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Fractional Chromatic Numbers of Incidence Graphs

by

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Dedication and Acknowledgements

Dedicated to Gregory, Gregory Jr., Rose, Mom, Dad and Robert

I am honored to acknowledge the many people in my life who have made getting to this point possible. It is not possible to give the following individuals enough thanks, but this is a start.

First, and foremost, I must thank God for instilling in me the desire, strength, motivation and for providing the earthly support to achieve this goal. He heard countless prayers in gratitude and struggle through these past years and I am sure He responded to each with exactly what would help me be successful.

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Abstract

In 1993, Brualdi and Massey defined the incidence graph of G, $Inc(G)$, to be the graph whose vertices are the set of incidences - pairs of the form (u, e) where u is a vertex of G and e is an edge of G containing u as an endpoint - and where two incidences, (u, e) and (v, f), are adjacent if (i) $u = v$, (ii) $e = f$ or (iii) $uv = e$ or $uv = f$. They determine the incidence chromatic number, $\chi_i(G) = \chi(Inc(G))$, of several classes of graphs and use this work to study the strong chromatic index of a certain bipartite graph H which is associated to G. Much work has been done in computing and bounding the incidence chromatic number of graphs. Of particular interest, in 2012, Yang defined the fractional incidence chromatic number of a graph to be $\chi_f(Inc(G))$; that is, the fractional chromatic number of the incidence graph of G.

In what follows, we generalize many known bounds on the incidence chromatic number to bounds on the fractional incidence chromatic number. By providing a lower bound on the fractional incidence chromatic number which provides equality when $Inc(G)$ is vertex transitive and giving a sufficient condition for when $Inc(G)$ is vertex transitive, we are able to compute the fractional incidence chromatic number of several families of graphs. Further, we generalize the bounds for the union, Cartesian product and join of two graphs obtained by Sun and Shiu along with the bounds for the lexicographic and direct products of two graphs and a bound involving the star arboricity and the edge chromatic number obtained by Yang. We also show that these bounds are all tight. Given these bounds, along with another bound involving the square of a graph, we show that $\chi_f(C_{3n}[K_\ell]) = 3\ell$ for $n \geq 1$ and $\ell \geq 2$. Finally, using the Strong Perfect Graph Theorem, we show that $Inc(G)$ is perfect precisely when G has circumference at most 3; that

is, when a longest cycle in G has length at most 3. As a result, we compute $\chi_f(Inc(G))$ in this case. We end with a discussion on some future work.

Contents

List of Figures

Chapter 1

Introduction

1.1 Incidence Graphs

Incidence graphs were introduced by Brualdi and Massey ([6]) as part of their effort to improve bounds for the strong chromatic index of graphs[1]. They defined the *incidence graph* of a graph G, denoted $Inc(G)$, to be the graph with

$$V(Inc(G)) = \{(u, e) \mid u \in V(G), e \in E(G) \text{ and } u \text{ is an endpoint of } e\}$$

and

$$E(Inc(G)) = \{(u, e)(v, f) \mid u = v, e = f, uv = e \text{ or } uv = f\}.$$

Example. Here is C_5 along with its incidence graph $Inc(C_5)$ drawn out explicitly. (We will consider another way to view $Inc(G)$ in the next section.)

[1] All graphs are considered to be finite and simple, unless otherwise noted.

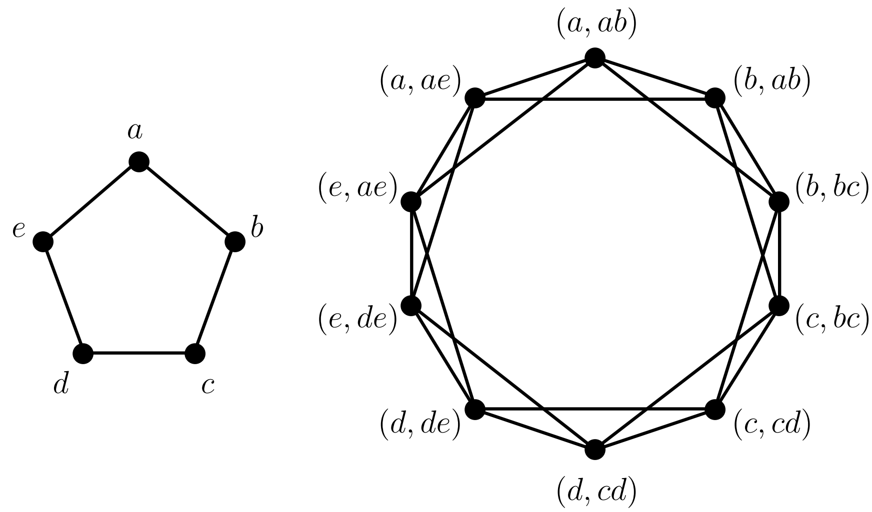


Figure 1.1: C_5 and $Inc(C_5)$ Explicitly

They went on, in their introductory paper, to achieve their goal of improving the bound for the strong chromatic index of a particular class of graphs, namely the class of bipartite graphs whose cycle lengths are all divisible by 4. Here, the strong chromatic index refers to the fewest number of colors required to properly color the edges of the given graph so that each color class is an induced matching of the graph. This is a slight strengthening of the classical chromatic index, or edge chromatic number. They obtain their improved bound by appealing to what they defined as the incidence chromatic number for the graph associated to each of the graphs in this particular class. The incidence chromatic number of G is denoted by $\chi_i(G)$ or $\chi(Inc(G))$. As a result of Brualdi and Massey's work, in the two decades since its definition there has been interest in calculating and finding bounds for the incidence chromatic number of graphs.

Upon studying the incidence chromatic number of graphs, Brualdi and Massey made several key observations. First, they proved that every graph has incidence chromatic number at least $\Delta + 1$, where Δ is the maximum degree of the graph, by showing that we can always find a clique of this size in $Inc(G)$. They go on to calculate the incidence chromatic number of complete graphs, trees with at least two vertices and complete bipartite graphs. It is worth noting that the first two of these calculations work out to be $\Delta+1$, where Δ is the maximum degree of the graph, so in particular the lower bound they proved is tight. However, the third example, for complete bipartite graphs, works out to be $\Delta+2$. They conjectured that the incidence chromatic number of any graph is at most $\Delta+2$. This conjecture became known as the Incidence Coloring Conjecture. It was disproved by Guiduli ([16]) in 1997 when he showed that the Paley graphs have the property that their incidence chromatic numbers are at least $\Delta+\Omega(\log\Delta)$.

In terms of progress on a general upper bound for all graphs, Brualdi and Massey proved that the incidence chromatic number of a graph is at most 2Δ. Guiduli, in the same paper where he disproved the Incidence Coloring Conjecture, proved another upper bound for the incidence chromatic number, namely that

$$\chi(Inc(G)) \leq \Delta + 20\log\Delta + 84.$$

Further, Yang ([37]) exibited an upper bound which appeals to the star arboricity of a graph, denoted $\mathrm{st}(G)$. Note that the star arboricity of a graph is the minimum number of star forests required to cover all the edges of G. Yang proved that the incidence chromatic number of a graph is at most $\mathrm{st}(G)+\Delta(G)$ for Class 1 graphs and is at most $\mathrm{st}(G)+\Delta(G)+1$ for Class 2 graphs. (Here, Class 1 and Class 2 refer to the classes distinguished under Vizing's Theorem ([35]).) Shiu and Sun ([28]) show that $\chi(Inc(G)) \leq \chi(G^2)$. Finally, Sun and Shiu ([31]) proved another lower bound and showed that the incidence chromatic number of G is at least $\frac{2|E(G)|}{|V(G)|-\gamma(G)}$, where $\gamma(G)$ is the domination number of G. This last bound will be particularly useful for us later.

Others have put forth new information regarding bounds on specific classes of graphs. In [36], Wang, Chen and Pang compute the incidence chromatic number of Halin graphs and outerplanar graphs with large enough maximum degree. In [11], Hosseini Dolama, Sopena and Zhu exhibit upper bounds for k-degenerate graphs, K_4-minor free graphs and planar graphs. In [25], Maydanskiy shows that the incidence chromatic number of a cubic graph is at most 5, and that this bound is tight. In [10], Hosseini Dolama and Sopena improve their previous bound for 3-degenerate graphs and produce upper bounds for graphs with various conditions on the maximum average degree and the maximum degree. In [31], Sun and Shiu develop bounds for the incidence chromatic number of the union, Cartesian product and join of two graphs. They further show that their bounds are tight. In [37], Yang develops bounds for the incidence chromatic number of the direct and lexicographic products of two graphs. In [30], Sun completes a characterization for the incidence chromatic number of cubic graphs by providing necessary and sufficient conditions for when the incidence chromatic number of a cubic graph is 4.

One final note on what is known about incidence graphs. In 2012, Hartke and Helleloid ([19]) proved a reconstruction algorithm which, given any graph H, either produces a graph G with no isolated vertices such that $Inc(G) \cong H$ or determines that no such graph exists. Note that the condition that G has no isolated vertices is sort of irrelevant since any isolated vertex of G has no effect on $Inc(G)$. The algorithm provided runs in linear time.

1.2 Visualizing $Inc(G)$

Given the definition of the incidence graph associated to a graph G, we can always think about the graph $Inc(G)$ explicitly; that is, we can always draw out the graph of $Inc(G)$, as in Figure 1.1. However, these graphs get quite large very quickly. To see this, simply consider the number of vertices of $Inc(G)$, i.e. the number of incidences of G. Since an incidence is a pair consisting of an edge and an endpoint of that edge, and since each edge has two endpoints, there are $2|E(G)|$ incidences. Hence,

$$|V(Inc(G))| = 2|E(G)|.$$

So, for example, we already saw above that $Inc(C_5)$ has 10 vertices, and we can also now calculate that $Inc(K_n)$ has $n(n-1)$ vertices. The edges of $Inc(G)$ make the graph even more complicated. In fact, we can show that

$$\deg_{Inc(G)}(u, uv) = 2\deg_G(u) + \deg_G(v) - 2.$$

To see this, observe that (u, uv) is adjacent in $Inc(G)$ to

- $\deg_G(u) - 1$ vertices of the type (u, uw), where $w \neq v$
- (v, uv); the only vertex which shares the same edge component as (u, uv)
- $\deg_G(u) - 1$ vertices of the type (w, uw), where $w \neq v$
- $\deg_G(v) - 1$ vertices of the type (v, vx), where $x \neq u$.

Adding all these adjacencies gives the desired degree of (u, uv) in $Inc(G)$. Hence, we would like another way of viewing the incidence graph associated to G.

As it turns out, we can view the incidence graph within the original graph. Each edge of G represents two vertices in the incidence graph, $Inc(G)$, one for each endpoint. Let $S(G)$ be the graph obtained from G by replacing each edge of G by a pair of oppositely oriented edges. This gives us a way of viewing the vertices of $Inc(G)$ by associating the edge uv oriented from u to v to the incidence (u, uv). Now, we need a way of recognizing adjacencies from this viewpoint.

Lemma 1.2.1. *Two oriented edges of G, as described above, are adjacent in $Inc(G)$ precisely when they fall into one of the following categories:*

1. *The two oriented edges are oriented away from a common vertex.*

2. *The two oriented edges form an oriented 2-cycle.*

3. *The two oriented edges form an oriented path on three vertices.*

Proof. Recall from the definition of the incidence graph that two incidences are adjacent precisely when they satisfy one of the following conditions:

1. The vertex components of the incidences are the same; that is, (u, e) is adjacent to (u, f) where $e \neq f$.

2. The edge components of the incidences are the same; that is, (u, e) is adjacent to (v, e) where $u \neq v$.

3. The two (distinct) vertex components of the incidences are the endpoints of one of the (distinct) edges associated to the incidences; that is, (u, uv) is adjacent to (v, vw).

Using the observation above, these precisely line up with the categories of the lemma; namely

1. The incidences (u, e) and (u, f) are both oriented away from their common vertex, u.

2. The incidences have the same edge component, so they are represented by the two ways to orient the edge e in G. Hence, they form an oriented 2-cycle.

3. The incidence (u, uv) is oriented from u to v and the incidence (v, vw) is oriented from v to w. Hence, we have an oriented path on three vertices from u to v to w.

Thus, the result follows. □

This provides us with another way of viewing $Inc(G)$ which will be useful later on, although we will use the two methods interchangably. Let's now compare this method of visualizing $Inc(G)$ with our previous method of drawing it out explicitly by looking at our example of $G = C_5$.

Example. Illustrated here is the same C_5 as before, alongside $Inc(C_5)$ being visualized using oppositely oriented arcs in $S(G)$.

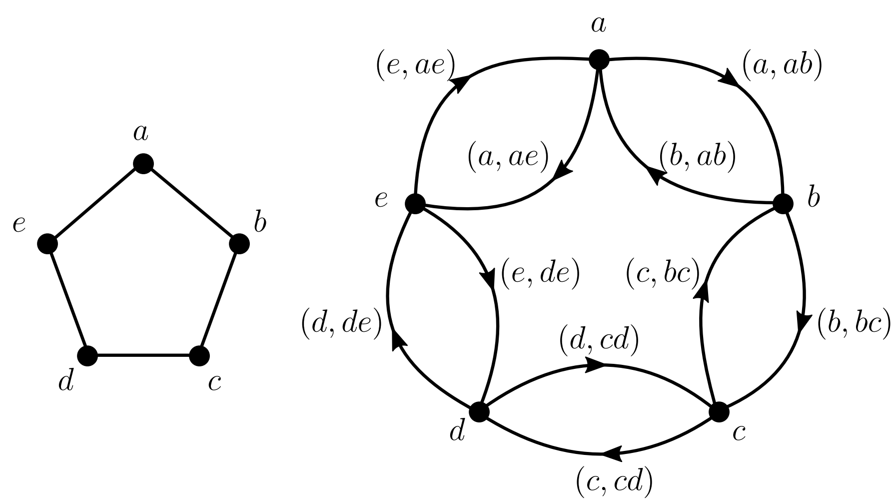


Figure 1.2: C_5 and $Inc(C_5)$ within $S(C_5)$

1.3 The Fractional Chromatic Number

Recall that the chromatic number of a graph G, denoted $\chi(G)$, is the smallest number of colors needed to color the vertex set of the graph so that no two adjacent vertices are given the same color. This is a way to assign to each graph a natural number. To generalize this idea, one can ask whether or not there is a way to assign to every graph a nonnegative rational number which, in some way, respects the chromatic number of the graph. One way to do this is with the fractional chromatic number. (Note: In Chapter 5, we will briefly discuss another way to generalize the chromatic number to obtain nonnegative rational numbers.

We will also discuss a possible new generalization.)

The fractional chromatic number of a graph G, denoted $\chi_f(G)$, can be viewed as a generalization of the usual chromatic number. The concept first appeared in 1973 in a paper of Hilton, Rado and Scott ([21]). In this paper, they prove that $\chi_f(G) < 5$ for all planar graphs. This was a step towards proving what they called the "weak" four colour conjecture; namely, that if G is planar, then $\chi_f(G) \leq 4$. Of course, as it will become clear, since $\chi_f(G) \leq \chi(G)$, this conjecture is now resolved by the Four Color Theorem (see [1] and [2]).

Before giving a precise definition for the fractional chromatic number, informally, we should think about this as assigning a collection of colors to each vertex of the graph, with a weight associated to each color, which satisfies some restriction relating to adjacent vertices. In practice, the usefulness of this concept is most easily seen with an application. In scheduling a set of meetings, for example, one is often interested in finding the least amount of time necessary to fit all of the meetings in. Nobody wants to be in meetings for longer than they have to! To do this, we can associate to the set of meetings a graph where the meetings are the vertices and two vertices are made to be adjacent precisely when the two meetings must be attended by a common member, and hence the meetings cannot occur at the same time. Coloring in the classical sense will provide the number of meeting times which will be necessary to hold all the meetings in the shortest amount of time, without the allowance for a break in the meeting. Fractionally coloring this graph will allow for the shortest amount of time while possibly forcing breaks in the meetings. We will see a more explicit example below. For now, let's move on to the precise definition.

There are, in fact, three ways in which we can define the fractional chromatic number of a graph G. Recall that there are three ways in which we can define the chromatic number in the classical sense. We will draw the analogy between these two invariants through their definitions. First, we define a *k-coloring* of G to be an assignment of precisely one of k colors to each vertex of G such that adjacent vertices are assigned different colors. Then, we can define $\chi(G)$ to be the smallest number, k, such that there exists a k-coloring of G. That is,

$$\chi(G) = \min\{k \mid G \text{ can be } k\text{-colored}\}.$$

This definition assumes that each vertex will be colored with precisely one color. Relaxing this condition, we can define an *(a, b)-coloring* of G to be an assignment of precisely b out of a colors to each vertex such that if two vertices are adjacent, they are assigned disjoint sets of colors. Then, we define

$$\chi_b(G) = \min\{a \mid G \text{ can be } (a, b)\text{-colored}\}.$$

Finally, we can define

$$\chi_f(G) = \inf\left\{\frac{\chi_b(G)}{b}\right\}.$$

This is the original definition given by Hilton, Rado and Scott. Note that

$$\chi_1(G) = \chi(G),$$

and so this is one way to see that $\chi_f(G) \leq \chi(G)$.

Example. Here is a $(5, 2)$-coloring of C_5.

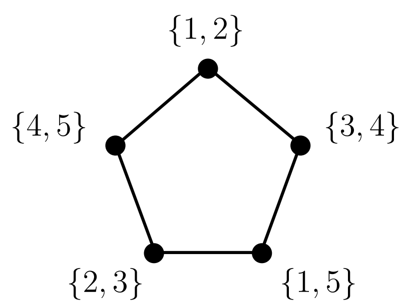


Figure 1.3: A $(5, 2)$-coloring of C_5

Therefore, this shows that $\chi_f(C_5) \leq \frac{5}{2}$. In fact, this is actually optimal.

Next, we can define $\chi(G)$ as the solution to the following integer program. Let $V(G) = \{v_1, \ldots, v_n\}$ be the vertices of G and let $\mathcal{I} = \{I_1, \ldots, I_k\}$ be the set of all independent sets of G; that is, all the subsets of the vertices in which the elements are pairwise nonadjacent. Define the vertex-independent set matrix A, whose rows are indexed by $V(G)$ and whose columns are indexed by $\mathcal{I}$, as

$$A_{i,j} = \begin{cases} 1 & v_i \in I_j \\ 0 & \text{otherwise} \end{cases}.$$

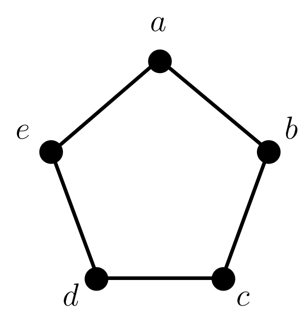
Then, $\chi(G)$ is the optimal solution to the integer program

$$\min \mathbf{1}^t\mathbf{x} \qquad \text{subject to} \qquad A\mathbf{x} \geq \mathbf{1}, \mathbf{x} \in \mathbb{Z}^k, \mathbf{x} \geq \mathbf{0}.$$

Note that the constraint $A\mathbf{x} \geq \mathbf{1}$ guarantees that every vertex gets a color (or assigned to some independent set, since the color classes are independent sets). The objective function $\min \mathbf{1}^t\mathbf{x}$ asks for the minimum possible number of colors (or independent sets) needed to satisfy the constraints. Relaxing the constraint that $\mathbf{x} \in \mathbb{Z}^k$ to $\mathbf{x} \in \mathbb{R}^k$, we get a linear program whose optimal solution is $\chi_f(G)$. Now, the constraint $A\mathbf{x} \geq \mathbf{1}$ guarantees that every vertex gets a full weight of colors (or independent sets), but it need not be fully one color (or in an independent set with weight at least 1). The objective function now asks for the minimum of the sum of the weights of the independent sets over all situations satisfying the constraint. Since any feasible solution to the integer program is also a feasible solution to the linear program, we again see that $\chi_f(G) \leq \chi(G)$.

This definition follows from a general method used in fractional graph theory. Namely, many integer invariants of graphs can be phrased in terms of the solution to an integer program. (Other examples include the clique number, covering number and packing number.) Any time we can write an invariant in this way, we can relax the integer constraint and obtain the fractional version of the invariant. (That is, we obtain the fractional clique number, the fractional covering number and the fractional packing number). We have done this process with the chromatic number (to obtain the fractional chromatic number) here. Finally, note that although given the input for a linear program, the optimal solution can be computed in polynomial time, it is still the case that calculating $\chi_f(G)$ is NP-complete in general. This is due to the fact that finding all the independent sets of a graph is an NP-hard problem (see [15]).

Example. Consider again $G = C_5$.



Then,

$$V(C_5) = \{a, b, c, d, e\}$$

and

$$\mathcal{I} = \{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, c\}, \{a, d\}, \{b, d\}, \{b, e\}, \{c, e\}\}.$$

Further,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

An optimal solution to this linear program is then given by

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

So, each singleton independent set gets weight 0 and each two-element independent set gets weight 1/2. In comparison to the previous example, there is a correlation between these weights and the colors found in the $(5, 2)$-coloring of C_5. Namely, the independent sets correspond to particular colors as follows:

- $\{a, c\} \leftrightarrow$ the color 1
- $\{a, d\} \leftrightarrow$ the color 2
- $\{b, d\} \leftrightarrow$ the color 3
- $\{b, e\} \leftrightarrow$ the color 4
- $\{c, e\} \leftrightarrow$ the color 5

Since we have obtained an optimal solution, we see that $\chi_f(C_5) = \frac{5}{2}$.

Finally, we have the notion of a homomorphism between two graphs.

Definition. A *homomorphism* between two graphs G and H is a map $f : G \to H$ such that if $uv \in E(G)$, then $f(u)f(v) \in E(H)$. If there exists a homomorphism from G to H, we write $G \to H$.

Then, we can define

$$\chi(G) = \min\{k \mid G \to K_k\},$$

where K_n is the complete graph on n vertices. Similarly, we can define

$$\chi_f(G) = \inf\left\{\frac{r}{s} \mid G \to K(r, s)\right\},$$

where $K(r, s)$ is a *Kneser graph*; that is, a graph whose vertices are the s-element subsets of an r-element set and whose edges are between vertices whose corresponding s-element subsets are disjoint. Note that $K(n, 1) \cong K_n$ and so again, we can see that $\chi_f(G) \le \chi(G)$.

Example. We can define a homomorphism $C_5 \to K(5, 2)$ as follows.

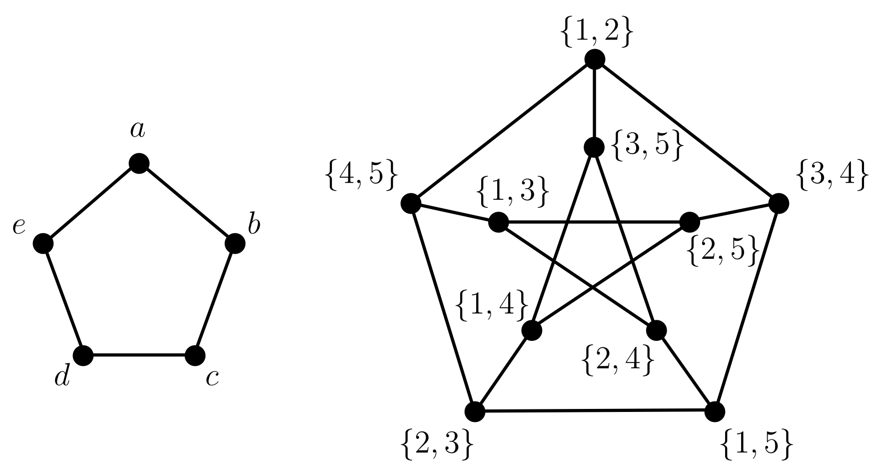


Figure 1.4: A Homomorphism: $C_5 \to K(5, 2)$

Given the labels on the graphs above, define a homomorphism

$$\varphi : C_5 \to K(5, 2)$$

by

$$\varphi(a) = \{1,2\}, \varphi(b) = \{3,4\}, \varphi(c) = \{1,5\}, \varphi(d) = \{2,3\}, \varphi(e) = \{4,5\}.$$

In fact, this is an inclusion homomorphism where C_5 can be seen as induced by the outer vertices of $K(5,2)$. This shows that since $C_5 \to K(5,2)$, then $\chi_f(C_5) \leq \frac{5}{2}$. Again, the colors associated to each vertex are the same as in the previous two studies of $\chi_f(C_5)$.

Each of these definitions has their benefits. We will utilize this last definition the most in what follows. This method is discussed in [27]. We will revisit the concept of the Kneser graphs and complete graphs serving as target graphs for coloring homomorphisms in Chapter 5.

To further see the connection between the fractional chromatic number and the usual chromatic number, there are a series of wonderful results and conjectures. This first one is a conjecture regarding the direct product of two graphs. See Chapter 3 for the precise definition of the direct product.

Conjecture 1.3.1 (Hedetniemi, [20])**.** *If G and H are graphs, then*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

Note that it is easy to see that

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}.$$

It is the reverse inequality that is difficult. In 2011, Zhu ([38]) proved that the fractional version of Hedetniemi's conjecture is true. That is, he showed the following.

Theorem 1.3.2. *If G and H are graphs, then*

$$\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}.$$

The next two results involve the lexicographic product. See Chapter 3 for its precise definition.

Proposition 1.3.3 ([17]). *If G and H are graphs,*

$$\chi(G[H]) \leq \chi(G)\chi(H).$$

Observe that this inequality can be strict. For example, consider $C_5[K_2]$. Then, $\chi(C_5) = 3$ and $\chi(K_2) = 2$, however there exists a 5-coloring of $C_5[K_2]$. This phenomenon does not occur in the fractional case.

Proposition 1.3.4 (Gao and Zhu, [14]). *If G and H are graphs,*

$$\chi_f(G[H]) = \chi_f(G)\chi_f(H).$$

Getting the equality in the fractional cases of the results regarding the direct and lexicographic products comes from the Strong Duality Theorem of linear programming which says, in this particular setting, that since the fractional chromatic number and the fractional clique number are dual problems,

$$\chi_f(G) = \omega_f(G).$$

Thus, the fact that we are allowing this relaxation into fractional graph theory is precisely what allows us to complete the opposite inequalities in these proofs.

The next result is a well-known inequality which holds for all graphs.

Proposition 1.3.5. *For any graph G,*

$$\chi(G) \geq \frac{|V(G)|}{\alpha(G)}.$$

Here, $\alpha(G)$ is the independence number of G which is the size of the largest independent set - a set of vertices which are all pairwise nonadjacent. In fractional coloring, we have the same result, with an extra piece of information. It is this idea that we will use repeatedly throughout the rest of what follows.

Theorem 1.3.6. *For any graph G,*

$$\chi_f(G) \geq \frac{|V(G)|}{\alpha(G)},$$

and there is equality when G is vertex transitive.

See Chapter 2 for a precise definition of vertex transitivity.

Example. Fractional chromatic numbers for a few vertex transitive graphs, along with their chromatic numbers for comparison.

1. $\chi_f(K_n) = n = \chi(K_n)$

2. $\chi_f(C_n) = \begin{cases} 2 & n \text{ even} \\ 2 + \frac{1}{k} & n \text{ odd} \end{cases}, n \geq 3.$
   This is equal to the chromatic number of C_n, if n is even. The chromatic number of odd cycles is 3, and so these are not the same in this case.

3. $\chi_f(P) = 5/2$, where P is the Petersen graph. The chromatic number of the Petersen graph is 3.

In this previous example, the Petersen graph can be recognized as part of the class of Kneser graphs. In particular, $P = K(5, 2)$. In 1978, Kneser's conjecture was proved. Namely, the following was shown.

Theorem 1.3.7 (Lovasz [24])**.** *If $r \geq 2s$, then*

$$\chi(K(r, s)) = r - 2s + 2.$$

In the fractional case,

Proposition 1.3.8. *If $r \geq 2s$, then*

$$\chi_f(K(r, s)) = \frac{r}{s}.$$

Proof. This result follows from Theorem 1.3.6 since Kneser graphs are vertex transitive. A simple counting argument shows that $K(r,s)$ has $\binom{r}{s}$ vertices and the Erdős-Ko-Rado Theorem ([12]) implies that

$$\alpha(K(r,s)) = \binom{r-1}{s-1}.$$

Combining these facts, we obtain the desired result. □

Thus, we know that we can obtain any rational number at least 2 as the fractional chromatic number of some graph. Note that we also saw above that $\chi_f(K_1) = 1$, so we can obtain any rational number equal to 1 or greater than or equal to 2. In fact, this is all you can obtain. This compares with the chromatic number in the sense that we can obtain any natural number greater than or equal to 1 as the chromatic number of a graph; namely, $\chi(K_n) = n$, for $n \geq 1$.

Finally, observe that $\chi_f(G)$ can be very close to $\chi(G)$; in fact, they can be equal as we have seen. On the other hand, they can also be arbitrarily far apart! Following an example in [27], consider $K(3n,n)$ for $n \geq 2$. Then, by Theorem 1.3.7,

$$\chi(K(3n,n)) = 3n - 2n + 2 = n + 2,$$

and by Proposition 1.3.8,

$$\chi_f(K(3n,n)) = \frac{3n}{n} = 3.$$

So, as n increases, the gap between $\chi_f(K(3n,n))$ and $\chi(K(3n,n))$ becomes arbitrarily large. Thus, the fractional chromatic number has very interesting properties.

1.4 Graph Perfection

We call a graph G *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$, where $\omega(H)$ is the clique number of H, the size of the largest clique. For example, even cycles of length at least 4 and the cycle of length 3 are perfect while odd cycles of length at least 5 are not perfect.

In 1961, Berge ([3]) studied a class of graphs which are now called Berge graphs. In order to define the class of Berge graphs, we need to first define what a hole and antihole are in a graph. A *hole* is defined to be an induced subgraph which is isomorphic to a cycle of length at least 4. An *antihole* is an induced subgraph whose complement is a hole in the complement of the whole graph. A graph is called *Berge* if it contains no holes or antiholes of odd length. Berge made two conjectures regarding perfect graphs in this paper, both of which have since been resolved. The first, now called the Perfect Graph Theorem, was resolved by Lovász in 1972. It is stated as follows:

Theorem 1.4.1 (Perfect Graph Theorem, [23]). *If G is perfect, then $\overline{G}$, the complement of G, is also perfect.*

The second, now called the Strong Perfect Graph Theorem, was resolved by Chudnovskey, Robertson, Seymour and Thomas in 2002, and was published by the group in 2006. It is stated as follows:

Theorem 1.4.2 (Strong Perfect Graph Theorem, [8]). *A graph G is perfect if and only if it is Berge.*

This theorem not only resolves a long standing conjecture, but it provides a way to test whether or not a graph is perfect. In fact, Chudnovsky et. al. ([7]) provide a polynoomial time algorithm for recognizing Berge graphs. We will use the Strong Perfect Graph Theorem to classify when $Inc(G)$ is perfect in Chapter 4.

1.5 Connections

In 2012, Yang ([37]) defined the fractional incidence chromatic number to be the fractional chromatic number of the incidence graph, $\chi_f(Inc(G))$. In this paper, he noted that Guiduli's results regarding the incidence chromatic number can be generalized to results regarding the fractional incidence chromatic number and observed that

$$\chi_f(Inc(G)) \leq \Delta + 20 \log \Delta + 84,$$

where Δ is the maximum degree of the graph. He further showed that Paley graphs also satisfy the property that

$$\chi_f(Inc(G)) \geq \Delta + \Omega(\log \Delta).$$

He goes on the study the fractional incidence chromatic number of C_5 and the circulant graphs. In particular, he computes that $\chi_f(Inc(C_5)) = \frac{10}{3}$ and he shows that for the circulant graph G_d^k, if $d \leq k/4$, then

$$\frac{2dk}{\left\lfloor \frac{2dk}{2d+1} \right\rfloor} \leq \chi_f(Inc(G_d^k)) \leq \frac{2dk}{2d \left\lfloor \frac{k}{2d+1} \right\rfloor}.$$

In Chapter 2, we introduce a technique involving vertex transitivity and the domination number to calculate $\chi_f(Inc(G))$ for a handful of common graph families. We also introduce the main technique used to extend the bounds on incidence chromatic numbers when combining two graphs in some way to bounds on the fractional incidence chromatic number. We will state and prove some of the more basic results, postponing some others for Chapter 3.

In Chapter 3, we study the fractional incidence chromatic number of graphs combined via the direct and lexicographic products. We also extend a bound which uses the star arboricity and edge chromatic number. All the bounds obtained are tight. We use the result of the lexicographic product along with a bound involving the square of a graph to compute new information about the fractional incidence chromatic number of a new infinite family of graphs.

In Chapter 4, we state and prove a characterization result for the perfectness of incidence graphs by appealing to Theorem 1.4.2. We further go on to compute the (fractional) incidence chromatic number of these graphs.

Finally, in Chapter 5, we consider the class of graphs $C_n[K_\ell]$, and more specifically $C_{2k+1}[K_\ell]$. We prove a characterization of when there is a homomorphism between two graphs in this second class. We give some bounds on the fractional incidence chromatic number of the graphs in these classes and pose some questions and possible ways to obtain a precise calculation. We also explore the difficulties and questions regarding this work and the possibility of using this class of graphs as the target class of graphs under homomorphisms for a new coloring generalization of the chromatic number. Finally, we discuss some properties held by G and $Inc(G)$ when $Inc(G)$ is perfect. We lastly pose some future work on trying to identify these classes of perfect graphs as classes which have previously been identified or showing that we have identified new classes of perfect graphs.

Chapter 2

Transitivity and Graph Homomorphisms

2.1 Arc Transitivity

The first goal of this chapter is to give one way of computing the fractional incidence chromatic number of some graphs. This will give us some examples to work with moving forward. To do this, we will determine a sufficient condition on G for when $Inc(G)$ is vertex transitive, so that we can use a variant of Theorem 1.3.6 to directly compute the fractional incidence chromatic number of G. Let's start with a definition.

Definition. An *automorphism* of a graph G is a bijective map $g : G \to G$ such that $uv \in E(G)$ if and only if $g(u)g(v) \in E(G)$.

Example. Here is an automorphism of the Petersen graph, $K(5, 2)$. This automorphism can be defined by permuting the elements of the set $\{1, \ldots, 5\}$ where $1 \leftrightarrow 2$ and the rest of the elements are fixed.

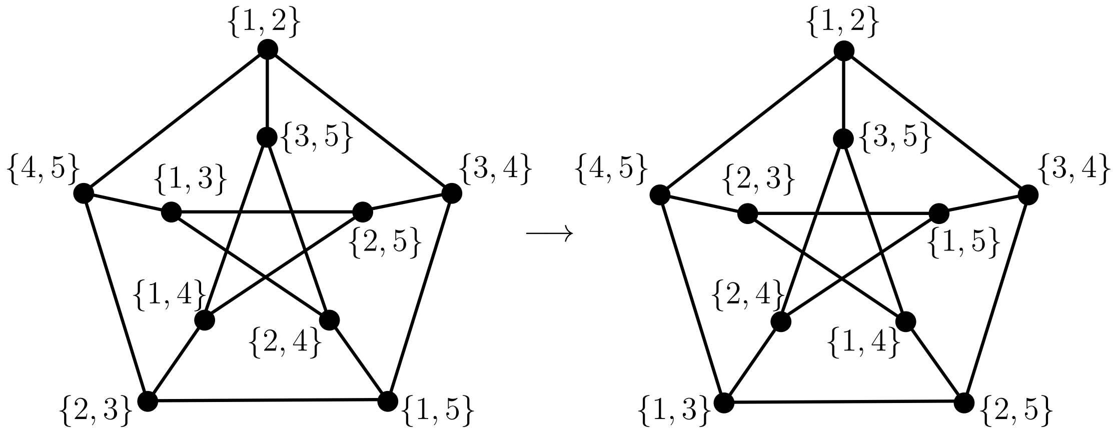


Figure 2.1: An Automorphism $K(5, 2) \to K(5, 2)$

Lemma 2.1.1. *Every automorphism of G induces an automorphism of $Inc(G)$.*

Proof. Let $\sigma \in \text{Aut}(G)$. Define $\tilde{\sigma} : Inc(G) \to Inc(G)$ by

$$\tilde{\sigma}(u, uv) = (\sigma(u), \sigma(u)\sigma(v)).$$

Note that since σ is an automorphism, $uv \in E(G)$ implies that $\sigma(u)\sigma(v) \in E(G)$. So, $(\sigma(u), \sigma(u)\sigma(v)) \in V(Inc(G))$.

Now, let's show that $\tilde{\sigma}$ is an automorphism. First, let's show that it is a bijection. Suppose $\tilde{\sigma}(u, uv) = \tilde{\sigma}(w, wx)$. Then, by definition of $\tilde{\sigma}$,

$$(\sigma(u), \sigma(u)\sigma(v)) = (\sigma(w), \sigma(w)\sigma(x)).$$

Then, $\sigma(u) = \sigma(w)$ and since σ is a bijection (and hence injective), $u = w$. Further, this also implies that $\sigma(v) = \sigma(x)$ and so $v = x$. Thus, $(u, uv) = (w, wx)$. So, $\tilde{\sigma}$ is injective.

Now, let $(y, yz) \in V(Inc(G))$. Then, since $y, z \in V(G)$, it follows from the bijectivity of σ that there exist unique $u, v \in V(G)$ such that $\sigma(u) = y$ and $\sigma(v) = z$. Note that $yz \in E(G)$ implies that $\sigma(u)\sigma(v) \in E(G)$. Thus, since σ is an automorphism, $uv \in E(G)$. So, $(u, uv) \in V(Inc(G))$. Then,

$$\tilde{\sigma}(u, uv) = (\sigma(u), \sigma(u)\sigma(v)) = (y, yz).$$

Thus, $\tilde{\sigma}$ is surjective and hence is bijective.

Now, we show that $\tilde{\sigma}$ preserves edges and nonedges. Suppose

$$(u, uv)(w, wx) \in E(Inc(G)).$$

Then, we have three cases:

1. If $u = w$, then $\sigma(u) = \sigma(w)$. Therefore,

$$\tilde{\sigma}(u, uv)\tilde{\sigma}(w, wx) = (\sigma(u), \sigma(u)\sigma(v))(\sigma(w), \sigma(w)\sigma(x)) \in E(Inc(G)).$$

2. If $uv = wx$, then either $u = w$ or $u = x$. If $u = w$, then the incidences are not distinct and hence cannot be adjacent. So, we may assume that $u = x$. Then, we also know that $v = w$. So,

$$\sigma(u) = \sigma(x), \sigma(v) = \sigma(w) \text{ and } \sigma(u) \neq \sigma(w).$$

Therefore,

$$\sigma(u)\sigma(v) = \sigma(x)\sigma(w) = \sigma(w)\sigma(x),$$

where the last equality holds because we are only concerned about the endpoints of the edge, not an orientation of the edge. So,

$$\tilde{\sigma}(u, uv)\tilde{\sigma}(w, wx) = (\sigma(u), \sigma(u)\sigma(v))(\sigma(w), \sigma(w)\sigma(x)) \in E(Inc(G)).$$

3. If $uw = uv$ (without loss of generality), then $v = w$ and so $\sigma(v) = \sigma(w)$. Thus, $\sigma(u)\sigma(w) = \sigma(u)\sigma(v)$. Therefore,

$$\tilde{\sigma}(u, uv)\tilde{\sigma}(w, wx) = (\sigma(u), \sigma(u)\sigma(v))(\sigma(w), \sigma(w)\sigma(x)) \in E(Inc(G)).$$

Therefore, $\tilde{\sigma}$ preserves edges.

To see that it also preserves nonedges, suppose (u, uv) is not adjacent to

(w, wx) in $Inc(G)$. Then, $u \neq w$ by the first property of adjacency in incidence graphs. Further, the third property of adjacency in incidence graphs tells us that $uw \neq uv$ and $uw \neq wx$. So, since each of these pairs of edges have a common endpoint, we conclude that $w \neq v$ and $u \neq x$. Now, consider

$$\tilde{\sigma}(u, uv) = (\sigma(u), \sigma(u)\sigma(v))$$

and

$$\tilde{\sigma}(w, wx) = (\sigma(w), \sigma(w)\sigma(x)).$$

Then, using the bijectivity of σ,

- $u \neq w$ implies that $\sigma(u) \neq \sigma(w)$, so the first condition for adjacency in $Inc(G)$ fails.
- $u \neq w$ and $u \neq x$ implies that $\sigma(u) \neq \sigma(w)$ and $\sigma(u) \neq \sigma(x)$. Thus, $\sigma(u)\sigma(v) \neq \sigma(w)\sigma(x)$. Hence, the second condition for adjacency in $Inc(G)$ fails.
- $w \neq v$ and $u \neq x$ implies that $\sigma(w) \neq \sigma(v)$ and $\sigma(u) \neq \sigma(x)$. This in turn implies that $\sigma(u)\sigma(w) \neq \sigma(u)\sigma(v)$ and $\sigma(u)\sigma(w) \neq \sigma(x)\sigma(w)$. Thus, the third condition of adjacency in $Inc(G)$ fails.

Hence, $\tilde{\sigma}(u, uv)$ is not adjacent to $\tilde{\sigma}(w, wx)$. Therefore, $\tilde{\sigma}$ is an automorphism of $Inc(G)$. □

Here is a toy example to get a feel of what we are looking for, and also to see why Lemma 2.1.1 is useful. Note that the graph $K_{n_1,n_2,\ldots,n_k}$ is called a *complete multipartite graph*. It consists of a vertex set which can be partitioned into k partite sets of sizes $n_1, n_2, \ldots, n_k$ such that there are no edges between vertices in the same partite set and every edge between two vertices in different partite sets appears. The graph discussed in the lemma below is the complete multipartite graph with $k \geq 2$ and where all the partite sets have the same size.

Lemma 2.1.2. *In $Inc(K_{n,n,\ldots,n})$, where there are at least 2 partite sets in $K_{n,n,\ldots n}$, there is an automorphism sending $(u, uv) \mapsto (v, uv)$ for every pair of adjacent vertices $u, v \in V(K_{n,n,\ldots,n})$.*

Proof. Consider $K_{n,n,\ldots,n}$ and assume there are at least 2 partite sets. Let u and v be vertices of $K_{n,n,\ldots,n}$ such that $uv \in E(K_{n,n,\ldots,n})$. Then, $u \in U$, $v \in V$ where U, V are different partite sets of $K_{n,n,\ldots,n}$. Then, since $|U| = |V| = n$, we can find a bijection $\varphi : U \to V$ which maps $u \mapsto v$. This induces an automorphism of $K_{n,n,\ldots,n}$ where

$$\sigma(w) = \begin{cases} \varphi(w) & w \in U \\ \varphi^{-1}(w) & w \in V \\ w & \text{else} \end{cases}$$

Since φ is a bijection, so is σ. Now, suppose $xy \in E(K_{n,n,\ldots,n})$. Then, there exist distinct partite sets X and Y of the vertex set of $K_{n,n,\ldots,n}$ such that $x \in X$ and $y \in Y$. If X and Y are both distinct from U and V, then

$$\sigma(x)\sigma(y) = xy \in E(K_{n,n,\ldots,n}).$$

Without loss of generality, if $X = U$ and Y is distinct from U and V, then

$$\sigma(x)\sigma(y) = \varphi(x)y \in E(K_{n,n,\ldots,n})$$

since $\varphi(x) \in V$ and $y \in Y \neq V$. If, without loss of generality, $X = V$ and Y is distinct from U and V, then

$$\sigma(x)\sigma(y) = \varphi^{-1}(x)y \in E(K_{n,n,\ldots,n})$$

since $\varphi^{-1}(x) \in U$ and $y \in Y \neq U$. Finally, if $X = U$ and $Y = V$, then

$$\sigma(x)\sigma(y) = \varphi(x)\varphi^{-1}(y) \in E(K_{n,n,\ldots,n})$$

since $\varphi(x) \in V$ and $\varphi^{-1}(y) \in U$. So, σ preserves edges. Now, suppose xy is not an edge of $K_{n,n,\ldots,n}$. Then, x, y are in the same partite set. If $x, y \in U$, then $\sigma(x) = \varphi(x)$ and $\sigma(y) = \varphi(y)$ are both elements of V. If $x, y \in V$, then $\sigma(x) = \varphi^{-1}(x)$ and $\sigma(y) = \varphi^{-1}(y)$ are both elements of U. Finally, if $x, y \in X$ where $X \neq U$ and $X \neq V$, then $\sigma(x) = x$ and $\sigma(y) = y$ are both elements of X. Therefore, in all cases, $\sigma(x)\sigma(y) \notin E(K_{n,n,\ldots,n})$ as x and y land in the same partite set under the map σ. Therefore, σ is an automorphism. By Lemma 2.1.1, σ induces an automorphism $\tilde{\sigma}$ of $Inc(K_{n,n,\ldots,n})$. In particular, recall that $u \in U$, $v \in V$ and $\varphi(u) = v, \varphi^{-1}(v) = u$. Then,

$$\tilde{\sigma}(u, uv) = (\sigma(u), \sigma(u)\sigma(v)) = (\varphi(u), \varphi(u)\varphi^{-1}(v)) = (v, vu) = (v, uv),$$

as desired. □

Recall that our current goal is to determine when $Inc(G)$ is vertex transitive. Let's now define this precisely.

Definition. A graph G is called *vertex transitive* if for every pair of vertices $u, v \in V(G)$, there is an automorphism of G, say φ, such that $\varphi(u) = v$.

Examples. The complete graphs K_n are vertex transitive by permuting the vertices. The Kneser graphs $K(r, s)$ are also vertex transitive by permuting the elements of $\{1, \ldots, r\}$. The cycles C_n are vertex transitive by rotation of the vertices. (Note that this completes the calculations given in Chapter 1, where we used the fact that K_n, C_n and $K(r, s)$ are vertex transitive without proof.) Paths with more than two vertices are not vertex transitive since there is no way to define an automorphism which takes an end vertex to an interior vertex.

Similarly, we can define edge transitivity.

Definition. A graph G is called *edge transitive* if for every pair of edges uv, xy in $E(G)$, there is an automorphism of G which sends the endpoints of uv to the endpoints of xy.

Examples. Again, the complete graphs, Kneser graphs and cycles are all edge transitive. Further, $K_{n,n,\ldots,n}$ and the hypercubes Q_r are also edge transitive. Note that the hypercubes Q_r is defined to be the graph with vertex set being $\{0,1\}^r$; that is, the r-tuples whose coordinates come from the set $\{0,1\}$. Two vertices are said to be adjacent precisely when their corresponding r-tuples differ in exactly one coordinate.

Example. There are graphs which are vertex transitive, but not edge transitive. One particular example is given below.

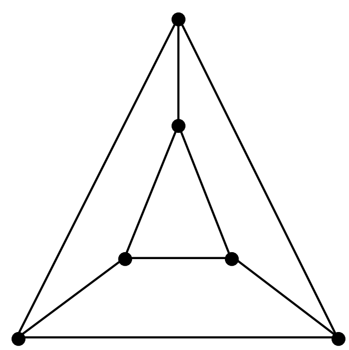


Figure 2.2: A Vertex Transitive, Non-Edge Transitive Graph

Further, there are graphs which are edge transitive, but not vertex transitive. Stars, $K_{1,n}$, are one particular example. Here is $K_{1,5}$.

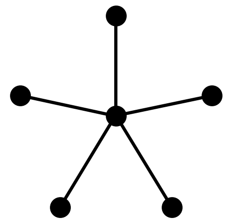


Figure 2.3: An Edge Transitive, Non-Vertex Transitive Graph

Proposition 2.1.3. *Let G be an edge transitive graph. If $Inc(G)$ has the property that there is an automorphism γ_{uv} of $Inc(G)$ sending $(u, uv) \mapsto (v, uv)$ for any $uv \in E(G)$, then $Inc(G)$ is vertex transitive.*

Proof. Let $(u, uv), (w, wx) \in V(Inc(G))$. Since G is edge transitive, there exists an automorphism of G, say σ, which maps $uv \mapsto wx$. Then, σ induces an automorphism on $Inc(G)$ by Lemma 2.1.1. Namely,

$$\tilde{\sigma}(u, uv) = (\sigma(u), \sigma(u)\sigma(v)) = \begin{cases} (w, wx) & \text{if } \sigma(u) = w \\ (x, xw) = (x, wx) & \text{if } \sigma(u) = x \end{cases}.$$

If $\sigma(u) = w$, we are done. If $\sigma(u) = x$, then

$$\gamma_{wx}\tilde{\sigma}(u, uv) = \gamma_{wx}(x, wx) = (w, wx)$$

and $\gamma_{wx}\tilde{\sigma}$ is an automorphism of $Inc(G)$ as it is a composition of automorphisms of $Inc(G)$. Therefore, $Inc(G)$ is vertex transitive. □

Corollary 2.1.4. *$Inc(K_{n,n,\ldots,n})$ is vertex transitive.*

Proof. This follows directly from Lemma 2.1.2 and Proposition 2.1.3, since $K_{n,n,\ldots,n}$ is edge transitive. □

Proposition 2.1.5. *$Inc(K(r, s))$ is vertex transitive.*

Proof. Since $K(r,s)$ is edge transitive, it suffices to show that there exists an automorphism γ_{uv} of $Inc(K(r,s))$ sending $(u,uv) \mapsto (v,uv)$ for any $uv \in E(G)$ by appealing to Proposition 2.1.3. Recall that each vertex of $K(r,s)$ is an s-element subset of $[r] = \{1,2,\ldots,r\}$, and two vertices are adjacent precisely when their corresponding s-element subsets are disjoint. Therefore, we may define a permutation on $[r]$ which sends $u \mapsto v$, which is idempotent, as follows. Let $u = \{u_1,\ldots,u_s\}$ and $v = \{v_1,\ldots,v_s\}$ be vertices of $K(r,s)$. Then, define an element of $S_{[r]}$ by $\rho = \prod_{i=1}^{s}(u_i v_i)$. Since $uv \in E(G)$,

$$\{u_1,u_2,\ldots,u_s\} \cap \{v_1,v_2,\ldots,v_s\} = \emptyset.$$

So ρ is a product of disjoint cycles. Note that $\rho(u) = v$ and $\rho(v) = u$. Further, since we know that $\mathrm{Aut}(K(r,s)) = S_{[r]}$, ρ is an automorphism of $K(r,s)$. Hence, by Lemma 2.1.1, ρ induces an automorphism $\tilde{\rho}$ of $Inc(K(r,s))$. Then,

$$\tilde{\rho}(u,uv) = (\rho(u),\rho(u)\rho(v)) = (v,vu) = (v,uv).$$

Therefore, $K(r,s)$ satisfies the conditions for Proposition 2.1.3 and hence $Inc(K(r,s))$ is vertex transitive. □

Corollary 2.1.6. *$Inc(K_n)$ is vertex transitive.*

Proof. Since $K_n \cong K(n,1)$, Proposition 2.1.5 implies that $Inc(K_n)$ is vertex transitive. □

Proposition 2.1.7. *$Inc(Q_r)$ is vertex transitive.*

Proof. Since Q_r is edge transitive, it suffices to show that Q_r satisfies the flipping condition of Proposition 2.1.3. Recall that the vertices of Q_r are r-tuples with coordinates from $\{0,1\}$. Further, if $uv \in E(Q_r)$, then u,v differ in exactly one component, say $1 \le i \le r$. Then, we can define $\varphi_i \in \mathrm{Aut}(Q_r)$ by

$$\varphi_i(x) = \varphi_i(x_1,\ldots,x_i,\ldots,x_r) = (x_1,\ldots,1-x_i,\ldots,x_r).$$

So,

$$\varphi_i(u) = (u_1, \dots, 1 - u_i, \dots, u_r) = (v_1, \dots, v_i, \dots, v_r) = v.$$

Similarly, $\varphi_i(v) = u$. Thus, φ_i induces an automorphism on $Inc(Q_r)$ by Lemma 2.1.1, denoted $\tilde{\varphi}_i$. Then,

$$\tilde{\varphi}_i(u, uv) = (\varphi_i(u), \varphi_i(u)\varphi_i(v)) = (v, vu) = (v, uv).$$

Therefore, $Inc(Q_r)$ is vertex transitive, since it satisfies the conditions of Proposition 2.1.3. □

The desired variant of Theorem 1.3.6 that we will want is the following. It involves the domination number of G, denoted $\gamma(G)$, which is the minimum number of vertices in a dominating set; that is, a set such that any vertex of G is either in the set or adjacent to a vertex in the set.

Theorem 2.1.8. *For any graph G,*

$$\chi_f(Inc(G)) \geq \frac{2|E(G)|}{|V(G)| - \gamma(G)},$$

with equality when $Inc(G)$ is vertex transitive.

Proof. By Theorem 1.3.6, the result will follow if we can argue that

$$\alpha(Inc(G)) = |V(G)| - \gamma(G).$$

By Lemma 1.2.1, it follows that independent sets of $Inc(G)$ correspond bijectively to star forests of G, where the edges are oriented toward the centers of the stars. In particular, the size of the independent set of $Inc(G)$ corresponds to the number of edges in the corresponding star forest of G. So, a maximum independent set of $Inc(G)$ corresponds to a maximum star forest, with respect to the number of edges covered. Ferneyhough, Haas, Hanson and MacGillivray ([13]) showed that the number of edges in a maximum star forest is $|V(G)| - \gamma(G)$. Therefore, $\alpha(Inc(G)) = |V(G)| - \gamma(G)$. So,

$$\chi_f(Inc(G)) \geq \frac{2|E(G)|}{|V(G)| - \gamma(G)}$$

with equality when $Inc(G)$ is vertex transitive. □

This theorem is a generalization of the analogous result for the incidence chromatic number of a graph, given by Sun and Shiu ([31]).

Note that we can obtain equality without $Inc(G)$ being vertex transitive. For example, consider $G = P_4$. Then,

$$\chi_f(Inc(P_4)) = 3,$$

which we will show in Chapter 4, and the bound gives that

$$\chi_f(Inc(P_4)) \geq \frac{2(3)}{4-2} = 3.$$

Observe that $Inc(P_4) = P_6^2$, which is not vertex transitive as it is not regular.

Using this inequality, and the previous results, we can calculate the fractional incidence chromatic numbers of the following infinite families.

Example. Suppose $K_{n,n,\ldots,n}$ has $k \geq 2$ partite sets. If $n > 1$, then we can choose two vertices, each in a distinct partite set and so $\gamma(K_{n,n,\ldots,n}) = 2$. Hence, we compute that

$$\chi_f(Inc(K_{n,n,\ldots,n})) = \frac{2|E(K_{n,n,\ldots,n})|}{|V(K_{n,n,\ldots,n})| - 2} = \frac{k(k-1)n^2}{kn-2}.$$

In particular, for the complete bipartite graphs,

$$\chi_f(Inc(K_{n,n})) = \frac{2n^2}{2n-2} = \frac{n^2}{n-1}.$$

If $n = 1$, then $\gamma(K_{n,n,\ldots,n}) = 1$ and

$$\chi_f(Inc(K_{n,n,\ldots,n})) = \frac{k(k-1)}{k-1} = k.$$

So, in particular,

$$\chi_f(Inc(K_{n,n})) = 2.$$

Note that if $n = 1$, then $K_{n,n,\ldots,n} = K_k$, so this calculation should line up with the next example.

Example. Let $n \geq 2$. Since $\gamma(K_n) = 1$,

$$\chi_f(Inc(K_n)) = \frac{2\binom{n}{2}}{n-1} = \frac{2(\frac{n(n-1)}{2})}{n-1} = n.$$

This lines up with the previous example since we always assumed that $k \geq 2$ there. Note that if $n = 1$, then $Inc(K_1)$ is empty and so $\chi_f(Inc(K_1)) = 0$.

Example. Consider $K(r,s)$. Observe that

$$|V(K(r,s))| = \binom{r}{s}.$$

Further,

$$|E(K(r,s))| = \frac{1}{2}\binom{r}{s}\binom{r-s}{s}.$$

A universal calculation of $\gamma(K(r,s))$ is unknown. The most recent work on this is the work of Östergård, Shao and Xu from 2015 ([26]). From this work, we thus have the following results.

- Since $\gamma(K(2s,s)) = \frac{1}{2}\binom{2s}{s}$. Therefore,

$$\chi_f(Inc(K(2s,s))) = \frac{\binom{2s}{s}\binom{2s-s}{s}}{\binom{2s}{s} - \frac{1}{2}\binom{2s}{s}} = \frac{\binom{2s}{s}\binom{s}{s}}{\frac{1}{2}\binom{2s}{s}} = 2.$$

  This makes sense since $K(2s,s)$ is a collection of disjoint edges.

- If $r \geq s^2 + s$, then $\gamma(K(r,s)) = s + 1$. Then,

$$\chi_f(Inc(K(r,s))) = \frac{\binom{r}{s}\binom{r-s}{s}}{\binom{r}{s} - (s+1)}.$$

- If $s \geq 3$ and $\frac{3}{4}s^2 + s \leq r < s^2 + s$, then

  $$\gamma(r, s) = s + 1 + \left\lceil \frac{s^2 + s - r}{\lfloor s/2 \rfloor} \right\rceil.$$

  Then,

  $$\chi_f(Inc(K(r, s))) = \frac{\binom{r}{s}\binom{r-s}{s}}{\binom{r}{s} - \gamma(K(r, s))}.$$

- Lastly, we can compute the fractional incidence chromatic number of the Petersen graph, $K(5, 2)$, as follows. If $r \geq 2s + 1$, then

  $$\gamma(K(r + 1, s)) \leq \gamma(K(r, s)).$$

  So, since $5 \geq 2(2) + 1$, it follows that

  $$\gamma(K(5, 2)) \geq \gamma(K(6, 2)).$$

  Since $6 \geq 2^2 + 2$, it follows that $\gamma(K(6, 2)) = 2 + 1 = 3$. However, it is not hard to exhibit a dominating set for $K(5, 2)$ with size 3 (consider taking the vertices $\{1, 2\}, \{1, 5\}, \{2, 5\}$ - these form a dominating set of size 3, for example). Therefore, $\gamma(K(5, 2)) = 3$. Thus,

  $$\chi_f(Inc(K(5, 2))) = \frac{\binom{5}{2}\binom{5-2}{2}}{\binom{5}{2} - 3} = \frac{30}{10 - 3} = \frac{30}{7}.$$

Example. Consider Q_n. Observe that

$$|V(Q_n)| = 2^n$$

and

$$|E(Q_n)| = 2^{n-1}n.$$

As with $K(r, s)$, a universal calculation of $\gamma(Q_n)$ is unknown. Here is a summary of the known results, and what they imply about the fractional incidence chromatic number of the hypercube.

- If $n = 2^k - 1$ for some integer k, then $\gamma(Q_n) = \frac{2^n}{n} + 1$ ([18]). So, in this case,

$$\chi_f(Inc(Q_n)) = \frac{2^n n}{2^n - \left(\frac{2^n}{n} + 1\right)} = \frac{2^n n^2}{2^n n - 2^n - n} = \frac{2^n n^2}{(n-1)2^n - n}.$$

- In [4], [32], and [33], it is shown that if $n = 2^k - 1$ or if $n = 2^k$, then

$$\gamma(Q_n) = 2^{n-k}.$$

  Thus,

$$\chi_f(Inc(Q_n)) = \frac{2^n n}{2^n - 2^{n-k}}.$$

Unfortunately, other than a few small explicit examples, not much else is known about the domination number of hypercubes.

All of these graphs have the following fundamental property.

Definition. A graph G is *arc-transitive* if for any pair of arcs, i.e. directed edges, uv and wx in G, there is an automorphism σ of G such that $\sigma(u) = w$ and $\sigma(v) = x$. This is also sometimes called *1-arc-transitive* and *flag transitive*.

Theorem 2.1.9. *If G is arc-transitive, then $Inc(G)$ is vertex transitive.*

Proof. Let G be arc-transitive and let $(u, uv), (w, wx) \in V(Inc(G))$. Then, consider uv and wx as arcs in G. By definition of arc-transitivity, there exists an automorphism of G, σ, such that $\sigma(u) = w$ and $\sigma(v) = x$. By Lemma 2.1.1, σ induces an automorphism $\tilde{\sigma}$ on $Inc(G)$, defined by

$$\tilde{\sigma}(y, yz) = (\sigma(y), \sigma(y)\sigma(z)).$$

Then,

$$\tilde{\sigma}(u, uv) = (\sigma(u), \sigma(u)\sigma(v)) = (w, wx).$$

Therefore, $Inc(G)$ is vertex transitive. □

Remark. Note that the complete multipartite graphs, Kneser graphs, complete graphs and hypercubes are all arc-transitive. Further, it is not hard to see that all arc-transitive graphs are edge transitive.

Corollary 2.1.10. *If G is arc-transitive, then*

$$\chi_f(Inc(G)) = \frac{2|E(G)|}{|V(G)| - \gamma(G)}.$$

Example. Observe that cycles are also arc-transitive. Then, since $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ for $n \geq 3$, it follows that

$$\chi_f(Inc(C_n)) = \frac{2n}{n - \lceil n/3 \rceil}.$$

2.2 Extending Bounds

The second goal of this chapter is to describe the main technique we use for proving bounds on the fractional incidence chromatic number. Generally speaking we will extend the technique of using graph homomorphisms into complete graphs, which provides bounds for the chromatic number of a graph, to using graph homomorphisms into Kneser graphs, which proves bounds for the fractional chromatic number of a graph.

In proofs regarding the chromatic number and combining two graphs, the homomorphisms used are into complete graphs, which, in terms of colors, means that every vertex is assigned precisely one color. Gluing two colorings is not too difficult in this case, as one does not need to worry about the number of colors being assigned to each vertex. In fractional coloring, given two graphs, their optimal colorings may correspond to vertices of each individual graph receiving a different sized subset of colors; that is, they have optimal homomorphisms into Kneser graphs whose vertices correspond to different sized sets. To remedy this, we would like to use the following result of Stahl ([29]).

Proposition 2.2.1. *Let r, s be relatively prime numbers such that $1 \leq 2s < r$. Then $K(mr, ms) \to K(nr, ns)$ for integers m, n if and only if n is a multiple of m.*

Note that if $2s = r$, then $2\ell s = \ell r$ and so $K(\ell r, \ell s)$ is a disjoint collection of edges for any $\ell \in \mathbb{N}$. Given any two graphs of this type, we can find a homomorphism between them. The issue here is that we cannot guarantee that r and s are relatively prime. However, we can remove this condition.

Corollary 2.2.2. *For any r, s such that $1 \leq 2s < r$, $K(mr, ms) \to K(nr, ns)$ for integers m, n if and only if n is a multiple of m.*

Proof. Given r, s such that $1 \leq 2s < r$, we can write

$$r = dr' \qquad \text{and} \qquad s = ds'$$

where $d = \gcd(r, s)$. Then, r', s' are relatively prime. Further,

$$1 \leq 2s < r$$

implies that

$$1 \leq 2ds' < dr'.$$

Since r' and s' are positive integers and $r, s \geq 1$ are integers, it follows that

$$1 \leq 2s' < r'.$$

Now, by Proposition 2.2.1, we see that

$$K(mr, ms) = K(mdr', mds') \to K(ndr', nds') = K(nr, ns)$$

for integers m, n if and only if nd is a multiple of md. However, this occurs precisely when n is a multiple of m. $\square$

In fact, we will not need the full strength of this. We will only need the following special case, which we extend to include the $2s = r$ case by the comments above.

Corollary 2.2.3. *Let r, s be integers such that $1 \leq 2s \leq r$. Then,*

$$K(r, s) \to K(nr, ns)$$

for any positive integer n.

Proof. The result follows from Corollary 2.2.2 since any integer n is a multiple of 1. □

2.2.1 The Union

Definition. Given two graphs G and H, their *union*, denoted $G \cup H$, is defined to be the graph such that

$$V(G \cup H) = V(G) \cup V(H)$$

and

$$E(G \cup H) = E(G) \cup E(H).$$

Here, we allow for overlapping of the vertex and edge sets of G and H.

Proposition 2.2.4 ([31])**.** *Let G_1 and G_2 be graphs. Then,*

$$\chi(Inc(G_1 \cup G_2) \leq \chi(Inc(G_1)) + \chi(Inc(G_2)).$$

Theorem 2.2.5. *Let G_1 and G_2 be graphs. Then,*

$$\chi_f(Inc(G_1 \cup G_2)) \leq \chi_f(Inc(G_1)) + \chi_f(Inc(G_2)).$$

Proof. Let G_1 and G_2 be graphs. If $E(G_1) \cap E(G_2) \neq \emptyset$, then remove the shared edges from G_1. That is, replace $E(G_1)$ with $E(G_1) \setminus (E(G_1) \cap E(G_2))$. Note that this has no effect on $Inc(G_1 \cup G_2)$ as we have not changed $G_1 \cup G_2$. Thus, we may assume that G_1 and G_2 are disjoint on their edge-sets. Then, for $i = 1, 2$, let

$$\sigma_i : Inc(G_i) \to K(r_i, s_i),$$

where $\chi_f(Inc(G_i)) = \frac{r_i}{s_i}$. If one of the G_i, say G_1, is totally disconnected, then $G_1 \cup G_2 \cong G_2$ along with some nonnegative number of isolated vertices. Therefore, $Inc(G_1 \cup G_2) \cong Inc(G_2)$, since isolated vertices do not contribute any information to the associated incidence graph. So,

$$\chi_f(Inc(G_1 \cup G_2)) = \chi_f(Inc(G_2)) = \chi_f(Inc(G_1)) + \chi_f(Inc(G_2))$$

since $\chi_f(Inc(G_1)) = 0$, as $Inc(G_1)$ is the empty graph, i.e., it has no vertices.

Now, suppose that both G_1 and G_2 have at least one edge. This implies that $K(r_i, s_i)$ must have an edge for $i = 1, 2$. In particular, this implies that $r_i \geq 2s_i$ for $i = 1, 2$. Therefore, by Corollary 2.2.3, we can extend these homomorphisms to

$$\tilde{\sigma}_1 : Inc(G_1) \to K(s_2 r_1, s_1 s_2)$$

and

$$\tilde{\sigma}_2 : Inc(G_2) \to K(s_1 r_2, s_1 s_2),$$

where the $s_2 r_1$ labels on the vertices of $K(s_2 r_1, s_1 s_2)$ are disjoint from the $s_1 r_2$ labels used on $K(s_1 r_2, s_1 s_2)$. Although these may not be the same Kneser graph, the important thing to note is that they both have vertices whose associated subsets have the same size, namely $s_1 s_2$. Now, define a fractional coloring homomorphism $\sigma : Inc(G_1 \cup G_2) \to K(s_2 r_1 + s_1 r_2, s_1 s_2)$ as follows:

$$\sigma(u, uv) = \begin{cases} \tilde{\sigma}_1(u, uv) & uv \in E(G_1) \\ \tilde{\sigma}_2(u, uv) & uv \in E(G_2) \end{cases}.$$

Now, we must check that this is, in fact, a homomorphism. Note that this is a well-defined function because we have eliminated any overlap in the edge sets of G_1 and G_2. Now, let (u, uv) and (x, xy) be adjacent in $Inc(G_1 \cup G_2)$. If uv and xy are both edges in some G_i, then the incidences are adjacent in $Inc(G_i)$. Therefore, recalling that the vertices of Kneser graphs are sets,

$$\sigma(u, uv) \cap \sigma(x, xy) = \tilde{\sigma}_i(u, uv) \cap \tilde{\sigma}_i(x, xy) = \emptyset,$$

since $\tilde{\sigma}_i$ being a homomorphism implies that $\tilde{\sigma}_i(u, uv)$ and $\tilde{\sigma}_i(x, xy)$ are adjacent in the target graph of $\tilde{\sigma}_i$ and hence are disjoint by the definition of Kneser graphs. Therefore,

$$\sigma(u, uv)\sigma(x, xy) \in E(K(s_2r_1 + s_1r_2, s_1s_2)).$$

If, without loss of generality, $uv \in E(G_1)$ and $xy \in E(G_2)$, then

$$\sigma(u, uv) \cap \sigma(x, xy) = \tilde{\sigma}_1(u, uv) \cap \tilde{\sigma}_2(x, xy) = \emptyset,$$

since $\tilde{\sigma}_1$ uses different labels than $\tilde{\sigma}_2$. Thus,

$$\sigma(u, uv)\sigma(x, xy) \in E(K(s_2r_1 + s_1r_2, s_1s_2)).$$

Therefore, σ is a homomorphism. So,

$$\begin{aligned}\chi_f(Inc(G_1 \cup G_2)) &\leq \frac{s_2r_1 + s_1r_2}{s_1s_2} = \frac{s_2r_1}{s_1s_2} + \frac{s_1r_2}{s_1s_2} \\ &= \frac{r_1}{s_1} + \frac{r_2}{s_2} = \chi_f(Inc(G_1)) + \chi_f(Inc(G_2)),\end{aligned}$$

as desired. □

Example. The bound in Theorem 2.2.5 is tight. Consider C_4 decomposed into two edge disjoint copies of $2K_2$. That is,

$$C_4 \cong 2K_2 \cup 2K_2.$$

We can see this pictorially as follows.

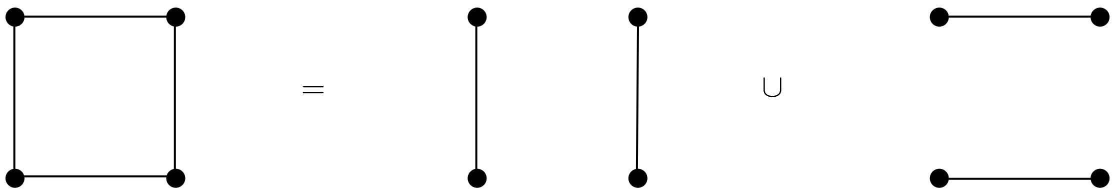


Figure 2.4: C_4 Decomposed as a Union

Then, we know that $\chi_f(Inc(C_4)) = 4$ and $\chi_f(Inc(2K_2)) = 2$. So,

$$\chi_f(Inc(C_4)) = \chi_f(Inc(2K_2)) + \chi_f(Inc(2K_2)).$$

Note that the bound does not provide equality since we can alternatively partition C_4 into two disjoint length two paths. So,

$$\chi_f(Inc(C_4)) \leq \chi_f(Inc(P_3)) + \chi_f(Inc(P_3)).$$

However, we have already seen that $\chi_f(Inc(C_4)) = 4$, but $\chi_f(Inc(P_3)) = 3$, which we will see in Chapter 4.

2.2.2 The Cartesian Product

Definition. The *Cartesian product* of G and H, denoted $G \square H$, is defined to be the graph with

$$V(G \square H) = V(G) \times V(H)$$

and

$$E(G \square H) = \{(g,h)(g',h') \mid (g = g' \text{ and } hh' \in E(H)) \text{ or } (gg' \in E(G) \text{ and } h = h')\}.$$

Example. The graphs of $K_2 \square K_2$ and $P_3 \square P_3$ are drawn below, respectively.

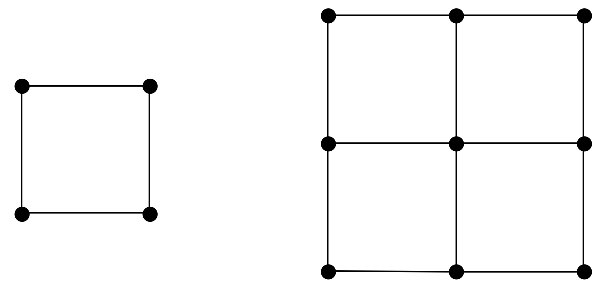


Figure 2.5: $K_2 \square K_2$ and $P_3 \square P_3$

Proposition 2.2.6 ([31]). *Let G_1 and G_2 be graphs. Then,*

$$\chi(Inc(G_1 \square G_2)) \leq \chi(Inc(G_1)) + \chi(Inc(G_2)).$$

Theorem 2.2.7. *Let G_1 and G_2 be graphs. Then,*

$$\chi_f(Inc(G_1 \square G_2)) \leq \chi_f(Inc(G_1)) + \chi_f(Inc(G_2)).$$

Proof. By the definition of the Cartesian product, we can decompose the edges of $G_1 \square G_2$ as the edges where $gg' \in E(G_1)$ and $h = h' \in V(G_2)$ and the edges where $g = g' \in V(G_1)$ and $hh' \in E(G_2)$. The edges of the first type induce a subgraph H_1 which is $|V(G_2)|$ copies of G_1. Similarly, the edges of the second type induce a subgraph H_2 which is $|V(G_1)|$ copies of G_2. So, we can write $G_1 \square G_2 \cong H_1 \cup H_2$. So, by Theorem 2.2.5,

$$\begin{aligned}
\chi_f(Inc(G_1 \square G_2)) &= \chi_f(Inc(H_1 \cup H_2)) \\
&\leq \chi_f(Inc(H_1)) + \chi_f(Inc(H_2)) \\
&= \chi_f(Inc(|V(G_2)|G_1)) + \chi_f(Inc(|V(G_1)|G_2)) \\
&= \chi_f(Inc(G_1)) + \chi_f(Inc(G_2)),
\end{aligned}$$

as desired. $\square$

Example. The bound in Theorem 2.2.7 is tight. Consider $C_4 = K_2 \Box K_2$, which is pictured above. Then, since $\chi_f(Inc(C_4)) = 4$ and $\chi_f(Inc(K_2)) = 2$, we see that

$$\chi_f(Inc(C_4)) = \chi_f(Inc(K_2)) + \chi_f(Inc(K_2)).$$

Note that the bound does not provide equality in all cases. Consider $Q_3 = C_4 \Box K_2$. We can show using transitivity that

$$\chi_f(Inc(Q_3)) = 4.$$

However,

$$\chi_f(Inc(C_4)) = 4 \qquad \text{and} \qquad \chi_f(Inc(K_2)) = 2.$$

2.2.3 The Join

Definition. The *join* of two graphs G and H, $G \vee H$, is the graph with vertex set

$$V(G \vee H) = V(G) \coprod V(H)$$

and edge set

$$E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}.$$

Example. Below $\overline{K_3} \vee \overline{K_3} \cong K_{3,3}$ and $P_3 \vee K_2$ are drawn out, respectively.

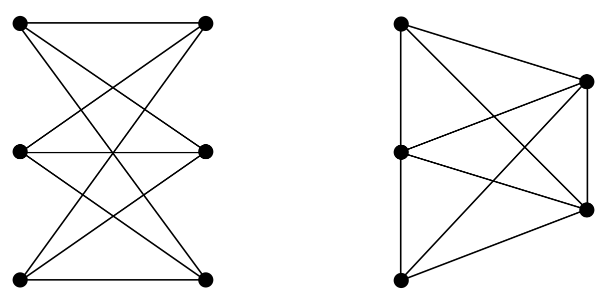


Figure 2.6: $\overline{K_3} \vee \overline{K_3}$ and $P_3 \vee K_2$

Example. The wheel graphs, $W_n \cong C_n \vee K_1$ for $n \geq 3$, is another example of the join of two graphs. For example, here is W_5.

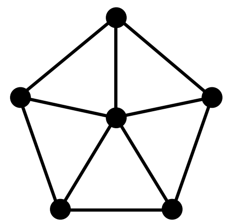


Figure 2.7: Wheel Graph, $W_5 \cong C_5 \vee K_1$

Proposition 2.2.8 ([31]). *Let G_1 and G_2 be graphs. Suppose $|V(G_1)| = m$ and $|V(G_2)| = n$, where $m \geq n \geq 2$. Then,*

$$\chi(Inc(G_1 \vee G_2)) \leq \min\{m+n, \max\{\chi(Inc(G_1)), \chi(Inc(G_2))\} + m + 2\}.$$

Theorem 2.2.9. *Let G_1 and G_2 be graphs. Suppose that $|V(G_1)| = m$ and $|V(G_2)| = n$, where $m \geq n \geq 2$. Then,*

$$\chi_f(Inc(G_1 \vee G_2)) \leq \min\{m+n, \max\{\chi_f(Inc(G_1)), \chi_f(Inc(G_2))\} + \chi_f(Inc(K_{m,n}))\}.$$

Proof. Since $G_1 \vee G_2$ has $m+n$ vertices, it is a subgraph of $K_{m+n} \cong K(m+n, 1)$. Therefore, there is the inclusion homomorphism

$$G_1 \vee G_2 \to K(m+n, 1),$$

and so

$$\chi_f(Inc(G_1 \vee G_2)) \leq \frac{m+n}{1} = m+n.$$

On the other hand, we can decompose $G_1 \vee G_2$ as $G_1 \cup K_{m,n} \cup G_2$, where G_1 and G_2 are not just edge disjoint, but vertex disjoint as well! In particular, this means that there are no adjacencies in the incidence graph between incidences of G_1 and incidences of G_2; so we may use the same labels on these incidences. To this end, let

$$\sigma_1 : Inc(G_1) \to K(r_1, s_1), \sigma_2 : Inc(G_2) \to K(r_2, s_2)$$

and

$$\sigma_3 : Inc(K_{m,n}) \to K(r_3, s_3),$$

where $\chi_f(Inc(G_i)) = \frac{r_i}{s_i}$ and $\chi_f(Inc(K_{m,n})) = \frac{r_3}{s_3}$. Note that if G_1 and G_2 have no edge, then $G_1 \vee G_2 = K_{m,n}$ and we are done. If at least one of G_1 or G_2 has an edge, then the homomorphism corresponding to the maximum of $\chi_f(Inc(G_i))$ can be extended. So, by Corollary 2.2.3, we can extend these homomorphisms to

$$\tilde{\sigma}_1 : Inc(G_1) \to K(s_2 s_3 r_1, s_1 s_2 s_3), \tilde{\sigma}_2 : Inc(G_2) \to K(s_1 s_3 r_2, s_1 s_2 s_3)$$

and

$$\tilde{\sigma}_3 : Inc(K_{m,n}) \to K(s_1 s_2 r_3, s_1 s_2 s_3).$$

(We will see below that only the homomorphism corresponding to the maximum of $\chi_f(Inc(G_i))$ will be relevant.) Note that each of these target Kneser graphs have vertices whose corresponding subsets have equal size, namely they all have size $s_1 s_2 s_3$. Now, let

$$R = \max\{s_2 s_3 r_1, s_1 s_3 r_2.$$

Then, there are inclusion homomorphisms which extend $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ to

$$\hat{\sigma}_1 : Inc(G_1) \to K(R, s_1 s_2 s_3), \hat{\sigma}_2 : Inc(G_2) \to K(R, s_1 s_2 s_3).$$

Now, define $\sigma : Inc(G_1 \vee G_2) \to K(R + s_1 s_2 r_3, s_1 s_2 s_3)$ by

$$\sigma(u, uv) = \begin{cases} \hat{\sigma}_i(u, uv) & uv \in E(G_i) \\ \tilde{\sigma}_3(u, uv) & uv \in E(K_{m,n}) \end{cases},$$

where the $\hat{\sigma}_1$ and $\hat{\sigma}_2$ use the same labels, and $\tilde{\sigma}_3$ uses labels disjoint from the ones used for $\hat{\sigma}_1$ and $\hat{\sigma}_2$. Then, given two adjacent incidences of $Inc(G_1 \vee G_2)$, we have three cases. First, both edges can come from G_1, from G_2 or from $K_{m,n}$, in which case the incidences are adjacent in the associated incidence graph of G_i or $K_{m,n}$ and so they receive disjoint labels and hence their images under σ are adjacent. Second, one edge can be in some G_i and the other one is in $K_{m,n}$, in which case the fact that the defining fractional coloring homomorphisms used different labels shows that the incidences receive disjoint labels. Finally, if one edge came from G_1 and the other came from G_2, then they could not have been adjacent in the first place. Thus, this is in fact a homomorphism. Hence,

$$\begin{aligned}
\chi_f(Inc(G_1 \vee G_2)) &\leq \frac{R + s_1 s_2 r_3}{s_1 s_2 s_3} = \frac{R}{s_1 s_2 s_3} + \frac{s_1 s_2 r_3}{s_1 s_2 s_3} \\
&= \frac{\max\{s_2 s_3 r_1, s_1 s_3 r_2\}}{s_1 s_2 s_3} + \frac{r_3}{s_3} \\
&= \max\left\{\frac{s_2 s_3 r_1}{s_1 s_2 s_3}, \frac{s_1 s_3 r_2}{s_1 s_2 s_3}\right\} + \chi_f(Inc(K_{m,n})) \\
&= \max\left\{\frac{r_1}{s_1}, \frac{r_2}{s_2}\right\} + \chi_f(Inc(K_{m,n})) \\
&= \max\{\chi_f(Inc(G_1)), \chi_f(Inc(G_2))\} + \chi_f(Inc(K_{m,n})).
\end{aligned}$$

Therefore,

$$\chi_f(Inc(G_1 \vee G_2)) \leq \min\{m+n, \max\{\chi_f(Inc(G_1)), \chi_f(Inc(G_2))\} + \chi_f(Inc(K_{m,n}))\},$$

as desired. □

Example. This is tight in regards to both bounds within the minimum. Consider $G_1 \cong K_m$ and $G_2 \cong K_n$. Then, $G_1 \vee G_2 \cong K_m \vee K_n \cong K_{m+n}$. And so,

$$\chi_f(Inc(G_1 \vee G_2)) = \chi_f(Inc(K_{m+n})) = m + n.$$

On the other hand, consider $G_1 \cong \overline{K_m}$ and $G_2 \cong \overline{K_n}$. Then, $G_1 \vee G_2 \cong K_{m,n}$. So, since G_1 and G_2 have no edges, their incidence graphs are empty. Hence, $\chi_f(Inc(G_i)) = 0$. Therefore,

$$\chi_f(Inc(G_1 \vee G_2)) = \max\{0, 0\} + \chi_f(Inc(K_{m,n})) = \chi_f(Inc(K_{m,n})).$$

So the bounds are tight.

Corollary 2.2.10. *Let G_1 and G_2 be graphs such that $|V(G_1)| = n = |V(G_2)|$. Then,*

$$\chi_f(Inc(G_1 \vee G_2)) \leq \min\left\{2n, \max\{\chi_f(Inc(G_1)), \chi_f(Inc(G_2))\} + \frac{n^2}{n-1}\right\}.$$

Chapter 3

Extending Bounds

The goal of this chapter is to use the techniques described in Chapter 2, specifically in §2.2, to extend the known results regarding the incidence chromatic number of the direct and lexicographic products (due to Yang [37]) to bounds on the fractional incidence chromatic number. Further, we will extend bounds on the incidence chromatic number using other graph invariants - such as the star arboricity and edge chromatic number of a graph - to bounds on the fractional incidence chromatic number.

3.1 The Direct Product

Definition. The *direct product* of G and H, denoted $G \times H$, is defined to be the graph with

$$V(G \times H) = V(G) \times V(H)$$

and

$$E(G \times H) = \{(g, h)(g', h') \mid gg' \in E(G) \text{ and } hh' \in E(H)\}.$$

The direct product is sometimes called the *categorical product* since it has projections into each of the factors which are homomorphisms.

Example. In comparison to the Cartesian product, drawn below is $K_2 \times K_2$ and $P_3 \times P_3$, respectively.

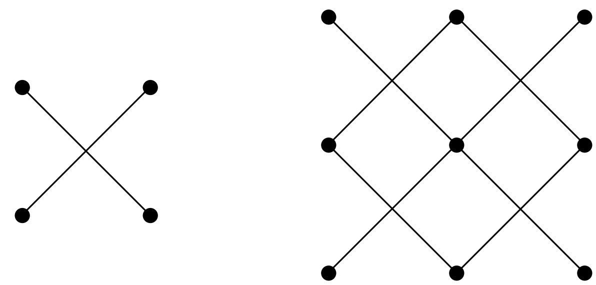


Figure 3.1: $K_2 \times K_2$ and $P_3 \times P_3$

We want to generalize the following result.

Proposition 3.1.1 (Yang [37]). *Let G and H be graphs. Then,*

$$\chi(Inc(G \times H)) \leq \min\{\Delta(H)\chi(Inc(G)), \Delta(G)\chi(Inc(H))\}.$$

The appropriate generalization is as follows.

Theorem 3.1.2. *Let G and H be graphs. Then,*

$$\chi_f(Inc(G \times H)) \leq \min\{\Delta(H)\chi_f(Inc(G)), \Delta(G)\chi_f(Inc(H))\}.$$

Proof. It suffices, by symmetry, to show that

$$\chi_f(Inc(G \times H)) \leq \Delta(H)\chi_f(Inc(G)).$$

Let

$$\sigma : Inc(G) \to K(r, s)$$

be a homomorphism where $\chi_f(Inc(G)) = \frac{r}{s}$. If G has no edges, then $G \times H$ has no edges and so $\chi_f(Inc(G \times H)) = 0$, since $Inc(G \times H)$ is empty. Further,

$\chi_f(Inc(G)) = 0$ since $Inc(G)$ is empty. Thus, the result follows. Similarly, if H has no edges, then $G \times H$ has no edges and so $\chi_f(Inc(G \times H)) = 0$, since $Inc(G \times H)$ is empty. Further, $\Delta(H) = 0$, so the result follows. Now, assume that G and H both have at least one edge. So, in particular, $r \geq 2s$ and $\Delta(H) \geq 1$. Let

$$i = ((v, w), (v, w)(v_1, w_1)) \in V(Inc(G \times H)).$$

Then, $(v, w)(v_1, w_1) \in E(G \times H)$ implies that $vv_1 \in E(G)$. (Also, it implies that $ww_1 \in E(H)$, but this is not relevant to us in the context of this proof. It would, however, be relevant in the symmetric argument to show that $\chi_f(Inc(G \times H))$ is at most $\Delta(G)\chi_f(Inc(H))$.) So, $(v, vv_1) \in V(Inc(G))$. Hence, define

$$\tilde{\sigma} : Inc(G \times H) \to K(\Delta(H)r, s)$$

by assigning to each vertex in the collection of vertices

$$I_{v,w,v_1} = \{((v, w), (v, w)(v_1, w_1))\}$$

a distinct and arbitrarily assigned subset of $[\Delta(H)r]$ of the form

$$(\sigma(v, vv_1)_k + nr)_{k=1}^{s}$$

with $n = 0, \ldots, \Delta(H) - 1$. To see why this makes sense, recall that $\sigma(v, vv_1)$ is an s-element subset of $[r]$. So, by $\sigma(v, v_1)_k$, we mean the k^{th} element of this s-element set. This map, $\tilde{\sigma}$, arbitrarily assigns to each incidence of the form above (of which there are at most $\Delta(H)$ ways to choose w_1 to fill in the edge $(v, w)(v_1, w_1)$, since this requires that $ww_1 \in E(H)$, and w is fixed) some nonnegative integer shift of the color assignment given by $\sigma(v, v_1)$. Now, to see that this is a homomorphism, consider two adjacent vertices of $Inc(G \times H)$, say, i from before and

$$j = ((x, y), (x, y)(x_1, y_1)) \in V(Inc(G \times H)).$$

Recall and observe that since $\sigma(v, vv_1) \subseteq [r]$, it follows that

$$\tilde{\sigma}(i) \subseteq [r + n_i r] \setminus [r + (n_i - 1)r]$$

if $n_i > 0$ and

$$\tilde{\sigma}(i) \subseteq [r]$$

if $n_i = 0$, under the random assignment of the shifting, denoted by n_i here. A similar relationship holds for $\tilde{\sigma}(j)$.

Now, we consider the three cases of adjacency in the incidence graph. We want to show that $\tilde{\sigma}(i)\tilde{\sigma}(j) \in E(K(r, s))$. Observe that it suffices to show that $(v, vv_1)(x, xx_1) \in E(Inc(G))$. If this is the case, then since σ is a homomorphism,

$$\sigma(v, vv_1)\sigma(x, xx_1) \in E(K(r, s))$$

and so

$$\sigma(v, vv_1) \cap \sigma(x, xx_1) = \emptyset$$

by definition of the Kneser graph. So, upon the random shifting assigned by $\tilde{\sigma}$, we want to show that $\tilde{\sigma}(i) \cap \tilde{\sigma}(j) = \emptyset$. To see this, recall that $\sigma(v, vv_1), \sigma(x, xx_1) \subseteq [r]$. The random shifting assigned by $\tilde{\sigma}$ is always given by some integer multiple of r, say $n_i r$ and $n_j r$ for i and j respectively. Suppose, without loss of generality, that $n_i \leq n_j$. If

$$c \in \tilde{\sigma}(i) \cap \tilde{\sigma}(j)$$

then it follows that

$$c = c_i + n_i r = c_j + n_j r$$

for some $c_i \in \sigma(v, vv_1)$ and some $c_j \in \sigma(x, xx_1)$ and thus

$$c_i = c_j + (n_j - n_i)r.$$

If $n_i = n_j$, then

$$c_i = c_j$$

and thus,

$$c_i = c_j \in \sigma(v, vv_1) \cap \sigma(x, xx_1) = \emptyset.$$

This is a contradiction. So, $n_i < n_j$. It follows that

$$c_i > r,$$

which implies that $c_i \notin \sigma(v, vv_1)$ which is a contradiction. Therefore,

$$\tilde{\sigma}(i) \cap \tilde{\sigma}(j) = \emptyset$$

and so $\tilde{\sigma}(i)\tilde{\sigma}(j) \in E(K(\Delta(H)r, s))$. Now, the goal is to show that (v, vv_1) and (x, xx_1) are adjacent in $Inc(G)$ by considering the three ways for the incidences i and j to be adjacent.

- Suppose $(v, w) = (x, y)$. Then, in particular, $v = x$. Therefore, (v, vv_1) and (x, xx_1) are adjacent in $Inc(G)$.

- Suppose $(v, w)(v_1, w_1) = (x, y)(x_1, y_1)$. Then, since i and j are distinct vertices, it follows that $(v, w) = (x_1, y_1)$. So, in particular, it follows that $v = x_1$. Thus,

  $$(v, vv_1) = (x_1, x_1v_1)$$

  and (x_1, x_1v_1) is adjacent to (x, xx_1). So, (v, vv_1) is adjacent to (x, xx_1).

- Finally, without loss of generality, suppose $(v, w)(x, y) = (v, w)(v_1, w_1)$. Then, $(x, y) = (v_1, w_1)$. In particular, this implies that $x = v_1$. So, $(x, xx_1) = (v_1, v_1x_1)$, which is adjacent to (v, vv_1).

Therefore, $\tilde{\sigma}$ is a homomorphism and so

$$\begin{aligned}\chi_f(Inc(G \times H)) \leq \frac{\Delta(H)r}{s} &= \Delta(H)\chi_f(Inc(G)) \\ &= \min\{\Delta(H)\chi_f Inc(G)), \Delta(G)\chi_f(Inc(H))\},\end{aligned}$$

as desired. □

Example. This bound is tight. Consider $K_2 \times K_2 \cong 2K_2$, which is drawn in Figure 3.1. So, since $\chi_f(Inc(K_2 \times K_2)) = \chi_f(2K_2) = 2$, $\chi_f(Inc(K_2)) = 2$ and $\Delta(K_2) = 1$,

$$\chi_f(Inc(K_2 \times K_2)) = \min\{\Delta(K_2)\chi_f(Inc(K_2)), \Delta(K_2)\chi_f(Inc(K_2))\}.$$

Definition. The *strong product* of G and H, denoted $G \boxtimes H$, is defined as the graph with

$$V(G \boxtimes H) = V(G) \times V(H)$$

and

$$E(G \boxtimes H) = E(G \square H) \cup E(G \times H).$$

Example. Again, for comparison, drawn below are $K_2 \boxtimes K_2$ and $P_3 \boxtimes P_3$, respectively.

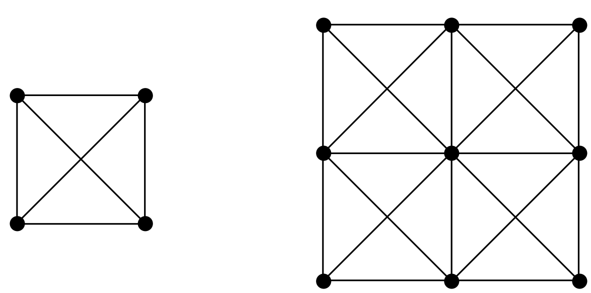


Figure 3.2: $K_2 \boxtimes K_2$ and $P_3 \boxtimes P_3$

Note that $K_2 * K_2$, where $*$ stands for any of the three graph products we've defined so far, has the useful property that the shape of the graph is precisely the symbol used to represent the product. This is helpful when trying to distinguish between these three products.

Corollary 3.1.3. *Let G and H be graphs. Then,*

$$\chi_f(Inc(G \boxtimes H)) \leq \chi_f(Inc(G)) + \chi_f(Inc(H)) + \min\{\Delta(H)\chi_f(Inc(G)), \Delta(G)\chi_f(Inc(H))\}.$$

Proof. Let G, H be graphs. Then, as per the definition, we can partition the edges of $G \boxtimes H$ into two disjoint sets; namely $E(G \Box H)$ and $E(G \times H)$. Thus, we may write

$$G \boxtimes H = (G \Box H) \cup (G \times H).$$

Thus, by Theorem 2.2.5,

$$\chi_f(Inc(G \boxtimes H)) \leq \chi_f(Inc(G \Box H)) + \chi_f(Inc(G \times H)).$$

Using Theorem 2.2.7 and Theorem 3.1.2, we see that

$$\chi_f(Inc(G \boxtimes H)) \leq \chi_f(Inc(G)) + \chi_f(Inc(H)) + \min\{\Delta(H)\chi_f(Inc(G)), \Delta(G)\chi_f(Inc(H))\}.$$

□

Example. Observe that the bound for the strong product is also tight. Consider the following example. Let $G = \overline{K_n}$ and let H be any nonempty graph. Then, $G \boxtimes H = nH$ since there are no edges in the direct product of these two graphs since G has no edges. Further, note that

$$\chi_f(Inc(G)) = 0$$

and since $\Delta(G) = 0$, it follows that

$$\min\{\Delta(H)\chi_f(Inc(G)), \Delta(G)\chi_f(Inc(H))\} = 0.$$

Therefore, the bound gives

$$\chi_f(Inc(G \boxtimes H)) = \chi_f(Inc(H)).$$

Since $\chi_f(Inc(nH)) = \chi_f(Inc(H))$, we see that the bound is tight for this example.

3.2 The Lexicographic Product

Definition. The *lexicographic product* of G and H, denoted $G[H]$ or $G \circ H$, is defined to be the graph with

$$V(G[H]) = V(G) \times V(H)$$

and

$$E(G[H]) = \{(g,h)(g',h') \mid gg' \in E(G) \text{ or } (g = g' \text{ and } hh' \in E(H))\}.$$

This is sometimes called the *wreath product* in the literature.

Example. Consider the graphs $K_2[K_2]$, $P_3[P_2]$, $P_2[P_3]$, drawn below respectively.

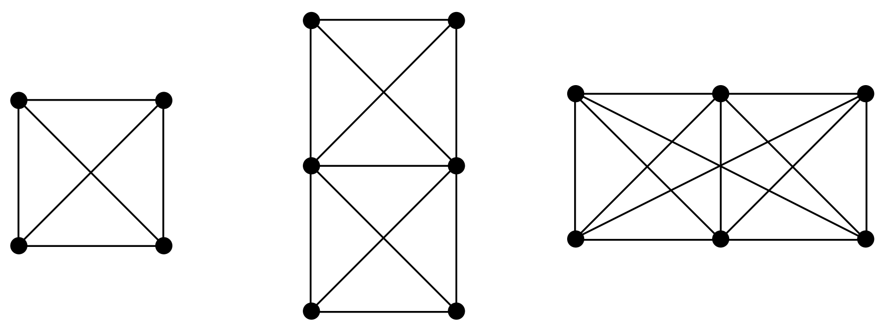


Figure 3.3: $K_2[K_2]$, $P_3[P_2]$ and $P_2[P_3]$

Observation. Note that, unlike the Cartesian, direct and strong products, the lexicographic product is *not* commutative! For example, see $P_3[P_2]$ and $P_2[P_3]$ in Figure 3.3. These graphs are not isomorphic. One way to see this is to observe that one of the graphs has a vertex of degree 3, while the other does not.

Proposition 3.2.1 ([37])**.** *Let G and H be graphs. Then,*

$$\chi(Inc(G[H])) \leq |V(H)|\chi(Inc(G)) + \chi(Inc(H)).$$

Theorem 3.2.2. *Let G and H be graphs. Then,*

$$\chi_f(Inc(G[H])) \leq |V(H)|\chi_f(Inc(G)) + \chi_f(Inc(H)).$$

Proof. Let

$$\sigma_1 : Inc(G) \to K(r_1, s_1) \qquad \text{and} \qquad \sigma_2 : Inc(H) \to K(r_2, s_2)$$

be homomorphisms such that $\chi_f(Inc(G)) = \frac{r_1}{s_1}$ and $\chi_f(Inc(H)) = \frac{r_2}{s_2}$. If G has no edges, then $G[H]$ is $|V(G)|$ copies of H. So,

$$\chi_f(Inc(G[H])) = \chi_f(Inc(H)).$$

Since G has no edges, $\chi_f(Inc(G)) = 0$ as $Inc(G)$ is empty. Thus, the result follows. If H has no edges, then $Inc(G[H])$ has all vertices of the form

$$((u, v), (u, v)(x, v))$$

with $ux \in E(G)$. We may use the argument which follows, where σ_2 is ignored, in this case. So, assume that G and H both contain at least one edge. Using Corollary 2.2.3, extend σ_1 and σ_2 to homomorphisms

$$\tilde{\sigma}_1 : Inc(G) \to K(r_1 s_2, s_1 s_2) \qquad \text{and} \qquad \tilde{\sigma}_2 : Inc(H) \to K(r_2 s_1, s_1 s_2).$$

Further, assume that $\tilde{\sigma}_2$ uses labels from $[r_2 s_1]$ and assume that $\tilde{\sigma}_1$ uses labels from $[r_1 s_2 + r_2 s_1] - [r_2 s_1]$. We will be doing an arbitrary shift, as in the proof of the direct product, on $\tilde{\sigma}_1$, which is why we want the labels used on $\tilde{\sigma}_1$ to be larger than the labels used on $\tilde{\sigma}_2$. This will make writing down the map easier and hence make the proof that we have defined a homomorphism easier as well. Recall that $((u, v), (u, v)(x, y)) \in V(Inc(G[H]))$ implies that $(u, v)(x, y) \in E(G[H])$. By definition of the lexicographic product, this means that either

(1) $ux \in E(G)$, or

(2) $u = x$ in G *and* $vy \in E(H)$.

Thus, either

(1) $(u, ux) \in Inc(G)$, or

(2) $(v, vy) \in Inc(H)$.

Call the incidences of $G[H]$ satisfying the conditions (1) Type 1 and the incidences satisfying the conditions (2) Type 2. Now, define

$$\sigma : Inc(G[H]) \to K(|V(H)|r_1 s_2 + r_2 s_1, s_1 s_2)$$

by setting

$$\sigma((u, v), (u, v)(x, y)) = (\tilde{\sigma}_1(u, ux)_k + n_u r_1 s_2)_{k=1}^{s_1 s_2}$$

distinctly and arbitrarily for $n_u = 0, \ldots, |V(H)| - 1$ if $((u, v), (u, v)(x, y))$ is a Type 1 incidence and

$$\sigma((u, v), (u, v)(x, y)) = \tilde{\sigma}_2(v, vy)$$

if $((u, v), (u, v)(x, y))$ is a Type 2 incidence. For the Type 1 incidences, observe that from any fixed incidence, say $((u, v), (u, v)(x, y))$, the only adjacent incidences of Type 1 are $((u, v), (u, v)(x, z))$ for any $z \in V(H) \setminus \{y\}$. Hence, we do need $|V(H)|$ distinct shifts of $\tilde{\sigma}_1(u, uv)$.

We must show that σ is a homomorphism. Let

$$i = ((u_1, v_1), (u_1, v_1)(x_1, y_1)) \qquad \text{and} \qquad j = ((u_2, v_2), (u_2, v_2)(x_2, y_2))$$

be adjacent in $Inc(G[H])$. Let's start with a lemma.

Lemma 3.2.3. *If i and j are both Type 1 incidences with (u_1, u_1x_1) and (u_2, u_2x_2) being adjacent, then $\sigma(i)$ and $\sigma(j)$ are adjacent.*

Proof. Suppose i and j are Type 1 incidences and suppose that (u_1, u_1x_1) and (u_2, u_2x_2) are adjacent. Then,

$$\tilde{\sigma}_1(u_1, u_1x_1) \cap \tilde{\sigma}_1(u_2, u_2x_2) = \emptyset.$$

Suppose $c \in \sigma(i) \cap \sigma(j)$. Then, for some (distinct) $n_i, n_j \in \{0, \dots, |V(H)| - 1\}$,

$$c = c_i + n_i r_1 s_2 = c_j + n_j r_1 s_2,$$

where

$$c_i \in \tilde{\sigma}_1(u_1, u_1x_1), c_j \in \tilde{\sigma}_1(u_2, u_2x_2).$$

Suppose, without loss of generality, that $n_i \leq n_j$. Then,

$$c_i = c_j + (n_j - n_i) r_1 s_2.$$

If $n_i = n_j$, then

$$c_i = c_j \in \tilde{\sigma}_1(u_1, u_1x_1) \cap \tilde{\sigma}_1(u_2, u_2x_2).$$

This is a contradiction. Thus, $n_i < n_j$. In this case, $c_i > r_1 s_2 + r_2 s_1$. This is also a contradiction, since $c_i \in \tilde{\sigma}_1(u_1, u_1x_1)$. Therefore, $\sigma(i) \cap \sigma(j) = \emptyset$. Hence, $\sigma(i)$ and $\sigma(j)$ are adjacent. □

Now, we consider the three cases for adjacency in an incidence graph.

- Suppose $(u_1, v_1) = (u_2, v_2)$. Then, $u_1 = u_2$ and $v_1 = v_2$.

  * If i and j are both Type 1 incidences, then since $u_1 = u_2$, (u_1, u_1x_1) and (u_2, u_2x_2) are adjacent. Thus, by Lemma 3.2.3, $\sigma(i)$ and $\sigma(j)$ are adjacent.

  * If i and j are both Type 2 incidences, then

$$\sigma(i) = \tilde{\sigma}_2(v_1, v_1y_1)$$

and

$$\sigma(j) = \tilde{\sigma}_2(v_2, v_2y_2).$$

Since $v_1 = v_2$, (v_1, v_1y_1) and (v_2, v_2y_2) are adjacent and so their images under $\tilde{\sigma}_2$ are adjacent; that is, they receive disjoint sets of labels. Hence, $\sigma(i), \sigma(j)$ are adjacent. Note that there is no need to worry about the shifting in this case.

* If, without loss of generality, i is Type 1 and j is Type 2, then, $\sigma(i)$ and $\sigma(j)$ come from disjoint sets of labels and so are disjoint. Thus, they are adjacent.

- Suppose $(u_1, v_1)(x_1, y_1) = (u_2, v_2)(x_2, y_2)$. Then, since the incidences are distinct, we know that

$$(u_1, v_1) = (x_2, y_2) \qquad \text{and so} \qquad u_1 = x_2, v_1 = y_2$$

and

$$(x_1, y_1) = (u_2, v_2) \qquad \text{and so} \qquad x_1 = u_2, y_1 = v_2.$$

Then,

* If i and j are both Type 1 incidences, then since $x_1 = u_2$, we can conclude that $(u_1, u_1x_1) = (u_1, u_1u_2)$ and (u_2, u_2x_2) are adjacent. So, by Lemma 3.2.3, $\sigma(i)$ and $\sigma(j)$ are adjacent.

* If i and j are both Type 2 incidences, then

$$\sigma(i) = \tilde{\sigma}_2(v_1, v_1y_1) \qquad \text{and} \qquad \sigma(j) = \tilde{\sigma}_2(v_2, v_2y_2).$$

Since $y_1 = v_2$, we can conclude that $(v_1, v_1y_1) = (v_1, v_1v_2)$ and (v_2, v_2y_2) are adjacent. Hence, their images under $\tilde{\sigma}_2$ are adjacent; that is, they receive disjoint sets of labels. Hence, $\sigma(i)$ and $\sigma(j)$ are disjoint and hence adjacent. Again, there is no need to worry about the random shifting.

* If, without loss of generality, i is Type 1 and j is Type 2, then $\sigma(i)$ and $\sigma(j)$ come from disjoint sets of labels and so are disjoint. Thus, they are adjacent.

- Suppose, without loss of generality, that $(u_1, v_1)(u_2, v_2) = (u_1, v_1)(x_1, y_1)$. Then, $(u_2, v_2) = (x_1, y_1)$ and so we see that $u_2 = x_1$ and $v_2 = y_1$. Then, since this was exactly the information used in the previous case (although we had more information), the same proof shows that $\sigma(i)$ and $\sigma(j)$ are adjacent in this case as well.

Therefore, σ is a homomorphism. So,

$$\begin{aligned}\chi_f(Inc(G[H])) &\leq \frac{|V(H)|r_1 s_2 + r_2 s_1}{s_1 s_2} = \frac{|V(H)|r_1}{s_1} + \frac{r_2}{s_2} \\ &= |V(H)|\chi_f(Inc(G)) + \chi_f(Inc(H))\end{aligned}$$

as claimed. □

Example. The bound obtained in Theorem 3.2.2 is tight. Consider $G = \overline{K_n}$ and let H be any nonempty graph. Then,

$$G[H] = \overline{K_n}[H] = nH.$$

Since

$$\chi_f(Inc(\overline{K_n}[H])) = \chi_f(Inc(nH)) = \chi_f(Inc(H))$$

and

$$|V(H)|\chi_f(Inc(\overline{K_n})) + \chi_f(Inc(H)) = \chi_f(Inc(H)),$$

we see that the bound is tight.

3.3 Star Arboricity and Edge Coloring

Shifting gears slightly, we now provide some further generalizations of bounds using other graph invariants. The first involves the star arboricity and the edge chromatic number of the graph.

Definition. A *star* is a specific example of a tree in which there is a distinguished vertex, called the center, which is adjacent to all other vertices in the graph. The *star arboricity* of a graph G, denoted $\mathrm{st}(G)$, is the minimum number of star forests (a forest in which every induced subtree is a star) needed to cover the edges of G.

Example. If S is a star, then precisely one star can cover all the edges. Thus, we need only one star forest to cover the edges of S, and so $\mathrm{st}(S) = 1$.

If T is a tree which is not a star, then by choosing any root for the tree and letting the vertices on the even levels be the centers for one star forest and the vertices on the odd levels be the centers for another star forest, and by choosing the edges below the given vertices, we can find that $\mathrm{st}(T) \leq 2$ for any tree. Note that since T is not a star, it is a connected graph with vertices on at least three levels. Since the individual stars within a star forest cannot share vertices, we see that T requires at least two star forests to cover its edges. So $\mathrm{st}(T) \geq 2$. Therefore, $\mathrm{st}(T) = 2$.

Definition. Given a graph G, we can define its *edge chromatic number*, denoted $\chi'(G)$, to be the least number of colors needed to color the edges such that if two edges share an endpoint, the edges receive different colors.

The following result boils down the computation of $\chi'(G)$ to deciding between one of two possibilities.

Theorem 3.3.1 (Vizing [35]). *Let G be a graph with maximum degree $\Delta(G)$.*

Then,

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

The graphs for which $\chi'(G) = \Delta(G)$ are collectively called the *Class 1* graphs. The remaining graphs, which have $\chi'(G) = \Delta(G) + 1$, are called the *Class 2* graphs.

Example. The even complete graphs are in Class 1 and the odd complete graphs are in Class 2. The Petersen graph is also a Class 2 graph.

Unfortunately, the task of deciding whether a simple graph is in Class 1 or Class 2 is an NP-complete problem (see [22]).

Now, back to the (fractional) incidence chromatic number, the following bound is known.

Proposition 3.3.2 (Yang [37])**.** *If G is any graph, then*

$$\chi(Inc(G)) \leq st(G) + \chi'(G).$$

We can generalize this to the following.

Proposition 3.3.3. *If G is any graph, then*

$$\chi_f(Inc(G)) \leq st(G) + \chi'_f(G).$$

Proof. Let $\mathrm{st}(G) = t$ be the star arboricity of G and let $\chi'_f(G) = \frac{r}{s}$. Recall that $Inc(G)$ can be viewed as a directed graph obtained from G where each edge is replaced by two oppositely oriented arcs as discussed in §1.2. Call this directed graph $S(G)$. So, there are two edge disjoint copies of G whose edges need to be colored so that edges corresponding to the same edge of G receive different colors, edges forming a directed path of length 2 receive different colors, and edges with the same tail receive different colors. So, clearly, the "color classes" of

$S(G)$ (which correspond to color classes of $Inc(G)$) are the directed star forests of G. So, take the t star forests and color each of them with one of the t disjoint s-element subsets from $[ts]$. This star forest covers one copy of G, so there is another (directed) copy of G to edge color. Fractionally color the remaining edges corresponding to a coloring witnessing $\chi'_f(G) = \frac{r}{s}$, using a color set disjoint from $[ts]$. Then, adjacent in the directed graph in terms of the incidence graph implies that the edges are incident in the undirected copy of G, and so this is a valid edge coloring. Thus, we have colored $Inc(G)$ using $ts + r$ colors and labeling with sets of size s. That is, we have defined a $(ts + r, s)$-coloring of $Inc(G)$. Therefore,

$$\chi_f(Inc(G)) \leq \frac{ts + r}{s} = t + \frac{r}{s} = \mathrm{st}(G) + \chi'_f(G).$$

□

Example. This bound is tight. Consider C_4. Then,

$$\mathrm{st}(C_4) = 2, \chi'_f(C_4) = 2, \text{ and } \chi_f(Inc(C_4)) = 4.$$

To continue the comparison, consider the use of Vizing's Theorem ([35]).

Remark. By the fractional version of Vizing's Theorem ([27]), namely, that

$$\Delta(G) \leq \chi'_f(G) \leq \Delta(G) + 1,$$

we see that

$$\chi_f(Inc(G)) \leq \mathrm{st}(G) + \chi'_f(G) \leq \mathrm{st}(G) + \Delta(G) + 1.$$

Remark. It can be shown that $\chi'_f(G) = \Delta(G) + 1$ precisely when $G = K_{2n+1}$ for $n \geq 1$ (see [27], [29]). Note that such a characterization is not known for $\chi'(G)$. In the case of $G = K_{2n+1}$ for $n \geq 1$,

$$\chi_f(Inc(G)) \leq \mathrm{st}(G) + \Delta(G) + 1,$$

and in *all* other cases,

$$\chi_f(Inc(G)) < \mathrm{st}(G) + \Delta(G) + 1.$$

Note that we can't do any better in general since

$$\chi'_f(G) \in [\Delta(G), \Delta(G) + 1]$$

unlike the nonfractional case where $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$. However, $\chi'_f(G)$ can be computed in polynomial time ([27])!

Example. Observe that the only planar graph with $\chi'_f(G) = \Delta(G) + 1$ is K_3. Then,

$$\chi'_f(K_3) = 3$$

by the previous remark and

$$\mathrm{st}(K_3) = 2$$

since K_3 is not a star, but can be covered with two stars which necessarily overlap (take a maximal star as one star forest and the remaining edge as the other star forest). So,

$$\chi_f(Inc(K_3)) \leq 2 + 3 + 1 = 6.$$

However, we know that $\chi_f(Inc(K_3)) = 3$. So, for *all* planar graphs,

$$\chi_f(Inc(G)) < \mathrm{st}(G) + \Delta(G) + 1.$$

3.4 Another Bound and a Computation

In this section, we prove another bound on the fractional incidence chromatic number of the lexicographic product using the square of a graph. As a result we will calculate precisely the fractional incidence chromatic number of another infinite family of graphs.

The following result can be found in [37] without proof. We provide a proof here for completeness.

Lemma 3.4.1. *For any graph G,*

$$\chi_f(Inc(G)) \leq \chi_f(G^2).$$

Proof. Let $\phi : G^2 \to K(r,s)$ be an optimal fractional coloring of G^2. Define $\phi_* : Inc(G) \to K(r,s)$ by

$$\phi_*(u, uv) = \phi(v).$$

Then, we want to show that ϕ_* is a homomorphism. Suppose $(u, uv), (x, xy)$ are distinct, adjacent vertices in $Inc(G)$. We consider each of the three ways for these two vertices to be adjacent in $Inc(G)$.

- If $u = x$, then since the vertices are distinct, it follows that $v \neq y$. Thus, since in G there is a path $v - u - y$ (since we know that uv and xy are edges and $u = x$) we know that $vy \in E(G^2)$. Therefore,

  $$\phi_*(u, uv)\phi_*(x, xy) = \phi(v)\phi(y) \in E(K(r,s)),$$

  since ϕ is a homomorphism.

- If $uv = xy$, then since the vertices are distinct, we know that $u = y$ and $v = x$. So,

  $$\phi_*(u, uv)\phi_*(x, xy) = \phi(v)\phi(y) = \phi(x)\phi(y) \in E(K(r,s)),$$

  since $xy \in E(G) \subseteq E(G^2)$.

- Finally, without loss of generality, assume that $ux = uv$. Then $x = v$. So,

  $$\phi_*(u, uv)\phi_*(x, xy) = \phi(v)\phi(y) = \phi(x)\phi(y) \in E(K(r,s)),$$

  since $xy \in E(G) \subseteq E(G^2)$.

Therefore ϕ_* is a homomorphism and so we conclude that

$$\chi_f(Inc(G)) \leq \frac{r}{s} = \chi_f(G^2).$$

□

We will need the following result to argue that this bound is tight.

Lemma 3.4.2. *$Inc(C_n) \cong C_{2n}^2$ for $n \geq 3$.*

Proof. Let

$$V(C_n) = \{u_0, \ldots, u_{n-1}\}$$

where

$$E(C_n) = \{u_i u_{i+1} \mid i = 0, n-1\}$$

for indices taken modulo n. Then,

$$V(Inc(C_n)) = \{(u_i, u_i u_{i+1}), (u_i, u_i u_{i-1})\}$$

again where indices are taken modulo n. Order the vertices lexicographically clockwise around C_{2n}^2; that is,

$$(u_i, u_i u_j) < (u_k, u_k u_\ell)$$

precisely when $i < k$ or when $i = k$ and $j < \ell$. Under this bijection of the vertex sets, it is not hard to see that the edges and nonedges are preserved. □

Example. See Figure 1.1 for the example where $Inc(C_5) \cong C_{10}^2$.

Example. Note that the bound in Lemma 3.4.1 is tight. For example, consider $G = K_3 = C_3$. By Lemma 3.4.2, $Inc(G) = C_6^2$. So,

$$\chi_f(Inc(K_3)) = 3 = \chi_f(C_6^2).$$

Further, we can get strict inequality also. Consider the Petersen graph. Recall that

$$\chi_f(Inc(K(5,2))) = \frac{30}{7}$$

and observe that $K(5,2)^2 = K_{10}$ since the diameter, or length of the longest shortest path, of $K(5,2)$ is 2. Thus,

$$\chi_f(K(5,2)^2) = \chi_f(K_{10}) = 10.$$

So, this is an example where

$$\chi_f(Inc(G)) < \chi_f(G^2).$$

Lemma 3.4.3. *If G has no isolated vertices, then*

$$G[H]^2 = G^2[K_{|V(H)|}].$$

Proof. Suppose G has no isolated vertices. Then, $G[H]^2$ has the same vertex set as $G^2[K_{|V(H)|}]$ since G has the same vertex set as G^2 and H has the same vertex set as $K_{|V(H)|}$. We now must compare the edge sets. Suppose $(u,v)(x,y)$ is an edge in $G[H]^2$. Then, either (u,v) and (x,y) are adjacent but in different factors of H corresponding to different vertices of G or they are adjacent but in the same factor of H corresponding to a single vertex of G. In the first case, this can only happen if $\text{dist}_G(u,x) \leq 2$. That is, this can happen precisely when $ux \in E(G^2)$. In the second case, since there is no factor of H which is isolated (since G has no isolated vertices) there is a path of length 2 from (u,v) to (x,y) in $G[H]$ by following (u,v) to a vertex in an adjacent (in terms of G) factor of H and following that vertex by (x,y). Therefore, corresponding to each vertex of G, there is a copy of $K_{|V(H)|}$ since each pair of vertices within the factor of H is within distance 2. Therefore,

$$E(G[H]^2) = E(G^2[K_{|V(H)|}]).$$

Hence, the result follows. □

Corollary 3.4.4. *If G has no isolated vertices, then*

$$\chi_f(Inc(G[H])) \leq |V(H)|\chi_f(G^2).$$

Proof. Applying Lemma 3.4.1, Lemma 3.4.3 and Proposition 1.3.4 we see that

$$\begin{aligned}\chi_f(Inc(G[H])) &\leq \chi_f(G[H]^2) = \chi_f(G^2[K_{|V(H)|}]) \\ &= \chi_f(G^2)\chi_f(K_{|V(H)|}) = \chi_f(G^2)|V(H)|\end{aligned}$$

□

Corollary 3.4.5. *If G has k isolated vertices, then*

$$\chi_f(Inc(G[H])) \leq \min\{|V(H)|\chi_f(G^2), |V(H)|\chi_f(Inc(G)) + \chi_f(Inc(H))\}.$$

Proof. Observe that since G has k isolated vertices, then

$$G[H] = (\tilde{G} \coprod kK_1)[H] = \tilde{G}[H] \coprod kK_1[H] = \tilde{G}[H] \coprod kH,$$

where $\tilde{G}$ is the graph obtained from G by removing the isolated vertices. Hence, in particular, $\tilde{G}$ has no isolated vertices. So,

$$Inc(G[H]) = Inc(\tilde{G}[H] \coprod kH) = Inc(\tilde{G}[H]) \coprod kInc(H).$$

Further, observe that $Inc(H)$ is an induced subgraph of $\tilde{G}[H]$ and so when coloring we can reuse colors from $Inc(\tilde{G}[H])$ on all k copies of $Inc(H)$. Thus, combining Corollary 3.4.4 and Theorem 3.2.2, we see that

$$\begin{aligned}\chi_f(Inc(G[H])) &= \chi_f(Inc(\tilde{G}[H]) \coprod kInc(H)) \\ &= \chi_f(Inc(\tilde{G}[H])) \\ &\leq \min\{|V(H)|\chi_f(\tilde{G}^2), |V(H)|\chi_f(Inc(\tilde{G})) + \chi_f(Inc(H))\} \\ &= \min\{|V(H)|\chi_f(G^2), |V(H)|\chi_f(Inc(G)) + \chi_f(Inc(H))\},\end{aligned}$$

where the final equality follows because the only thing remaining to color in G^2 is a collection of isolated vertices which can reuse the labels already used on $\tilde{G}^2$, and $Inc(\tilde{G})$ is actually the same graph as $Inc(G)$ as the graphs G and $\tilde{G}$ have the same edge sets; the difference is a collection of isolated vertices, which do not contribute to the incidence graph. □

Finally, we are almost ready for the computation mentioned above. With one last remark about the domination number of the lexicographic product of two graphs, we will be ready for the computation.

Remark. It is not hard to show that if H is a graph such that $\gamma(H) = 1$, that is, H has a universal vertex, then

$$\gamma(G[H]) = \gamma(G)$$

for any graph G. This fact will be useful in the calculation below.

Example. Consider $C_{3n}[K_\ell]$ for $n \geq 1$, $\ell \geq 2$. Then, by Theorem 2.1.8,

$$\begin{aligned}\chi_f(Inc(C_{3n}[K_\ell])) &\geq \frac{2|E(C_{3n}[K_\ell])|}{|V(C_{3n}[K_\ell])| - \gamma(C_{3n}[K_\ell])} \\ &= \frac{3n\ell(2\ell + \ell - 1)}{3nl - \gamma(C_{3n})} \\ &= \frac{3n\ell(3\ell - 1)}{3n\ell - \lceil \frac{3n}{3} \rceil} \\ &= \frac{3n\ell(3\ell - 1)}{3n\ell - n} = \frac{3n\ell(3\ell - 1)}{n(3\ell - 1)} = 3\ell\end{aligned}$$

Further, by Corollary 3.4.5,

$$\begin{aligned}\chi_f(Inc(C_{3n}[K_\ell])) &\leq \min\{|V(K_\ell)|\chi_f(C_{3n}^2), |V(K_\ell)|\chi_f(Inc(C_{3n})) + \chi_f(Inc(K_\ell))\} \\ &= \min\left\{\ell\left(\frac{3n}{n}\right), \ell(3) + \ell\right\} \\ &= \min\{3\ell, 4\ell\} = 3\ell,\end{aligned}$$

where the middle equality follows from Theorem 1.3.6, since C_{3n}^2 is vertex transitive by rotation. Therefore,

$$\chi_f(Inc(C_{3n}[K_\ell])) = 3\ell.$$

Note that $C_{3n}[K_\ell]$ is a $(3\ell - 1)$-regular graph and so

$$\chi_f(Inc(C_{3n}[K_\ell])) = \Delta + 1.$$

Remark. The previous example is one where the bound

$$\chi_f(Inc(G[H])) \leq |V(H)|\chi_f(G^2)$$

is better than

$$\chi_f(Inc(G[H])) \leq |V(H)|\chi_f(Inc(G)) + \chi_f(Inc(H)).$$

Chapter 4

Characterization of Perfect Incidence Graphs

In this chapter, we will provide a characterization on G for which $Inc(G)$ is perfect. This will allow us to calculate the (fractional) incidence chromatic numbers of such graphs. The general proof method uses the Strong Perfect Graph Theorem (Theorem 1.4.2).

4.1 Excluding Graphs with Large Cycles

In this section, we will exclude graphs with subgraphs isomorphic to cycles of length at least 4 from having their incidence graphs being perfect.

Lemma 4.1.1. *If H is any subgraph of G, then $Inc(H)$ is an induced subgraph of $Inc(G)$.*

Proof. Let H be a subgraph of G. Then, consider the subgraph of $Inc(G)$ induced by the vertices of $Inc(H)$. We wish to show that this induced subgraph is precisely

$Inc(H)$. Suppose $(u, uv)(x, xy)$ is an edge of the subgraph induced by the vertices in $Inc(H)$. Then, in $Inc(G)$, we have three cases.

- If $u = x$, then since u, x are necessarily vertices of H, it follows that $(u, uv)(x, xy) \in E(Inc(H))$.
- If $uv = xy$, then since in order for $(u, uv), (x, xy) \in V(Inc(H))$, it must be the case that $uv, xy \in E(H)$. Thus, $(u, uv)(x, xy) \in E(Inc(H))$.
- Finally, if $ux = uv$ (without loss of generality), then since $u, v, x \in V(H)$, it follows that $ux = uv \in E(H)$. Thus, $(u, uv)(x, xy) \in E(Inc(H))$.

Therefore, $Inc(H)$ is an induced subgraph of $Inc(G)$. □

Proposition 4.1.2. *If G contains a subgraph isomorphic to C_n for $n \geq 4$, then $Inc(G)$ is not perfect.*

Proof. We will consider the cases where n is even and when n is odd separately. Let's first consider the case where n is even. Then, considering $Inc(C_n) = C_{2n}^2$ with vertices labeled $0, \dots, 2n-1$ consecutively, we can find an induced cycle of length $n+1$, which is odd. Namely, the cycle induced by the vertices labeled

$$1, 2, 4, 6, \dots, n, n+1, n+3, \dots, 2n-1.$$

is a cycle of length $n+1$.

Now, if n is odd, using the same labeling on $Inc(C_n) = C_{2n}^2$, the cycle induced by the vertices labeled

$$1, 3, 5, 7, \dots, 2n-1$$

is an cycle of length n. Hence, in both cases, we have found an induced odd cycle of length at least 5. So, by the Strong Perfect Graph Theorem (Theorem 1.4.2), it follows that $Inc(C_n)$ is not perfect. Further, by Lemma 4.1.1, since $Inc(C_n)$ is an induced subgraph of $Inc(G)$, it follows that an induced odd cycle can be found in $Inc(G)$. So, $Inc(G)$ cannot be perfect. □

Definition. The *circumference* of a graph G is the length of a longest cycle in G.

Thus, we may rephrase Proposition 4.1.2 to say that if G has circumference at least 4, then $Inc(G)$ is not perfect. Moving forward in the next section, we consider graphs with circumference at most 3.

4.2 Characterization

In this section we show that all graphs with no subgraph isomorphic to a cycle of length at least 4 have perfect incidence graphs. The following is an exercise from [9].

Lemma 4.2.1. *For $k \geq 2$, if G is k-connected and has at least $2k$ vertices, then G contains a cycle of length at least $2k$.*

Proof. Let $k \geq 2$. Let G be k-connected on at least $2k$ vertices. Then, in particular, G is connected. Since

$$\delta(G) \geq k \geq 2,$$

where $\delta(G)$ is the minimum degree of G, it follows that G is not a tree and hence G contains a cycle. In fact, it is not hard to see that G contains a cycle of length at least $\delta(G) + 1 \geq k + 1$. (Take a longest path in G, and let x be one of the endpoints. Then, necessarily, all of the neighbors of x must lie on the path. Let y be the neighbor of x farthest along on the path. Then, following the path from x to y and then following the edge from y back to x results in a cycle on at least $\delta(G) + 1$ vertices.) Let C be a longest cycle in G and suppose that $|V(C)| < 2k$. Let $v \in V(G)$ be a vertex not on C. Consider the sets $N_G(v)$ and $V(C)$. By the remarks above, $|N_G(v)| \geq k$ and $|V(C)| \geq k+1$. If $S \subseteq V(G)$ is a subset of vertices which separates $N_G(v)$ and $V(C)$, it must have size at least k. Otherwise, this set would be a set of size less than k which disconnects v from some vertex of C. This contradicts that G is k-connected. Therefore, Menger's Theorem implies that there are k disjoint paths from $N_G(v)$ to $V(C)$. Note that if any two of these paths end at the same vertex of C, removing the endpoints of all these paths disconnects v from some vertex of C. Since we would have removed less than k vertices, this contradicts that G is k-connected. So, all endpoints are distinct. Since $|V(C)| < 2k$, the Pigeonhole Principle implies that there exists a pair of adjacent vertices on the cycle serving as endpoints to these paths. Let c_1, c_2 be these vertices. Let v_1, v_2 be the corresponding endpoints of these paths in $N_G(v)$ and call the corresponding paths P_1 and P_2. If $v_1 = v_2$, then replacing the edge c_1c_2 in C by the two paths, P_1 and P_2, this results in a longer cycle, which is a contradiction. If $v_1 \neq v_2$, then replacing edge c_1c_2 in C with the two paths P_1 and P_2 along with the edges to v results in a longer cycle, again a contradiction. Therefore, $|V(C)| \geq 2k$, as desired. □

Lemma 4.2.2. *If G is connected on at least 4 vertices and has circumference at most 3, then G has a cut vertex.*

Proof. Suppose not. Then, G must be 2-connected. So, by Lemma 4.2.1, since G is 2-connected on at least 4 vertices, G contains a subgraph which is isomorphic to C_n, where $n \geq 4$. This contradicts that G has circumference at most 3. □

Lemma 4.2.3. *Let v be a cut vertex of G. Then, the set of vertices*

$$C_v = \{(v, uv) \mid u \in N_G(v)\}$$

is a cut set of $Inc(G)$ which is a clique of size $\deg_G(v)$.

Proof. Let v and C_v be as above. It is clear to see that C_v is a clique of size $\deg_G(v)$. Let $G_1, G_2, \ldots, G_n$ be the components of $G - v$. Then, the components of $Inc(G) - C_v$ are of the form $H_i = Inc(G_i + v) - C_v$ for $i = 1, \ldots, n$, where $G_i + v$ is the subgraph of G induced by the vertices of G_i along with v. To see this, first note that each of these components are nonempty because there is at least one vertex in each G_i adjacent to v, call it u_i. Then, $(u_i, u_i v) \in V(H_i)$. Now, it suffices to show that H_j is disconnected from H_k when $j \neq k$. It is clear that they are vertex disjoint. Consider a shortest path from H_j to H_k in $Inc(G)$. The last vertex on the path in H_j must have the form (u, uv) for some $u \in V(G_j)$ since otherwise there would be some edge out of G_j not containing v. This contradicts that v is a cut vertex. Similarly, the first vertex on the path in H_k must have the form $(u', u'v)$ for some $u' \in V(G_k)$. The only other vertices remaining are those in C_v, so the only way for H_j and H_k to be connected is for (u, uv) and $(u', u'v)$ to be adjacent. Since $u \in V(G_j)$ and $u' \in V(G_k)$, it follows that $u \neq u'$ and also that $uv \neq u'v$. Further, $v \neq u$ and $v \neq u'$. So, $uu' \neq uv$ and $uu' \neq u'v$. Therefore, these two vertices in $Inc(G)$ are not adjacent. Hence, this path must pass through C_v. Therefore, C_v is a cut set of $Inc(G)$. □

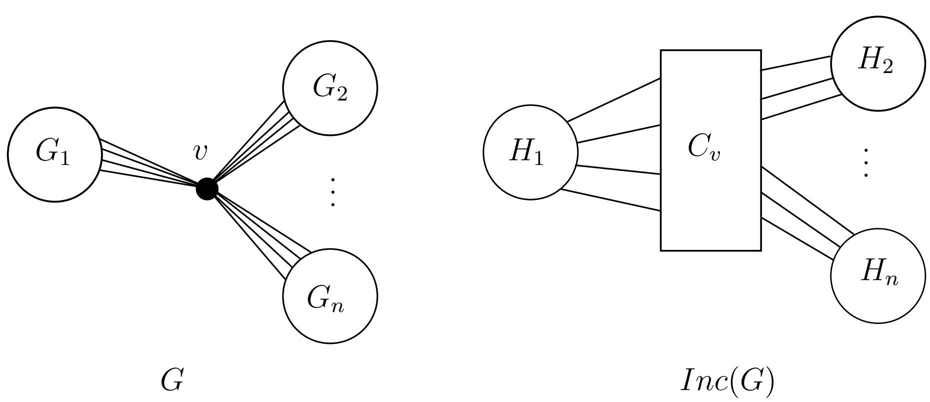


Figure 4.1: Illustration of the Induced Cut Sets

Theorem 4.2.4. *If G has circumference at most 3, then $Inc(G)$ is perfect.*

Proof. Note that if G is not connected, then

$$G = \coprod A_i$$

where A_i are all connected components of G. So,

$$Inc(G) = \coprod Inc(A_i).$$

Such a graph is perfect precisely when all components are perfect. Thus, we will assume G is connected moving forward.

By the Strong Perfect Graph Theorem (Theorem 1.4.2), it suffices to show that $Inc(G)$ contains no odd induced cycles (odd holes) of length at least 5 and that $Inc(G)$ contains no complements of odd induced cycles (odd antiholes) of length at least 5. Let's take care of a few base cases first. If G has a single vertex, then $G = K_1$ and since G has no edges, $Inc(G)$ is empty. Hence, it is vacuously perfect. If G has two vertices, then since G is connected, $G = K_2$. So, $Inc(G) = Inc(K_2) = K_2$, which is perfect. Finally, if G has three vertices, then we have two choices; as G is connected, either $G = P_3$ or $G = C_3$. Then, either $Inc(G) = Inc(P_3) = P_4^2$ or $Inc(G) = Inc(C_3) = C_6^2$, both of which are perfect. Note that the perfection of P_4^2 and C_6^2 results from the Perfect Graph Theorem (Theorem 1.4.1) since $\overline{P_4^2}$ and $\overline{C_6^2}$ are both bipartite and hence perfect.

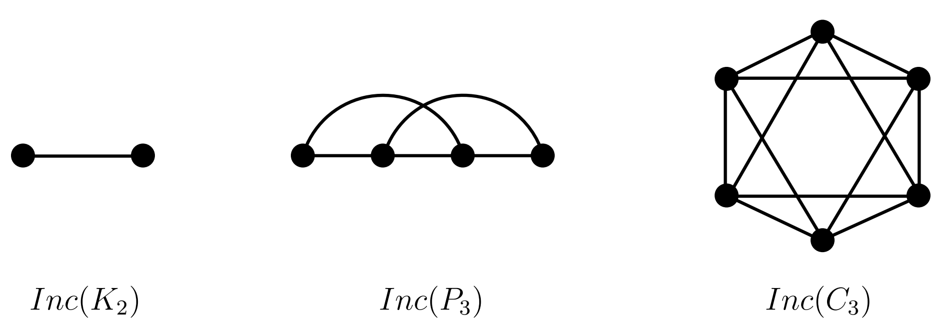


Figure 4.2: $Inc(G)$ for small G

Now, we may assume moving forward that G has at least 4 vertices. Let's argue that the rest are perfect by contradiction. Suppose there were a counterexample. Then, let G be a minimal counterexample; that is, let G be such that $|V(G)| + |E(G)|$ is minimal and $Inc(G)$ is not perfect. In particular, this implies that every proper subgraph H of G has the property that $Inc(H)$ is perfect. Since G has at least 4 vertices, and has circumference at most 3, we know that G must have a cut vertex by Lemma 4.2.2. For each cut vertex v of G, let C_v be the corresponding cut set of $Inc(G)$ and let $H_{v,i} = Inc(G_{v,i} + v) - C_v$ be the component of $Inc(G) - C_v$ corresponding to the component $G_{v,i}$ of $G - v$. As before, $G_{v,i} + v$ is the subgraph of G induced by $V(G_{v,i}) \cup \{v\}$. Since $Inc(G)$ is not perfect, it follows that there must exist an odd hole of length at least 5 or an odd antihole of length at least 5. We will show that neither of these situations can occur.

Lemma 4.2.5. *Odd holes and odd antiholes (of length at least* 5*) cannot lie completely in any* $H_{v,i}$ *or* C_v*.*

Proof. Observe that $H_{v,i}$ is an induced subgraph of $Inc(G_{v,i} + v)$. Since $G_{v,i} + v$ has fewer vertices than G, the minimality of G implies that $Inc(G_{v,i}+v)$ is perfect. Therefore, $H_{v,i}$ does not contain an odd hole or odd antihole of length at least 5 as a subgraph. Similarly, $C_v \cong K_{\deg_G(v)}$, which is perfect, since $\overline{C_v}$ is a set of isolated vertices and so is perfect. Thus, C_v does not contain any odd hole or odd antihole of length at least 5 as a subgraph. □

So, any odd hole or odd antihole must cross through C_v, but not be completely contained in C_v. Now, let's restrict where an odd hole or odd antihole can lie.

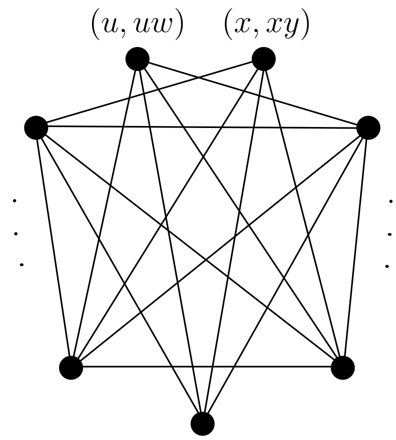
Lemma 4.2.6. *Any odd hole in* $Inc(G)$ *must be completely contained in some* $H_{v,j} \cup C_v$*; that is, an odd hole cannot have vertices in more than one component* *of* $Inc(G) - C_v$.

Proof. Observe first than any odd hole can contain at most two vertices of C_v. If one contained at least 3 vertices, then this would result in a chord since C_v is a clique. This is a contradiction. If two vertices of C_v are on the odd hole, then they must be consecutive on the cycle for the same reason. Now, observe that if the odd hole has vertices in more than one component $H_{v,i}$, then it would need to include a pair of nonadjacent vertices of C_v, since C_v is a cut set. This results in a chord, which is a contradiction. □

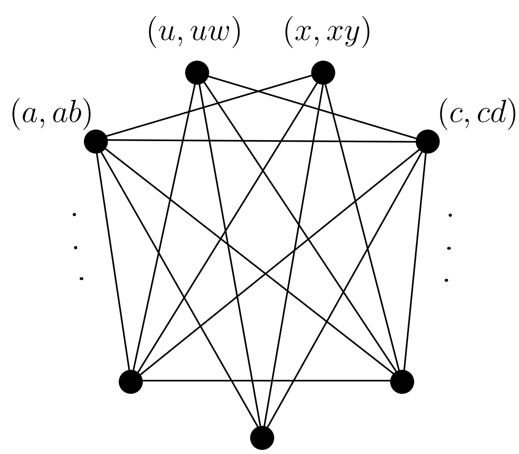
We have an analogous result for odd antiholes.

Lemma 4.2.7. *Any odd antihole in* $Inc(G)$ *must be completely contained in some* $H_{v,k} \cup C_v$*; that is, an odd antihole cannot have vertices in more than one component of* $Inc(G) - C_v$.

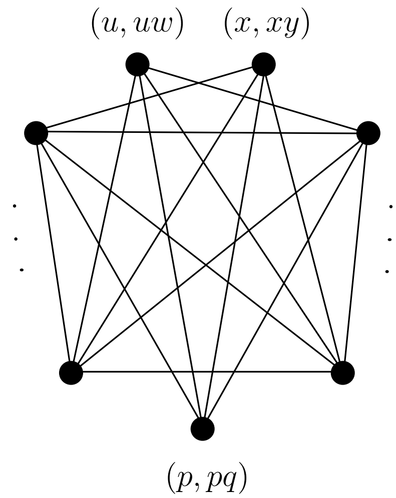
Proof. Suppose there was an odd antihole that contained vertices from two different components, say H_{v,i_1} and H_{v,i_2}. Let $(u, uw) \in V(H_{v,i_1})$ and $(x, xy) \in V(H_{v,i_2})$ be two vertices of the odd antihole. Then, by the proof of Lemma 4.2.3, (u, uw) and (x, xy) are not adjacent in $Inc(G)$. So, they must be consecutive vertices around the odd antihole. (Note that this argument shows that we cannot have vertices from three or more components of $Inc(G) - C_v$ in the odd antihole because these vertices would need to be pairwise nonadjacent. However, the size of a maximum independent set of an odd antihole is 2. So, this cannot happen.)



Observe that since $C_v \cong K_{\deg_G(v)}$, the vertices of C_v can appear on less than half the vertices of the odd antihole. Otherwise, the pigeonhole principle implies that two of them must be consecutive along the antihole and hence are nonadjacent, which is a contradiction. So, there is some vertex (p, pq), distinct from (u, uw) and (x, xy), on the odd antihole which is not in C_v, since there are at least 5 vertices on the odd antihole. Then, necessarily, $(p, pq) \in V(H_{v,i_1})$ or $(p, pq) \in V(H_{v,i_2})$. Note that we may always find this vertex (p, pq) such that (p, pq) is not consecutive along the odd antihole to (u, uw) and (x, xy). Let (a, ab) and (c, cd) be the vertices consecutive to (u, uw) and (x, xy), respectively, along the odd antihole.



First, suppose both (a, ab) and (c, cd) are not in C_v. Then, it must be the case that (a, ab) is in H_{v,i_2} as it must be adjacent to (x, xy) and not adjacent to (u, uw). Similarly, (c, cd) must be a vertex of H_{v,i_1}. However, (a, ab) and (c, cd) are adjacent in the antihole since they are not consecutive along the odd antihole. This is a contradiction since H_{v,i_1} and H_{v,i_2} are disconnected. So, at least one of (a, ab) or (c, cd) must be in C_v. Now, assume (without loss of generality) that (a, ab) is a vertex of C_v. Then, the next consecutive vertex cannot be in C_v, since then we would have two consecutive vertices of the odd antihole which are adjacent. Hence, we may choose this vertex as (p, pq).



As noted above, (p, pq) is either in H_{v,i_1} or it is in H_{v,i_2}. However, this vertex is suppose to be adjacent to both (u, uw) and (x, xy). If (p, pq) is in H_{v,i_1}, then it is not adjacent to (x, xy) which is a contradiction. Similarly, if (p, pq) is in H_{v,i_2}, then it is not adjacent to (u, uw) which is a contradiction. Therefore, we cannot have an odd antihole of length at least 5 containing vertices from more than one distinct component of $Inc(G) - C_v$. □

In light of Lemma 4.2.6 and Lemma 4.2.7, given any odd hole or odd antihole, for each cut vertex v of G, we can find precisely one component of $G - v$, $G_{v,k}$, such that the odd hole or odd antihole is completely contained in

$$H_{v,k} \cup C_v.$$

Now, observe that if there is some cut vertex v^* of G such that the odd hole or odd antihole contains vertices of G_{v^*,k^*} and H is the subgraph of G induced by

$$G_{v^*,k^*} \cup \{v^*\} \cup N_G(v^*),$$

then the odd hole or odd antihole is contained in $Inc(H)$. If $H \neq G$, then since H is a proper subgraph of G, $Inc(H)$ is perfect. So, this is a contradiction.

Moving forward, we may now assume that for every cut vertex v, the subgraph of G induced by

$$G_{v,k} \cup \{v\} \cup N_G(v)$$

is equal to G. Let's now reduce several times to the most basic case possible.

Lemma 4.2.8. *If $G_{v,i} \neq G_{v,k}$, then $G_{v,i} = K_1$ or $G_{v,i} = K_2$.*

Proof. We know that $G_{v,i}$ is nonempty and connected, as a component of $G - v$. Suppose $G_{v,i}$ has at least 3 vertices. Then, $G_{v,i}$ contains a path on 3 vertices. Since all of these vertices are necessarily neighbors of v by the comment above, this path on 3 vertices creates a subgraph of G isomorphic to $P_3 \vee K_1$. However, $P_3 \vee K_1$ contains a subgraph isomorphic to C_4. Hence, G has a subgraph isomorphic to C_4. This contradicts that G has circumference 3. Therefore, $G_{v,i}$ has at most 2 vertices. Recalling that $G_{v,i}$ is connected, the result follows. □

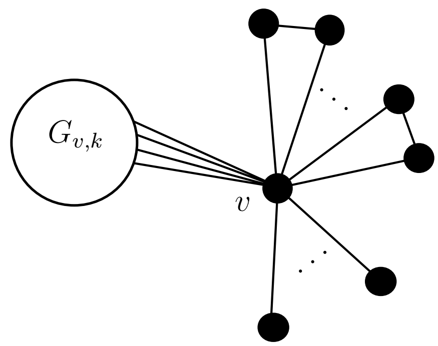


Figure 4.3: Reduction #1 for G

Lemma 4.2.9. *In fact, if $G_{v,i} \neq G_{v,k}$, then we may assume that $G_{v,i} = K_1$.*

Proof. Suppose $G_{v,i} \neq G_{v,k}$ and $G_{v,i} \cong K_2$. Let $V(G_{v,i}) = \{a, b\}$. Then, by Lemma 4.2.3, in particular, the vertices

$$(a, ab), (b, ab) \in V(H_{v,i})$$

are not used in the odd hole or odd antihole. These two vertices correspond to the edge $ab \in E(G_{v,i})$. Thus, the odd hole or odd antihole is contained in $Inc(G-ab)$. Since $G - ab$ is a proper subgraph of G, $Inc(G - ab)$ is perfect by the minimality of G. Hence, we have a contradiction. Therefore, all $G_{v,i} \neq G_{v,k}$ are such that $G_{v,i} \cong K_1$. □

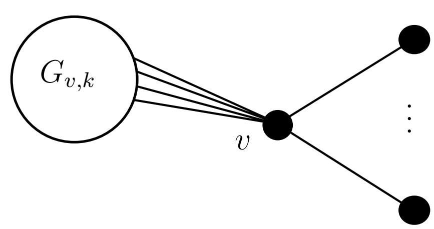


Figure 4.4: Reduction #2 for G

Now, if any vertex of C_v of the form (v, vu) where u is a vertex of $G_{v,i}$, a component different from $G_{v,k}$, is left unused in the odd hole/antihole, then since we also know that the corresponding vertex (u, vu) is also unused as it lies in $H_{v,i} \neq H_{v,k}$, it follows that the odd hole/antihole is contained in $Inc(G - vu)$, which is perfect by the minimality of G. Thus, we may assume that every vertex of the form (v, vu) where u lies outside of $G_{v,k}$ is used in the odd hole/antihole.

We need another lemma before continuing. This lemma will allow us to further reduce the cases we need to consider.

Lemma 4.2.10. $|N_{H_{v,k}}(v, vu')| = 1$ *or* 2*, where* u' *is a neighbor of* v *outside of* $V(G_{v,k})$*.*

Proof. Observe that

$$N_{H_{v,k}}(v, vu') = \{(u, uv) \mid u \in V(G_{v,k})\}.$$

(All other vertices of the form (v, e) are in C_v, and so are not in $H_{v,k}$. The edge vu' is not an edge in $G_{v,k}+v$. Since u' is not in $G_{v,k}$, $vu \neq vu'$ for any $u \in V(G_{v,k})$. So, the only remaining way for a vertex of $H_{v,k}$ to be adjacent to (v, vu') is to satisfy the condition that the edge has one endpoint at v. This is precisely what is captured above.) Now, suppose this set has at least three vertices in it. Observe that these vertices are pairwise nonadjacent. To see this, if they are distinct, they must have different vertex components by the definition of $N_{H_{v,k}}(v, vu')$ above. Hence, they must also have different edge components (since the vertices u will be different). And, since v is a common endpoint to all of the edges being represented in the incidences and v does not appear as the vertex component in this set, no two vertices of this neighborhood can be adjacent. However, these vertices are in the same component of $Inc(G) - C_v$. So, the vertex components of these vertices lie in the same component $G_{v,k}$ of $G - v$. Thus, there are paths between them in $G_{v,k}$. If there is some collection of 3 vertices taken from the vertex components which form a path in $G_{v,k}$, then taking these three vertices along with v, we can find a subgraph of G which is isomorphic to C_4, which contradicts the circumference of G being at most 3. On the other hand, if no such collection of three vertices exists, there is at least one pair of vertices that require a path of length at least 2 to connect them in $G_{v,k}$. Taking that path and the edges from the vertices to v, we can form a subgraph of G isomorphic to a cycle of length at least 4. Again, this contradicts the circumference of G being at most 3. Therefore, this neighborhood can have at most 2 vertices. Through this argument, note that we have also shown that if there are two such vertices in $N_{H_{v,k}}(v, vu')$, then their vertex components in $G_{v,k}$ must be adjacent. Finally, since $H_{v,k}$ is nonempty, there is at least one vertex in this neighborhood, and the result follows. □

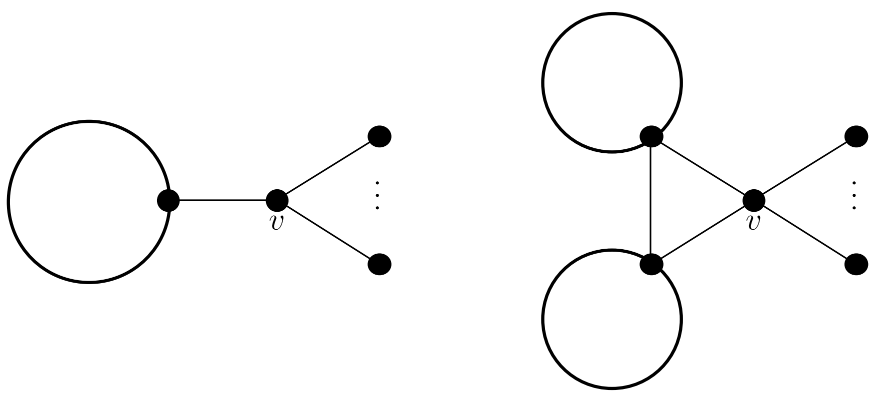
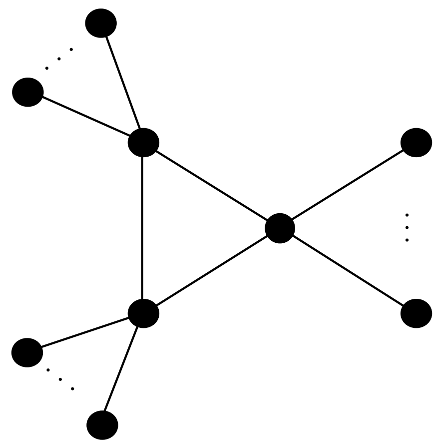
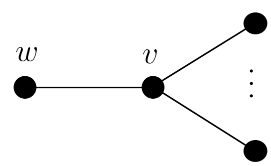


Figure 4.5: Cases: Neighborhood has size 1 or 2

Lemma 4.2.11. *We need only consider the case where G is the graph obtained from a triangle with a collection of vertices hanging from each vertex of the triangle. Namely,*



Proof. Given the cut vertex v, we know by Lemma 4.2.10 that $|N_{H_{v,k}}(v, vu')| = 1$ or 2, where u' is a neighbor of v outside of $G_{v,k}$. If $|N_{H_{v,k}}(v, vu')| = 1$, then by the proof of Lemma 4.2.10, v has precisely one neighbor, say w in $G_{v,k}$. If the $\deg_G(w) = 1$, then G is a star.



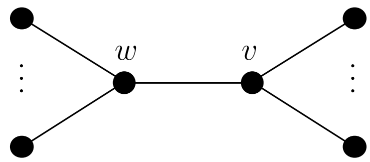
Notice that the star is an induced subgraph of the graph described in the statement of the lemma.

If $\deg_G(w) > 1$, then w is a cut vertex in G. By our previous assumption, we thus know that the subgraph H of G induced by

$$G_{w,\ell} \cup \{w\} \cup N_G(w)$$

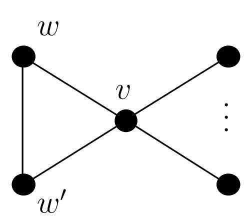
where $G_{w,\ell}$ is the component of $G-w$ that induces the component $H_{w,\ell}$ of $Inc(G)-C_w$ which intersects the odd hole or antihole which we have assumed exists, must be equal to all of G. Since we know that the odd hole or antihole contains vertices of the form (v, vu) where u lies outside of $G_{v,k}$, we can assume, by appealing to

Lemma 4.2.6 and Lemma 4.2.7, along with Lemma 4.2.9, that all components of $G - w$ other than $G_{w,\ell}$ is isomorphic to K_1. Therefore, G is isomorphic to the graph obtained from an edge by hanging a collection of vertices from each endpoint of the edge.



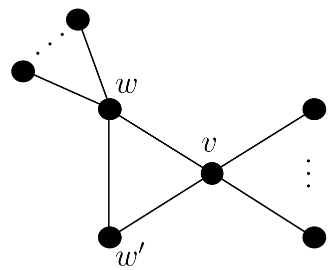
Again, note that this graph is an induced subgraph of the graph described in the statement of the lemma.

Now, suppose $|N_{H_{v,k}}(v, vu')| = 2$. Recall that in the proof of Lemma 4.2.10, we showed that if there were two vertices in this neighborhood, then the vertex components in G must be adjacent. So, let w, w' be the neighbors of v in $G_{v,k}$. Necessarily, $ww' \in E(G)$. If $\deg_G(w) = \deg_G(w') = 2$, then G is a triangle formed by w, w', v, with a collection of vertices neighboring v.



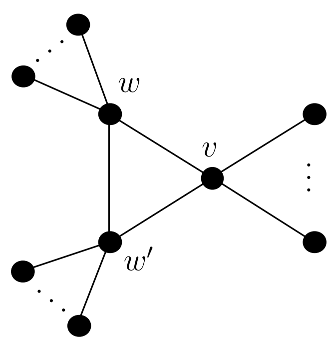
This is an induced subgraph of the graph described in the statement of the lemma.

If $\deg_G(w) > 2$ and $\deg_G(w') = 2$ (without loss of generality), then w is a cut vertex of G and we may apply the same argument as above to show that G must be the graph obtained from a triangle by adjoining vertices to v and w.



Again, this is an induced subgraph of the graph described in the statement of the lemma.

Finally, if $\deg_G(w) > 2$ and $\deg_G(w') > 2$, then both w and w' are cut vertices of G and by the argument above, we may assume that G is obtained from a triangle created by the vertices w, w', v by joining a collection of vertices from each of the three vertices.



This is precisely the graph described in the statement of the lemma. Therefore, it suffices, moving forward, to assume that G is such a graph, since by Lemma 4.1.1 and the definition of perfect, if $Inc(G)$ is perfect, then $Inc(H)$ is perfect where H is a subgraph of G. □

Our goal now is to show that $Inc(G)$ is perfect. We will do this by showing that it does not contain an odd hole or an odd antihole of length at least 5. Let's consider the odd holes of length at least 5 first. Let the vertices of the triangle of G be v, w and w'. Let $\{v_i\}, \{w_i\}$ and $\{w'_i\}$ denote the pairwise nonadjacent vertices adjacent to v, w and w' respectively.

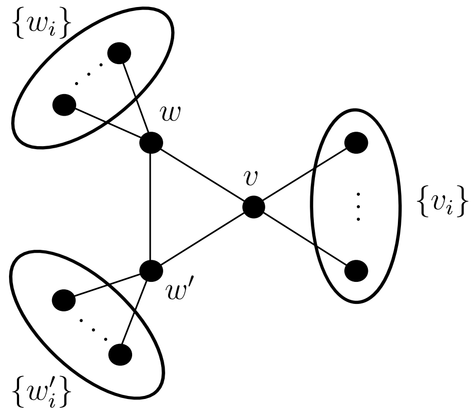
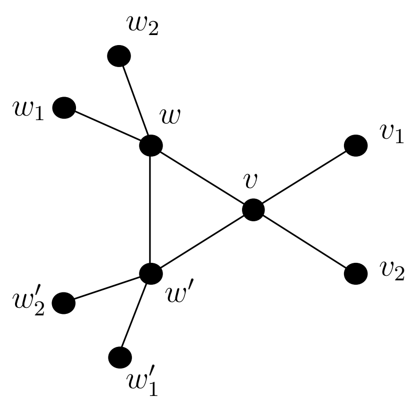


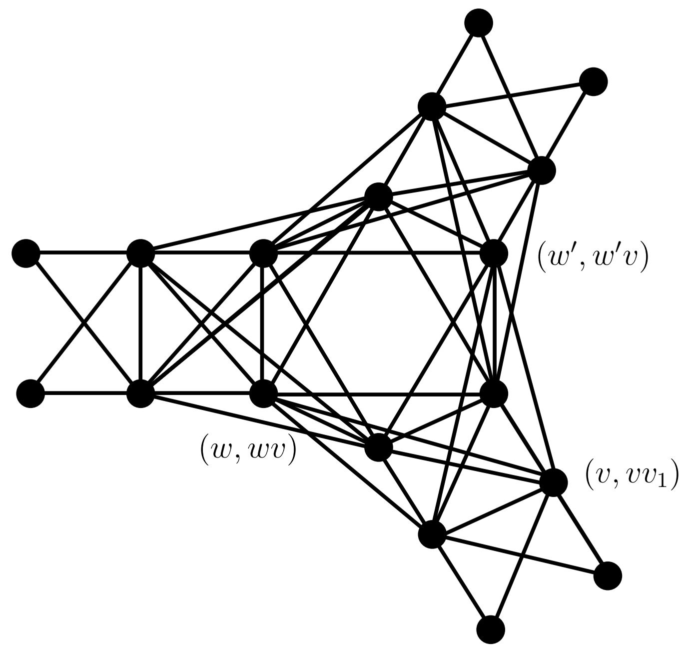
Figure 4.6: Reduction #3 of G

Lemma 4.2.12. *$Inc(G)$ has no odd hole of length at least* 5.

Proof. Observe that by Lemma 4.2.6, we cannot use any vertex of $Inc(G)$ of the form (v_i, vv_i), (w_i, ww_i) or $(w'_i, w'w'_i)$ as these do no lie in the correct component. Observe that since C_v, C_w and $C_{w'}$ are complete subgraphs of $Inc(G)$ by Lemma 4.2.3, we cannot contain more than two vertices from each in any odd hole contained in $Inc(G)$. Therefore, we may reduce to the case where there are exactly two vertices joined to each vertex of the triangle.



This graph is small enough that we can draw out $Inc(G)$ explicitly.



If no vertex of the form (v, vv_i), (w, ww_i) or $(w', w'w'_i)$ is in the odd hole, then the odd hole is contained in $Inc(C_3) \cong C_6^2$, which is perfect. Thus, this is a contradiction. So, by the symmetry of G, assume that (v, vv_1) is a vertex on the odd hole. Then, the available neighbors which can appear on an odd hole consecutive to (v, vv_1) are

$$(v, vv_2), (v, vw), (v, vw'), (w, wv), (w', w'v).$$

Note that in order to extend the odd hole, we must find two vertices which are not adjacent, but which are both adjacent to (v, vv_1). The only possible choice is (w, wv) and $(w', w'v)$ and we can no longer choose any other neighbor of (v, vv_1). Since we are looking for an *odd* hole, we need to find a pair of (distinct) vertices such that one is adjacent to (w, wv) and the other is adjacent to $(w', w'v)$ and such that neither of the vertices are adjacent to (v, vv_1). Observe that any common neighbor of (w, wv) and $(w', w'v)$ cannot be chosen as a vertex on the odd hole since this would create an induced C_4 inside the odd hole which is a contradiction.

So, after removing all common neighbors of (w, wv) and $(w', w'v)$, which consist of

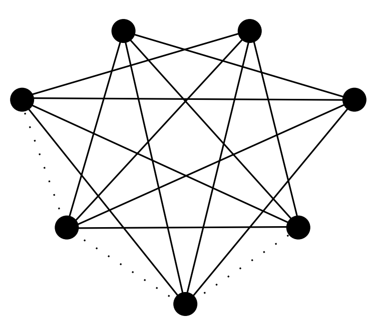
$$(w, ww'), (w', ww'), (v, vw), (v, vw'), (v, vv_2)$$

we see that we have disconnected the graph. Therefore, we cannot complete the rest of the vertices on the odd hole, and so there is no odd hole of length at least 5 in $Inc(G)$. □

Now, let's focus on excluding the odd antiholes of length at least 5.

Lemma 4.2.13. *$Inc(G)$ has no odd antihole of length at least* 5.

Proof. Recall that all the vertices of the form (v, vu) such that u is not in $G_{v,k}$ must be in the odd antihole. But, this set of vertices is a clique and all are adjacent to the vertices (w, wv) and $(w', w'v)$. This is the closed neighborhood in $H_{v,k} \cup C_v$ of any (v, vu) such that u is not in $G_{v,k}$. Recall that since we know that the odd antihole must go through $H_{v,k}$ and not intersect any other $H_{v,i}$, these are the only vertices we need to consider. Fix some (v, vv_i). Since we know there are no odd holes by Lemma 4.2.12 and hence no induced subgraph isomorphic to $\overline{C_5} \cong C_5$, we need to show that there is no odd antihole of length at least 7. Thus, adjacent to (v, vv_i), we need to use at least 4 neighbors of (v, vv_i). Note that given any four consecutive vertices around an odd antihole, we see that there are 3 nonedges seen consecutively.



However, within the neighborhood of (v, vv_i), the only pair of nonadjacent vertices we have is (w, wv) and $(w', w'v)$. Thus, we cannot find four vertices such that there are 3 nonadjacent pairs. So, we cannot complete an odd antihole. Therefore, $Inc(G)$ has no odd antihole of length at least 5. □

Therefore, by Lemma 4.2.12 and Lemma 4.2.13, $Inc(G)$ cannot contain an odd hole or an odd antihole of length at least 5. Hence, $Inc(G)$ is perfect by the Strong Perfect Graph Theorem. This contradicts that G is a minimal counterexample. Thus, no counterexample exists. So, if G has circumference at most 3, then $Inc(G)$ is perfect, as desired.

□

4.3 Another Computation

The goal of this section is to compute the (fractional) incidence chromatic number of $Inc(G)$ when G has circumference at most 3; that is, when G is perfect.

Lemma 4.3.1. *If G is perfect, then $\chi_f(G) = \chi(G) = \omega(G)$.*

Proof. As described in Chapter 1, we know that $\chi_f(G) \leq \chi(G)$. The reverse inequality is not always true as we have seen in some examples. When G is perfect, by definition we have that $\chi(G) = \omega(G)$. By linear duality, we have that $\chi_f(G) = \omega_f(G)$. And finally, since the (linear) program associated to the (fractional) clique number is a maximization problem, we see that $\omega_f(G) \geq \omega(G)$. Thus, putting these together, we have the following.

$$\chi_f(G) = \omega_f(G) \geq \omega(G) = \chi(G) \geq \chi_f(G).$$

Therefore, $\chi_f(G) = \chi(G) = \omega(G)$. □

Proposition 4.3.2. *If G has circumference at most 3, then*

$$\chi_f(Inc(G)) = \Delta(G) + 1.$$

Proof. We already know from Brualdi and Massey's work ([6]) that

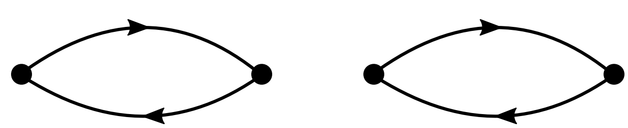
$$\omega(Inc(G)) \geq \Delta(G) + 1.$$

Suppose that there is a clique with size bigger than $\Delta(G)+1$. Note that the clique of size $\Delta(G)+1$ defined by Brualdi and Massey contains the vertices

$$\{(u, uv_i)\}_{i=1}^{\Delta(G)} \cup \{(v_k, uv_k)\},$$

where $\deg_G(u) = \Delta(G), \{v_i\}_{i=1}^{\Delta(G)}$ are all the neighbors of u in G and $v_k \in \{v_i\}_{i=1}^{\Delta(G)}$ is some particular vertex. Then, in any bigger clique we must have a different combination of incidences. So, one of the following cases must hold.

- There are two distinct pairs of edges, each with the same edge component.

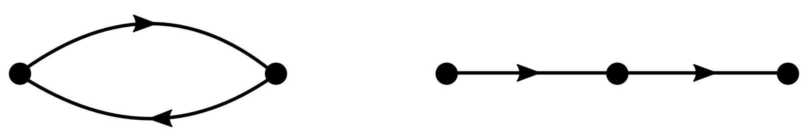


  Either these pairs of edges share an endpoint or they don't. If they don't, the only adjacent edges are within the pairs representing the same edge. So, choosing an arc from each pair gives nonadjacent vertices in $Inc(G)$. If they do share an endpoint, then the two edges directed towards the common endpoint are not adjacent. Therefore, these cannot be in a clique.

- There are two pairs of edges which form directed paths on 3 vertices.

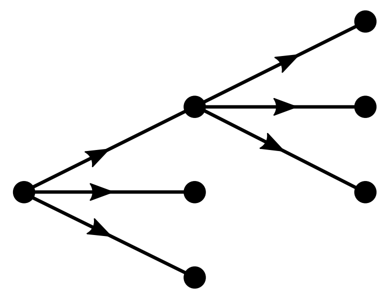
Either these two edges have an edge overlap or they don't. If they overlap completely so that the two directed paths have the same vertex set but are (necessarily) oriented in opposite directions, then we can resort to the previous case. If they overlap on one edge, then we can find two edges which do not share any endpoints, in which case those edges are not adjacent. Finally, if they don't overlap on any edge, then we can find edges which do not share endpoints and again these are not adjacent. Therefore, we cannot have a clique in this case.

- There are two pairs of edges, one which represents the same edge oriented in either direction and one which is a directed path on 3 vertices.



  If the edges don't overlap at all, then we can find a pair of edges which do not share an endpoint and so cannot be adjacent. If the edges do overlap, depending on the orientations, either there are two edges oriented towards the same vertex and so these cannot be adjacent, or we have the case described above in the allowed case. In this second case, we haven't exhibited a bigger clique.

- Finally, there are two vertices of G which have more than one edge oriented away from them.



There cannot be any overlap of the edges, since this would mean that the two vertices were the same. Thus, we must be able to find a pair of edges which do not share endpoints. Therefore, we do not have a clique.

Therefore, we cannot have a clique of size larger than $\Delta(G)+1$, and so

$$\omega(Inc(G)) = \Delta(G) + 1.$$

So, by the perfectness of $Inc(G)$ given by Theorem 4.2.4, we see that

$$\chi_f(Inc(G)) = \chi(Inc(G)) = \omega(Inc(G)) = \Delta(G) + 1.$$

□

Chapter 5

Coloring Via Homomorphisms and Other Future Work

In this final chapter, we will discuss a few questions which are posed as interesting future work. There are, of course, questions regarding the fractional incidence chromatic number of classes of graphs which were studied, but for which no precise calculation was obtained. There are also questions which relate to the perfectness of incidence graphs and the properties that these graphs have. Finally, we pose questions based on where interest in the fractional chromatic number started. Namely, we are interested in coloring problems at large and ultimately want to know if we are able to define a new coloring problem using a class of graphs along with graph homomorphisms. This is where we start.

5.1 New Target Graphs for Coloring

Recall that we started here with talking about the chromatic number and then moved to the fractional chromatic number. Much of what we have said about the fractional chromatic number has come from this idea of using the Kneser graphs as the target graphs of homomorphisms instead of the complete graphs. This raises a natural question. Are there other ways in which we can define classes of graphs to serve as targets of graph homomorphisms for a coloring? There is at least one other right now, namely, circular coloring and using the circulant graphs.

Definition. An *r-circular coloring* of G assigns to each vertex of G a point on a circle of circumference r so that adjacent vertices receive points separated by distance at least 1 in each direction around the circle. The *circular chromatic number*, $\chi_c(G)$, of G is the smallest r such that G has an r-circular coloring.

Observation. We can view a circle of circumference r as the interval $[0, r)$. Then, an r-circular coloring is a function $c : V(G) \to [0, r)$ such that if $uv \in E(G)$, then $1 \leq |c(u) - c(v)| \leq r - 1$.

The following is an alternative definition of the circular chromatic number, given by Bondy and Hell [5].

Definition. Let G_d^k be the graph whose vertex set is $\mathbb{Z}_k = \{0, 1, \dots k - 1\}$ and whose edge set is $\{ij : d \leq |i-j| \leq k-d\}$. Then, a (k, d)-*coloring* of a graph G is equivalent to a homomorphism of G into G_d^k. The *circular chromatic number* of G is then the smallest ratio k/d such that there exists a homomorphism $G \to G_d^k$.

Remark. In the definition above, k is the number of colors used and d is the notion of distance.

Observation. Note that a $(k, 1)$-coloring of G is just a proper k-coloring of G, since $G_1^k = K_k$. Thus, the circular chromatic number is a generalization of the chromatic number.

The following is a well-known result of Vince [34].

Theorem 5.1.1. *For any finite graph G,*

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

There are a couple remarks to make about this result. First, two proofs are given in the literature; one given by Vince ([34]) and another by Bondy and Hell ([5]). The proof in [5] uses graph homomorphisms. Second, note that the circular chromatic number does contrast with the fractional chromatic number in that it must lie quite close to $\chi(G)$, whereas $\chi_f(G)$ can be arbitrarily far away from $\chi(G)$.

So, now we know that the circulant graphs G_d^k is a class of graphs which is used for the targets of coloring homomorphisms. Can we define another? Here is an attempt.

Motivation. Consider the class of graphs,

$$\mathscr{C} = \{C_{2k+1}[K_\ell] \mid k, \ell \in \mathbb{N}\}.$$

We want to try and find a way to associate some element of an ordered set X to each graph G using this class of graphs. This would give rise to a different type of coloring than proper coloring, fractional coloring and circular coloring. Given this association, we could define a new version of the chromatic number.

Unfortunately, there are some difficulties when considering this class of graphs. Recall that

$$\chi(K_n) = \chi_f(K_n) = \chi_c(K_n) = n.$$

We would want this version of the chromatic number evaluated on K_n to be n, or some other "maximum" type element in X. In the classical, fractional and circular chromatic numbers, we can recognize K_n as one of the target graphs for the associated class of graphs under the graph homomorphisms. That is, K_n is clearly a complete graph, it can be recongized as $K(n, 1)$ for the fractional coloring and it can be recognized as G_1^n for circular coloring. Unfortunately, the only complete graphs that we can find in $\mathscr{C}$ is $K_{3\ell} = C_3[K_\ell]$. Although, certainly, given any complete graph K_n, there is an inclusion homomorphism into every $C_{2k+1}[K_n]$.

In the same light, we would want this version of the chromatic number evaluated on an independent set to be 1, or some "minimum" type of element in X.

Now, consider the property of being vertex transitive.

Proposition 5.1.2. *If G and H are vertex transitive graphs, then $G[H]$ is vertex transitive.*

Proof. Suppose G and H are vertex transitive graphs. Let $(g, h), (g', h') \in V(G[H])$. Then, let φ be an automorphism of G such that $\varphi(g) = g'$ and let ψ be an automorphism of H such that $\psi(h) = h'$. (Note: These automorphisms exist by the definition of vertex transitivity on G and H.) Define $f : G[H] \to G[H]$ by

$$f(u, v) = (\varphi(u), \psi(v)).$$

We want to show this is an automorphism of $G[H]$ which sends (g, h) to (g', h'). First, observe that

$$f(g, h) = (\varphi(g), \psi(h)) = (g', h'),$$

as desired. Now, suppose $(g_1, h_1)(g_2, h_2) \in E(G[H])$. We want to show that

$$f(g_1, h_1)f(g_2, h_2) = (\varphi(g_1), \psi(h_1))(\varphi(g_2), \psi(h_2))$$

is an edge in $G[H]$. We have two cases. First, suppose that $g_1g_2 \in E(G)$. Then, $\varphi(g_1)\varphi(g_2) \in E(G)$ since φ is an automorphism on G. Thus,

$$f(g_1, h_1)f(g_2, h_2) \in E(G[H]),$$

as desired. Now, suppose that $g_1 = g_2$ and $h_1h_2 \in E(H)$. Then, $\varphi(g_1) = \varphi(g_2)$ and $\psi(h_1)\psi(h_2) \in E(H)$. Thus,

$$f(g_1, h_1)f(g_2, h_2) \in E(G[H])$$

again, as desired. The same proof above with φ and ψ replaced by φ^{-1} and ψ^{-1} will conclude the proof that f is an automorphism of $G[H]$. Therefore, $G[H]$ is vertex transitive. □

Observation. Since C_{2k+1} and K_ℓ are vertex transitive,

$$C_{2k+1}[K_\ell]$$

is vertex transitive by Proposition 5.1.2. We want $\mathscr{C}$ to have this property because we want the ability to permute the colors and have what we would define as the chromatic number with respect to this coloring to be the preserved. Note that the class of complete graphs, Kneser graphs and circulant graphs are all vertex transitive, so this property holds in those cases.

We have observed several characteristics that a target class of graphs for a coloring homomorphism must have. It would be interesting to know which other properties are required. Further, are there any properties that these target graphs possess which are not necessary?

5.2 Classification of Homomorphisms in $\mathscr{C}$

Recall the following result.

Lemma 5.2.1. *Let G and H be graphs. If $G \to H$, then $\chi(G) \leq \chi(H)$.*

Proof. Let G and H be graphs. Suppose $f : G \to H$ is a homomorphism. Additionally, let $c : H \to K_{\chi(H)}$ be a (coloring) homomorphism. Then,

$$c \circ f : G \to K_{\chi(H)}$$

is a homomorphism. Hence,

$$\chi(G) \leq \chi(H),$$

as desired. □

Therefore, there is a relationship between the existence of homomorphisms and the chromatic numbers of graphs. Note that in the proof of this result, it didn't really matter that we were considering the classical chromatic number. Replacing $K_{\chi(H)}$ by $K(r, s)$ or G_d^k where $\chi_f(H) = \frac{r}{s}$ or $\chi_c(H) = \frac{k}{d}$, the analogous results for the fractional chromatic number and the circular chromatic number would follow. Hence, it would be useful for us to know when there exists a homomorphism between two graphs from $\mathscr{C}$. That is, we would like to know what conditions on k, k', ℓ and ℓ' we must impose in order to ensure that

$$C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell'}].$$

This will give us an indication of what type of ordering we want on the set X, which will define the new coloring if it exists. We will use the chromatic number of these graphs, along with some other integer invariants of the graphs, to construct the desired conditions on k, k', ℓ and ℓ'.

Proposition 5.2.2 ([17])**.** $\chi(C_{2k+1}[K_\ell]) = 2\ell + \lceil \frac{\ell}{k} \rceil$.

First, we have the following result classifying when there is a homomorphism $K_\ell \to K_{\ell'}$.

Lemma 5.2.3. *$K_\ell \to K_{\ell'}$ if and only if $\ell \leq \ell'$.*

Proof. If $K_\ell \to K_{\ell'}$, then Lemma 5.2.1 implies that

$$\ell = \chi(K_\ell) \leq \chi(K_{\ell'}) = \ell'.$$

Conversely, if $\ell \leq \ell'$, then

$$\chi(K_\ell) = \ell \leq \ell' = \omega(K_{\ell'}).$$

So, by Lemma 5.2.6, $K_\ell \to K_{\ell'}$. □

Let's now consider the following two technical lemmas.

Lemma 5.2.4. *If $k < k'$, then under a homomorphism*

$$g : C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell'}],$$

there are two consecutive $K_{\ell'}$-layers, that is, two subgraphs isomorphic to $K_{\ell'}$ which correspond two consecutive vertices of $C_{2k'+1}$, which are not in the image of g.

Proof. Let $k < k'$. Suppose

$$V(C_{2k+1}) = \{0, 1, \ldots, 2k\}, V(C_{2k'+1}) = \{0, 1, \ldots, 2k'\}$$

$$V(K_\ell) = \{v_1, \ldots, v_\ell\}, V(K_{\ell'}) = \{v'_1, \ldots, v'_{\ell'}\}.$$

By applying an automorphism (which exists since $C_{2k'+1}[K_{\ell'}]$ is vertex-transitive), we may assume that $g(0, v_1) = (0, v'_1)$. Suppose there exist

$$(i, v_j), (i', v_{j'}) \in V(C_{2k+1}[K_\ell])$$

such that

$$g(i, v_j) = (k', v'_m) \qquad \text{and} \qquad g(i', v_{j'}) = (k' + 1, v'_{m'}).$$

Then,

$$\begin{aligned} k &\geq \operatorname{dist}((0,v_1),(i,v_j)) \geq \operatorname{dist}(g(0,v_1),g(i,v_j)) \\ &= \operatorname{dist}((0,v_1'),(k',v_m')) = k'. \end{aligned}$$

So, we see that $k \geq k'$. However, we assumed that $k < k'$, thus we have a contradiction. So, the $K_{\ell'}$-layer associated to the vertex k' in $C_{2k'+1}$ is not in the image of $C_{2k+1}[K_\ell]$ under g. Similarly, we have that

$$\begin{aligned} k &\geq \operatorname{dist}((0,v_1),(i',v_{j'})) \geq \operatorname{dist}(g(0,v_1),g(i',v_{j'})) \\ &= \operatorname{dist}((0,v_1'),(k'+1,v_{m'}')) = k'. \end{aligned}$$

So, again, we see that $k \geq k'$. Thus we have a contradiction. Hence, the $K_{\ell'}$-layer associated to the vertex $k'+1$ in $C_{2k'+1}$ is not in the image of $C_{2k+1}[K_\ell]$ under g. Finally, observe that the $K_{\ell'}$-layers associated to the vertices k' and $k'+1$ are consecutive in $C_{2k'+1}[K_{\ell'}]$ since k' and $k'+1$ are adjacent in $C_{2k'+1}$. Thus, the result follows. □

Lemma 5.2.5. *Let $\ell, k \in \mathbb{N}$. Then,*

$$\left\lceil \frac{\ell}{k} \right\rceil \leq 2\left\lceil \frac{\ell}{2k} \right\rceil \leq \left\lceil \frac{\ell}{k} \right\rceil + 1.$$

Proof. Let $\ell, k \in \mathbb{N}$. Then there exists some integer n such that

$$n < \frac{\ell}{2k} \leq n+1.$$

Thus,

$$\left\lceil \frac{\ell}{2k} \right\rceil = n+1$$

and hence

$$2\left\lceil \frac{\ell}{2k} \right\rceil = 2(n+1) = 2n+2.$$

If

$$n < \frac{\ell}{2k} \le n + \frac{1}{2},$$

then

$$2n < \frac{\ell}{k} \le 2n + 1.$$

So,

$$\left\lceil \frac{\ell}{k} \right\rceil = 2n + 1.$$

Hence,

$$\left\lceil \frac{\ell}{k} \right\rceil \le 2 \left\lceil \frac{\ell}{2k} \right\rceil = \left\lceil \frac{\ell}{k} \right\rceil + 1.$$

If

$$n + \frac{1}{2} < \frac{\ell}{2k} \le n + 1,$$

then

$$2n + 1 < \frac{\ell}{k} \le 2n + 2.$$

So,

$$\left\lceil \frac{\ell}{k} \right\rceil = 2n + 2.$$

Hence,

$$\left\lceil \frac{\ell}{k} \right\rceil = 2 \left\lceil \frac{\ell}{2k} \right\rceil \le \left\lceil \frac{\ell}{k} \right\rceil + 1.$$

Thus, the result follows. □

We now need to introduce a few basic lemmas which will used heavily in the classification of when homomorphisms within $\mathscr{C}$ exist.

Lemma 5.2.6. *Let G and H be graphs. If $\chi(G) \le \omega(H)$, then $G \to H$.*

Proof. Let G and H be graphs. Suppose $\chi(G) \le \omega(H)$. By definition of the chromatic number, we have a homomorphism $c : G \to K_{\chi(G)}$. We can also view $K_{\chi(G)}$ as a subgraph of $K_{\omega(H)}$ since $\chi(G) \le \omega(H)$. Thus, we can extend c (via an inclusion homomorphism) to find a homomorphism $c^* : G \to K_{\omega(H)}$. Now, since $\omega(H)$ is the size of the largest clique in H, we can view $K_{\omega(H)}$ as a subgraph of H. So, inclusion gives us a homomorphism $f : K_{\omega(H)} \to H$. So, $f \circ c^* : G \to H$ is a homomorphism. Thus, $G \to H$, as desired. □

Lemma 5.2.7. *Let G and H be (simple) graphs. If $G \to H$, then $\omega(G) \leq \omega(H)$.*

Proof. Let G and H be graphs. Suppose $f : G \to H$ is a homomorphism. Then, observe that under a homomorphism, the images of the vertices of a clique in G must be distinct. If not, say $u, v \in V(G)$ are in the same clique such that $f(u) = f(v)$ but $u \neq v$. Then, since $uv \in E(G)$ (as the vertices are in a clique), it follows that $f(u)f(v) \in E(H)$. However, H has no loops, and so this is a contradiction. Since all pairs of vertices in the largest clique of G are adjacent, it follows by definition of a homomorphism that all pairs of vertices in the image of the largest clique of G are adjacent. Hence, we have found a clique in H with at least $\omega(G)$ vertices. Hence, $\omega(G) \leq \omega(H)$. □

Lemma 5.2.8. *Suppose $G_1 \to G_2$ and $H_1 \to H_2$. Then, $G_1[H_1] \to G_2[H_2]$.*

Proof. Let $g : G_1 \to G_2$ and $h : H_1 \to H_2$. Then, define $f : G_1[H_1] \to G_2[H_2]$ by

$$f(u, v) = (g(u), h(v)).$$

We want to show that f is a homomorphism. Suppose $(u_1, v_1)(u_2, v_2) \in E(G_1[H_1])$. Then, we have two cases. First, suppose that $u_1u_2 \in E(G_1)$. Then, since g is a homomorphism, $g(u_1)g(u_2) \in E(G_2)$. So, $(g(u_1), h(v_1))(g(u_2), h(v_2)) \in E(G_2[H_2])$. Thus, it follows that $f(u_1, v_1)f(u_2, v_2) \in E(G_2[H_2])$. Next, suppose that $u_1 = u_2$ and $v_1v_2 \in E(H_1)$. Then, $g(u_1) = g(u_2)$ and since h is a homomorphism, $h(v_1)h(v_2) \in E(H_2)$. So, $(g(u_1), h(v_1))(g(u_2), h(v_2)) \in E(G_2[H_2])$. Thus, it follows that $f(u_1, v_1)f(u_2, v_2) \in E(G_2[H_2])$. Therefore, f is a homomorphism, as desired. □

Theorem 5.2.9. *Let* $k, k', \ell, \ell' \in \mathbb{N}$*. Then,*

1. *If* $\ell > \ell'$, $C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell'}]$ *if and only if* $\ell' > \frac{1}{3}\left(2\ell + \left\lceil \frac{\ell}{k} \right\rceil\right)$.
2. *If* $\ell \leq \ell'$, $C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell'}]$ *if and only if one of the following holds*
   (a) $k \geq k'$*; or*
   (b) $k < k'$ *and* $\ell' \geq \ell + \frac{1}{2}\left\lceil \frac{\ell}{k} \right\rceil$

Proof. Let $k, k', \ell, \ell' \in \mathbb{N}$.

1. Suppose $\ell > \ell'$. We have four cases.
   (a) Suppose $k > 1$ and $k' > 1$. Then,
   $$\omega(C_{2k+1}[K_\ell]) = 2\ell > 2\ell' = \omega(C_{2k'+1}[K_{\ell'}]).$$
   So, by Lemma 5.2.6, $C_{2k+1}[K_\ell] \not\to C_{2k'+1}[K_{\ell'}]$.
   (b) Suppose $k = 1$ and $k' = 1$. Then,
   $$\omega(C_{2k+1}[K_\ell]) = 3\ell > 3\ell' = \omega(C_{2k'+1}[K_{\ell'}]).$$
   So, by Lemma 5.2.6, $C_{2k+1}[K_\ell] = K_{3\ell} \not\to K_{3\ell'} = C_{2k'+1}[K_{\ell'}]$.
   (c) Suppose $k = 1$ and $k' > 1$. Then,
   $$\omega(C_{2k+1}[K_\ell]) = 3\ell > 3\ell' > 2\ell' = \omega(C_{2k'+1}[K_{\ell'}]).$$
   So, by Lemma 5.2.6, $C_{2k+1}[K_\ell] = K_{3\ell} \not\to C_{2k'+1}[K_{\ell'}]$.
   (d) Suppose $k > 1$ and $k' = 1$. Then, $C_{2k'+1}[K_{\ell'}] = C_3[K_{\ell'} = K_{3\ell'}$. Thus, by definition of $\chi(C_{2k+1}[K_\ell])$, implies that
   $$C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell'}] = K_{3\ell'}$$

if and only if $\chi(C_{2k+1}[K_\ell]) \leq 3\ell'$. Since $\chi(C_{2k+1}[K_\ell]) = 2\ell + \lceil \frac{\ell}{k} \rceil$ by Proposition 5.2.2, then $C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell'}]$ if and only if

$$2\ell + \left\lceil \frac{\ell}{k} \right\rceil \leq 3\ell' \qquad \Leftrightarrow \qquad \ell' \geq \frac{1}{3}\left(2\ell + \left\lceil \frac{\ell}{k} \right\rceil\right).$$

Hence, this result follows.

2. Suppose $\ell \leq \ell'$. We consider two cases.

   (a) Suppose $k \geq k'$. Then, $C_{2k+1} \to C_{2k'+1}$, by folding paths of length three into one edge until we reach a cycle of length $2k'+1$, and $K_\ell \to K_{\ell'}$ by Lemma 5.2.3. Hence, by Lemma 5.2.8, $C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell'}]$.

   (b) Suppose $k < k'$. We can write $\ell' = \ell + i$ for some $i \in \mathbb{Z}_{\geq 0}$. Observe that $\chi(C_{2k+1}[K_\ell]) \leq \omega(C_{2k'+1}[K_{\ell'}])$ if and only if $i \geq \frac{1}{2}\lceil \frac{\ell}{k} \rceil$. Thus, in this case, Lemma 5.2.6 implies that $C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell'}]$. Conversely, suppose $i < \frac{1}{2}\lceil \frac{\ell}{k} \rceil \leq \lceil \frac{\ell}{2k} \rceil$, where the second inequality follows from Lemma 5.2.5. Then, it suffices to show that

$$C_{2k+1}[K_\ell] \nrightarrow C_{2k'+1}\left[K_{\ell + \lceil \frac{\ell}{2k} \rceil - 1}\right].$$

   Indeed, if $j < \lceil \frac{\ell}{2k} \rceil - 1$ and $C_{2k+1}[K_\ell] \to C_{2k'+1}[K_{\ell+j}]$, then by Lemma 5.2.8 (since $C_{2k'+1} \to C_{2k'+1}$ by the identity homomorphism and

$$K_{\ell+j} \to K_{\ell + \lceil \frac{\ell}{2k} \rceil - 1}$$

   by Lemma 5.2.3), we could extend to a homomorphism

$$C_{2k+1}[K_\ell] \to C_{2k'+1}\left[K_{\ell + \lceil \frac{\ell}{2k} \rceil - 1}\right],$$

   which would be a contradiction. So, by way of contradiction, suppose there were such a homomorphism, say

$$g : C_{2k+1}[K_\ell] \to C_{2k'+1}\left[K_{\ell + \lceil \frac{\ell}{2k} \rceil - 1}\right].$$

Then, by Lemma 5.2.4, we can define a homomorphism

$$h : g(C_{2k+1}[K_\ell]) \to K_{2\ell+2\lceil \frac{\ell}{2k} \rceil - 2}$$

as follows. Let

$$V(C_{2k'+1}) = \{0, 1, \dots, 2k'\}$$

where the vertex 0 corresponds to a $K_{\ell+\lceil \frac{\ell}{2k} \rceil - 1}$-layer which is in the image of g. Further, let

$$V(K_{\ell+\lceil \frac{\ell}{2k} \rceil - 1}) = \{v_1, \dots, v_{\ell+\lceil \frac{\ell}{2k} \rceil - 1}\}$$

and

$$V(K_{2\ell+2\lceil \frac{\ell}{2k} \rceil - 2}) = \{v'_1, \dots, v'_{2\ell+2\lceil \frac{\ell}{2k} \rceil - 2}\}.$$

Then, define

$$h(u, v_n) = \begin{cases} v'_n & \text{dist}_{C_{2k+1}}(0, u) \text{ is even} \\ v'_{n+\ell+\lceil \frac{\ell}{2k} \rceil - 1} & \text{dist}_{C_{2k+1}}(0, u) \text{ is odd} \end{cases}.$$

Now, we want to show that h is a homomorphism. Suppose

$$(u, v_n)(u', v_m) \in E(g(C_{2k+1}[K_\ell])) \subseteq E(C_{2k'+1}[K_{\ell+\lceil \frac{\ell}{2k} \rceil - 1}]).$$

Then, we have two cases.

i. Suppose $uu' \in E(C_{2k'+1})$. Then, $u \neq u'$ and by definition of h, $h(u, v_n) = v'_n$ if and only if $h(u', v_m) = v'_{m+\ell+\lceil \frac{\ell}{2k} \rceil - 1}$. Certainly, these are distinct and hence adjacent in $K_{2\ell+2\lceil \frac{\ell}{2k} \rceil - 2}$.

ii. Suppose $u = u'$ and $v_n v_m \in E(K_{\ell+\lceil \frac{\ell}{2k} \rceil - 1})$. Then, $v_n \neq v_m$ (hence $n \neq m$) and so

$$h(u, v_n) = v'_n \text{ (or } v'_{n+\ell+\lceil \frac{\ell}{2k} \rceil - 1})$$

and

$$h(u', v_m) = v'_m \text{ (or } v'_{m+\ell+\lceil \frac{\ell}{2k} \rceil -1}),$$

which are distinct and hence adjacent in $K_{2\ell+2\lceil \frac{\ell}{2k} \rceil -2}$.

Therefore, h is a homomorphism. So, by composition,

$$h \circ g : C_{2k+1}[K_\ell] \to K_{2\ell+2\lceil \frac{\ell}{2k} \rceil -2}$$

is a homomorphism. However,

$$\begin{aligned}
\chi(K_{2\ell+2\lceil \frac{\ell}{2k} \rceil -2}) &= 2\ell + 2\left\lceil \frac{\ell}{2k} \right\rceil - 2 \\
&< 2\ell + 2\left\lceil \frac{\ell}{2k} \right\rceil - 1 \\
&\leq 2\ell + \left\lceil \frac{\ell}{k} \right\rceil \\
&= \chi(C_{2k+1}[K_\ell]).
\end{aligned}$$

So, this is a contradiction to Lemma 5.2.1. Hence,

$$C_{2k+1}[K_\ell] \nrightarrow C_{2k'+1}\left[K_{\ell+\lceil \frac{\ell}{2k} \rceil -1}\right].$$

Therefore, the result follows.

□

Again, we would like the ability to show that these graphs can be used as the targets for a new type of coloring problem, or find some property that this class of graphs has that the complete, Kneser and circulant graphs all do have which prevents them from being a successful candidate for such a class of graphs.

5.3 Questions Regarding $C_n[K_\ell]$

Moving away from using the class $\mathscr{C}$ as the target graphs for a new coloring problem, there are other questions to ask about this class, and an even slightly more general class where we do not restrict to odd cycles. Recall that in Chapter 3, we computed that

$$\chi_f(Inc(C_{3n}[K_\ell])) = 3\ell.$$

The necessary piece of information that we needed here was that 3 divided $3n$. Let's look at what we do know about the fractional incidence chromatic number of $C_n[K_\ell]$. By Theorem 2.1.8, we know that

$$\begin{aligned}\chi_f(Inc(C_n[K_\ell])) &\geq \frac{2|E(C_n[K_\ell])|}{|V(C_n[K_\ell])| - \gamma(C_n[K_\ell])} \\ &= \frac{n\ell(2\ell + \ell - 1)}{n\ell + \lceil \frac{n}{3} \rceil} \\ &= \frac{n\ell(3\ell - 1)}{n\ell + \lceil \frac{n}{3} \rceil}.\end{aligned}$$

From Corollary 3.4.5, we know that

$$\begin{aligned}\chi_f(Inc(C_n[K_\ell])) &\leq \min\{\ell\chi_f(C_n^2), \ell\chi_f(Inc(C_n)) + \chi_f(Inc(K_\ell))\} \\ &= \min\left\{\ell\left(\frac{n}{\lfloor n/3 \rfloor}\right), \ell\left(\frac{2n}{n - \lceil n/3 \rceil}\right)\right\}\end{aligned}$$

It is not clear that from this we can say anything. One possible direction would be to consider whether or not $C_n[K_\ell]$ is arc transitive. This is a reasonable question for a couple of reasons. First, in Chapter 3, when we showed the equality of the known upper and lower bounds when 3 divides n, we also showed that the fractional incidence chromatic number is equal to the lower bound. This occurs when $Inc(G)$ is vertex transitive, although there is no reason to suspect that this occurs exclusively when this is the case. We argued in Chapter 2 that if G is arc transitive, then $Inc(G)$ is vertex transitive. There may be other ways in which

$Inc(G)$ may be vertex transitive - which would be interesting to know.

It is also reasonable to study whether or not $C_n[K_\ell]$ is arc transitive since both C_n and K_ℓ are arc transitive. It would be interesting to know, and also quite useful, if whenever G, H are arc transitive, then $G[H]$ is arc transitive. Note that if this were the case, then we would know how to compute $\chi_f(Inc(C_n[K_\ell]))$.

Finally, can we use the homomorphisms from the previous section to further improve these bounds on $C_{2k+1}[K_\ell]$?

5.4 Properties of G and $Inc(G)$ when $Inc(G)$ is Perfect

Lastly, in Chapter 4, we classified when $Inc(G)$ is perfect. Having realized a subclass of perfect graphs, we would like to know what other properties this class has and whether or not this class can be recognized as precisely a class of perfect graphs which has already been studied in the literature. Further, this class of graphs with circumference at most 3 is also perfect, so we would like to ask the same questions about this class of graphs.

For example, let's consider the class of graphs with circumference at most 3. This class of graphs has the following properties.

- Not bipartite graphs - some of the graphs have triangles, hence odd cycles.
- Chordal graphs - there are no cycles of length at least 4; however, there are graphs which are chordal and have circumference at least 4.
- Block graphs - all 2-connected blocks are cliques; namely they are either edges or triangles in our case. However, there are graphs which are block graphs but have circumference at least 4.

- Not interval graphs - branching out from a triangle causes problems when trying to recognize the vertices as intervals which overlap when the vertices are adjacent.

- Not comparability graphs - this is for similar reasons as why these graphs are not interval graphs. A triangle with a single edge hanging from each vertex is a forbidden induced subgraph of a comparability graph.

- Perfectly orderable graphs - this is because chordal graphs are a subclass and we know the graphs with circumference at most 3 are chordal.

There are other subclasses of perfect graphs which have not yet been examined. For example, it is not known whether or not these graphs are line graphs of bipartite graphs.

As for the class of perfect incidence graphs, we can run the same analysis.

- Not bipartite graphs - $Inc(C_3)$ contains a triangle.

- Not chordal graphs - $Inc(C_3)$ has an induced C_4.

- Not block graphs - $Inc(C_3)$ is 2-connected, but is not complete.

It has not been studied whether or not these graphs are interval graphs, comparability graphs, perfectly orderable graphs, or line graphs of bipartite graphs. It would be interesting to study these properties, and any other properties defining subclasses of perfect graphs.

Bibliography

[1] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21:429–490, 1977.

[2] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21:491–567, 1977.

[3] C. Berge. Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 10:114, 1961.

[4] E.R. Berlekamp. *Algebraic Coding Theory.* (revised) edition, 1984.

[5] J.A. Bondy and P. Hell. A note on the star chromatic number. *J. Graph Theory*, 14:479–482, 1990.

[6] R.A. Brualdi and J.J.Q. Massey. Incidence and strong edge colorings of graphs. *Discrete Math.*, 122:51–58, 1993.

[7] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković. Recognizing Berge graphs. 25(2):143–186, 2005.

[8] M. Chudnovsky, N. Roberson, P. Seymour, and R. Thomas. The strong perfect graph theorem. 164:51–229, 2006.

[9] R. Diestel. *Graph Theory*. Springer-Verlag, second edition, 2000.

[10] M. Hosseini Dolama and E. Sopena. On the maximum average degree and the incidence chromatic number of a graph. *Discrete Math. and Theoret. Comput. Sci.*, 7:203–216, 2005.

[11] M. Hosseini Dolama, E. Sopena, and X. Zhu. Incidence coloring of k-degenerated graphs. *Discrete Math.*, 283(1-3):121–128, 2004.

[12] P. Erdős, C. Ko, and R. Rado. Intersection theorem for systems of finite sets. *Quarterly J. of Math. Oxford. Second Series*, 12:313–320, 1961.

[13] S. Ferneyhough, R. Haas, D. Hanson, and G. MacGillivray. Star forests, dominating sets and Ramsey-type problems. 245:255–262, 2002.

[14] G. Gao and X. Zhu. Star-extremal graphs and the lexicographic product. *Discrete Math.*, 152:147–156, 1996.

[15] M. Grötschel, L. Lovász, and A. Schrijver. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica*, 1:169–197, 1981.

[16] B. Guiduli. On incidence coloring and star arboricity of graphs. *Discrete Math.*, 163:275–278, 1997.

[17] R. Hammack, W. Imrich, and S. Klavžar. *Handbook of Graph Products*. CRC Press, Inc., Boca Raton, FL, USA, 2nd edition edition, 2011.

[18] F. Harary, J.P. Hayes, and H. Wu. A survey of the theory of hypercube graphs. 15(4):277–289, 1988.

[19] GS.G. Hartke and SG.T. Helleloid. Reconstrgraphs a graph from its arc incidence graph. *Graphs Combin.*, 28(5):637–652, 2012.

[20] S. Hedetniemi. Homomorphisms of graphs and automata. Technical Report 03105-44-T, University of Michigan, 1966.

[21] A.J.W. Hilton, R. Rado, and S.H. Scott. A (< 5)-color theorem for planar graphs. *Bull. London Math. Soc.*, 5:302–306, 1973.

[22] I. Holyer. The NP-completeness of edge-coloring. *SIAM J. Computing*, 10:718–720, 1981.

[23] L. Lovász. A characterization of perfect graphs. 13(2):95–98, 1972.

[24] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25:319–324, 1978.

[25] M. Maydanskiy. The incidence coloring conjecture for graphs of maximum degree 3. *Discrete Math.*, 292:259–266, 2005.

[26] P.R.J. Östergård, Z. Shao, and X. Xu. Bounds on the domination number of Kneser graphs. 9:197–205, 2015.

[27] E. Scheinerman and D. Ullman. *Fractional Graph Theory. A Rational Approach to the Theory of Graphs*. Dover Publications, Inc., Mineola, NY, 2011.

[28] W.C. Shiu and P.K. Sun. Invalid proofs on incidence coloring. *Discrete Math.*, 308(24):6575–6580, 2008.

[29] S. Stahl. n-tuple colorings and associated graphs. *J. Combinatorial Theory Ser. B*, 20(2):185–203, 1976.

[30] P.K. Sun. Incidence coloring of regular graphs and complement graphs. *Taiwanese Journal of Math.*, 16(6):2289–2295, 2012.

[31] P.K. Sun and W.C. Shiu. Some results on incidence coloring, star arboricity and domination number. *Australasian Journal of Combin.*, 54:107–114, 2012.

[32] T.M. Thompson. *From Error-Correcting Codes through Sphere Packing to Simple Groups*. 1983.

[33] G.J.M. van Wee. Improved sphere bounds on the covering radius of codes. 34:237–245, 1988.

[34] A. Vince. Star chromatic number. *J. Graph Theory*, 12:551–559, 1988.

[35] V.G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz*, 3:25–30, 1964.

[36] S.D. Wang, D.L. Chen, and S.C. Pang. The incidence coloring number of Halin graphs and outerplanar graphs. *Discrete Math.*, 256:397–405, 2002.

[37] D. Yang. Fractional incidence coloring and star arboricity of graphs. *Ars Combin.*, 105:213–224, 2012.

[38] X. Zhu. Fractional Hedetniemi's conjecture is true. *European J. Combin.*, 32(7):1168–1175, 2011.