The Peter-Weyl Theorem and Generalizations

By John Bergan

Faculty Advisor: David Pollack
Department of Mathematics and Computer Science
Wesleyan University, Middletown CT
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Introduction

Throughout this paper we assume $G$ to be a compact Hausdorff topological group, and that $L^p(G)$ functions are complex-valued. The Peter-Weyl Theorem states that $L^2(G)$ is the completion (closure) of a direct sum of of certain subspaces $\mathcal{M}_\rho$ of $L^2(G)$. Each $\mathcal{M}_\rho$ is isomorphic to the direct sum of finitely many copies of the representation space $V_\rho$ of an irreducible representation $\rho : G \to GL(V_\rho)$. That is,

$$L^2(G) = \bigoplus \mathcal{M}_\rho.$$  

Since the $\mathcal{M}_\rho$ will be Hilbert spaces, this is also a Hilbert space direct sum which we denote by $\hat{\bigoplus}$. For Hilbert spaces $H_1, H_2, \ldots$ with respective norms $\|\cdot\|_1, \|\cdot\|_2, \ldots$

$$\hat{\bigoplus}_n H_n = \left\{ (h_1, h_2, \ldots) \in \prod_n H_n : \sum_n \|h_n\|_n^2 < \infty \right\}.$$  

The space $\hat{\bigoplus}_n H_n$ is then a Hilbert space with inner product

$$\langle (h_1, h_2, \ldots), (h_1', h_2', \ldots) \rangle = \sum_n \langle h_n, h_n' \rangle_n$$

where $\langle \cdot, \cdot \rangle_n$ is an inner product on $H_n$. It can be shown that $\hat{\bigoplus}_n H_n \cong \bigoplus_n H_n$, and the Peter-Weyl Theorem will state that as representations,

$$L^2(G) \cong \bigoplus_{[\rho]} V_\rho^{\dim V_\rho}.$$  

All necessary background information and notation will be explained within this paper. However, the reader should already be familiar with $L^p$ spaces and inner product spaces. Not too much background is required for topological groups, but one should
certainly know that by definition the maps \((g, h) \mapsto gh\) and \(g \mapsto g^{-1}\) are continuous. Therefore translations \(g \mapsto hg\) are homeomorphisms and neighborhoods of arbitrary points can be viewed as translates of neighborhoods of the identity of \(G\). We also point out that finite groups are compact groups endowed with the discrete topology and counting measure. So all results proved in this paper apply to finite groups as well, although they may be proved more directly. In fact, the Peter-Weyl theorem for finite groups is established with elementary methods. For a proof, see [3], page 18. The closure of the direct sum is itself in this case because there are only finitely many non-isomorphic representations of a finite group ([3], page 19). More details will follow in Section 3.

We will need established results from both representation theory and functional analysis. Most proofs will be given here, but some results, especially in functional analysis, are well established and will simply be stated. We will also use Haar measure extensively, which will be discussed but not proven to exist. The regular representation will also be discussed in its own section. The Peter-Weyl Theorem will then be proven as a collection of propositions and theorems which closely follows those used in [1] by Asif Zaman. Broadly, though, this is a two stage attack: 1.) construct a subspace \(M\) of continuous functions called matrix coefficients that satisfy certain orthogonality relations in \(L^2(G)\), and 2.) show that \(M\) is dense in the space of continuous functions which will imply \(M\) is dense in \(L^2(G)\). How this proves the theorem will be elaborated within.

1 Haar Measure

It is a well known result that \(G\) admits a positive Borel measure \(\mu\), called the Haar measure, which satisfies \(\mu(gX) = \mu(X) = \mu(Xg)\) for all \(g \in G\) and all \(\mu\)-measurable
sets $X$. This measure is unique up to constant multiplication, so we will use the unique 
mu satisfying $\mu(G) = 1$. This allows us to integrate over $G$ which proves to be extremely 
useful. We can also consider $L^2(G)$ with respect to $\mu$ along with the standard inner 
product on $L^2(G)$ given by

$$\langle f_1, f_2 \rangle_{L^2} = \int_G f_1(g) \overline{f_2(g)} \, d\mu(g).$$

From this point on we will write $dg$ instead of $d\mu(g)$ for convenience. We also note that 
since $g \mapsto hg$ is continuous, a substitution yields

$$\int_G f(g) \, dg = \int_G f(hg) \, d(hg) = \int_G f(hg) \, dg,$$

and similarly,

$$\int_G f(g) \, dg = \int_G f(gh) \, d(gh) = \int_G f(gh) \, dg,$$

for any $\mu$-measurable function $f$. We will not prove the existence or uniqueness of Haar 
measure here, though we will prove the following simple but necessary proposition.

Recall that for a Borel measure, all continuous functions are measurable.

**Proposition 1.1.** Let $\eta : G \to G$ be an automorphism or an antiautomorphism, i.e., 
$\eta(gh) = \eta(g)\eta(h)$ for all $g, h \in G$ or $\eta(gh) = \eta(h)\eta(g)$ for all $g, h \in G$. If $\eta^{-1}$ 
continuous and $\eta^2$ is the identity map on $G$, then $\mu(\eta(X)) = \mu(X)$ for all $\mu$-measurable 
sets $X$.

**Proof.** Define $\mu_\eta(X) = \mu(\eta(X))$. Let $X$ be a $\mu$-measureable set. Since $\eta^{-1}$ is continuous, 
we know $\eta(X)$ is $\mu$-measurable. Therefore $\mu_\eta$ is defined on all $\mu$-measurable sets $X$. It is also a measure: $\mu_\eta(\emptyset) = \mu(\eta(\emptyset)) = \mu(\emptyset) = 0$, and for disjoint $\mu$-measurable
sets \( \{X_1, X_2, \ldots \} \), injectivity of \( \eta \) gives \( \mu_\eta (\bigcup_{n=1}^\infty X_n) = \)

\[
\mu \left( \eta \left( \bigcup_{n=1}^\infty X_n \right) \right) = \mu \left( \bigcup_{n=1}^\infty \eta(X_n) \right) = \sum_{n=1}^\infty \mu(\eta(X_n)) = \sum_{n=1}^\infty \mu_\eta(X_n).
\]

Furthermore, if \( g \in G \) and \( \eta \) is an automorphism, then

\[
\mu_\eta(gX) = \mu(\eta(gX)) = \mu(\eta(g)\eta(X)) = \mu(\eta(X)) = \mu_\eta(X), \text{ and}
\]

\[
\mu_\eta(Xg) = \mu(\eta(Xg)) = \mu(\eta(X)\eta(g)) = \mu(\eta(X)) = \mu_\eta(X).
\]

Note that this is still true if we assume \( \eta \) is an antiautomorphism. Thus \( \mu_\eta \) is \( G \)-invariant implying \( \mu_\eta = \alpha \mu \) for some \( \alpha \in \mathbb{R}^+ \). If \( \alpha = 1 \), we’re done. We find

\[
\mu(X) = \mu(\eta^2(X))
= \mu(\eta(\eta(X)))
= \mu_\eta(\eta(X))
= \alpha \mu(\eta(X))
= \alpha \mu_\eta(X)
= \alpha^2 \mu(X)
\]

which implies \( \alpha = 1 \) since \( \mu \) is positive. \( \square \)

This result holds for automorphisms and antiautomorphisms of order greater than 2, but we omit this since the map \( g \mapsto g^{-1} \) will be the only map for which we need the proposition. In this case we have shown that \( \mu(X) = \mu(\{x^{-1} : x \in X\}) \).
2 Representation Theory

The major motivation for representation theory was to realize abstract groups as matrix groups; thus allowing linear algebra to be applied to groups. Therefore a representation of a group $G$ is generally defined to be a homomorphism from $G$ to $GL(V)$, the general linear group of some vector space $V$. However, this definition may be generalized or specified to suit one's purposes. For example, one may allow $V$ to be a Hilbert space or restrict $V$ to be over some specific field. For this paper, though, we will use the following definition.

**Definition 2.1.** A representation of a group $G$ is a continuous homomorphism $\rho : G \rightarrow GL(V)$, where $V$ is a finite dimensional complex vector space. The vector space $V$ is called the representation space of the representation $\rho$, and $\text{dim} V$ is the dimension of $V$ as a vector space.

We sometimes abuse notation and call $V$ the representation of $G$. Identifying $GL(V)$ with a subspace of $\mathbb{C}^{(\text{dim} V)^2}$, $\rho$ is continuous with respect to the standard euclidean topology on $GL(V)$.

Notice that a representation gives us an action of $G$ on $V$ via $gv = \rho(g)(v)$ since

$$1_Gv = \rho(1_G)(v) = 1_{GL(V)}(v) = v,$$

and

$$(gh)v = \rho(gh)(v) = \rho(g)\rho(h)(v) = \rho(g)(hv) = g(hv).$$

It's not hard to see that a linear action of $G$ on $V$ also gives us a representation of $G$. Thus a continuous linear action of $G$ on $V \cong \mathbb{C}^{\text{dim} V}$ is equivalent to our definition of a representation, and we will often use this definition with notation

$$gv = \rho(g)(v).$$
Definition 2.2. Let $\rho : G \to GL(V_\rho)$ and $\pi : G \to GL(V_\pi)$ be representations of $G$. A morphism between $\rho$ and $\pi$ is a linear map $f : V_\rho \to V_\pi$ such that $f \circ \rho(g) = \pi(g) \circ f$.

Or equivalently in terms of group action, $f(gv) = gf(v)$.

If $f$ is invertible, we say that $\rho$ and $\pi$ are isomorphic and write $V_\rho \cong V_\pi$. We denote $[\rho]$ as the class of all representations of $G$ that are isomorphic to $\rho$.

We denote by $Hom(V_\rho, V_\pi)$ the space of all morphisms from $V_\rho$ to $V_\pi$.

We can also consider $V \oplus W$ for representations $V$ and $W$ of $G$. Let $gv$ and $gw$ be the actions of $G$ on $V$ and $W$, respectively. Then we have a linear action of $G$ on $V \oplus W$ by $g(v, w) = (gw, gv)$, and the inclusion map $i : V \to V \oplus W$ is a morphism since $i(gv) = (gv, 0) = g(v, 0) = gi(v)$.

Definition 2.3. Let $\rho : G \to GL(V)$ be a representation. A subspace $U \subset V$ is called a subrepresentation if it’s invariant under $G$, i.e., if $gu = \rho(g)(u) \in U$ for all $g \in G$ and $u \in U$. A representation $V$ is then called irreducible if $\{0\}$ and $V$ are the only subrepresentations of $V$.

Some examples of subrepresentations are the kernel and image of a morphism $f : V_\rho \to V_\pi$. For if $v$ is in the kernel of $f$, then

$$f(\rho(g)(v)) = \pi(g)(f(v)) = \pi(g)(0) = 0,$$

implying $\rho(g)(v)$ is also in the kernel. And if $w = f(v)$ is in the image of $f$, then

$$\pi(g)(w) = \pi(g)(f(v)) = f(\rho(g)(v)),$$

implying $\pi(g)(w)$ is in the image of $f$. Thus the kernel and image of $f$ are $G$-invariant which makes them subrepresentations. Also, if $f : V_\rho \to V_\rho$ is a morphism, then
the eigenspaces of $f$ are subrepresentations. For if $v \in V_{\rho}$, then $f(v) = \lambda v$ implies $f(\rho(g)(v)) = \rho(g)(f(v)) = \rho(g)(\lambda v) = \lambda \rho(g)(v)$.

**Lemma 2.4** (Schur’s Lemma). Let $G$ be any group, and let $V$ and $W$ be irreducible representations of $G$. Any morphism $f : V \to W$ is either 0 or an isomorphism. Furthermore, if $W = V$, then $f(v) = \lambda v$ for some $\lambda \in \mathbb{C}$.

**Proof.** We know the kernel of $f$ is a subrepresentation of the irreducible $V$. So it’s either $V$ or $\{0\}$. If it’s $V$, $f$ is the zero map. Otherwise $f(V)$ is isomorphic to a nonzero subspace of the irreducible $W$. Since $f(V)$ is a subrepresentation, we conclude $f(V) = W$ and that $f$ is an isomorphism.

Now assume $W = V$. If $f \equiv 0$, clearly $f(v) = 0v$. Otherwise, $f$ has a nontrivial eigenvalue $\lambda$, and $\{v \in V : f(v) = \lambda v\}$ is a nonzero subspace of $V$. It’s also a nontrivial subrepresentation which makes it all of $V$. Thus $f(v) = \lambda v$ as desired. \(\square\)

**Corollary 2.5.** If $V$ and $W$ are irreducible representations of $G$, then

$$
\dim \text{Hom}(V, W) = \begin{cases} 
0 & \text{if } V \ncong W \\
1 & \text{if } V \cong W 
\end{cases}
$$

**Proof.** If $V \ncong W$, then we know the only morphism is the zero map. Therefore $\dim \text{Hom}(V, W) = 0$. Now suppose $V \cong W$ via the isomorphism $f : V \to W$. Let $f_1 \in \text{Hom}(V, W)$. Then $f^{-1} \circ f_1 : V \to V$ is a morphism, so that $(f^{-1} \circ f_1)(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. Hence $f_1(v) = f(\lambda v) = \lambda f(v)$ which shows $\text{Hom}(V, W) = \mathbb{C}f$ and $\dim \text{Hom}(V, W) = 1$. \(\square\)

**Corollary 2.6.** An irreducible representation of an abelian group $G$ is one dimensional.

**Proof.** Let $\rho : G \to V$ be a representation of $G$. Since $G$ is abelian, for $g, h \in G$ we
have
\[ \rho(g)(\rho(h)v) = \rho(gh)(v) = \rho(hg)(v) = \rho(h)\rho(g)(v). \]

Therefore \( \rho(g) : V \to V \) is an isomorphism of representations for all \( g \in G \). We then have \( \rho(g)(v) = \lambda_g v \) for some \( \lambda_g \in \mathbb{C} \) by Schur’s lemma. This means any nontrivial subspace of \( V \) is \( G \)-invariant, and thus a subrepresentation of \( V \). Then since \( V \) is irreducible, it must have no nontrivial subspaces which makes \( V \) one-dimensional. \( \square \)

Inner products on the representation spaces will also play an important role. A **Hermitian** inner product on a complex vector space \( V \) is a complex-valued bilinear form on \( V \) which is positive definite, and conjugate linear in the second variable.

**Definition 2.7.** Let \( V \) be a representation of \( G \), and let \( \langle \cdot , \cdot \rangle \) be a Hermitian inner product on \( V \). Then \( \langle \cdot , \cdot \rangle \) is called **\( G \)-invariant** if \( \langle gu, gv \rangle = \langle u, v \rangle \) for all \( g \in G \) and all \( u, v \in V \). A representation together with a \( G \)-invariant inner product is called a **unitary representation**.

Let \( \rho : G \to GL(V) \) be a unitary representation. We can choose an orthonormal basis for \( V \), and the \( G \)-invariant inner product means that \( \rho(g) \) is a **unitary** matrix. A unitary matrix \( A \) satisfies \( AA^* = I \), where \( A^* \) is the conjugate transpose of \( A \). The following theorem shows that all representations of a compact group are unitarizable.

**Theorem 2.8.** For any representation \( \rho : G \to GL(V) \) of a compact group \( G \) there exists a \( G \)-invariant Hermitian inner product on \( V \).

**Proof.** Let \( \langle \cdot , \cdot \rangle \) be any Hermitian inner product on \( V \) and define
\[ \langle v, w \rangle_\rho = \int_G \langle gv, gw \rangle \, dg. \]

This is well defined since \( G \) is compact and \( g \mapsto \langle gv, gw \rangle \) is a composition of continuous functions, and hence continuous. Then since \( \langle \cdot , \cdot \rangle \) is linear in the first variable and
conjugate linear in the second, one can see that $\langle \cdot, \cdot \rangle_\rho$ is too. The inner product $\langle \cdot, \cdot \rangle_\rho$ is also positive definite. This follows since $\langle gv, gv \rangle$ is non-negative for all $v \in V$ and $g \in G$. And if $\langle v, v \rangle_\rho = 0$, then $\langle gv, gv \rangle = 0$ for almost all $g \in G$. By continuity, $\langle gv, gv \rangle = 0$ for all $g \in G$ and in particular for the identity of $G$. So $\langle v, v \rangle = 0$ implying $v = 0$. Furthermore, $\langle v, w \rangle_\rho$ is $G$-invariant: for $h \in G$,

$$\langle hv, hw \rangle_\rho = \int_G \langle ghv, ghw \rangle \, dg = \int_G \langle gv, gw \rangle \, dg = \langle v, w \rangle_\rho$$

Later on we will also show that this $G$-invariant inner product is unique up to scalar multiplication. We can immediately see the utility of a $G$-invariant inner product in the next proposition. This proposition is not true over fields of characteristic greater than zero, so we reiterate that we are only considering finite dimensional complex representations.

**Proposition 2.9** (Mashke). Let $V$ be a representation of the compact group $G$. If $U$ is a subrepresentation of $V$, then there exists a subrepresentation $W$ of $V$ such that $V = U \oplus W$.

**Proof.** Choose a $G$-invariant inner product $\langle \cdot, \cdot \rangle$ on $V$, and let $W$ be the orthogonal complement of $U$ in $V$. Then $V = U \oplus W$, and we just need to show that $W$ is $G$-invariant. Well, for $u \in U$ and $w \in W$, we have $\langle u, gw \rangle = \langle g^{-1}u, w \rangle = 0$ since $g^{-1}u \in U$. We conclude $gw \in W$ so that $W$ is $G$-invariant as desired. 

This proves that $V$ is a direct sum of irreducible subrepresentations. For if $U = \{0\}$ is the only proper subrepresentation of $V$, then $V$ is irreducible. Otherwise, $V = U \oplus W$ for proper subrepresentations $U$ and $V$, and $0 < \dim U, \dim W < \dim V < \infty$. It
follows by induction on dimension that $V$ decomposes into a direct sum of irreducible subrepresentations. The representation $V$ is said to be **decomposable**.

For a complex vector space $V$, we now introduce the conjugate space $\tilde{V}$. As an additive group, $\tilde{V}$ is equal to $V$, and we identify \( \tilde{v} \in \tilde{V} \) with $v \in V$. Scalar multiplication in $\tilde{V}$ is then defined by $\alpha \tilde{v} = \overline{\alpha}v$. If $V$ is a representation of $G$ with action $gv$, we can define an action of $G$ on $\tilde{V}$ by $g\tilde{v} = \overline{g}v$. It is in fact an action since

\[
1_G \tilde{v} = \overline{1}_G v = \tilde{v} \quad \text{and} \quad (gh)\tilde{v} = \overline{(gh)}v = \overline{g}\overline{(hv)} = g(h\overline{v}).
\]

Therefore $\tilde{V}$ is a representation of $G$ called the **conjugate representation** of $V$. It’s important to notice that $\tilde{V} \cong V$ as representations. The dual space $V^*$, consisting of all linear functionals $\ell : V \to \mathbb{C}$, is another representation related to $V$. The representation is given by the action $g\ell(v) = \ell(g^{-1}v)$.

**Lemma 2.10.** If $V$ is a representation of a compact group $G$, then $\tilde{V} \cong V^*$.

**Proof.** Let $\langle \cdot, \cdot \rangle$ be a $G$-invariant inner product on $V$. Define $f : \tilde{V} \to V^*$ by $f(\tilde{v}) = \langle \cdot, \tilde{v} \rangle$. Since $V$ is finite dimensional, $f$ is a bijection. Additivity of $f$ is immediate since $\tilde{v} + \tilde{w} = \tilde{v} + \tilde{w}$. And for $\alpha \in \mathbb{C}$, $f(\alpha \tilde{v}) = f(\overline{\alpha}v) = \langle \cdot, \overline{\alpha}v \rangle = \alpha \langle \cdot, v \rangle = \alpha f(\tilde{v})$. Thus $f$ is linear. Furthermore, for any $w \in V$ we have

\[
f(g\tilde{v})(w) = f(\overline{g}v)(w) = \langle w, g^{-1}v \rangle = g\langle w, v \rangle = g f(\tilde{v})(w).
\]

Thus $f$ is an isomorphism of representations. \qed

**Proposition 2.11.** If $G$ is compact, then the $G$-invariant inner product of an irreducible representation of $G$ is unique up to scalar multiplication by a positive real number.
Proof. Let $\rho : G \to GL(V)$ be an irreducible representation of $G$, let $\langle \cdot, \cdot \rangle$ be a $G$-invariant inner product on $V$, and let $f : \widetilde{V} \to V^*$ be the isomorphism in Lemma 2.10.

Suppose $\langle \cdot, \cdot \rangle'$ is another $G$-invariant inner product on $V$. Then $f' : \widetilde{V} \to V^*$ defined by $f'(\tilde{v}) = \langle \cdot, v \rangle'$ is also an isomorphism. Since $f, f' \in \text{Hom}(\widetilde{V}, V^*)$ and $	ext{dim Hom}(\widetilde{V}, V^*) = 1$ by Corollary 2.5, we find $f = \alpha f'$ for some $\alpha \in \mathbb{C}$. Therefore $\langle \cdot, \cdot \rangle = \alpha \langle \cdot, \cdot \rangle'$. Since $\langle v, v \rangle, \langle v, v \rangle' > 0$ for all $v \neq 0$, we conclude $\alpha$ is real and positive. 

In Section 5 we will discuss complex functions on $G$ called matrix coefficients. They are of the form $\langle \rho(g)v_1, v_2 \rangle_{\rho}$ where $\rho : G \to GL(V_\rho)$ is a representation, where $\langle \cdot, \cdot \rangle_{\rho}$ is a $G$-invariant inner product on $V_\rho$, and $v_1, v_2 \in V_\rho$. It will be shown that matrix coefficients are in $L^2(G)$. The following theorem gives us orthogonality relations between matrix coefficients in $L^2(G)$ that are crucial to the Peter-Weyl Theorem.

**Theorem 2.12 (Orthogonality Relations).** Let $V_\rho$ and $V_\pi$ be nonisomorphic irreducible representations of $G$ with respective $G$-invariant inner products $\langle \cdot, \cdot \rangle_{\rho}$ and $\langle \cdot, \cdot \rangle_{\pi}$. Then for $a, v \in V_\rho$ and $b, w \in V_\pi$

$$\int_G \overline{\langle ga, v \rangle_{\rho}} \langle gb, w \rangle_{\pi} \, dg = 0.$$ 

**Proof.** Fix $a$ and $b$. Let $\psi : V_\rho \times \widetilde{V_\pi} \to \mathbb{C}$ be the map defined by

$$\psi(v, \tilde{w}) = \int_G \overline{\langle ga, v \rangle_{\rho}} \langle gb, w \rangle_{\pi} \, dg.$$ 

We show $\psi$ is a bilinear form. Linearity of $\langle \cdot, \cdot \rangle_{\rho}$ clearly gives linearity of $\psi$ in the variable $v$. For linearity in $\tilde{w}$, additivity follows from the fact that $V_\pi = \widetilde{V_\pi}$ as a group;
\( \tilde{w} + \tilde{w}_0 \) is identified with \( w + w_0 \). Furthermore, for \( \alpha \in \mathbb{C} \),

\[
\psi(v, \alpha \tilde{w}) = \psi(v, \tilde{\alpha}w) = \int_G \langle ga, v \rangle_\rho \langle gb, \alpha w \rangle_\pi \, dg = \alpha \psi(v, \tilde{w})
\]

by conjugate linearity of \( \langle \cdot, \cdot \rangle_\pi \).

The function \( \psi \) is also \( G \)-invariant. For \( h \in G \),

\[
\psi(hv, h\tilde{w}) = \psi(hv, \tilde{hw}) = \int_G \langle ga, hv \rangle_\rho \langle gb, hw \rangle_\pi \, dg = \int_G \langle h^{-1}ga, v \rangle_\rho \langle h^{-1}gb, w \rangle_\pi \, dg = \psi(v, \tilde{w})
\]

by the \( G \)-invariance of the inner products and Haar measure.

Therefore, \( \psi \) induces a linear map \( f : V_\rho \rightarrow \tilde{V}_\pi^* \) given by \( f(v) = \psi(v, \cdot) \). Then \( f \) is also a morphism, since for all \( \tilde{w} \in \tilde{V}_\pi \) and \( h \in G \),

\[
f(hv) (\tilde{w}) = \psi(hv, \tilde{w}) = \psi(v, h^{-1}\tilde{w}) = h \psi(v, \tilde{w}) = hf(v)(\tilde{w}).
\]

But we know \( V_\pi \cong \tilde{V}_\pi \cong \tilde{V}_\pi^* \) by Lemma 2.10, so we have a morphism

\[
V_\rho \rightarrow \tilde{V}_\pi^* \cong V_\pi.
\]

This map is trivial by Schur’s lemma which implies \( f \) is too. Therefore \( \psi \) is trivial and the proof is complete. \( \square \)
Early in algebra, we learn from Cayley’s Theorem that finite groups can be viewed as a set of permutations, namely translations. Finite groups also have a natural representation where the action is built on translation. For a group $G$, we define the group ring $\mathbb{C}[G]$ to be the set of all finite formal sums $\sum_{g \in G} \lambda_g g$ where $\lambda_g \in \mathbb{C}$. It’s also a vector space, and when $G$ is finite, $\mathbb{C}[G]$ becomes a natural representation of $G$ under the action $g(\sum_{h \in G} \lambda_h h) = \sum_{h \in G} \lambda_h (gh)$. This is called the regular representation. This representation of $G$ is particularly important because it decomposes into a direct sum of all the non-isomorphic irreducible representations of $G$ with each summand having multiplicity the degree of the representation. For a proof, see [3], page 18. Notice that this is precisely the Peter-Weyl theorem for finite groups. For in this case, $L^2(G)$ is the space of all complex functions on $G$. Thus $L^2(G) \cong \mathbb{C}[G]$ via the isomorphism $f \mapsto \sum_{g \in G} f(g)g$.

But for infinite groups, $L^2(G)$ is not isomorphic to $\mathbb{C}[G]$, and $\mathbb{C}[G]$ becomes far more complicated. There are further difficulties regarding continuity, so one might think to replace $\mathbb{C}[G]$ with the space of continuous functions. However, the function that takes 1 at some $g \in G$ and 0 elsewhere is a basis element of $\mathbb{C}[G]$, but not continuous in the infinite case. This function is in the $L^p(G)$ spaces, though, as the zero function. Then since $L^2(G)$ also has the added structure of an inner product space, it’s a natural choice for a representation space analogous to $\mathbb{C}[G]$. Knowing the structure of $L^2(G)$ as a representation is certainly important in its own right anyway. So for all groups $G$ endowed with Haar measure, we now define $L^2(G)$ to be the regular representation of $G$. Notice this is not a finite dimensional representation of $G$. However, this will not be an issue since the results of Section 2 will only be applied to finite dimensional subrepresentations of $L^2(G)$. We may define the action as translation on the left or on
the right by

\[ L_g f(h) = f(g^{-1}h) \quad \text{or} \quad R_g f(h) = f(hg), \]

respectively. But in the compact case they are isomorphic. To see why, consider the linear map \( A \) on \( L^2(G) \) given by \( f(g) \mapsto f(g^{-1}) \). Since \( g \mapsto g^{-1} \) is continuous, Proposition 1.1 tells us

\[
\int_G |Af(g)|^2 \, dg = \int_G |f(g^{-1})|^2 \, dg = \int_G |f(g)|^2 \, d(g^{-1}) = \int_G |f(g)|^2 \, dg.
\]

Thus \( A \) is an operator on \( L^2(G) \) and it’s clearly bijective. Also, for all \( h \in G \),

\[
A(R_{h}f)(g) = Af(gh) = f(((gh)^{-1})) = f(h^{-1}g^{-1}) = L_{h}f(g^{-1}) = L_{h}Af(g),
\]

so that \( A \) is an isomorphism between representations. Furthermore,

\[
\langle Af_1, Af_2 \rangle_{L^2(G)} = \int_G Af_1(g) \overline{Af_2(g)} \, dg = \int_G f_1(g^{-1}) \overline{f_2(g^{-1})} \, dg = \int_G f_1(g) \overline{f_2(g)} \, d(g^{-1}) = \langle f_1, f_2 \rangle_{L^2}
\]

where the last line follows from Proposition 1.1. Thus \( A \) preserves orthogonality relations in \( L^2(G) \).

Though \( L^2(G) \) will now be the object of interest, it is still instructive to consider
\( \mathbb{C}[G] \). It is a ring with a multiplication \( * \) given by

\[
\left( \sum_g \lambda_g g \right) * \left( \sum_h \mu_h h \right) = \sum_{g,h} \lambda_g \mu_h g h = \sum_k \left( \sum_{gh=k} \lambda_g \mu_h \right) k = \sum_g \left( \sum_h \lambda_{gh^{-1}} \mu_h \right) g
\]

So viewing \( \mathbb{C}[G] \) as complex functions on \( G \), we see \( * \) is in fact convolution of functions with finite support. We know convolution is also defined on the \( L^1 \) functions by

\[
(f_1 * f_2)(g) = \int_G f_1(gh^{-1}) f_2(h) \, dh,
\]

and we will eventually see that the continuous functions on \( G \) are contained in \( L^2(G) \subset L^1(G) \) for compact \( G \). Therefore \( * \) is also defined on \( L^2(G) \), and it should come as no surprise later to see that convolution will play a large role in the proof of the Peter-Weyl theorem.

## 4 Functional Analysis

### 4.1 Definitions and Classic Theorems

For topological spaces \( X \) and \( Y \), let \( C(X, Y) \) denote the space of continuous functions from \( X \) to \( Y \). Recall that when \( X \) is compact and \( Y \) is \( \mathbb{R} \) or \( \mathbb{C} \), then \( C(X, Y) \) is equipped with the sup norm \( \| \cdot \|_\infty \) given by \( \| f \|_\infty = \sup_{x \in X} |f(x)| \).

**Definition 4.1.** Let \( X \) be a space and \( (Y, d) \) be a metric space. Let \( \mathcal{F} \subset C(X, Y) \). Then \( \mathcal{F} \) is said to be equicontinuous at \( x_0 \) if given \( \epsilon > 0 \), there is a neighborhood \( U \) of
such that for all \( x \in U \) and all \( f \in \mathcal{F} \),

\[
d(f(x), f(x_0)) < \epsilon
\]

If \( \mathcal{F} \) is equicontinuous at \( x_0 \) for each \( x_0 \in X \), \( \mathcal{F} \) is simply said to be **equicontinuous** [4].

**Definition 4.2.** If \((Y, d)\) is a metric space, then \( \mathcal{F} \subset C(X, Y) \) is said to be **pointwise bounded** under \( d \) if for each \( x \in X \), the subset

\[
\mathcal{F}_x = \{ f(x) | f \in \mathcal{F} \}
\]

of \( Y \) is bounded under \( d \) [4].

**Theorem 4.3** (Ascoli’s Theorem). Let \( X \) be a compact space; let \((\mathbb{R}^n, d)\) denote euclidean space with the euclidean metric; give \( C(X, \mathbb{R}^n) \) the corresponding uniform topology (induced by the sup norm \( \| \cdot \|_\infty \)). A subspace \( \mathcal{F} \) of \( C(X, \mathbb{R}^n) \) has compact closure if and only if \( \mathcal{F} \) is equicontinuous and pointwise bounded under the metric \( d \).

**Proof.** See [4], page 278.

**Theorem 4.4** (Urysohn lemma). Let \( X \) be a normal space; let \( A \) and \( B \) be disjoint closed subsets of \( X \). Let \([a, b]\) be a closed interval in \( \mathbb{R} \). Then there exists a continuous map \( f : X \to [a, b] \) such that \( f(x) = a \) for every \( x \in A \), and \( f(x) = b \) for every \( x \in B \).

**Proof.** See [4], page 207.

We may apply the previous theorem to \( G \) since a compact Hausdorff space is in fact normal. See [4], page 202.
4.2 Some Necessary Results

Let \( \| \cdot \|_1 \), \( \| \cdot \|_2 \), and \( \| \cdot \|_\infty \) be the standard norms for the complex spaces \( L^1(G) \), \( L^2(G) \), and \( L^\infty(G) \), respectively. Let \( \langle \cdot , \cdot \rangle_{L^2} \) be the standard inner product on \( L^2(G) \) with respect to the normalized Haar measure \( \mu \) on \( G \).

**Proposition 4.5.** If \( f \) is \( \mu \)-measurable, then \( \| f \|_1 \leq \| f \|_2 \leq \| f \|_\infty \).

**Proof.**

\[
(\| f \|_2)^2 = \int_G |f(g)|^2 \, dg \\
= \int_G [f(g)]^2 \cdot 1 \, dg \\
= \| f^2 \cdot 1 \|_1 \\
\leq \| f^2 \|_\infty \| 1 \|_1 \quad \text{(Hölder's Inequality)} \\
= (\| f \|_\infty)^2 \mu(G) \\
= (\| f \|_\infty)^2
\]

so that \( \| f \|_2 \leq \| f \|_\infty \). Also, by the Cauchy-Schwarz inequality

\[
\| f \|_1 = \int_G |f(g)| \, dg = \int_G |f(g)| \cdot 1 \, dg = \langle |f| , 1 \rangle_{L^2} \leq \| f \|_2 \| 1 \|_2 = \| f \|_2.
\]

The previous Proposition will be used throughout the paper. It also shows that \( L^\infty(G) \subset L^2(G) \subset L^1(G) \) which does not hold for arbitrary measure spaces. The corollary to the following proposition will be critical to the final proof of the Peter-Weyl Theorem.

**Proposition 4.6.** For \( 1 \leq p < \infty \), \( C(G, \mathbb{C}) \) is dense in \( L^p(G) \).
Proof. See [6], page 68. \[\Box\]

**Corollary 4.7.** If $M$ is dense in $C(G, \mathbb{C})$, then $M$ is dense in $L^2(G)$.

**Proof.** Let $f \in L^2(G)$ and let $\epsilon > 0$. We need $\sigma \in M$ such that $\|f - \sigma\|_2 < \epsilon$. Choose $\phi \in C(G, \mathbb{C})$ such that $\|f - \phi\|_2 < \epsilon/2$. Since $M$ is dense in $C(G, \mathbb{C})$, choose $\sigma \in M$ such that $\|\phi - \sigma\|_\infty < \epsilon/2$. Using Proposition 4.5 we find

$$\|f - \sigma\|_2 \leq \|f - \phi\|_2 + \|\phi - \sigma\|_2 \leq \|f - \phi\|_2 + \|\phi - \sigma\|_\infty < \epsilon,$$

as desired. \[\Box\]

**Lemma 4.8.** Let $e$ be the identity of $G$. For any open set $V$ containing $e$, there exists an open $U \subset V$ such that $e \in U$ and $U^2 \subset V$.

**Proof.** By definition, the map $(g, h) \mapsto gh$ is continuous. Then since $e \in V$ and $V$ is open, there exists an open set $A \times B \subset G \times G$ such that $(e, e) \in A \times B$ and $AB \subset V$. Set $U = A \cap B$. Then $U^2 \subset AB \subset V$ as desired. \[\Box\]

**Proposition 4.9.** Let $\phi \in C(G, \mathbb{C})$ and let $\epsilon > 0$. Then there exists an open set $U$ containing the identity of $G$ such that $|\phi(xg) - \phi(g)| < \epsilon$ for all $g \in G$ and all $x \in U$.

**Proof.** Let $e$ be the identity of $G$. For every $g \in G$, there exists an open set $V_g$ containing $e$ such that $|\phi(vg) - \phi(g)| < \epsilon/2$ for all $v \in V_g$. By Lemma 4.8, choose $U_g \subset V_g$ such that $e \in U_g$ and $U_g^2 \subset V_g$. Then $\{U_g\}_{g \in G}$ is an open cover of $G$, and we obtain a finite subcover $\{U_{g_i}\}_{i=1}^n$. We claim

$$U = \bigcap_{i=1}^n U_{g_i}$$

is the desired open set containing $e$. \[18\]
Let \( g \in G \). Then for some \( i \), we have \( g \in U, g_i \subset V \). Therefore \( g = v g_i \) for some \( v \in V \), and we have

\[
|\phi(g) - \phi(g_i)| = |\phi(v g_i) - \phi(g_i)| < \epsilon/2.
\]

Now let \( x \in U \). Then \( x g \in U U, g_i \subset U^2 g_i \subset V \). Therefore \( x g = v' g_i \) for some \( v' \in V \), and we have

\[
|\phi(xg) - \phi(g)| = |\phi(v' g_i) - \phi(g)| < \epsilon/2.
\]

We can now conclude that

\[
|\phi(xg) - \phi(g)| \leq |\phi(xg) - \phi(g_i)| + |\phi(g_i) - \phi(g)| < \epsilon
\]

for all \( g \in G \) and all \( x \in U \).

\[
\square
\]

4.3 Hilbert Space

Now let \( H \) be a Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \).

**Definition 4.10.** A linear operator \( T : H \to H \) is called **compact** if the closure of the image of the closed unit ball is compact.

**Definition 4.11.** A linear operator \( T : H \to H \) is called **self-adjoint** if \( T = T^* \). That is, for all \( x, y \in H \), \( \langle Tx, y \rangle = \langle x, Ty \rangle \).

**Theorem 4.12** (Spectral Theorem). If \( T \) is a compact self-adjoint operator on \( H \), and \( \{\lambda_1, \lambda_2, \ldots\} \) are the distinct eigenvalues of \( T \) with corresponding eigenspaces \( \{M_{\lambda_1}, M_{\lambda_2}, \ldots\} \), then \( H \) is an orthogonal Hilbert Space direct sum of the \( M_{\lambda_n} \) and
$M_{\lambda_n}$ is finite dimensional for $\lambda_n \neq 0$. Furthermore, each $\lambda_n$ is real and $\lambda_n \to 0$ as $n \to \infty$.

**Proof.** See [5], page 47-50. □

Also recall the Pythagorean identity $\| \sum_{n=0}^{\infty} v_n \|^2 = \sum_{n=0}^{\infty} \| v_n \|^2$ for a convergent series of orthogonal vectors $v_n \in H$.

### 4.4 Convolution Operator

Let $\phi$ be a continuous complex-valued function on $G$. Recalling the convolution operation $*$ from Section 3 we define the convolution operator $T_\phi$ by

$$T_\phi f(x) = \int_G \phi(xg^{-1}) f(g) \, dg.$$ 

The operator $T_\phi$ and the following propositions will be used extensively in the main proof of the Peter-Weyl Theorem.

**Proposition 4.13.** For any continuous complex-valued function $\phi$ on $G$, $T_\phi$ is a bounded (thus continuous) operator on $L^2(G)$.

**Proof.** We need a constant $M$ such that $\| T_\phi f \|_2 \leq M \| f \|_2$ for all $f \in L^2(G)$. Notice
that for any \( x \in G \),

\[
|T_\phi f(x)| = \left| \int_G \phi(xg^{-1})f(g) \, dg \right| \\
\leq \int_G \left| \phi(xg^{-1})f(g) \right| \, dg \\
= \int_G \left| \phi_x(g)f(g) \right| \, dg \quad \text{(where } \phi_x(g) = \phi(xg^{-1}) \text{)} \\
= \|\phi_x f\|_1 \\
\leq \|\phi_x\|_\infty \|f\|_1 \quad \text{(Hölder’s Inequality)}
\]

Therefore \( \sup_{x \in G} |T_\phi f(x)| \leq \|\phi\|_\infty \|f\|_1 \), i.e., \( \|T_\phi f\|_\infty \leq \|\phi\|_\infty \|f\|_1 \). By Proposition 4.5, we then have

\[
\|T_\phi f\|_2 \leq \|\phi\|_\infty \|f\|_2.
\]

This gives us the desired constant since \( \|\phi\|_\infty \) is independent of \( f \). \( \square \)

The next proposition will allow us to apply the Spectral Theorem to \( T_\phi \) in the main theorem.

**Proposition 4.14.** For any continuous complex-valued function \( \phi \) on \( G \), \( T_\phi \) is a compact operator on \( L^2(G) \). Furthermore, if \( \phi(g^{-1}) = \overline{\phi(g)} \), then \( T_\phi \) is self-adjoint with respect to the inner product, \( \langle \cdot, \cdot \rangle_{L^2} \).

**Proof.** We first show that \( T_\phi \) is a compact operator on \( L^2(G) \), i.e., the closure of \( \mathcal{F} = \{ T_\phi f : \|f\|_2 \leq 1 \} \) in \( L^2(G) \) is compact in \( L^2(G) \). To that end we first show that the closure of \( \mathcal{F} \) in \( C(G, \mathbb{C}) \) is compact in \( C(G, \mathbb{C}) \). By Ascoli’s theorem we just have to
show that $\mathcal{F}$ is pointwise bounded and equicontinuous ( Note: a priori we don’t know if $\mathcal{F} \subset C(G, \mathbb{C})$, but showing equicontinuity will imply this). Notice that the proof of Proposition 4.13 tells us that $\sup_{x \in G} |T_\phi f(x)| \leq \|\phi\|_\infty \|f\|_1$. Since $\|f\|_1 \leq \|f\|_2 \leq 1$, we see that $\mathcal{F}$ is in fact pointwise bounded.

Now for equicontinuity. Fix $\epsilon > 0$. By Proposition 4.9, there exists an open set $U$ containing the identity of $G$ such that

$$|\phi(xh) - \phi(h)| < \epsilon$$

for all $h \in G$ and all $x \in U$. So for $T_\phi f \in \mathcal{F}$, we have

$$|T_\phi f(xh) - T_\phi f(h)| = \left| \int_G \phi(xhg^{-1})f(g) \, dg - \int_G \phi(hg^{-1})f(g) \, dg \right| = \left| \int_G [\phi(xhg^{-1}) - \phi(hg^{-1})]f(g) \, dg \right| \leq \int_G |\phi(xhg^{-1}) - \phi(hg^{-1})| |f(g)| \, dg < \epsilon \int_G |f(g)| \, dg = \epsilon \|f\|_1 \leq \epsilon \|f\|_2 \leq \epsilon$$

for all $h \in G$ and all $x \in U$. That is, $|T_\phi f(y) - T_\phi f(h)| < \epsilon$ for all $y$ in the open set $Uh$ and for all $T_\phi f \in \mathcal{F}$. Thus $\mathcal{F}$ is equicontinuous.

By Ascoli’s Theorem we conclude that the closure of $\mathcal{F}$ in $C(G, \mathbb{C})$ is compact in $C(G, \mathbb{C})$. But since $C(G, \mathbb{C})$ is a metric space we know that the closure of $\mathcal{F}$ in $C(G, \mathbb{C})$ is sequentially compact in $C(G, \mathbb{C})$. Since $\|f\|_2 \leq \|f\|_\infty$, we find the closure of $\mathcal{F}$ in $C(G, \mathbb{C})$ is sequentially compact in $L^2(G)$ and thus compact. So the closure of
\[ \mathcal{F} \text{ in } L^2(G) \text{ is the same as the closure of } \mathcal{F} \text{ in } C(G, \mathbb{C}), \text{ making it compact as desired.} \]

Now suppose \( \phi(g^{-1}) = \overline{\phi(g)} \). Then

\[
\langle T_\phi f_1, f_2 \rangle_{L^2} = \int_G T_\phi f_1(g) \overline{f_2(g)} \, dg \\
= \int_G \left( \int_G \phi(gh^{-1}) f_1(h) \, dh \right) \overline{f_2(g)} \, dg \\
= \int_G \left( \int_G \overline{\phi(gh^{-1})} f_1(h) \, dh \right) \overline{f_2(g)} \, dg \\
= \int_G \left( \int_G \overline{\phi(gh^{-1})} f_1(h) \overline{f_2(g)} \, dh \right) \, dg \quad \text{(Fubini)} \\
= \int_G f_1(h) \left( \int_G \phi(gh^{-1}) \overline{f_2(g)} \, dg \right) \, dh \\
= \int_G f_1(h) \left( \int_G \overline{\phi(gh^{-1})} f_2(g) \, dg \right) \, dh \\
= \int_G f_1(h) \left( \int_G \overline{\phi(hg^{-1})} f_2(g) \, dg \right) \, dh \\
= \int_G f_1(h) \left( \int_G \phi(hg^{-1}) f_2(g) \, dg \right) \, dh \\
= \int_G f_1(h) T_\phi f_2(h) \, dh \\
= \langle f_1, T_\phi f_2 \rangle_{L^2} \\
\]

\[ \blacksquare \]

**Proposition 4.15.** Given an open set \( U \) containing the identity \( e \) of \( G \), there exists a non-negative \( \phi \in C(G, \mathbb{C}) \) supported in \( U \) which satisfies \( \int_G \phi(g) \, dg = 1 \) and \( \phi(g^{-1}) = \overline{\phi(g)} \).

**Proof.** Since \( G \) is Hausdorff, \( \{e\} \) is closed. Since \( G \) is normal, there exists an open set \( V \) containing \( e \) such that \( \overline{V} \subseteq U \). Then \( G \setminus U \) and \( \overline{V} \) are disjoint closed sets. By the Urysohn lemma, there exists a continuous \( \phi_0 : G \to [0, 1] \) satisfying \( \phi_0(G \setminus U) = \{0\} \) and \( \phi_0(\overline{V}) = \{1\} \), i.e., \( \phi_0 \) is supported on \( U \). 

\[ 23 \]
Now set \( \phi_1(g) = \phi_0(g)\phi_0(g^{-1}) \). Then \( \phi_1(g^{-1}) = \phi_1(g) = \overline{\phi_1(g)} \) since \( \phi(g) \in [0, 1] \). Notice \( \phi_1 \) is still supported in \( U \). Finally, set \( \phi(g) = \frac{\phi_1(g)}{\int_G \phi_1(g) dg} \).

**Proposition 4.16.** For \( \lambda \in \mathbb{C} \), let \( M_\lambda \) be the corresponding eigenspace for \( T_\phi \). Then for each \( f \in M_\lambda, f_g \in M_\lambda \), where for each \( g \in G \) \( f_g(x) = f(xg) \).

**Proof.** We have \( T_\phi f_g(x) = \int_G \phi(xh^{-1})f(hg) dh \). Since the map \( h \mapsto hg^{-1} \) is continuous for fixed \( g \in G \), we substitute to get

\[
T_\phi f_g(x) = \int_G \phi(xh^{-1})f(hg) dh \\
= \int_G \phi(xhg^{-1})^{-1}f(hg^{-1}g) d(hg^{-1}) \\
= \int_G \phi(xgh^{-1})f(h) d(hg^{-1}) \\
= \int_G \phi(xgh^{-1})f(h) dh \quad \text{(Invariance of Haar Measure)} \\
= T_\phi f(xg) \\
= \lambda f(xg) \\
= \lambda f_g(x)
\]

Thus \( f_g \) is an eigenvector for \( \lambda \) so that \( f_g \in M_\lambda \).

\[\square\]

### 5 Matrix Coefficients

We now begin the first stage of attack. Let \( \rho : G \to GL(V_\rho) \) be a finite dimensional representation, and let \( \langle \cdot, \cdot \rangle_\rho \) be a \( G \)-invariant inner product on \( V_\rho \).

**Definition 5.1.** A **matrix coefficient** is a function \( \sigma : G \to \mathbb{C} \) of the form \( \sigma(g) = \langle \rho(g)v_1, v_2 \rangle_\rho \), where \( v_1, v_2 \in V_\rho \).
More generally, matrix coefficients are defined as functions of the form \( \ell(\rho(g)v_1) \)
where \( \ell \in V_\rho^* \), the space of linear functionals, and \( v_1 \in V_\rho \). However, since we only consider cases where \( V_\rho \) is finite dimensional, we know all linear functionals are of the form \( \langle \cdot, v_2 \rangle \) for some \( v_2 \in V_\rho \). Thus \( \ell(\rho(g)v_1) = \langle \rho(g)v_1, v_2 \rangle_\rho \) as above. We will use the latter notation, though, when it’s convenient. Notice that when \( \langle \cdot, \cdot \rangle \) is the standard inner product on \( \mathbb{C}^n \) and \( v_1, v_2 \) are standard basis vectors, \( \langle \rho(g)v_1, v_2 \rangle \) is a coefficient of the matrix \( \rho(g) \). Hence the term matrix coefficients.

Matrix coefficients will be the main thrust of the Peter-Weyl Theorem. Being the composition of continuous functions they are continuous, and therefore also in \( L^2(G) \). Our ultimate objective is to show that they are dense in \( L^2(G) \). So let \( M \) be the set of matrix coefficients of \( G \). The first thing to show is that \( M \) is a subspace of \( C(G, \mathbb{C}) \subset L^2(G) \).

**Proposition 5.2.** The set \( M \) is closed under pointwise addition and scalar multiplication.

**Proof.** Let \( \rho_1 : G \to V_1 \) and \( \rho_2 : G \to V_2 \) be representations of \( G \) with respective matrix coefficients \( \ell_1(\rho_1(g)v_1) \) and \( \ell_2(\rho_2(g)v_2) \) where \( v_1 \in V \) and \( v_2 \in V \). Then \( \rho_1 \oplus \rho_2 : G \to V_1 \oplus V_2 \) is also a representation of \( G \), and \( \ell_1 \oplus \ell_2 \) is a linear functional on \( V_1 \oplus V_2 \). Then \( (\ell_1 + \ell_2)(\rho_1 \oplus \rho_2)(g)(v_1, v_2) \) is also a matrix coefficient of \( G \), and equal to \( \ell_1(\rho_1(g)v_1) + \ell_2(\rho_2(g)v_2) \) as desired.

Now let \( \alpha \in \mathbb{C} \). Then

\[
\alpha \ell_1(\rho_1(g)v_1) = \ell_1(\alpha \rho_1(g)v_1) \\
= \ell_1(\rho_1(g)(\alpha v_1)) \quad \text{(since } \rho(g) \in GL(V))
\]
is a matrix coefficient since $\alpha v_1 \in V_1$.  

We now consider certain subspaces of $\mathcal{M}$.

**Definition 5.3.** For an irreducible representation $\rho : G \to GL(V_\rho)$, define the subspace

$$\mathcal{M}_\rho = \text{Span}\{ \sigma \in \mathcal{M} : \sigma(g) = \langle \rho(g)v_1, v_2 \rangle_\rho ; v_1, v_2 \in V_\rho \}.$$ 

An element of $\mathcal{M}_\rho$ is then called a **matrix coefficient of the representation** $\rho$.

**Proposition 5.4.** Let $[\rho]$ denote an equivalence class of isomorphic representations of $G$. If $\pi \in [\rho]$, then $\mathcal{M}_\pi = \mathcal{M}_\rho$.

**Proof.** Let $\psi : V_\rho \to V_\pi$ be the isomorphism satisfying $\pi(g) = \psi \rho(g) \psi^{-1}$. Let $\langle \cdot, \cdot \rangle_\rho$ be the $G$-invariant inner product on $V_\rho$. Then we may define the $g$-invariant inner product $\langle \cdot, \cdot \rangle_\pi$ on $V_\pi$ by $\langle v_1, v_2 \rangle_\pi = \langle \psi^{-1}v_1, \psi^{-1}v_2 \rangle_\rho$. Then for $\sigma \in \mathcal{M}_\pi$ and some $v_1, v_2 \in V_\pi$,

$$\sigma(g) = \langle \pi(g)v_1, v_2 \rangle_\pi = \langle \psi^{-1}\pi(g)v_1, \psi^{-1}v_2 \rangle_\rho = \langle \rho(g)\psi^{-1}v_1, \psi^{-1}v_2 \rangle_\rho$$

so that $\sigma \in \mathcal{M}_\rho$. Thus $\mathcal{M}_\pi \subset \mathcal{M}_\rho$. By a symmetric argument, $\mathcal{M}_\rho \subset \mathcal{M}_\pi$ so that $\mathcal{M}_\pi = \mathcal{M}_\rho$.  

Therefore we will denote by $\mathcal{M}_{[\rho]}$ the space $\mathcal{M}_\rho$ for any $\rho \in [\rho]$. Notice that the proof of Proposition 5.2 together with Proposition 2.9 (Mashke) shows us that a matrix coefficient of a reducible representation is a sum of matrix coefficients of irreducible representations. This implies $\mathcal{M}$ is the span of the union of the $\mathcal{M}_{[\rho]}$. So by Proposition 2.12 (Schur Orthogonality), we have

$$\mathcal{M} = \bigoplus_{[\rho]} \mathcal{M}_{[\rho]}$$
in $L^2(G)$, where the sum is orthogonal with respect to the $L^2(G)$ inner product $\langle f_1, f_2 \rangle_{L^2} = \int_G f_1(g) \overline{f_2(g)} \, dg$.

**Proposition 5.5.** Let $V_\rho \otimes V_\rho^*$ be the representation of $G$ under the action
\[ g(v_1 \otimes \langle \cdot, v_2 \rangle_\rho) = gv_1 \otimes \langle \cdot, v_2 \rangle_\rho. \]
Then
\[ \mathcal{M} \cong \bigoplus_{[\rho]} V_\rho \otimes V_\rho^* \cong \bigoplus_{[\rho]} V_\rho^{\dim V_\rho}. \]

**Proof.** We show that each corresponding summand is isomorphic to get the result. The map $f : V_\rho \otimes V_\rho^* \to \mathcal{M}_{[\rho]}$ given by $v_1 \otimes \langle \cdot, v_2 \rangle_\rho \mapsto \langle \rho(g)v_2, v_1 \rangle_\rho$ is a bijection. It’s also a morphism: for $h \in G$,
\[
\begin{align*}
    f(h(v_1 \otimes \langle \cdot, v_2 \rangle_\rho)) &= f(hv_1 \otimes \langle \cdot, v_2 \rangle_\rho) \\
    &= \langle \rho(g)v_2, hv_1 \rangle_\rho \\
    &= \langle \rho(h^{-1}g)v_2, v_1 \rangle_\rho \\
    &= \langle \rho(g)v_2, v_1 \rangle_\rho \\
    &= hf(v_1 \otimes \langle \cdot, v_2 \rangle_\rho).
\end{align*}
\]
This proves the first isomorphism. Now consider the second isomorphism. Let $n = \dim V^* = \dim V$. Let $\{\ell_i\}_{i=1}^n$ be a basis for $V_\rho^*$ so that $V_\rho^* = \bigoplus_{i=1}^n \mathbb{C}\ell_i$. Then as vector spaces,
\[ V_\rho \otimes V_\rho^* \cong \bigoplus_{i=1}^n V_\rho \otimes \mathbb{C}\ell_i \cong \bigoplus_{i=1}^n V_\rho = V_\rho^{\dim V_\rho}. \]
Since $V_\rho$ is preserved in each isomorphism, the $G$-action on $V_\rho$ is also preserved and we have an isomorphism of representations.

To end this section, we prove one more result about matrix coefficients which will be needed to finish the proof that $\mathcal{M}$ is dense in $C(G; \mathbb{C})$, and ultimately dense in
Let \( f : G \to \mathbb{C} \), and define \( f_g : G \to \mathbb{C} \) by \( f_g(x) = f(xg) \). If \( \{f_g : g \in G\} \) spans a finite dimensional vector space, then \( f \) is called \textbf{right G-finite}.

**Theorem 5.7.** Let \( f : G \to \mathbb{C} \). Then \( f \) is a matrix coefficient if and only if it is right \( G \)-finite.

**Proof.** First suppose \( f \) is a matrix coefficient of some finite dimensional representation \( \rho \), i.e., \( f(g) = \langle \rho(g)v_1, v_2 \rangle_{\rho} \) for some \( v_1, v_2 \in V \). Then

\[
   f_g(x) = f(xg) = \langle \rho(xg)v_1, v_2 \rangle_{\rho} = \langle \rho(x)\rho(g)v_1, v_2 \rangle_{\rho}.
\]

Since \( \rho(g)v_1 \in V \), this shows \( f_g \) is a matrix coefficient and in \( \mathcal{M}_\rho \). Thus \( \{f_g : g \in G\} \) is contained in the finite dimensional \( \mathcal{M}_\rho \cong V_{\rho}^{\dim V_{\rho}} \) which makes \( f \) right \( G \)-finite.

Now suppose \( f \) is right \( G \)-finite, let \( V \) be the finite dimensional vector space spanned by \( \{f_g : g \in G\} \). Then \( G \) acts on basis elements of \( V \) via \( hf_g = f_{gh} \), and the action extends linearly to all of \( V \). This defines a finite dimensional representation \( \rho : G \to GL(V) \) where \( \rho(h)f_g = f_{gh} \). Let \( \ell \) be the linear functional on \( V \) such that \( \ell(v) = v(1) \). Then for all \( g \in G \),

\[
   \ell(\rho(g)f_1) = \ell(f_g) = f_g(1) = f(g),
\]

showing that \( f \) is a matrix coefficient of \( \rho \). \( \square \)
6 The Peter-Weyl Theorem

Now for the second, and final, stage of attack; we show $\mathcal{M}$ is dense in $L^2(G)$. For then we will have

$$L^2(G) = \mathcal{M} = \bigoplus_{[\rho]} \mathcal{M}_{[\rho]} \cong \widehat{\bigoplus_{[\rho]} \mathcal{M}_{[\rho]}},$$

and be ready to prove the Peter-Weyl Theorem. To accomplish this, we will show that $\mathcal{M}$ is dense in $C(G, \mathbb{C})$. Then by Corollary 4.7, we can conclude that $\mathcal{M}$ is dense in $L^2(G)$.

**Theorem 6.1.** The matrix coefficients of $G$ are dense in $C(G, \mathbb{C})$.

**Proof.** Let $f \in C(G, \mathbb{C})$. For $\epsilon > 0$, we need to find a matrix coefficient of $G$ within $\epsilon$ of $f$ under the sup norm $\| \cdot \|_{\infty}$. By Proposition 4.9, we know there exists an open set $U$ containing the identity of $G$ such that

$$|f(xh) - f(h)| < \epsilon/2$$

for all $h \in G$ and all $x \in U$. By Proposition 4.15 there exists a non-negative $\phi \in$
$C(G, \mathbb{C})$ supported in $U$ satisfying $\int_G \phi(g) \, dg = 1$. So given $h \in G$ we have

$$|T_{\phi} f(h) - f(h)| = \left| \int_G \phi(hg^{-1}) f(g) \, dg - f(h) \right|$$

$$= \left| \int_G \phi(h(g^{-1}h)^{-1}) f(g^{-1}h) \, dg - f(h) \right| \quad (g \mapsto g^{-1}h)$$

$$= \left| \int_G \phi(g) f(g^{-1}h) \, dg - f(h) \right|$$

$$= \left| \int_G \phi(g) f(g^{-1}h) \, dg - \int_G f(h) \phi(g) \, dg \right|$$

$$\leq \int_G \phi(g) ||f(g^{-1}h) - f(h)|| \, dg$$

$$= \int_U \phi(g) |f(g^{-1}h) - f(h)| \, dg$$

$$< \epsilon/2 \int_U \phi(g) \, dg$$

$$= \epsilon/2$$

Therefore since $T_{\phi} f$ and $f$ are continuous,

$$\|T_{\phi} f - f\|_\infty = \sup_{h \in G} |T_{\phi} f(h) - f(h)| < \epsilon/2.$$

We will use this fact at the end of the proof. Call it fact 1.

Now we also know $\phi$ satisfies $\phi(g^{-1}) = \overline{\phi(g)}$. So $T_{\phi}$ is a compact self-adjoint operator on $L^2(G)$ by Proposition 4.14. Thus Theorem 4.12(Spectral Theorem) tells us that $L^2(G)$ is a direct orthogonal sum of the eigenspaces $M_{\lambda_n}$ of $T_{\phi}$, and $M_{\lambda_n}$ is finite dimensional for $\lambda_n \neq 0$. Furthermore, each $\lambda_n$ is real and $\lambda_n \to 0$ as $n \to \infty$.

Since also $f \in L^2(G)$, we therefore have

$$f = f_0 + \sum_{n=1}^{\infty} f_{\lambda_n}$$
where each $f_{\lambda_n} \in M_{\lambda_n}$ and $f_0 \in M_0 = \ker T_\phi$. Since $\{f_{\lambda_n}\}$ is orthogonal, the Pythagorean identity gives us

$$\|f\|_2^2 = \|f_0\|_2^2 + \sum_{n=1}^{\infty} \|f_{\lambda_n}\|_2^2 < \infty.$$ 

So we may choose $N$ sufficiently large so that

$$\sum_{n=N}^{\infty} \|f_{\lambda_n}\|_2^2 \left( \frac{\epsilon/2}{\|\phi\|_{\infty}} \right)^2.$$ 

Let $f^N = \sum_{n=N}^{\infty} f_{\lambda_n}$. Then by Proposition 4.5

$$\|f^N\|_1^2 \leq \|f^N\|_2^2 = \left\| \sum_{n=N}^{\infty} f_{\lambda_n} \right\|_2^2 = \sum_{n=N}^{\infty} \|f_{\lambda_n}\|_2^2 < \left( \frac{\epsilon/2}{\|\phi\|_{\infty}} \right)^2$$

so that

$$\|\phi\|_{\infty} \|f^N\|_1 < \epsilon/2.$$ 

This fact will also be used at the end of the proof. Call it fact 2.

Now let $\tilde{f} = f - f_0 - f^N$. Then $\tilde{f} \in \oplus_{n=1}^{N-1} M_{\lambda_n}$ which implies $T_\phi \tilde{f} \in \oplus_{n=1}^{N-1} M_{\lambda_n}$. Proposition 4.16 then tells us that the right $g$-translates of $T_\phi \tilde{f}$, $T_\phi \tilde{f}_g$, are also in $\oplus_{n=1}^{N-1} M_{\lambda_n}$, which is finite dimensional by the Spectral theorem. Thus the $T_\phi \tilde{f}_g$ span a finite dimensional vector space making $T_\phi \tilde{f}$ a matrix coefficient by Proposition 5.7.
We claim $T_\phi \tilde{f}$ is the desired matrix coefficient within $\epsilon$ of $f$. Observe

\[
\left\| f - T_\phi \tilde{f} \right\|_\infty \leq \left\| f - T_\phi f \right\|_\infty + \left\| T_\phi f - T_\phi \tilde{f} \right\|_\infty \\
< \epsilon/2 + \left\| T_\phi (f - \tilde{f}) \right\|_\infty \quad \text{(fact 1)} \\
= \epsilon/2 + \left\| T_\phi (f_0 + f^N) \right\|_\infty \\
= \epsilon/2 + \left\| T_\phi f_0 + T_\phi f^N \right\|_\infty \\
= \epsilon/2 + \left\| T_\phi f^N \right\|_\infty \\
\leq \epsilon/2 + \| \phi \|_\infty \left\| f^N \right\|_1 \quad \text{(see proof of 4.13)} \\
< \epsilon/2 + \epsilon/2 \quad \text{(fact 2)} \\
= \epsilon
\]

as desired. \(\square\)

**Theorem 6.2** (The Peter-Weyl Theorem). As a representation of $G$,

\[
L^2(G) \cong \bigoplus_{[\rho]} V^\dim V_\rho
\]

where the sum runs over all irreducible representation classes of $G$.

**Proof.** We know by Proposition 5.5 that $\bigoplus_{[\rho]} M_{[\rho]} \cong \bigoplus_{[\rho]} V^\dim V_\rho$ as representations. So

\[
\bigoplus_{[\rho]} M_{[\rho]} \cong \bigoplus_{[\rho]} V^\dim V_\rho
\]

with respect to the norm on $V^\dim V_\rho$ induced by a $G$-invariant inner product on $V^\dim V_\rho$.

We therefore conclude

\[
L^2(G) \cong \bigoplus_{[\rho]} V^\dim V_\rho
\]

by Theorem 6.1, and the proof is complete. \(\square\)

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We have only considered finite dimensional representations in this paper. One might wonder if anything changes when we allow representations over infinite-dimensional Hilbert spaces. The following shows that in fact nothing changes. Recall the action $R_g f(h) = f(hg)$ on $L^2(G)$.

**Corollary 6.3.** All irreducible complex representations of a compact group are finite dimensional.

**Proof.** Let $\pi : G \to GL(V)$ be an irreducible representation where $V$ is an infinite dimensional Hilbert space. Since $V$ has an inner product, we may still construct a $G$-invariant inner product $\langle \cdot, \cdot \rangle_\pi$ on $V$. We can then define a nontrivial linear map $f : V \to L^2(G)$ by $f(v) = \langle \pi(\cdot)v, w \rangle_\pi$, for some nonzero $w \in V$. We show $f$ is a morphism. Let $g, h \in G$. Then

$$f(\pi(g)v)(h) = \langle \pi(h)\pi(g)v, w \rangle_\pi = \langle \pi(hg)v, w \rangle_\pi = R_g\langle \pi(h)v, w \rangle_\pi = R_g f(v)(h)$$

as desired. The image of $f$ is thus a subrepresentation of $L^2(G)$, and it’s irreducible since $V$ is.

Notice that Proposition 2.12(Orthogonality Relations) depends only on a $G$-invariant inner product and not dimension. Therefore the image of $f$ is orthogonal to $M = \bigoplus_{[\rho]} M_{[\rho]}$, where the sum is over all non-isomorphic finite dimensional representations $\rho$. By continuity of $\langle \cdot, \cdot \rangle_\pi$ and the fact that $L^2(G)$ is a metric space, we then have the image of $f$ orthogonal to $M$. But we know $L^2(G) = \overline{M}$ by Peter-Weyl. This implies the image of $f$ is $\{0\}$ which is a contradiction. □

We conclude this section with an example. Let $G = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$ with Lebesgue measure. Since $S^1$ is abelian we know from Corollary 2.6 that its representations $\rho$ are one-dimensional, i.e., $\rho : S^1 \to GL(\mathbb{C}) \cong \mathbb{C}^\times$. Since compact
representations are also unitary, we know $1 = \rho(z)\rho(z)^* = \rho(z)\overline{\rho(z)} = |\rho(z)|^2$ which implies $\rho(z) \in S^1$. Therefore all representations of $S^1$ are continuous homomorphisms $S^1 \to S^1$. These are known to be the maps $z \mapsto z^n$ for $n \in \mathbb{Z}$, which are all non-isomorphic. Recalling that all $z \in \mathbb{C}$ can be written as $re^{ix}$ where $r \in \mathbb{R}$ and $x \in [0, 2\pi)$, these are the maps $e^{ix} \mapsto e^{nix}$ which we denote by $\rho_n$. We therefore have $\rho_n(e^{ix})(z) = ze^{nix}$ for all $z \in \mathbb{C}$.

We now need to construct matrix coefficients, and therefore need $S^1$-invariant inner products $\langle \cdot, \cdot \rangle_{\rho_n}$ on the representation spaces $V_{\rho_n}$. We have the standard $S^1$-invariant inner product on $\mathbb{C}$ given by $\langle z_1, z_2 \rangle = z_1 \overline{z_2}$, so we simply set $\langle \cdot, \cdot \rangle_{\rho_n} = \langle \cdot, \cdot \rangle$. For a matrix coefficient $\sigma_n : S^1 \to \mathbb{C}$ of $V_{\rho_n}$, we then have

$$\sigma_n(e^{ix}) = \langle \rho_n(e^{ix})z_1, z_2 \rangle_{\rho_n} = z_1 \overline{z_2} e^{nix},$$

so that $\mathcal{M}_{\rho_n} = \mathbb{C} e^{nix}$. So by Peter-Weyl,

$$L^2(S^1) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} e^{nix}.$$

We therefore have

$$L^2([0, 2\pi)) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} e^{nix}$$

since $\mathbb{R}/2\pi\mathbb{Z} \cong S^1$. This is the classic theory of Fourier series for periodic functions $f : \mathbb{R} \to \mathbb{C}$. The theory states that $f \in L^2([0, 2\pi))$ if and only if the Fourier series for $f$ converges to $f$ in the $L^2$ norm.

As we have said, this is a classical result. However, Peter-Weyl allows us to generalize Fourier series to any compact group including non-abelian groups. This is the power of the Peter-Weyl theorem.
References


