Lemma and Ludo arrive in another place.

"This," says Lemma, "is the 'Second Place.'"

"Is it the second closest to the village?" asks Ludo.

"No," says Lemma. "It's one of them."

"Are they all called the 'Second Place?'" asks Ludo.

"Three 'Second Places!'" says Lemma.

While Ludo begins walking heel-to-toe around the place, Lemma sits by the pool watching. The layout is identical to that of the First Place.

"Are you going to do this in every place you visit?" asks Lemma.

"This is the same as the First Place," says Ludo.

"I suppose you mean 'identical to'," says Lemma. "And I could have told you that."

"Then, how do you tell them apart?" asks Ludo.

"What do you mean?" asks Lemma.

"I mean, how do you know whether you're in the First Place or the Second Place?" asks Ludo.

"I named them!" says Lemma.

"But," says Ludo, "what about when you forget?"

"I don't," says Lemma.

Ludo doubts this, and, looking around at the different paths, says, "I've already forgotten which path we took into this place. We ought to have marked the path we took right away. Aren't there any markings on the paths or places?"

"Never for very long," responds Lemma. "They tend to go away when you aren't paying attention. Besides, they aren't very attractive, and would ruin the symmetry of the places. It wouldn’t be fair. Don't worry, I can lead you back."

Ludo takes turns standing on each of the three rocks, and looking down the opposite paths. Each one is perfectly straight.

"They're perfectly straight," says Lemma.

"Every path?" asks Ludo.

"Every path that I've seen," says Lemma.

"And the places?" asks Ludo.

"No," says Lemma, "I wouldn't call the places straight."

"I meant," asks Ludo, "are all of the places the same as this one and the last? Do they all have three rocks and three paths?"

"I doubt that I've seen every place," says Lemma, "but every place that I've seen has been like this one. If you like, I can show you another place that's got

something rather different.  Only for the moment, of course.  Soon it'll be just the same as the others."

Lemma begins walking down another of the paths. Ludo follows Lemma to another place, though Lemma chooses another path there and continues walking.

"You're fortunate to have arrived when you did," says Lemma.  "This is rather rare."

"What is it?" asks Ludo.

"A worm," says Lemma.

"Worms aren't particularly rare," says Ludo.

"No," says Lemma, "But this a particularly rare worm."

Lemma and Ludo continue through several more of the paths and places. Ludo quickly loses track of the paths that the two have taken.

"Which place is the worm in?" asks Ludo.

"The place that the worm is in," says Lemma.

"What is the number of the place?" asks Ludo.

"It doesn't have a number," says Lemma.

Ludo is surprised to hear this.

"Why doesn't it have a number?" asks Ludo.

"Because no one has numbered it," says Lemma.

"And why haven't you numbered it?" asks Ludo.

"I haven't had any need to," says Lemma.  "I remember where I've been."

"Then, why have you numbered other places?" asks Ludo.

"Those," says Lemma, "are numbers that Dilemma and I used to speak about the places that we had already visited. We could certainly both remember the places that we had been to, but it was easier for us to talk about them by numbering them. Since Dilemma disappeared, I haven't had any need of further numbering."

"Who is Dilemma?" asks Ludo.

"Only Dilemma, I believe," says Lemma.

"I mean, how did you know Dilemma?" asks Ludo.

"Dilemma and I used to explore new paths and places together," says Lemma. "Before Dilemma disappeared, we had been doing so for as long as either of us could remember."

"What happened to Dilemma?" asks Ludo.

"I don't know," says Lemma. "We were in the First Place, and then, Dilemma wasn't."

The two pass through another place.

"And you numbered all of the paths that you visited?" asks Ludo.

"Yes," says Lemma.

"And how many had you visited?" asks Ludo.

"I don't know," says Lemma.

"You mean you've forgotten the number," says Ludo.

"No," says Lemma. "I've never known the number."

"But," says Ludo, "there must have been a last place that the two of you gave a number, and so the number that you gave to that place would be the number of places that you'd already visited."

"Not at all," says Lemma. "The First Place is the last place that Dilemma and I reached before Dilemma disappeared."

"So," says Ludo, "you must have numbered the places backwards. Whichever place you'd most recently reached was the First Place. The one you'd reached before that was the Second Place, and so on. That is, you changed the numbering every time you reached a new place? Why didn't you call the place you started in the First Place, and then count each place as you reached it? Then you wouldn't have had to constantly renumber the places."

"Certainly," says Lemma. "But, neither Dilemma nor I could remember a place we had started in. So, the only way to number all of the places that we could recall having visited was to number them in reverse, beginning with the place that we had visited most recently."

"Aha!" says Ludo, "so, you forgot the place you started in."

"I suppose," says Lemma, "if we started in a place."

"You must have started somewhere," says Ludo. "How many places can you remember having visited?"

"I don't know," says Lemma.

"More than a hundred?" asks Ludo.

"Yes," says Lemma.

"More than a thousand?" asks Ludo.

"Yes," says Lemma. "I can remember visiting many more than that. In fact, I can remember having visited a greater number of places than any number I have considered so far."

"You can remember infinitely many places?" asks Ludo.

"Certainly not," says Lemma. "That isn't a number. Perhaps, I just haven't thought of a large enough number yet."

Lemma and Ludo arrive in the place with the worm that Lemma has seen. Ludo decides to measure the distances between the pool and rocks in this clearing, while Lemma searches for the worm.

"Here it is," says Lemma.

Ludo, content that the measurements of this place are similar to the measurements of the other places that Ludo has been to, walks over to Lemma. Lemma is lying down on Lemma's stomach, looking intently at something on the ground. Ludo kneels down next to Lemma, and sees a

worm. Both ends of the worm are in the ground, and only a small part of its body is visible to Ludo and Lemma. At first, Ludo thinks that the worm isn't moving. Looking closely, however, Ludo sees from a single crease dividing two segments of the worm's body that the worm is slowly moving out of one hole and into the other.

"A worm with large segments?" asks Ludo.

"It's infinitely long," says Lemma. "But, each segment moves out of the one hole and into the other twice as fast as the previous one. The last segment took only a week to cross from one hole to another. The next one should take only a few days. So, in the next week, we should be able to see the entire worm disappear!"

As Ludo watches, the crease reaches the next hole, and at exactly the same time, Ludo can make out another crease leaving the other hole.

"How do you know that will happen?" asks Ludo.

"I've seen them before," says Lemma. "I found this one here a few days ago."

"How many have you seen?" asks Ludo.

"As many as I can think to remember," says Lemma.

"I thought you said that they're rare," says Ludo.

"Yes," says Lemma, "one doesn't get to see them very often."

Lemma and Ludo sit for a while, examining the worm. Occasionally, Lemma leans in to sniff the worm. When it begins to get dark, the crease of the worm has only moved a small way out of the next hole. Ludo stands up.

"I ought to return to the village," says Ludo.

"You'll miss the end of the worm," says Lemma.

"I can't stay here until then!" says Ludo.

Lemma and Ludo make their way back to the First Place.

"Which path leads back to the village?" asks Ludo.

"I suppose they all do," says Lemma. "I've seen villagers come and go along all three."

"All three?" asks Ludo. "Then, what about the places that the paths lead to?"

"The villagers don’t make it to them," says Lemma. Lemma nods towards one of the paths. "Today, you arrived from that path."

Ludo starts following this path.

"Remember to come back," says Lemma, "to see the end of it."

Odel,

The structure of the garden seems to be as simple and as elegant as I could have hoped. Not only is it clear from what I've already said that the paths of the garden are nearly perfectly straight, and that they only meet one another in clearings of the garden. There also appear to be exactly three paths departing from every clearing in the garden. And, the three paths always depart at equal angles to one another. I have not yet had the patience to measure the lengths of the paths. Still, this is a very rigid structure, and there are two possible constructions of the garden that immediately arise from this structure.

The first possibility is that the garden forms a network of regular hexagons. That is, suppose that the paths of the garden all have equal length, and form the sides of regular hexagons. Then, if we think of the pools as being at the vertices of these hexagons, what would each clearing look like? It would have three paths meeting at the pool in the center, and the angle between each pair of paths would be equal. And, each path would lead directly from one clearing to another! Of course, since all of these hexagons would have the same size,

there could only be a small number of them that would fit in the region inside of the wall.

The second possibility is that the paths of the garden form branches. So, from a particular clearing, there are three paths leading out. At the end of each of these paths is a different clearing. In each of these clearings, there are two new paths that branch off to new clearings, and so on. In this case, each of the three paths leading out of a particular clearing leads to an entirely separate part of the garden. There would be no way to get from one of these three sections to another without going back through the clearing that connected all three of these.

This layout, which we might call a tree, seems to allow for the possibility of more clearings in the garden. On the one hand, none of the clearings that we get to from different paths are the same (unlike the case of the hexagons, where we could go in either direction around a hexagon to get to the same clearing at the opposite end). On the other hand, since the paths have no fixed length, the size of each branch of the garden could get smaller and smaller, so that more clearings could fit into the garden. If we imagine that the clearings and paths are infinitely small, then with this layout, infinitely many clearings and paths could fit

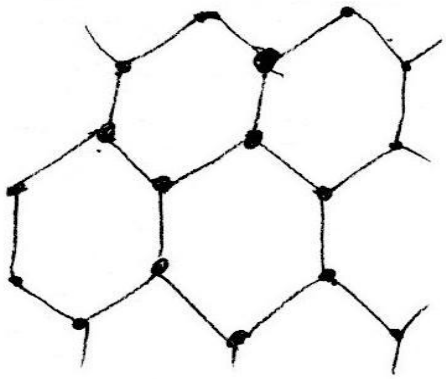
inside the region within the wall. We must insist, however, to be reasonable, that the paths and clearings are big enough to fit people, and so, although there can be many more clearings and paths than in the hexagonal framework, there could not be infinitely many.

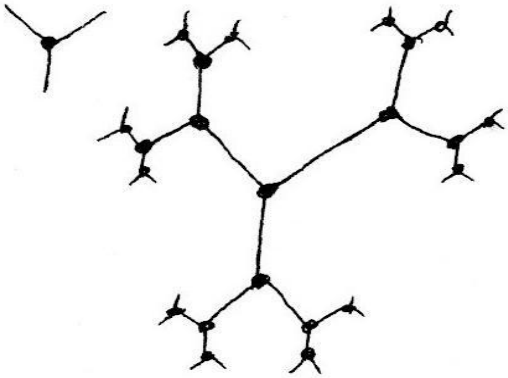
The garden, of course, could be a mixture of these two layouts, with some branches and some hexagons. Some of the clearings and paths could also be different from the ones that I've seen. Honestly, I cannot yet know whether one of these would be the actual layout of the garden. When I return, however, I will at least be able to measure the distances along some of the paths. If they are all the same, then there is a better chance that the garden is hexagonal. If they differ, then it will be more likely that the garden is a tree. In either case, the layout is significantly more elegant (as I pointed out to Milla) than a single clearing with three paths.

The question of how many clearings there are in the garden still remains. It would seem that since there are many clearings in the garden, it is more likely that the garden branches off into smaller sections. Yet, I would never have imagined that there were so many clearings in the garden as I have seen. So, I cannot trust to be able to estimate how many clearings could be in the garden. Nor can I know if I've really visited most of them.

Perhaps I've only visited a small fraction of the clearings. If there were infinitely many clearings, for example, then I would have visited approximately none of them. Of course, this none would be slightly greater than the none of the clearings that I had visited before arriving in Arenne.

Ludo





Odel,

Milla wanted to visit the garden with me today, but Molly's parents arrived unexpectedly at the house early in the morning. The villagers have decided to search the woods in the area, in case Molly has wandered into them. Since Milla lives so close to the woods, they asked if they might stop and rest at Milla's house throughout the day. So, Milla had to stay in the house.

I asked Milla whether the villagers would be searching for Molly in the garden. Milla said that the wall is too high for Molly to climb, and so it is very unlikely that Molly could be in the garden. Since the villagers tend to avoid the garden anyway, Milla doubts that any of the villagers would think it worth looking for the girl there.

The garden has been in the vicinity of Arenne for longer than anyone in the village can remember, and the villagers have yet to determine the layout of the garden. Moreover, the villagers are too frightened of getting lost in the garden to go near it. We know, however, that the garden is somewhere in the area surrounded by the wall, which is a bounded region; that is, we could see, on a map, a scaled-down image of the whole area inside of the wall. So, in a very basic way, we know that with

enough time and people we could determine exactly what the garden looks like. Since people take up and can see a certain amount of space around them, with enough people we could fill up the entire region bounded by the wall, and so determine every aspect of the garden.

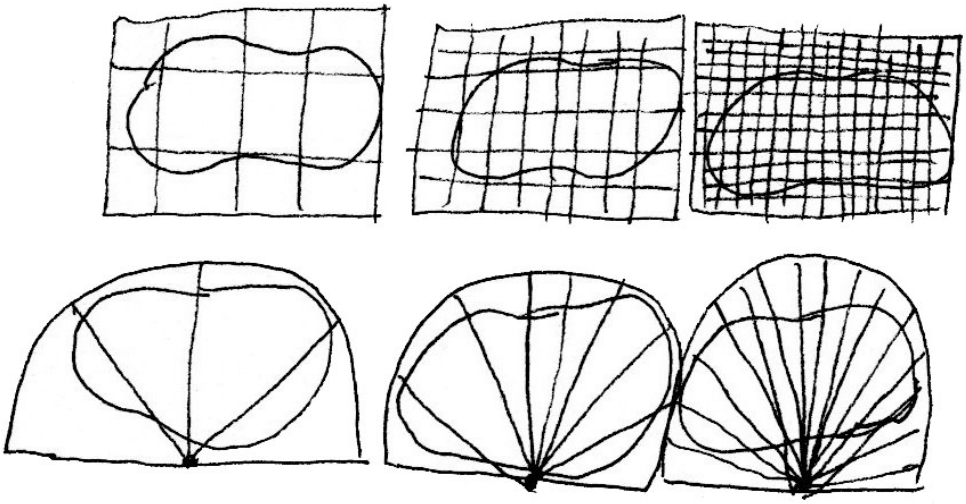
I have determined several methods, however, that would require only a few villagers and a few days. By measuring the distances that they have traveled, and the angles between their routes, the villagers can split up the garden into small regions to search. If they do not find Molly on the first pass, the villagers can divide the area within the wall into smaller regions to search, and do this again. The villagers can repeat this process, dividing the area within the wall into smaller and smaller pieces, until each piece is so small that the villagers must know where every part of the garden is. Moreover, the probability of finding Molly at each pass increases, and so it is not likely that the pieces would become as small as the area that a single person takes up.

I mentioned these methods to some of the villagers that were searching in the woods. The villagers responded, as Milla had predicted, that it was not likely that Molly could have gotten over the wall, and that

their time would be better spent searching for Molly where she is likely to be found.

I tried to return to the garden today, but failed again. I did not return until the evening, when the villagers had stopped searching for the day. Milla said that she has been asked to conduct a careful search of the school, to see if she could find any trace of Molly there. Milla invited me to accompany her to the school tomorrow, so that I might see some of Milo's sketches. She thinks that there might be something in these that will help us to determine the layout of the garden.

Ludo



Odel,

I accompanied Milla to the school today, and she showed me Milo's sketches. They indeed appeared to be rather old, and looked as though they had gotten wet. Only some of the sketches have "Milo" written on them.

In the pile were what appeared to be sketches of the wall of the garden. These were done from various perspectives, so that in some of the sketches, a small section of the wall took up the entire sheet of paper. Others seemed to depict aerial views of the wall, though these too were somewhat strange. In some, the wall seemed to be perfectly straight, and led off the end of the page, or else, appeared to be perfectly circular. In some, the wall would twist and turn around obstacles, perhaps trees in the woods, though there did not appear to be any sensible choice of the directions in which the wall would diverge. In others, the viewer was so far from the wall, that the entire wall appeared as a single point on the page!

There were also notations on the sketches that suggested that Milo was responsible for leading in the construction of the wall. There were various figures computed for the quantities of people, materials, and time that would be needed to build the wall. Not only

that, but Milo appears to have computed all different variations of these quantities. Milo noted, for example, that the longer the amount of time that was spent on building the wall, the less work that would need to be accomplished on the wall each day. So, if the people simply waited infinitely long, the wall would construct itself. Or else, if the wall were infinitely thin, then it wouldn't require any materials, and so the people building the wall wouldn't have to worry about carrying these materials. Not only that, but they could easily make the wall as high as they liked.

Milla pointed out there were also some diagrams, similar to the ones that I recently drew regarding the shape of the garden. Milo appears to have been interested in coverings of the plane with polygons. In particular, Milo was apparently interested in tessellations, coverings of the plane with tiles of repeated shape. Some of the diagrams were crossed out, and so Milo seems to have been experimenting, or searching for something in particular. Finally, there was a diagram of a hexagonal network. Milo's notations indicated that this was what Milo was looking for. Milla agreed with me that it seems likely that Milo was interested in the garden.

When I asked Milla where the sketches had come from, Milla said that it was Molly that found the sketches. She said that Molly returned after lunch one day, and brought the sketches to Milla. At first, Milla believed that they were drawings that Molly had done. She soon realized from the complexity of the notations on the pages that Molly could not have done them herself. When Milla asked where Molly got the sketches, Molly would only laugh and say that she got them from the well.

We spent a while searching in and around the school for any clues as to where Molly had gone. Searching for Molly with no idea as to where to look isn't very practical. By the time we finished, I had looked in some of the drawers, and Milla had checked on the roof for the girl. There was no trace of Molly, but Milla showed me some drawings that the girl had done herself. These showed the well, what appeared to be the garden (with the triangle outside of the circle), and a black and white dog.

Milla said these were fantasies prompted by Molly's discovery of the sketches. According to Milla, most of the children of the village are afraid of the garden, undoubtedly because of the frightening stories they're told by the older villagers. After Molly found Milo's drawings, however, she enjoyed pretending that that she

could play in the garden with Lemma, who Milla said is the dog in Molly's drawings. When Milla would ask how Molly could get into the garden, the girl would explain that it was easy, and that she had figured it out from Milo's maps that she had given to Milla. When Milla asked Molly about some of the computations done on the sketches, however, it was clear that the girl had not actually understood them.

I asked Milla if we might take Milo's sketches back to Milo's house, and she said that it would be a good idea. Milla thinks that Molly might have been right, at least, in that the sketches could be useful in determining how to get into the garden.

Ludo

Odel,

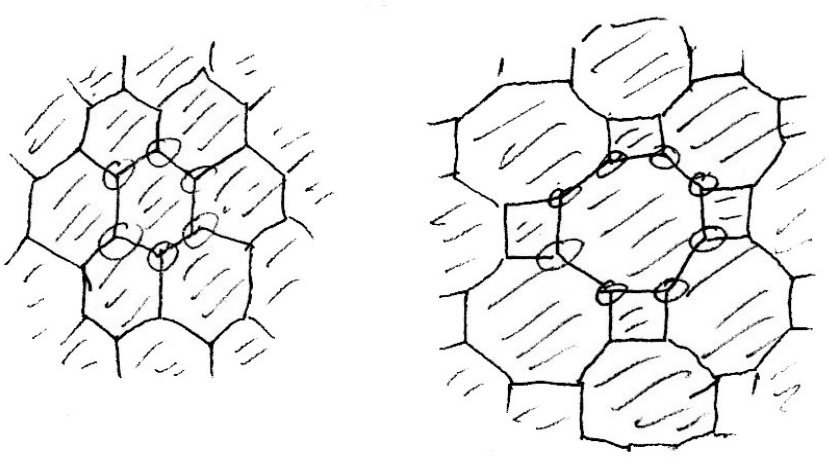
Milla and I have not been able to determine what Milo's intentions were in drawing the various sketches of the wall. There is no indication as to which construction was eventually used. I've determined, however, that Milo was correct in determining that the hexagonal network is an optimal arrangement of the garden. Recall that we saw that the paths of the garden determine different regions within the garden. We supposed then that the paths were straight, so that these regions would be polygons but, for the moment, we don't care whether the paths are straight. Suppose that we only want to arrange the paths in such a way, straight or curved, that all of the regions outlined by the paths have some fixed size. But, we want to do this with the smallest total amount of path in the garden. Then, it turns out that a hexagonal plan is the best way to do this!

It seems clear from the care taken in drawing the diagrams, and from the notations written alongside, that Milo was deeply interested in the mathematical properties of the drawings. It is likely, then, that Milo would have realized the hexagonal tessellation's relationship to the optimization problem I've stated. If this was

indeed the problem that Milo was investigating, then perhaps Milo designed the garden. After all, such a design would certainly have been convenient to build, since it requires the smallest amount of paths in the garden.

I have tried to determine, therefore, whether this might have been the solution to a different problem. For example, perhaps Milo only explored the garden, and was trying to determine, as we are, the overall structure of the garden. It occurred to me that Milo might have deduced this result from assuming that the structure of the garden was a tessellation, and observing that every clearing in the garden has three paths leading out of it. Is it the case that the hexagonal pattern is the only such tessellation? Not quite. Milla provided a construction consisting of octagons and squares.

Ludo



Odel,

Milla and I have decided that we ought to work together to determine the shape of the boundary of the garden. In order to be able to reach the garden consistently, we must have a clear picture first of the overall shape of the garden, and then of the particular ways of getting into and out of the garden. This will help us avoid some frustration in the future.

We decided first to determine the shape of the wall of the garden. We took the trail near Milla's house to the wall of the garden. We were able to climb atop the wall using one of the nearby trees. Then, I walked in one direction along the top of the wall, while Milla walked in the other. The wall must turn very slowly. In fact, I never noticed any curvature of the wall as I walked along it, and so I assumed for a long time that the wall of the garden must be rectangular. I eventually met Milla, however, walking in the opposite direction. I was surprised to hear that she, too, had not noticed any curvature in the wall. Yet we met each other! This tells us that the wall must form a loop, and so it must curve (though, too slowly for us to have noticed). This would normally suggest that the region within the wall is gigantic, yet it didn't take Milla and me too long to

walk around the wall, and so it cannot have been too large. In any case, the wall must have been constructed according to one of Milo's simpler designs, if any.

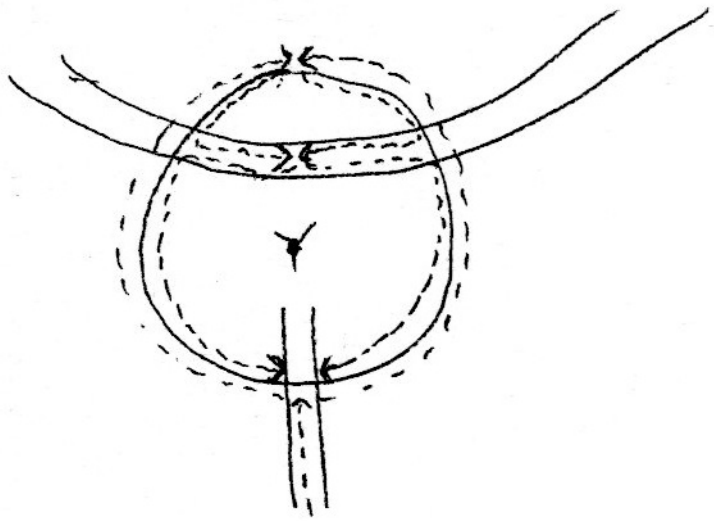
Milla and I had also each noticed, in walking around the wall, a trail that the wall crosses over. This, in addition to the trail that leads to Milla's house, suggests that there are indeed three trails leading from the wall towards the region inside the wall on one side, and into the woods outside of the wall on the other. Milla guesses that the two other trails lead out of the woods to nearby villages. We decided that we would each return to the trail that we had passed, and follow it on the inside of the wall. We expected that we would each arrive in a clearing of the garden, though we didn't know whether to expect that these would be the same clearing. We agreed that when we arrived in a clearing of the garden, we would wait for the other only for a while, since the other might be waiting in a different clearing.

We were, however, mistaken. We each returned to the trail that we had passed, and followed it. Neither of us came to a clearing. Instead, we met somewhere in the middle. Milla found this amusing. These two trails must form a single trail that crosses the region within the wall without passing through the garden. That is, the trail likely starts at the edge of the woods at some

nearby village, crosses the wall to the inside, crosses back to the outside, and leads to some nearby village on the other side of the woods.

We decided that in leaving the garden, we would each continue along the trail in the direction that we had started. That is, Milla would go in the direction that I had come, and I in the direction that she had come. We would then return along the wall to the trail that led back to Milla's house, meet each other there, and return. I did not mention to Milla that I strongly doubted that the wall could have been as straight in the direction that she had walked as it had been in the direction that I had walked. Apparently, it is.

Ludo



Ludo arrives accidentally in the garden. Ludo looks around for Ludo's friend, who was exploring with Ludo just before Ludo arrived on one of the paths leading to the First Place. Ludo sees Lemma in the place, who is lying down on Lemma's stomach in front of the water. Lemma jumps up when Lemma sees Ludo, and trots up to Ludo.

"Let's go," says Lemma. Lemma immediately sets off down one of the paths, and Ludo runs to catch up.

"You're very lucky," says Lemma. "I was certain you'd miss the end. Or, I suppose I should say 'I thought that I was certain you'd miss it.' I certainly thought you'd miss it."

"I tried to return," says Ludo. "But I had trouble. The way here from the village is difficult to find."

"You villagers," says Lemma, "have no sense of direction."

"Do you know the way then?" asks Ludo. "Have you visited the village? And, I'm not a villager."

"I," says Lemma, "have no interest in going to the village. It sounds ugly and poorly designed. I much prefer exploring the paths and places. You never know what a new place will look like, and yet when you arrive at one, it looks just as you might have hoped. I can't

understand why the villagers always seem horrified to find themselves here."

"How do they get here?" asks Ludo.

"I assume," says Lemma, "that they get here by accident."

"You've never asked?" asks Ludo.

"I prefer not to speak with the villagers," says Lemma. "I find their inconsistency tiresome."

"Then," says Ludo, "how do you know that they're villagers?"

"All they speak to each other about," says Lemma, "is getting back to the village."

Ludo and Lemma arrive in the place that the worm is in, and Ludo walks around the place while Lemma searches for the worm.

"Did you know," asks Ludo, "that the hexagonal structure of the paths is an optimal organization? It..."

"Of course," says Lemma, "it minimizes the total length of the paths for a fixed area of each region bounded by the paths. Come quickly!"

Ludo joins Lemma on the ground in front of the worm. The two ends of the worm are still in the ground, and Ludo and Lemma watch as a segment slowly moves from one side over to the other. The next moves twice as fast as

the last. It becomes increasingly difficult for Ludo to watch the individual creases move across from one hole into the next. Soon, Ludo cannot distinguish the separate segments of the worm, and finally, the worm is gone. Ludo and Lemma sit for a while, looking at the empty mouths of the two holes.

"Where does it go?" asks Ludo.

"Perhaps to another place," says Lemma.

"If it's infinitely long," says Ludo, "then it must inhabit infinitely many places. Otherwise, it would fill up all of the places it was in. Perhaps you've only ever seen this one worm."

"Unless," says Lemma, "I was mistaken, and it isn't infinitely long."

"How could that be?" asks Ludo. "Each segment has equal length, and there are infinitely many segments."

"It might be that the segments extend while they are out of the ground. So, while it is in the ground, the worm has infinitely many segments, but each one is shorter than the last, so that the entire worm is only finitely long. After all, when it enters the next hole, the worm must contract the front of its body to pull in the end. Then, each of these segments would extend momentarily as it was pulled from one hole to the next, before contracting again on the other side."

"It must not enjoy being out of the ground," says Ludo, "since it’s always hurrying back in."

Lemma stands to stretch and walk around the pool, while Ludo lies down, exhausted from running to the place with the worm.

"There ought," says Ludo, "to be a faster way to get here from the village. It's a pain having to find the way from the wall each time."

"Be careful," says Lemma. "Dilemma was looking for a faster way to travel as well, just before disappearing."

"To travel to the village?" asks Ludo.

"Certainly not," says Lemma. “The villagers didn't start appearing in the First Place until Dilemma disappeared from it. I doubt that Dilemma would have been able to tolerate them. Dilemma thought that there might be a way to travel between the places, without taking the paths."

"Did Dilemma find a way to do it?" asks Ludo.

"I don't know," says Lemma. "I didn't believe it was possible, until Dilemma disappeared. And then, when Molly started appearing in the First Place, without having taken any of the paths."

"Molly!" says Ludo.

"Yes," says Lemma. "Molly was peculiar, and not only because of her smell. Whenever I came upon Molly,

Molly would already be playing in the First Place, splashing the water from the pool, or digging in the dirt. I couldn't detect Molly's scent along any of the paths."

"But," says Ludo, "I thought you said that all of the villagers arrived along one of the three paths."

"I thought that they did," says Lemma, "and so I've assumed that Molly isn't a villager. Molly never had the difficulties that the villagers have in getting here."

"Molly is from the village," says Ludo, "and has disappeared. The villagers are searching the entire village for Molly, but without success."

"Molly would only tell me," says Lemma, "that Molly had come from the water. It's strange that Molly has disappeared from the village, because Molly hasn't been in the First Place recently. I assumed that wherever it was that Molly came from, Molly had decided to stay. I did think it unusually rude of Molly not to have said 'goodbye.'"

"I ought," says Ludo, "to figure out how Molly was able to get here from the village."

"Or else," says Lemma, "you could simply stay here. It would make it much easier to get here. It's not such a pain navigating the garden. And, there's much exploration of the garden to be done."

"Then," says Ludo, "why don't you explore? You're always near to the First Place, and so you cannot travel very far from it."

"It's best," says Lemma, "not to explore the paths and places alone," says Lemma.

"Because they're dangerous?" asks Ludo.

"Very dangerous!" says Lemma. "With no one to check one's thinking, one runs the terrible risk of missing a gap in one's reasoning. Imagine arranging a route to visit every place and path, and making one small mistake. Depending on the route that one has designed, one could end up missing an infinitely large number of the paths and places."

"Isn't there anyone else?" asks Ludo.

"There used to be," says Lemma. "Dilemma and I used to meet others that had wandered onto paths or into places. After Dilemma disappeared, I tried exploring on my own, expecting that I would occasionally meet someone else, with whom I might check my thinking. At first I thought that I was simply unfortunate in not seeing anyone. Soon, however, it became highly improbable that I was just experiencing the improbable. There doesn't seem to be anyone else anymore, except for the villagers in the First Place, and Molly."

Ludo stands up.

"I've forgotten someone," says Ludo, “my friend from the village."

"Just like a villager," says Lemma. "You might as well stay--your friend has probably already forgotten you, too."

"I must go," says Ludo.

"All right," says Lemma. "Then, go ahead."

Ludo looks around the place, uncertain of where to go. Lemma begins walking towards one of the paths.

"This way," says Lemma.

Odel,

I have now determined that because the garden does have a hexagonal structure, there is no perfect way of exploring the garden. That is, there is no route that we could take around the garden so that we could be sure of visiting each path exactly one time. In fact, any route that uses every path of the garden must visit one of the three paths connected to each clearing twice. This is clear when we consider the following.

Suppose that there is a clearing that we have not yet visited. The clearing has three paths that lead into it. We must take one of the paths to get into the clearing. We must take a second path to leave the clearing. Then, the only possible way for us to take the third path would be for us to take it from some other clearing back into this clearing. So, we would find ourselves inside of this clearing, having taken all three paths. The only way for us to leave this clearing again would be to take one of the paths for a second time. Thus, any route around the entire garden must retrace some paths.

This is troublesome. It means that if one wants to visit the entire garden, one cannot hope to do it in such a way that one can avoid visiting some of the paths

multiple times. Notice that if such routes did exist, then they would all take equivalent amounts of time to travel. This would simply depend on the total distance of paths in the garden, and the speed at which one travels. If we want to compare routes that do repeat some paths, however, we have to compare how often each route is forced to repeat paths in order to determine an optimal way of exploring the garden.

This would be true for other layouts of the garden as well. You've probably noticed from the proof I've just given that it isn't important that each clearing has three paths leading into it. What matters is that each clearing has an odd number of paths leading to it. So, the proof follows for any layout of the garden where some of the clearings have an odd number of paths. To avoid repeating paths, for each path we take into a clearing, we would like to take one that we haven't taken back out. If the number of paths leading to the clearing is odd, then we cannot pair paths that we take in with paths we take out. We will be left having taken the last path into the clearing, and be forced to repeat a path to get back out. If the number of paths is even, however, then every path we take into a clearing can be paired with one we take back out, and so we don't need to repeat paths.

(We must think of loops as two paths leading to the same clearing.)

I intended to show Milla that, in general, there is a way to visit every path of the garden exactly once precisely if all of the clearings have an even number of paths leading to them. Milla, however, is still upset that she was unable to get into the garden. Apparently, when we were separated the last time we were exploring the area around the garden, Milla tried to find the garden on her own. She followed one of the trails on the inside of the wall, and it took her to the wall. So, she walked around the inside of the wall to the next trail, and followed that one. She did this repeatedly, occasionally changing directions, and could not understand why all of the trails seemed to arrive at the wall. Recall, after all, that we had already determined that there are only three places where the wall crosses the trails. Milla eventually gave up, and returned to the house. She was jealous to hear that I had accidentally arrived in the garden.

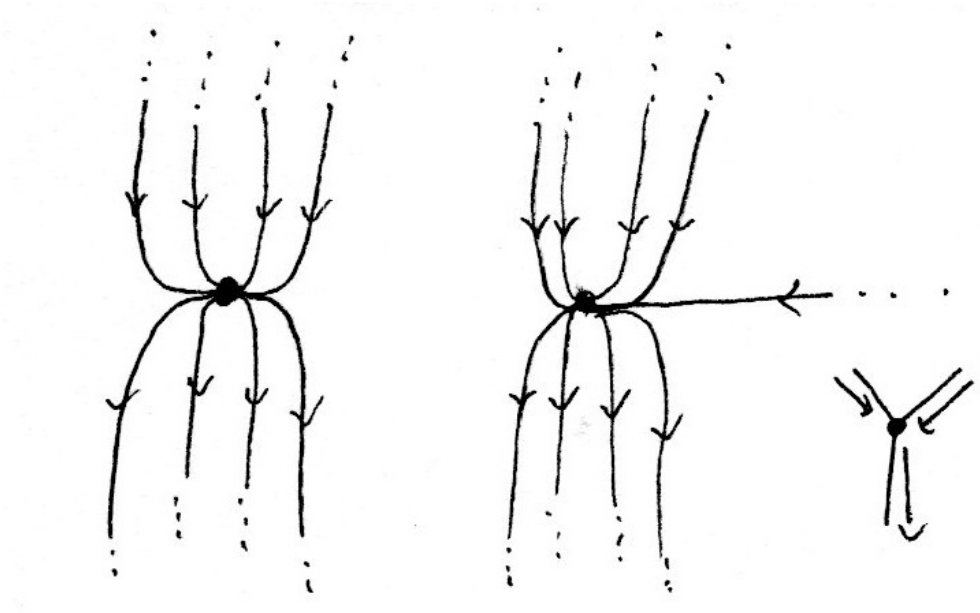
Milla says that she has been trying unsuccessfully to get into the garden for a long time. Milla said that she can remember that she was in the garden once when she was very young. Milla remembers being immensely happy,

though she cannot remember any details of what she had done in the garden that was so amusing.

When Molly showed Milla the sketches, Milla says that she was, in fact, even more excited than Molly. Though Milla did not expect that the sketches were maps, she hoped that she might be able to use them to determine how to enter the garden.

I mentioned to Milla that it is very likely that Molly was, in fact, able to get into the garden. I explained to Milla what I saw in the garden. It is unbelievable to me that this, along with what Milla has told me about Molly, could have been a coincidence. I pointed out that if the young girl could manage to get into the garden, then Milla probably could as well.

Ludo



Odel,

Milla decided to visit Molly's parents today, and I agreed to accompany her. Apparently, Molly's family lives towards the center of the village, near the well. The houses in this area look a bit different from the rest of the houses of the village, and Milla said that this is because they are much older. I noticed that there were no children playing around the well today. Milla says that this is because most of the children are now forbidden to play near the well. The villagers have associated the well with Molly's disappearance, even though Milla and Molly's parents did not find the girl there.

When we arrived, Milla's parents invited us into the house. They looked unhappy, though they thanked us for coming to visit. We sat for a while, and Molly and Milla's parents began discussing the search for Molly. They agreed that there is still hope of finding her. I pointed out that since there is evidence that Molly has been in the garden before, it is entirely possible that she is there now. That the villagers refuse to search out the area inside the wall and the garden, therefore, is a foolish omission. I added that it is possible that

in my investigations of the garden, I will happen to find Molly.

Molly's parents asked why I thought that Molly had been in the garden. I told them what I've mentioned to you about Molly and the garden. Molly's parents are familiar with the girl's interest in the garden. They said that Molly constantly pretended that she could get into the garden. Milla mentioned that Molly would tell her about playing in the garden with Lemma. Molly's parents said that Molly had said the same to them. They said that once, to be playful, they told Molly that they had seen Lemma as well, walking around the village. She replied, however, that they were clearly mistaken, because Lemma did not come to the village. She said that they must have seen Dilemma. When Molly's parents asked her who Dilemma was, the girl told them that this was the boy from the story. Presumably, Molly misunderstood someone's reference to the boy's dilemma. Molly imagined that Dilemma had been Lemma's companion, but that the two had had an argument about Dilemma visiting the village through the well. Dilemma became so upset, that, after going to the village, Dilemma forgot how to get back into the garden. Molly apparently confused the story, thinking that the boy was trapped outside of the garden, rather than inside of it.

Milla and I asked where Molly had gotten the idea of Dilemma going through the well, and Molly's parents told us that the girl had found some drawings done of the well. We asked to see them, and Molly's parents brought out what appeared to be more sketches that had been done by Milo. We mentioned that Molly had also brought such sketches to Milla. Like us, Molly's parents weren't sure where Molly had found the sketches. These looked much like the sketches that I have described. There were what appeared to be more depictions of the wall in various forms, though Molly's parents said that they believed that they were sketches of the well in the center of the village, and that Molly had claimed that they were. Looking closely, I noticed that there seemed to be some shading on the paper that might have suggested water, though I had supposed that these were only smudges. I tried to determine from Milo’s notations whether these were supposed to be of the wall or the well. In Milo's various computations, however, Milo had experimented with the wall or well having both gigantic and miniscule proportions, and so it was impossible to determine which of the two Milo had had in mind.

Molly's parents said that it was after finding these that Molly became obsessed with the well, and would spend hours pretending that she could go into the well. She

would often go there to talk into the well, and this was why Molly was so often to be found playing around the well.  They said that though Molly hadn't fallen into the well, they ought not to have let their daughter play so close to it.

Ludo

Odel,

Milla agreed, along with a few others of the villagers from Arenne, to visit some of the nearby villages. They hope that some of the villagers in the other villages will have seen the girl. Otherwise, they will, at least, be able to notify the neighboring villagers that the girl is gone, and to ask the villagers to search for her.

I decided, in the meantime, to visit the well in the center of the village. I realize that Molly probably could not have reached the garden through the well. Yet, for the reasons that I've mentioned, Molly has undeniably been able to enter the garden, and it is almost as difficult to believe that Molly could have gotten all the way over the wall and into the garden from the trail without anyone noticing as it is to believe that she could have gone through the well.

In any case, there is a chance that Milo was interested in the well. I thought that I might be able to find a spot around the well where Molly had noticed the sketches. I looked for some sort of secret compartment of the well, or for any loose parts, but I couldn't find any. It seems that since Milo was so skilled, Milo could have placed a tunnel or secret

passage in the well that would lead all the way to the garden. I could not find any trace of this either. I considered the possibility that it would be at the bottom of the well, but a young girl probably could not have survived falling to the bottom, let alone managed to climb back out. Moreover, this would still not account for the sheer distance that Molly would have to cover in order to get to into the garden.

When none of the villagers were around, I called out into the well. There was no response, except, of course, for my echo.

Ludo

Ludo follows a path to the First Place, and, looking around, sees that Lemma isn't there. Ludo walks up to the edge of the pool in the center of the place and looks into the water, but cannot see the bottom of the pool. Ludo bends down next to the water and reaches in, but cannot feel the bottom. Ludo then sits at the edge of the water, and begins sliding into the water. Ludo has still not reached the bottom of the pool when Ludo is entirely in the water. Ludo takes a breath and swims downwards.

It is too dark in the water for Ludo to see, but Ludo reaches out for the bottom of the pool. Ludo touches it, and has time briefly to feel around the bottom before needing to surface for air. When Ludo comes above the water, Ludo cannot remember from which direction Ludo has come to get into the First Place. Ludo sees Lemma standing at the edge of the water, watching Ludo.

"What did you find?" asks Lemma.

"Nothing of interest," says Ludo.

"If you're looking for something of interest," says Lemma, "I've discovered an excellent proof..."

"Not anything of interest," says Ludo. "I'm looking for a particular thing of interest."

Ludo takes another breath, and dives once more for the bottom of the pool. Moving more quickly, Ludo is able to feel around more of the bottom. Since it is so dark, Ludo does not know whether Ludo is feeling the same part of the pool or a different one. Ludo returns to the surface, and sees Lemma walking around the pool.

"Could you stop moving?" asks Ludo. "You're making it impossible for me to keep track of where I'm searching."

Lemma turns and begins to walk out of the place.

"Stay!" says Ludo. "I meant that you could be more helpful if you would stop moving."

Lemma returns, and sits.

"I suppose you're right," says Lemma. "You ought to say what you mean, since, either way, I have to assume you mean what you say. What is it you're looking for?"

"A passageway," says Ludo, "that Molly might have been able to take here from the village."

Lemma stays sitting while Ludo dives several more times, without finding anything that might be a passage under the water. Ludo swims to the edge of the pool, close to where Lemma is sitting, and climbs out of the water.

"There's nothing," says Ludo.

"I wouldn't expect so," says Lemma. "I've already checked."

"For a passage?" asks Ludo.

"Yes," says Lemma. "Dilemma thought that there might be a way to travel through the water. The last time that I saw Dilemma, Dilemma was sitting leaning over the water, staring into it. Naturally, when Dilemma disappeared, I checked in the water."

"Molly," says Ludo, "claimed to have been able to get here through the well. I thought there might be an underwater passage from the well to the pool."

"Yes," says Lemma. "Molly told me that she came from the water. I mentioned that Dilemma had thought that there was a way to travel through the water, and Molly said this was true, and that she had seen Dilemma in the water. Molly tried to show me Dilemma in the water, but, of course, there wasn't anyone there. Molly had a peculiar imagination. Let's go for a walk?"

Ludo and Lemma rise, and begin walking along one of the three paths.

“Molly,” says Ludo, “claimed that you had an argument with Dilemma.”

“I never told Molly that,” says Lemma.

"Molly must have imagined it then," says Ludo.

"No," says Lemma. "I had an argument with Dilemma before Dilemma disappeared, though I didn't mention this to Molly."

"About traveling through the water," says Ludo.

"When we reached the First Place," says Lemma, "Dilemma became frustrated, and said that we moved too slowly, and that if we didn't find a faster way to go between places, then we would never be able to reach distant places."

"But," says Ludo, "you had already been between distant places. And given an arbitrary amount of time, you could have reached arbitrarily distant places."

"Dilemma argued," says Lemma, "that since we had seen the same kind of place so many times, we would probably only encounter more of the same types of place by walking along the paths. Dilemma worried that there were infinitely distant places that we wouldn't be able to reach by continuing in the same way."

"Why did this occur to Dilemma in the First Place?" ask Ludo.

"Why not?" asks Lemma. "I told Dilemma that there would be no way to reach infinitely distant places, because one couldn't cover an infinite distance in a finite amount of time. Dilemma pointed out that the worms are able to travel infinitely quickly, and stopped

speaking to me.  Dilemma began drawing all over the paths, searching for another way to travel, until Dilemma gave up on these drawings, and then became fixated on the water."

Ludo and Lemma pause in another place.  Lemma walks alongside Ludo, as Ludo walks heel-to-toe around the place.

"How could Molly have known?" asks Ludo.

"After I mentioned Dilemma," says Lemma, "Molly seemed aware of the argument without my having told Molly.  When I asked Molly how Molly knew, Molly would only laugh, and tell me not to worry, that Dilemma wasn't upset, and had only forgotten how to get back through the water."

Odel,

The villagers have stopped searching for Molly. I went with Milla to a meeting of the entire village, today, that was meant to decide how to proceed in the search for the missing girl. The meeting was held in the school. Along the way, I could see villagers coming from all over the village, and when we arrived, Milla said that there were members of every household present at the meeting. To begin the meeting, the villagers were asked whether anyone had found any new traces of Molly. Apparently, no one had.

The villagers added the places that they had most recently searched to a list that is being kept. The list was then read out loud. The villagers were asked whether there were any ideas of places that Molly could be in the village that had not yet been searched. A few of the villagers made suggestions of places that had not been specifically mentioned on the list, though other villagers responded that they had already checked these places once, or multiple times. None of the villagers could think of a place that had not been searched.

I pointed out that the villagers had not searched for Molly in the garden, and that this is obviously where the girl must be. The villagers asked how this is

obvious, and I replied that the girl is not in the village and that the girl must be somewhere nearby. The villagers asked how I thought such a young girl would have been able to climb the wall of the garden to get into the garden. I told the villagers that Molly might have been able to get into the garden by other means, though I avoided suggesting what I suspected these means were. The villagers dismissed this, and sat for a long time in silence. After a while, the villagers began leaving the school. Some of the villagers approached Molly's parents first before leaving. Milla stayed seated until all of the other villagers, except for Molly's parents, were gone. I left, and waited for her outside.

Milla seemed upset when she came out. We stopped by the well on our way back to the house, and Milla spent a while looking at the water. When we arrived back at the house, Milla took out and began analyzing Milo's sketches. She said that I was right, and that if Molly isn't anywhere else, then she must be somewhere in the garden. I helped Milla study the sketches of the wall, or the well, that Molly had given to Milla, along with those Molly's parents have given us. This was, perhaps, a mistake, because the two sets of sketches look very

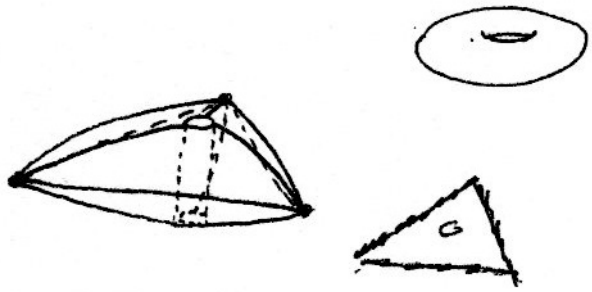
similar, and it's likely that we confused them in the process.

Looking at the sketches which could have depicted either the well or the wall, it occurred to me that the well and the wall might have the same function. I have certainly been able to go into the garden by going over the wall. Suppose, then, as Molly did, that it is also possible to get into the garden by going through the well. This suggests that there are two boundaries of the garden, the larger one being in the region of the wall, and the smaller being in the well. That is, they are two parts of the boundary between the village and the garden.

Then, the dihedral group does seem to apply to the boundary problem. Imagine that the village is on one side of the triangle, and the garden is on the other. Around the edge of the triangle is the wall of the garden, and in the center of the triangle on one side is the well, and in the center on the other side, is the pool of the garden. Then, there would be two ways of getting into the garden. We could do so by going through the water in the center, or by walking to the wall at one of the three edges of the triangle, climbing around the other face of the triangle, and continuing on to the pool in the clearing.

I mentioned this solution to Milla. She pointed out, however, that the water and ground have depth to them, and that we don't flip upside-down when we get over the wall. I suggested that we could think of the shape having an extra dimension, a puffed triangle with a tunnel in the center, like a torus. Milla asked how this would account for my having always reached the same clearing in the garden, as I've claimed, when I got there over the wall. This I couldn't solve. Of course, the theory would work if there were only the village one side, and a single clearing of the garden on the other. I suspect that Milla was frustrated with the search for Molly, because she asked where, if my solution were correct, I thought Molly would be.

Ludo



Odel,

Milla is gone. She woke me up last night, asking me to accompany her to the well. Milla said that she could not imagine how we might have reasoned incorrectly. She said that Molly must have found a simpler route to get into the garden, and that the only place in the village that could be the entryway to this route would be in the well. I had already told Milla that I had gone to the well, and searched for any trace of a passageway, or secret nook, without any success. She said that since everywhere else had been checked, however, we had no choice but to search the bottom of the well. I told Milla that I've checked under the water in the garden without finding a passageway. I pointed out that it would be extremely dangerous for either of us to try to enter the well, especially because it was so dark out. Milla argued that leaving out this possibility would be a foolish oversight, and that Molly was undoubtedly trapped in the well even as we spoke. She insisted that we go.

It was dark outside, though not as dark as I had expected, because it was a clear night. There were no lights in any of the houses as we made our way to the well. We first carefully checked the ground around the well again, feeling around with our fingers and tapping

it, hoping to find something without having to enter the well. We then proceeded to do the same with the part of the well that is above ground, and tried to pry out loose stones. We dropped the bucket into the well, but when we pulled it up, we didn't find anything strange.

Finally, Milla asked me to lower her into the well in the bucket. I know that you'll think that this was a foolish thing to do. It seemed terribly dangerous to me, but Milla insisted that if I didn't lower her into the well, then she would lower herself, which seemed even more dangerous. So, Milla wrapped her legs around the rope and sat on the bucket, and I lowered her down slowly, until I felt the tension on the rope release.

Looking into the well, all I could see was some of the light from the sky reflected on the surface of the water. I thought, for a moment, that I saw the image of a young girl on the water, but this, of course, must have been Milla's face under the water. I called out after a while, but there was no response, and I feared that Milla was gone. I then heard Milla call out, though, and ask me to raise the bucket. I lifted her back up.

Milla told me, when she climbed out of the well, that she hadn't found anything strange under the water. She said, however, that while she was under the water, she looked up, and was certain that she saw Molly's face

on the surface of the water. I pointed out that if Milla had been looking up, then she undoubtedly saw my face at the top of the well. Milla ignored this, and claimed that she had also seen the image of the black and white dog, Lemma. It seems more likely to me that Milla had simply seen the sky.

Milla argued all the way back to the house that Molly must be in the garden. While she was drying off at the house, she said that my own solution must have been correct, and that she must have seen Molly's reflection on the other side of the water. That is, Milla believed that the surface of the water was somehow the passage into the garden, and that that she must have been able to see Molly in the clearing, with Lemma. Milla asked me to try to lead her into the garden immediately, before Molly disappears again. I told Milla that it would difficult to find the way while it was so dark out. Milla argued that it's difficult to find the way anyway.

Milla and I made our way slowly along the trail, as it was especially difficult amongst the trees to see where we were going. Milla was convinced that we had lost the way, until she walked into the wall of the garden. She would not be deterred, however, and asked me to help her up to the top of the wall. I climbed over with Milla, and we followed the trail on the other side.

I'm not sure how long I walked before I walked into the wall myself. My eyes teared, and I asked Milla for something to wipe my head, which I felt was bleeding. I had lost Milla, however, somewhere along the trail. I followed the trail back, hoping to find Milla again, but found no trace of her before I came to the wall. I figured that there would be no use continuing to search in the dark, and I made my way back to the house.

I hoped that Milla would return once it became light out, but she still hasn't. Milla will undoubtedly be missed today, since she is supposed to teach the children. I'm going to go search for Milla.

Ludo

Ludo is relieved to arrive along one of the paths of the garden, and moves hurriedly towards the First Place. There, Dilemma is drawing in the path, demonstrating an excellent proof to Lemma. Lemma and Dilemma look up as Ludo approaches.

"Well done!" says Lemma.

"Milla!" says Ludo.

"Milla," says Lemma. "There's nothing uglier than an extra name. What's the point of a name if you're going to change it?"

"I've already told you," says Dilemma, "that I forgot my name. I certainly know better than to change names. There wouldn't be a point in calling something this, and then calling it that. If everything can be called everything, then nothing would really be called anything."

"Forgetting your name," says Lemma. "There must be something about the village that makes everyone so absent-minded."

"Yes," says Dilemma, "something in the water!"

"You've forgotten your name?" asks Ludo.

"I've remembered it," says Dilemma.

"Dilemma ought to thank you," says Lemma. "Dilemma had forgotten nearly everything when you found Dilemma in the village."

"Dilemma?" asks Ludo.

Dilemma jumps up onto one of the rocks in the place, and then jumps off towards one of the paths.

"Surely," says Dilemma, "you can work this out. Let's go for a walk?"

Lemma, Dilemma, and Ludo follow the path out of the First Place. Lemma and Dilemma walk closely on either side of Ludo.

"You're Dilemma," says Ludo.

"Right," says Dilemma.

"Then, you used to explore the garden with Lemma," says Ludo.

"Certainly," says Dilemma.

"But," says Ludo.

"But," says Lemma.

"But," says Ludo, "you thought that there would be a better way to travel between the places."

"Right," says Dilemma.

"And this method," says Ludo, "entailed going through the water in the pool in the center of the First Place."

"How quick!" says Lemma. "Not our ordinary villager."

"After you went through the water," says Ludo, "you arrived in the village."

"It must have been awful," says Lemma.

"Chaos," says Dilemma, "though I admit I found the variety I was looking for."

"You must have forgotten," says Ludo, "how it was that you managed to get into the village. And, I gather, you forgot that you're Dilemma."

"Pitiful," says Lemma.

"Then," says Ludo, "you tried to get back as Milla. But failed, until I arrived."

"I suppose so," says Dilemma.

"Then," says Ludo, "you saw Dilemma in the well."

"Lemma saw me as well," says Dilemma.

"I nearly drowned," says Lemma, "trying to search the water in the dark."

"Then," says Ludo, "I led you over the wall. And you figured out how to return?"

Yes," says Lemma. "I'm still a bit unclear on that point as well. How did you manage to find the way from the wall?"

"I didn't figure it out," says Dilemma. "I think that it must have been an accident. I knocked my head against the wall. And then, Ludo cleverly moved too far ahead of me, and I lost the trail. And then, I found the path."

"And now," says Lemma, "you're here. You know, Dilemma has forgotten all of the places and paths that we'd visited before Dilemma's disappearance. So, we've agreed that we'll have to start counting from the beginning, just like you wanted."

Ludo sits down on a rock as the three are passing through a place.

"But," says Ludo, "what about Molly?"

"That's right," says Dilemma. "Have you heard about the Molly problem?"

"Yes," says Lemma. "Ludo mentioned that the villagers are searching for Molly. I haven't seen Molly either. It's a peculiar problem."

"When I saw you in the well," says Dilemma, "I thought that I saw Molly as well. I assumed that Molly was here with you."

"I saw Molly's image in the water as well," says Lemma. “Molly seems to spend more time in the water than out.”

"I think," says Dilemma, "that Molly must have been telling the truth, and gone through the water in the well. If Molly didn't arrive in the First Place, then, perhaps, Molly is in another place."

"You know,” says Lemma, “if we assume Molly could only arrive in a pool, then we wouldn't need to visit all

of the paths to search for Molly. We would only have to visit every place at least once, and that's a much simpler problem. I think a spiral would manage to visit each place exactly once."

"I suppose," says Dilemma. "Though, it would be a shame to leave so many of the paths unexplored while visiting every place. And, it would certainly be sufficient to visit every path to have visited every place."

"Certainly," says Lemma.

"Will we necessarily find Molly?" asks Dilemma.

"If all of the paths and places are hexagonal," says Lemma, "then we can find a route to explore all of them. And so, we know that we will reach every path and place at some point."

"But," says Dilemma, "at any given point, we will only have searched finitely many places. Since there are infinitely many places, the probability that Molly will be in one of the ones that we have already visited is zero."

Lemma and Dilemma begin walking along one of the paths.

"So," says Lemma, "there will never be a moment when we have found Molly. Yet the process guarantees that

she'll be found, as long as we don't stop exploring new places?"

"But," says Ludo, "couldn't Molly be somewhere else?"

"An interesting problem," says Dilemma. "Are you coming?"

Odel,

Milla still hasn't returned from the garden. Some of the villagers approached me today. They think, now that Milla has also disappeared, and since I was staying with her, that my presence is related to her disappearance and to Molly's. I mentioned to the villagers that given the coincidence of these events with my stay in the village, it's very likely that they are. This seemed to upset the villagers, and I worry they intend me some harm. I've found a solution, however, which I intend to propose to them later.

It has occurred to me that the strange boundary of the village might be explained by the notion of a higher dimension. It is clear that Molly has been in the village, and that Molly has been in the garden. She must also, therefore, have crossed from one to the other. Let us assume, then, (and this doesn't seem to me to be a terrible assumption) that Molly has not left the vicinity of these locations. Yet, the villagers have searched all over the village without finding the girl. And, I have searched a large region around the garden with Lemma, without finding any trace of the girl outside of the clearing closest to the village.

Recall that Milla believed that she saw Molly's reflection on the surface of the water in the well. Suppose then, that she did not see Molly's reflection, but Molly herself. That is, perhaps the village is somehow on one side of the surface of the water, and the garden is on the other. Then, the way to get from one to the other would not really be through the water itself, but through the surface of the water. If Molly is on neither side, then she must be somewhere in the boundary region itself.

This seems to disagree with the definition I've already given you regarding the nature of a boundary. That is, since the boundary is not itself a location, how could Molly be located there? Recall, however, that the boundary is defined by the two sets that that it determines. If we think of the two sets as being the village and the garden, then it's true the boundary is not a proper location.

If we regard these two sets as being subsets of a larger set in a higher dimension, however, we might regard them as being, in fact, connected. Consider, for example, a line on the page. From the surface of the page, the line would certainly appear as a boundary that separates one side of the page from the other. From our three dimensional perspective, however, we see that the

line does not effectively create any sort of boundary at all, and that if we wanted to move a finger from one side of the page to the other, we could simply pick up our finger, and move it, without touching the line. Thus, while Molly appears to us have to been caught in an infinitely small boundary between the village and the garden, in fact, she's simply off the page. QED!

Ludo