Homology of Complete and Torsion Modules

By

Gabriel Valenzuela

Advisor: Mark Hovey,

Professor of Mathematics

Wesleyan University

Middletown, CT

May 2015

A Thesis in Mathematics

submitted in partial fulfillment of the

requirements for the degree of Doctorate of Philosophy

Para mi madre, Soledad.

Abstract

Let R be a Noetherian ring and I an ideal generated by a regular sequence. The category of I-adically complete R-modules is not Abelian, but it can be enlarged to an Abelian category of so-called L-complete modules. This category is an Abelian subcategory of the full category of R-modules, but it is not usually a Grothendieck category. It is well known that a Grothendieck category always has a derived category, however, this is much more delicate for arbitrary Abelian categories.

In this thesis, we will show that the derived category of the L-complete modules exists, and that it is in fact equivalent to a certain Bousfield localization of the full derived category of R. L-complete modules should be dual to I-torsion modules, which do form a Grothendieck category. We make this precise by showing that although these two Abelian categories are clearly not equivalent, they are derived equivalent. As an application, we will explain how this result can be used to effortlessly recover well known duality theorems currently found in the literature.

Acknowledgements

I would like to express my gratitude to Professor Mark Hovey for his patience and encouragement throughout my years at Wesleyan. It is because of your inspiration and passion that I became interested in homotopy theory, and the work in this document would not be possible without your constant support. I am also very thankful for the helpful comments of the members of my committee, Professors Christopher Rasmussen and Mehdi Khorami.

The faculty, administrative assistants and graduate students of our department are all central to our supportive community. I particularly want to thank Professors Wai Kiu Chan and David Pollack for your advise and for so many enjoyable conversations. To my first year classmates Abbey, Bonita, Brett, and James, I will always treasure our experiences throughout these 6 years. It has been a real pleasure to be part of this process with you. To Alicia, Freda, Leah and Noelle, thank you for listening to me during the most stressful moments while completing this document.

There are many things I am grateful for during my time at Wesleyan, but without a doubt, my life would not be same if I had not met you, Sonja. I thank you for your love throughout all these wonderful years together, and I hope that we will have many more to cherish.

Finally, I would like to thank my family, who have always been there for me. Mother, it was you who sparked my curiosity for science, and I would have never become a mathematician if it was not for all that you patiently taught me since my childhood. Thank you for your love and unconditional support.

Contents

CHAPTER I

Introduction

Like localization, completion is among the central tools for investigating commutative rings and their modules. Completion was introduced by Hensel in the context of number theory, in analogy to the local theory of complex functions of one variable. The success of his framework is evident, by results like the local-to-global principle of Hasse, which states that certain types of quadratic equations have solutions over $\mathbb{Q}$ if and only if they do so over every p-adic completion of $\mathbb{Q}$ and over $\mathbb{R}$, an important result towards the understanding of when a given integer can be represented as a sum of squares.

As discussed in [**Eis95**], completions are very useful in algebraic geometry. The basic objects of study in this field are algebraic varieties, which can be thought of as solution sets of systems of polynomial equations in several variables. They have a natural topology called the Zariski topology, but the open sets in the Zariski topology are not small enough to study the local properties of a variety. Completing the local ring at a point of the variety allows us to study the truly local information near that point.

The main motivation for the author comes from homotopy theory, more precisely, from the subarea called *chromatic homotopy theory*. This can roughly be described as trying to understand homotopy information of an object from its minimal building blocks; contrary to the case of number theory, it is not enough to localize at each prime p, but there is also a *chromatic level* $n \geq 0$ for any given prime number. Then, the "chromatic pieces" obtained by localizing at a fixed prime and chromatic level can be put together to recover all the homotopical data. This strategy has been very successful when applied to the computation of homotopy groups of spheres, for instance. The key fact for us is that the homotopy groups of these chromatic building blocks are always close to complete (in fact, they are L-complete; see below) as modules over certain associated ring. Moreover, the tools involved in the calculation of the homotopy groups can be written in terms of derived functors of completion, and so it is important to understand the homological algebra in this situation.

Let R be a commutative Noetherian ring and I an ideal generated by a finite set $\{x_0, x_1, \ldots, x_{n-1}\}$. For an R-module M, the I-adic completion of M is defined as the limit $M^{\wedge}_I = \lim_k M/I^k M$. We say that M is I-adically complete if the natural map $M \to M^{\wedge}_I$ is an isomorphism. As we will discuss in the first chapter, completion is in general neither left nor right exact as a functor on the category R-modules. As a consequence of this, the full subcategory of I-adically complete modules is not an Abelian category, i.e., it does not satisfy the minimal properties needed to define the basic constructions of homological algebra.

The solution comes from enlarging this category to include the so-called L-complete modules, which were introduced by Greenlees and May in [GM92]. If L_k denotes the k-th left derived functor of I-adic completion, then M is said to be L-complete if the associated natural map $M \to L_0M$ is an isomorphism. The full subcategory of L-complete modules $\widehat{\mathrm{Mod}}_R$ forms an Abelian category, so we can try to construct its derived category; this is the category obtained from chain complexes by forcing maps inducing isomorphisms in homology to be actual isomorphisms. The derived category $\mathrm{D}(\mathcal{A})$ of an Abelian category $\mathcal{A}$ was introduced by Verdier (following ideas of Grothendieck) as the natural place where to study the homological algebra of $\mathcal{A}$. The existence of the derived category is a delicate question in general, although it is well known that Grothendieck categories have derived categories.

However, we will see that filtered colimits fail to be exact in $\widehat{\mathrm{Mod}}_R$ which is one of the requirements for a category to be a Grothendieck category. To deal with this situation, we will use techniques from abstract homotopy theory. More precisely, we will work in the axiomatic stable homotopy framework of Hovey, Palmieri, and Strickland introduced in [HPS97]. In the same way that Abelian categories are an abstraction designed to mimic the properties of categories of modules, stable homotopy categories satisfy certain axioms that are inspired by classical stable homotopy theory. An example of such category is the derived category of a ring R, where all of our work takes place. Our primary objective is to show that $\mathrm{D}(\widehat{\mathrm{Mod}}_R)$ exists; we do this by showing that it is equivalent to a certain Bousfield localization of $\mathrm{D}(R)$.

Our method yields a natural connection between L-complete modules and I-power torsion modules. We say M is I-power torsion if each element $x \in M$ is annihilated by some power of I. It turns out that the category of I-power torsion modules $\mathrm{Mod}_R^{\mathrm{tor}}$ is a Grothendieck category, and so the existence of its derived category is guaranteed. Nonetheless, the nature of our method allows us to obtain a characterization of $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$ in the same way as we do for L-complete modules. Furthermore, it follows that while $\widehat{\mathrm{Mod}}_R$ and $\mathrm{Mod}_R^{\mathrm{tor}}$ are certainly not equivalent, they are derived equivalent; that is, $\mathrm{D}(\widehat{\mathrm{Mod}}_R) \cong \mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$. The latter follows easily once our machinery is set in place. This is one of the points we wish to emphasize; the homotopical techniques provide elegant and conceptually sound proofs of these result.

As an illustration of the power of this framework, we will show how important duality theorems of Grothendieck, Hartshorne and Lipman can be effortlessly recovered. Additionally, we will deduce a new criterion for when a module is L-complete.

CHAPTER II

Preliminaries

II.1. Completion and torsion on Mod_R

Let R be a Noetherian ring and I a finitely generated ideal. We will consider two natural functors on the category of R-modules, Mod_R. First, the *I-power torsion submodule of an R-module M* is defined as

$$\Gamma_I(M) = \{x \in M | I^k x = 0 \text{ for some } k\}.$$

The assignment $M \mapsto \Gamma_I(M)$ is functorial; for $f : M \to N$, the image of the restriction of f to $\Gamma_I(M)$ is contained in $\Gamma_I(N)$ so we can set $\Gamma_I(f) = f|_{\Gamma_I(M)}$. Moreover, it is easy to see that for $f, g : M \to N$, $\Gamma_I(f+g) = \Gamma_I(f) + \Gamma_I(g)$ so Γ_I is an *additive functor*. The inclusion gives a natural map $\Gamma_I M \to M$, and we say that M is *I-power torsion* (or simply torsion) if the latter is an isomorphism.

As an example, let $R = \mathbb{Z}$ and $I = (p)$ for some prime p. In this case, $\Gamma_{(p)}$ simply computes p^k-torsion for all $k \geq 1$. For instance, $\Gamma_{(p)}\mathbb{Z} = 0$ as $\mathbb{Z}$ is torsion free, and $\Gamma_{(p)}\mathbb{Z}/p^k = \mathbb{Z}/p^k$ since by construction every element of $\mathbb{Z}/p^k$ is p^k-torsion.

In order to understand the homological properties of Γ_I, we need to know how it behaves after applying it to short exact sequences. The proof of the following proposition is left to the interested reader.

Proposition II.1.1. $\Gamma_I : \mathrm{Mod}_R \to \mathrm{Mod}_R$ *is a left exact functor. This is, if*

$$0 \to K \to M \to N \to 0$$

is a short exact sequence of R-modules, then

$$0 \to \Gamma_I K \to \Gamma_I M \to \Gamma_I N$$

is exact.

It is easy to find examples where the right exactness is lost. Let $R = \mathbb{Z}$ and $I = (p)$ for some prime p as in the example above. Then, the p-power torsion of the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

is

$$0 \to 0 \to 0 \to \mathbb{Z}/p \to 0,$$

which is clearly not right exact.

On the other hand, the *I-adic completion of an R-module M* is given by

$$M \mapsto M_I^\wedge = \lim_k M/I^k M,$$

where the maps $M/I^{k+1}M \to M/I^kM$ are reduction modulo I^kM. The I-adic completion can be written as the composition of functors

$$M_I^\wedge = \lim_k M/I^kM = \lim_k (R/I^k \otimes M),$$

so it defines a functor on Mod_R. The maps $M \to M/I^kM$ induce a natural homomorphism $M \to M_I^\wedge$, and we say that M is I-adicaly complete (or simply complete) if the latter is an isomorphism.

For a familiar example, let $R = \mathbb{Z}$ and $I = (p)$ as before. Then, $\mathbb{Z}_{(p)}^\wedge = \lim_k Z/p^k = \mathbb{Z}_p$ is the ring of p-adic integers. Further, note that $p^k Z/p = 0$ for all $k \geq 1$. Hence, $(\mathbb{Z}/p)_{(p)}^\wedge = \lim_k Z/p = \mathbb{Z}/p$.

One can show that for any surjection $f : M \twoheadrightarrow N$, the induced map $f_I^\wedge : M \to N$ is also surjective [HS99]. Despite this, I-adic completion is still not right exact. Furthermore, it is also not left exact as we can see in the following example.

Example II.1.2 ([HS99, Appendix A])**.** Consider $\mathbb{Z} \to \mathbb{Q}$ as a map of Abelian groups. Since p is invertible in $\mathbb{Q}$, we have $p^k\mathbb{Q} = \mathbb{Q}$, for all $k \geq 1$. Thus, $\mathbb{Q}_{(p)}^\wedge = 0$ and the p-completion of $\mathbb{Z} \to \mathbb{Q}$ is $Z_p \to 0$. This shows that completion does not preserve monomorphisms.

Next, consider the short exact sequence

$$0 \to \bigoplus_k \mathbb{Z} \xrightarrow{\oplus_k p^k} \bigoplus_k \mathbb{Z} \to \bigoplus_k \mathbb{Z}/p^k \to 0.$$

By definition, the elements of the p-completion of an Abelian group M are sequences (a_k) of elements in M/p^kM such that $a_k \equiv a_l \mod p^k$, for all $k \leq l$. Thus, the sequence $(p, 0, \ldots), (p, p^2, 0, \ldots), \ldots$ gives an element of $(\bigoplus_k \mathbb{Z})^\wedge_{(p)}$ which we identify with $(p, p^2, p^3, \ldots)$. Note that $(p, p^2, p^3, \ldots)$ is not zero in the cokernel of $(\oplus_k p^k)^\wedge_{(p)}$ because there is no $x \in (\bigoplus_k \mathbb{Z})^\wedge_{(p)}$ with $(\oplus_k p^k)^\wedge_{(p)}(x) = (p, p^2, \ldots)$. However, the sequence $(p, 0, \ldots), (p, p^2, 0, \ldots), \ldots$ represents a sequence of zeros in $\bigoplus_k \mathbb{Z}/p^k$ so $(p, p^2, \ldots)$ is zero in $(\bigoplus_k \mathbb{Z}/p^k)^\wedge_{(p)}$. It follows that completion is not right exact.

When an additive functor is left (resp. right) exact, there is a universal way to complete the left (resp. right) exact sequence, obtained by applying the functor to a short exact sequence, into a long exact sequence using right (resp. left) derived functors. If the functor is neither right nor left exact, the situation is slightly more complicated, but still manageable as we will see. In the next section, we will review the basic homological algebra necessary for these constructions.

II.2. Homological algebra

II.2.1. Chain complexes and the category $\mathrm{Ch}(R)$**.** Let R be a commutative ring. A *chain complex* X is a sequence of R-linear maps

$$\cdots \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n--1} \xrightarrow{d_{n-1}} \cdots$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$. The maps d_n are called *differentials*. Note that $\mathrm{im}(d_{n+1}) \subseteq \ker(d_n)$ for all n. Thus, the *n-th homology module of X* can be defined as the R-module $H_n(X) = \ker(d_n)/\,\mathrm{im}(d_{n+1})$. The idea is that $H_n(X)$ measures the failure of exactness at the n-th spot of the complex.

The collection of chain complexes forms a category denoted by $\mathrm{Ch}(R)$. If X, Y are two chain complexes, a morphism $f : X \to Y$ is a sequence of R-linear maps $f_n : X_n \to Y_n$ such that the diagram

$$\begin{tikzcd} X_n \arrow[r, "d_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{n-1}"] \\ Y_n \arrow[r, "d_n"] & Y_{n-1} \end{tikzcd}$$

commutes for all n.

We can thus turn each homology into a covariant functor $H_n : \mathrm{Ch}(R) \to \mathrm{Mod}_R$; given $f : X \to Y$ in $\mathrm{Ch}(R)$, we get a map $H_n(f) : H_n(X) \to H_n(Y)$ induced by the restriction of f_n to its kernel. This R-linear map is often denoted as f_*. A morphism $X \to Y$ is called a *quasi-isomorphism* if the maps $H_n(X) \to H_n(Y)$ are all isomorphisms.

If we instead consider sequences of R-linear maps where the differentials rise the degree

$$\cdots \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots$$

with $d^n d^{n+1} = 0$, we get the notion of a *cochain complex*. Observe that for every cochain complex $X^\bullet$ there is a chain complex $X_\bullet$ where $(X_\bullet)_n = X^{-n}$. Thus, we can translate all the notions above by reindexing. For instance, *the n-th cohomology module of a cochain complex X* is defined by $H^n(X) = \ker(d^n)/\operatorname{im}(d^{n-1})$. Morphisms and quasi-isomorphisms are defined in a similar fashion and each H^n forms a functor on the category of cochain complexes, which we also denote by $\mathrm{Ch}(R)$.

The category of chain complexes has many formal properties. First, $\mathrm{Ch}(R)$ is an *additive category*, i.e.

(1) Every hom-set $\mathrm{Hom}(X, Y)$ has the structure of an Abelian group in such a way that composition distributes over addition.
(2) There is a *zero object*, this is, an object that is both initial and terminal.
(3) Finite *products* $\prod X_i$ and finite *coproducts* (direct sums) exist, both of which are defined degreewise. Moreover, the canonical map

$$X \bigoplus Y \to X \prod Y$$

is an isomorphism.

In fact, arbitrary products and coproducts can be defined degreewise. The same can be said about inverse limits and filtered colimits (direct limits).

On the other hand, it is not immediately clear how to define a tensor product from the one available from Mod_R. For $X, Y \in \mathrm{Ch}(R)$, consider the *double complex* given by $X_p \otimes Y_q$. If we imagine varying the first complex horizontally and the second vertically, we get two maps

$$d^h : X_p \otimes Y_q \xrightarrow{d \otimes \mathrm{id}} X_{p-1} \otimes Y_q \text{ and } d^v : X_p \otimes Y_q \xrightarrow{(-1)^p (\mathrm{id} \otimes d)} X_p \otimes Y_{q-1}$$

so that $d^h d^h = d^v d^v = d^v d^h + d^h d^v = 0$. We define the *total tensor product* to be the complex $X \otimes Y$ with

$$(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes Y_q,$$

and differential given by $d = d^h + d^v$ [Wei94]. The same construction applied to a general double complex $(C_{\bullet,\bullet}, d^h, d^v)$ is called the *total complex* $\mathrm{Tot}^{\oplus}(C_{\bullet,\bullet})$.

A similar construction can be carried to define an internal hom-complex. We start with the double complex $\mathrm{Hom}(X_p, Y_q)$

$$d^h : \mathrm{Hom}(X_p, Y_q) \xrightarrow{\mathrm{Hom}(d,\mathrm{id})} \mathrm{Hom}(X_{p+1}, Y_q), \text{ and}$$
$$d^v : \mathrm{Hom}(X_p, Y_q) \xrightarrow{(-1)^p \mathrm{Hom}(\mathrm{id},d)} \mathrm{Hom}(X_p, Y_{q-1})$$

so that $d^h d^h = d^v d^v = d^v d^h + d^h d^v = 0$ as well. Then, the *internal Hom*, $\mathrm{Hom}_\bullet(X, Y)$, is given by

$$(\mathrm{Hom}_\bullet(X, Y))_n = \prod_{p+q=n} \mathrm{Hom}(X_p, Y_q),$$

with differential $d = d^v + d^h$. For a general double complex $(C_{\bullet,\bullet}, d^h, d^v)$, this construction is also called the total complex, but it is denoted by $\mathrm{Tot}^{\prod}(C_{\bullet,\bullet})$.

The total tensor product together with the internal Hom make $\mathrm{Ch}(R)$ into a *closed symmetric monoidal category* (which will be discussed in Section III.2). The tensor product $-\otimes-$ and $Hom_\bullet(-,-)$ are bifunctors on $\mathrm{Ch}(R)\times\mathrm{Ch}(R)$, and we have a natural isomorphism

$$\mathrm{Hom}(X\otimes Y, Z)\cong \mathrm{Hom}(X, \mathrm{Hom}_\bullet(Y,Z))$$

in all three variables.

II.2.2. Derived functors. Recall that $P\in\mathrm{Mod}_R$ is called *projective* if for any surjection $M\to N$ and any map $P\to N$, there is a solution to the diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists"'] \arrow[d] \\ M \arrow[r, two heads] & N \end{tikzcd}$$

It can be shown that P is projective if and only if $\mathrm{Hom}(P,-)$ is an exact functor. Another characterization is that P is projective if and only if it is a summand of a free module. In particular, free modules are projective. We refer the reader to [**Wei94**] for proofs of these statements.

A *projective resolution* of M is a complex of projective modules P bounded below 0, and a map $\epsilon : P_0\to M$ such that the complex

$$\ldots \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M\to 0$$

is exact. Observe that ϵ can be completed to a chain map $P\to M$

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r, "d"] \arrow[d] & P_0 \arrow[d, "\epsilon"] \\ \cdots \arrow[r] & 0 \arrow[r] & M \end{tikzcd}$$

and then the condition above is equivalent to $P \to M$ being a quasi-isomorphism. For this reason, we sometimes refer to P as a projective approximation of M. It turns out that projective resolutions exist for every object in Mod_R [**Wei94**].

An *additive functor* $F : \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ is a functor so that every map $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$ is a group homomorphism. In particular, such a functor sends the zero map to itself, and so it does with the zero object. Hence, if X is a chain complex, so is $F(X)$, where F is applied to X degreewise.

Suppose now that F is right exact. Then, given a short exact sequence

$$0 \to K \to M \to N \to 0,$$

we would like to find a universal way to complete

$$F(K) \to F(M) \to F(N) \to 0$$

into a long exact sequence. It turns out that the solution is given by the left derived functors of F.

Let $F : \mathrm{Ch}(R) \to \mathrm{Ch}(R)$ be an additive functor. If M is an R-module, we define the *left derived functors* $L_k F$ *of* F by the formula

$$L_k F(M) = H_k(F(P)),$$

where P is a projective resolution of M. It is not obvious that $L_k(M)$ is independent of the projective resolution, less so that this defines functors $L_k F$:

$\mathrm{Mod}_R \to \mathrm{Mod}_R$. The following proposition summarizes the main properties of left derived functors.

Proposition II.2.1. *Each $L_k F : \mathrm{Mod}_R \to \mathrm{Mod}_R$ is an additive functor on the category of R-modules. Moreover, given a short exact sequence*

$$0 \to K \to M \to N \to 0,$$

we get a long exact sequence

$$\cdots \to L_1F(K) \to L_1F(M) \to L_1F(N) \to L_0F(K) \to L_0F(M) \to L_0F(N) \to 0$$

PROOF. See Chapter 2 of [**Wei94**]. □

Note that if F is right exact,

$$F(P_1) \to F(P_0) \xrightarrow{F(\epsilon)} F(M) \to 0$$

is also right exact. Thus, $L_0F = F$ in this situation.

Example II.2.2. Fix an R-module N and consider the functor $N \otimes - : \mathrm{Mod}_R \to \mathrm{Mod}_R$. This is clearly an additive functor and, in fact, it can be shown that $N \otimes -$ is right exact [**Wei94**]. The left derived functors $L_k(N \otimes -)$ are called *Tor groups*, and they are denoted by $\mathrm{Tor}_k(N, -)$. Note that if P is projective, then $\mathrm{Tor}_k(N, P) = 0$ for all $k \geq 1$. However, the converse is not true. We say that a module F is *flat* if the functor $F \otimes -$ is exact. For example, if S is a multiplicative subset of R, the localization $S^{-1}R$ is always a flat R-module [**Eis95**].

Dually, an R-module I is *injective* if for any monomorphism $M \hookrightarrow N$ and any map $M \to I$, the following diagram has a solution

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & N \arrow[dl, dashed, "\exists"] \\ I & \end{tikzcd}$$

Note that I is injective in Mod_R if and only if I is projective in Mod_R^{op}, the category consisting of the same objects but reversed arrows. Thus, I is injective if and only if $\mathrm{Hom}(-, I)$ is exact. Free modules, and in particular R itself, need not be injective. For example, injective objects in the category of Abelian groups coincide with divisible groups [**Wei94**]. Hence, $\mathbb{Z}$ is not injective as a $\mathbb{Z}$-module, but $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are.

An *injective resolution* I is a cochain complex of injective modules, together with a monomorphism $M \to I^0$ so that

$$0 \to M \to I^0 \to I^1 \to \cdots$$

is exact. Every module M in Mod_R can be approximated by an injective resolution [**Wei94**]. Thus, for an additive functor F, we can define the right derived functors R^kF as

$$R^kF(M) = H^i(F(I)),$$

where I is an injective approximation of M. Consider the additive functor F^{op} : $\mathrm{Mod}_R^{op} \to \mathrm{Mod}_R^{op}$. Then, I becomes a projective resolution of M in Mod_R^{op} and thus

$$R^k F(M) = (L_k F^{op})^{op}(M).$$

Therefore, all the results about left derived functors apply to right derived functors.

Example II.2.3. Consider $\mathrm{Hom}(N, -) : \mathrm{Mod}_R \to \mathrm{Mod}_R$ as a functor on Mod_R. One can show that this is an additive left exact functor [**Wei94**]. Its derived functors are called the *Ext groups*, and they are denoted by $\mathrm{Ext}^k(N, -)$. Using the fact that I is injective if and only if $\mathrm{Hom}(-, I)$ is exact, it follows that I is injective if and only if $\mathrm{Ext}^k(N, I) = 0$ for all $k \geq 1$ and all $N \in \mathrm{Mod}_R$.

II.2.3. The derived category of a ring. By construction, homology is an invariant on $\mathrm{Ch}(R)$ as chain isomorphisms are mapped to isomorphisms on Mod_R. However, there are many chain maps that induce isomorphisms on homology but are not invertible. For example,

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

is clearly a quasi-isomorphism but not an invertible map. Thus, homology is not an invariant strong enough to detect complexes up to isomorphism. Note that every complex X is by definition quasi-isomorphic to $H_*(X)$ viewed as a complex with $H_n(X)$ in degree n and all zero differentials. In view of this, if we are going to insist on using homology as our main invariant, one could argue for just remembering the homology of a complex. However, two complexes may have the same homology while not being quasi-isomorphic, as the example

$$\mathbb{C}[x,y] \oplus \mathbb{C}[x,y] \xrightarrow{(x,y)} \mathbb{C}[x,y] \text{ and } \mathbb{C}[x,y] \xrightarrow{0} \mathbb{C}$$

from Thomas [Tho01] illustrates. Therefore, the correct approach is to remember all the complexes up to quasi-isomorphism. This is the motivation for the following construction.

Consider the class of chain complexes with morphisms given by "roofs"

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

where f is a morphism and s is a quasi-isomorphism. In the same way that a/b represents ab^{-1} in $\mathbb{Q}$, the dotted arrow represents the morphism $s^{-1}f$. For this to work, we need to declare two roofs (s,f) and (t,g) to be equivalent if "$s^{-1}f = t^{-1}g$". Formally, we require the existence of an object V and quasi-isomorphisms $p : V \to Z$ and $q : V \to Z'$ such that the diagram

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & V \arrow[u, "p"'] \arrow[d, "q"] & Y \\ & Z' \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

commutes. Roofs can be composed in the following manner

$$\begin{tikzcd} & & U \arrow[dl, dashed, "u"'] \arrow[dr, dashed, "h"] & & \\ & Z \arrow[dl, "s"'] \arrow[dr, "f"] & & Z' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & W & & Y \end{tikzcd}$$

where u is a quasi-isomorphism and h is a chain map. It can be shown that the dotted arrows always exist [GM03].

Let $\mathrm{Ch}(R)[\simeq]^{-1}$ be the class of chain complexes together with morphisms between them given by the equivalence classes of roofs. By definition, it seems that $\mathrm{Ch}(R)[\simeq]^{-1}$ forms a category. However, there is a subtle set-theoretical issue. Given $X, Y \in \mathrm{Ch}(R)$, the collection of maps between them does not a priori need to form a set. This is because there is a proper class of objects Z that can be used to construct roofs. The following result guarantees that $\mathrm{Ch}(R)[\simeq]^{-1}$ is in fact a category.

THEOREM II.2.4. *There exists a category* $\mathrm{D}(R)$ *and a functor* $Q : \mathrm{Ch}(R) \to \mathrm{D}(R)$ *with the following properties:*

(1) *Q(f) is an isomorphism for any quasi-isomorphism* f.

(2) *Any functor* $F : \mathrm{Ch}(R) \to \mathcal{D}$ *transforming quasi-isomorphisms into isomorphisms factors uniquely through* $\mathrm{D}(R)$*:*

$$\begin{tikzcd} \mathrm{Ch}(R) \arrow[r, "Q"] \arrow[dr, "F"'] & \mathrm{D}(R) \arrow[d, "\exists !"] \\ & \mathcal{D} \end{tikzcd}$$

PROOF. See [Wei94, GM03]. □

This construction is completely analogous to the way $\mathbb{Q}$ is constructed from $\mathbb{Z}$ and thus we say $\mathrm{D}(R)$ is the *localization* of $\mathrm{Ch}(R)$ by the class of quasi-isomorphisms $[\simeq]$.

We will finish this section with a brief discussion of the derived category in more general settings. First, we observe that if $\mathcal{A}$ is an additive category, we can define chain complexes of objects in $\mathcal{A}$ in the same way. As with chain complexes of modules, they form a category that we will denote by $\mathrm{Ch}(\mathcal{A})$. Moreover, $\mathrm{Ch}(\mathcal{A})$ is also an additive category, however, this is not enough to define homology since we need kernels and images. The additional properties needed were introduced by Grothendieck in his "Tohuku paper", as [Gro57] is sometimes called; our exposition follows [Wei94] instead. Recall that in an additive category $\mathcal{A}$, a *kernel* of a morphism $f : B \to C$ is a map $i : A \to B$ so that $fi = 0$ and i is universal with respect to this property. Dually, a *cokernel* of f is a morphism $e : C \to D$ such that $ef = 0$. A map $i : A \to B$ is said to be *monic* if $ig = 0$ implies $g = 0$ for every map $g : A' \to A$ and $e : C \to D$ is an *epi* if $he = 0$ implies $h = 0$ for all $h : D \to D'$.

A category is then said to be *Abelian* if

(1) Every map in $\mathcal{A}$ has a kernel and cokernel.

(2) Every monic in $\mathcal{A}$ is the kernel of its cokernel.

(3) Every epi in $\mathcal{A}$ is the cokernel of its kernel.

If $\mathcal{A}$ is Abelian, then so is $\mathrm{Ch}(\mathcal{A})$ [**Wei94**], and we can repeat our discussion about the derived category by merely replacing Mod_R for $\mathcal{A}$. As we have mentioned, the fact that the construction we explained yields a category is not immediate. The following result gives conditions that guarantee the existence of the derived category.

THEOREM II.2.5. *If $\mathcal{A}$ is a* well-powered *Abelian category with a set of generators such that filtered colimits are exact, then* $\mathrm{D}(\mathcal{A})$ *exists.*

PROOF. See Remark 10.4.5 in Weibel's book [**Wei94**]. □

Let us explain the terms in the theorem above. A category is *well-powered* if the collection of subobjects of every object forms a set. Recall that a *subobject* of an object A is an equivalence class of monomorphisms $K \to A$ under the equivalence relation of factoring through each other. On the other hand, an object G is said to be a generator if every object A admits an epimorphism $\bigoplus_I G \to A$.

A *Grothendieck category* is an Abelian category with arbitrary coproducts, a generator, and exact filtered colimits. Such categories are examples where the theorem above applies because every Abelian category with a generator is well-powered [**VdB01**].

We finish our discussion with an example that illustrates how even for nice categories, missing only one of the properties above, the derived category fails to exist.

Example II.2.6 (Freyd [**Fre64**], Casacuberta, Neeman [**CN09**])**.** Let I be the class of all small ordinals, and let $R = \mathbb{Z}[I]$ be the polynomial ring generated by I. Observe that while the ring has a proper class of elements, we can still consider the category of R-modules $\mathcal{A}$ as follows. The objects of $\mathcal{A}$ are Abelian groups M together with endomorphism $\varphi_i : M \to M$ for each $i \in I$, such that all the φ_i commute. Note that the R-module homomorphisms $\mathrm{Hom}(M, N)$ form a set because they are contained by the set of Abelian groups between M and N. Moreover, this category has many nice properties; it has arbitrary coproducts and its filtered colimits are exact, however, it does not have a generator.

Let $\mathbb{Z}$ be the trivial R-module, i.e., we take all the φ_i equal to zero. Then, the authors construct a proper class of modules M_i for each $i \in I$ fitting in an exact sequence

$$0 \to \mathbb{Z} \to M_i \to \mathbb{Z} \to 0,$$

none of which are pairwise isomorphic. Thus, $\mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z})$ contains a proper class of elements. If $\mathrm{D}(\mathcal{A})$ was a category, then we would have $\mathrm{Hom}_{\mathrm{D}}(\mathcal{A})(\mathbb{Z}, \Sigma\mathbb{Z}) \cong \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z})$, which contradicts the categorical requirement of having hom-sets. Therefore, $\mathrm{D}(\mathcal{A})$ does not exist.

II.3. Derived functors of completion and torsion

The objective of this section is to introduce local cohomology and local homology in order to describe the derived functors of torsion and completion. Local cohomology was originally introduced by Grothendieck in his seminar at Harvard in 1961 [**Har67**]. While it was introduced in a completely geometric language, we will follow the commutative algebra approach of [**BS13**] because we do all of our work on the derived category.

Local homology, on the other hand, is much more recent. It was first introduced by Greenlees and May in their papers [**GM92, GM95**], where the connection with derived functors of completion was established.

There are several equivalent ways of working with local (co)homology. We will only use the point of view of Koszul complexes, which we describe in the following.

Throughout this section, we fix a Noetherian ring R and a finitely generated ideal $I = (x_0, x_1, \dots, x_{n-1})$.

II.3.1. Koszul complexes. Given x in R, we define $K^\bullet(x)$ to be the cochain complex

$$K^\bullet(x) = R \to R[x^{-1}],$$

where the non-zero modules are in degree 0 and 1. Observe that this is a complex of flat modules.

It will be useful to have a description of $K^\bullet(x)$ in terms of simpler complexes. We denote by $K_s^\bullet(x)$ the complex given by

$$K_s^\bullet(x) = R \xrightarrow{x^s} R,$$

where again the copies of R are in degree 0 and 1. In contrast to $K^\bullet(x)$, each $K_s^\bullet(x)$ is a complex of free modules.

There are chain maps $K_s^\bullet(x) \to K_{s+1}^\bullet(x)$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & R & \xrightarrow{x^s} & R & \longrightarrow & \cdots \\
 & & \| & & \downarrow x & & \\
\cdots & \longrightarrow & R & \xrightarrow{x^{s+1}} & R & \longrightarrow & \cdots
\end{array}$$

and since

$$R[x^{-1}] = \operatorname{colim}(R \xrightarrow{x} R \xrightarrow{x} \cdots),$$

we get the following.

Proposition II.3.1. $K^\bullet(x) = \operatorname{colim}_s K_s^\bullet(x)$, *where the maps* $K_s^\bullet(x) \to K_{s+1}^\bullet(x)$ *are as above.*

Now we are ready for our definition. The *Koszul cochain complex* $K^\bullet(I)$ is defined as the tensor product

$$K^\bullet(I) = K^\bullet(x_0) \otimes \cdots \otimes K^\bullet(x_{n-1}),$$

which is also a complex of flat modules.This is not really well-defined since the complex above depends on the choice of generators $x_0, \ldots, x_{n-1}$. However, we are only interested in $K^\bullet(I)$ up to quasi-isomorphism.

Proposition II.3.2. *Up to isomorphism, $K^\bullet(I)$ depends only on the ideal I.*

PROOF. This is Corollary 1.2 of [**GM95**]. □

Likewise, we can form the tensor product

$$K_s^\bullet(I) = K_s^\bullet(x_0) \otimes \cdots \otimes K_s^\bullet(x_{n-1}).$$

Note that each $K_s^\bullet(I)$ is a complex of free modules, and as colimits commute with tensor product, we get

Proposition II.3.3. *$K^\bullet(I) \cong \operatorname{colim}_s K_s^\bullet(I)$, where the cochain maps are the tensor product of the maps $K_s^\bullet(x) \to K_{s+1}^\bullet(x)$, for $i = 0, \ldots, n-1$.*

II.3.2. Local (co)homology and derived functors. Recall that I-torsion functor Γ_I is additive and left exact on Mod_R, so we can consider its right derived functors.

Definition II.3.4 (**Local cohomology**)**.** Following [**Har67**, **BS13**], we define the *k-th local cohomology functor*, H_I^k, to be the k-th right derived functor of Γ_I.

Note that $H^k(I) = 0$ for all $k \geq 1$, where I is any injective module.

First, it is not hard to express Γ_I and its derived functors in terms of more familiar derived functors.

Proposition II.3.5. *We have natural isomorphisms*

$$H^k(M) \cong \operatorname{colim}_s \operatorname{Ext}^k(R/I^s, M)$$

for $k \geq 0$, where the colimit is taken with respect to the corresponding maps induced by $R/I^{s+1} \to R/I^s$.

PROOF. It is easy to see that $\mathrm{Hom}(R/I^s, M) \cong \{x \in M | I^s x = 0\}$, so

$$\Gamma_I(M) \cong \mathrm{colim}_s \mathrm{Hom}(R/I^s, M).$$

Furthermore, since colimits are exact on Mod_R, it follows that

$$R^k(\mathrm{colim}_s \mathrm{Hom}(R/I^s, -)) \cong \mathrm{colim}_s R^k \mathrm{Hom}(R/I^s, -),$$

and the proof is complete. $\square$

Instead of using a different injective resolution of M every time we need to compute $H^k(M)$, it turns out that we can avoid this by introducing the Koszul complex in the following way.

THEOREM II.3.6 (Grothendieck [**Har67**]). *There are natural isomorphisms*

$$H^k_I(M) = H^k(K^\bullet(I) \otimes M)$$

for all $k \geq 0$.

Note that the formula above makes sense because $K^\bullet(I)$ is well-defined up to quasi-isomorphism (see Proposition II.3.2).

Next, we continue with the derived functors of I-adic completion. Recall that this functor is neither left nor right exact in general. Hence, it is not clear a priori whether to consider left or right derived functors. However, it turns out that the right derived functors of completion are trivial [**GM92**]. Thus, we concentrate on the left ones.

Following [**GM92**], we denote the *k-th left derived functor of I-adic completion* by L_k^I. As it was the case with local cohomology, there is also a way to relate L^k to more familiar derived functors. However, the fact that limits are not exact in Mod_R makes the situation slightly more difficult.

Proposition II.3.7. *There are short exact sequences*

$$0 \to \lim_s{}^1 \operatorname{Tor}_{k+1}(R/I^s, M) \to L_k^I(M) \to \lim_s \operatorname{Tor}_k(R/I^s, M) \to 0$$

for all $k \geq 0$.

PROOF. See Proposition 1.1 of [**GM92**]. □

Remark II.3.8. Notice how the derived functors of completion are dual to the derived functors on torsion in the sense that we can go from the former to the latter by changing the colimit with a limit, and the Ext out of R/I^s with Tor out of R/I^s.

Definition II.3.9 (Local homology). Dually to local cohomology, the *k-th local homology* of an R-module M is given by

$$H_k^I(M) = H_k(\operatorname{Hom}_R(PK^\bullet(I), M)),$$

where $PK^\bullet(I)$ is a projective approximation of $K^\bullet(I)$.

We have an analogous result to Theorem II.3.6.

THEOREM II.3.10 (Greenlees-May [GM92]). *We have natural isomorphisms*

$$L_k^I(M) \cong H_k^I(M) = H_k(\mathrm{Hom}_R(PK^\bullet(I), M)),$$

for all R-modules M.

Next, we continue with the definition of Čech (co)homology. We start by considering the augmentation map $K^\bullet(I) \to R$, with cofiber denoted by $C^\bullet(I)$. Using the explicit description of cofibers as mapping cones, it is clear that the cofiber of the map $PK^\bullet(I) \to R$, $PC^\bullet(I)$ is a projective approximation of $C^\bullet(I)$.

Definition II.3.11 (Čech (co)homology [GM95]). We define the Čech cohomology of an R-module as

$$\check{C}H_I^k(M) = H^k(C^\bullet(I) \otimes M),$$

while Čech homology is defined as

$$\check{C}H_k^I(M) = H_k(\mathrm{Hom}_R(PC^\bullet(I), M)).$$

These functors are relevant because the cofibration $PK^\bullet(I) \to R \to PC^\bullet(I)$ induces long exact sequences relating Čech (co)homology with local (co)homology that can be useful in computations.

II.3.3. L-complete modules. As pointed out before, the fact that the I-adic completion functor is neither left nor right suggests to replace it with its zeroth derived functor and study the homological algebra of the so called L-complete modules.

Let R be a commutative Noetherian ring, and I an ideal generated by a *regular sequence* $x_0, x_1, \ldots x_{n-1}$. Recall that $x_0, \cdots, x_{n-1}$ is a regular sequence if x_i is not a zero divisor of $R/(x_0, \cdots, x_{i-1})$, for $i = 0, \cdots, n-1$.

Using the description of L_0^I as $H_0^I(\mathrm{Hom}(PK^\bullet(I), M))$, it follows that the cochain map $PK^\bullet \to R$ induces a natural map $M \xrightarrow{\eta} L_0^I M$. Combining this with Proposition II.3.7, we get natural maps $M \xrightarrow{\eta} L_0^I M \twoheadrightarrow M_I^\wedge$ so that the composition $M \to M_I^\wedge$ is the usual transformation.

Definition II.3.12 (**L-complete modules**)**.** We say that M is *L-complete* if η is an isomorphism.

For instance $M_I^\wedge$ and $L_k^I M$ for all $k \geq 0$ are L-complete modules. Moreover, if M is finitely generated, then $L_0^I M = M_I^\wedge = R_I^\wedge \otimes M$ [**GM92**].

Let $\widehat{\mathrm{Mod}}_R$ be the full subcategory of L-complete modules. We have the following theorem summarizing the basic properties of $\widehat{\mathrm{Mod}}_R$. We will exploit these facts repeatedly throughout the document.

THEOREM II.3.13.

(1) *Every I-adically complete module is L-complete.*

(2) *If M is L-complete, then $L_k^I(M) = 0$ for $k \geq 1$.*

(3) $\widehat{\mathrm{Mod}}_R$ *is an Abelian subcategory of* Mod_R*, which is closed under extensions.*

(4) *If $\{M_k\}$ is a collection of L-complete modules, then $\prod_k M_k$ is L-complete. Furthermore, if they form an inverse system then $\lim_k M_k$ is L-complete, and if they form a tower, so is $\lim^1_k M_k$.*

(5) *If M is a flat module, then $L_0^I M = M_I^\wedge$ and $L_k^I M = 0$ for $k \geq 1$.*

PROOF. The proof of these results follows word-by-word from Theorem A.6 in [**Hov99**] except for (4) which is Corollary 1.3 in [**Hov04b**]. $\square$

CHAPTER III

Axiomatic stable homotopy theory

In the last chapter, we discussed the derived category of a ring R. This is an example of a more general phenomenon called a stable homotopy category. In this chapter, we recall the basics of stable homotopy categories from [**HPS97**], where they were introduced. In particular, we discuss Bousfield localization and colocalization, which will be crucial for our work. Before we do this, we will give a brief introduction to triangulated and closed symmetric monoidal categories.

III.1. Triangulated categories

Triangulated categories were introduced by Verdier as an abstraction of the derived category he had introduced in his thesis based on ideas of Grothendieck. Our exposition is based on Neeman's book [**Nee01**] and we refer to it for any further details. The derived category of a ring R is an additive category but it is not Abelian; kernels and cokernels rarely exist. Nonetheless, there is a natural notion that encodes exact sequences.

A *triangle* (X, Y, Z, u, v, w) in an additive category $\mathcal{C}$ with an additive automorphism $\Sigma : \mathcal{C} \to \mathcal{C}$ is a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X,$$

where vu and wv are zero. Morphisms of triangles are commutative diagrams

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

We will refer to the automorphism Σ as the *suspension functor*.

Definition III.1.1 (Triangulated Category). A *triangulated category* $\mathcal{C}$ is an additive category together with an additive automorphism $\Sigma : \mathcal{C} \to \mathcal{C}$ and a class of *distingushed triangles* satisfying the following conditions:

(1) Any triangle which is isomorphic to a distinguished triangle is a distinguished triangle. The triangle

$$X \xrightarrow{1} X \to 0 \to \Sigma X$$

is distinguished.

(2) For any morphism $f : X \to Y$, there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \to Z \to \Sigma X.$$

(3) Consider the two triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X,$$

and

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y.$$

If one of them is distinguished, then so is the other.

(4) For any commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] & \Sigma X \arrow[d, "\Sigma f"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & \Sigma X'
\end{tikzcd}$$

where the rows are distinguished triangles, there is a morphism $h : Z \to Z'$, *not necessarily unique*, which makes the diagram above commute.

(5) *Octahedral axiom*. Given $f : X \to Y$ and $g : Y \to Z$, and distinguished triangles (X, Y, U, f, a, r), (Y, Z, W, g, b, s) and (X, Z, V, gf, c, t), there exists a triangle (U, V, W, u, v, w) so that the diagram below commutes.

$$\begin{tikzcd}
W \arrow[rr, dashed, "t"] \arrow[rrrr, dashed, bend left, "w"] & & Y \arrow[rr, "a"] \arrow[dl, "gf"'] & & U \arrow[dl, dashed, "r"] \arrow[ddll, bend left] \\
& Z \arrow[ul, "c"] \arrow[dr, "b"] & & X \arrow[ul, "f"'] \arrow[ll, "g"] & \\
& & V \arrow[ur, dashed, "s"'] \arrow[uull, bend left, "v"] & &
\end{tikzcd}$$

where every dashed arrow $A \dashrightarrow B$ means $A \to \Sigma B$.

Remark III.1.2. Given a map $f : X \to Y$, we are supposed to think of the Z in the distinguished triangle (X, Y, Z) given by (2) above as playing the role of both the kernel and cokernel. We will refer to Z as the *cofiber of* f and to $\Sigma^{-1}Z$ as the *fiber of* f. Thus, if we think of $f : X \to Y$ and $g : Y \to Z$ as injections, the octahedral axiom is essentially giving us the familiar isomorphism $(X/Y)/(Y/Z) \cong X/Z$.

Example III.1.3. As we mentioned before, $\mathrm{D}(R)$ is the prototype of a triangulated category, with the suspension functor given by translation, $(\Sigma X)_n = X_{n-1}$. There is an explicit construction of the cofiber of every map called the *mapping cone*, but we will not need it in our proofs (see [**Wei94**] for a definition). However, we will need the following fact that can be easily verified using this construction.

Proposition III.1.4. *Let $\mathcal{A}$ be an Abelian category, and suppose that the derived category $\mathrm{D}(\mathcal{A})$ exists. Then, every short exact sequence*

$$0 \to X \to Y \to Z \to 0$$

of chain complexes fits in an exact triangle in $\mathrm{D}(\mathcal{A})$ isomorphic to the one given by (2) in the definition of a triangulated category. In particular, the cofiber of $X \to Y$ is isomorphic to Z in $\mathrm{D}(\mathcal{A})$.

PROOF. See Example 10.4.9 in [**Wei94**]. □

In terms of functors between triangulated categories, we are interested in the following kind.

Definition III.1.5. An *exact functor* $F : \mathcal{C} \to \mathcal{D}$ between triangulated categories is an additive functor that sends distinguished triangles to distinguished triangles. More precisely, there is a natural isomorphism $F\Sigma \cong \Sigma F$ so that if

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

is a distinguished triangle, so is

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{Fw} \Sigma FX.$$

Note that by definition Σ is always an exact functor.

III.2. Closed symmetric monoidal categories

The derived category also has a tensor product and an internal hom-object that originate from the ones we have defined at the chain complexes level. These are important as they allow us to perform some of the formal algebraic manipulations that we are used to doing with modules, for example. All of the following can be found in A.2 of [**HPS97**].

Definition III.2.1 (Closed symmetric monoidal category). A *closed symmetric monoidal category* is a category $\mathcal{C}$ together with

(1) A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative and commutative up to coherent natural isomorphism. We will refer to this functor as the *symmetric monoidal product* or simply as the tensor product.

(2) A *unit object* S, such that $S \otimes X \cong X$ up to coherent isomorphism.

(3) *Function objects* $F(-,-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ contravariant in the first variable and covariant in the second such that

$$\mathrm{Hom}(X \otimes Y, Z) \cong \mathrm{Hom}(X, F(Y, Z))$$

naturally in all three variables.

Example III.2.2. The category of sets is closed symmetric monoidal. The Cartesian product gives a symmetric monoidal product while the unit can be chosen to be any singleton. The set of functions $\mathrm{Hom}(A, B)$ are the function objects.

If R is a commutative ring, then the category of R-modules Mod_R is the prototypical closed symmetric monoidal category. The symmetric monoidal product is the tensor product and the function objects are simply given by the categorical hom-functor $\mathrm{Hom}_R(M, N)$.

Additionally, as we discussed by the end of Subsection II.2.1, $\mathrm{Ch}(R)$ inherits a closed symmetric monoidal structure from Mod_R.

Now suppose that $\mathcal{C}$ is also a triangulated category. Then, we require the closed symmetric monoidal structure to be *compatible* with the suspension functor in the following manner.

(1) The symmetric monoidal product preserves suspensions, i.e., there is a natural equivalence

$$e_{X,Y} : \Sigma X \otimes Y \to \Sigma(X \otimes Y).$$

Moreover, if $r_X : X \otimes S \to X$ denotes the unital equivalence, then

$$\begin{tikzcd}
\Sigma(X \otimes Y) \arrow[r, "e_{X,S}"] \arrow[dr, "r_{\Sigma X}"'] & \Sigma(X \otimes Y) \arrow[d, "\Sigma r_X"] \\
& \Sigma X
\end{tikzcd}$$

and if $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is the *associator*, then the following diagram commutes

$$\begin{tikzcd}
& (\Sigma X \otimes Y) \otimes Z \arrow[dl, "e_{X,Y} \otimes 1_Z"'] \arrow[dr, "a_{\Sigma X,Y,Z}"] & \\
\Sigma(X \otimes Y) \otimes Z \arrow[d, "e_{X \otimes Y,Z}"'] & & \Sigma X \otimes (Y \otimes Z) \arrow[d, "e_{X,Y \otimes Z}"] \\
\Sigma((X \otimes Y) \otimes Z) \arrow[rr, "\Sigma a_{X,Y,Z}"] & & \Sigma(X \otimes (Y \otimes Z))
\end{tikzcd}$$

Finally, we have isomorphisms $F(\Sigma X, Y) \cong \Sigma^{-1} F(X,Y)$ and $F(X, \Sigma Y) \cong \Sigma F(X,Y)$. These last two are redundant; they follow using the isomorphisms e, the adjunction in (3) of Definition III.2.1, and Yoneda's Lemma.

(2) The symmetric monoidal product is exact. This is, if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a distinguished triangle, and we use $e_{X,W}$ to identify $\Sigma X \otimes W$ with $\Sigma(X \otimes Y)$, then

$$X \otimes W \xrightarrow{f\otimes 1} Y \otimes W \xrightarrow{g\otimes 1} Z \otimes W \xrightarrow{h\otimes 1} \Sigma(X \otimes W)$$

is a distinguished triangle as well.

(3) The functor $F(-,-)$ is exact in the first variable in the same way, but it is only exact in the second variable up to a sign. If (X, Y, Z, f, g, h) is a distinguished triangle, then

$$\Sigma^{-1}F(X,W) \xrightarrow{-F(h,1)} F(Z,W) \xrightarrow{F(g,1)} F(Y,W) \xrightarrow{F(f,1)} F(X,W).$$

(4) Let $T : X \otimes Y \to Y \otimes X$ the twist map. If we write S^n for $\Sigma^n S$ for all n, we require the product of S^r and S^t to be graded-commutative

$$\begin{array}{ccc} S^r \otimes S^t & \xrightarrow{\cong} & S^{r+t} \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle (-1)^{rt}} \\ S^t \otimes S^r & \xrightarrow{\cong} & S^{r+t} \end{array}$$

Example III.2.3. The main example of a closed symmetric monoidal triangulated category to have in mind is the derived category of a commutative ring R. The symmetric monoidal product is the *total left derived tensor product* $\otimes^L$, which is the universal functor $\otimes : \mathrm{D}(R) \times \mathrm{D}(R) \to \mathrm{D}(R)$ so that if we see $M, N \in \mathrm{D}(R)$ as complexes concentrated in degree zero, then

$$H_i(M \otimes^L N) = \mathrm{Tor}_i(M, N),$$

for all $i \geq 0$. Likewise, the function objects are given by the *total right derived functor of* Hom_R F, which is the universal functor $F : \mathrm{D}(R) \times \mathrm{D}(R) \to \mathrm{D}(R)$ satisfying

$$H^i(F(M,N)) = \mathrm{Ext}^i(M,N)$$

for all modules M, N and $i \geq 0$. The existence of such functors is not automatic; it is a consequence of the existence of projective and injective resolutions for arbitrary objects [**Wei94**].

Finally, we define the type of functors that respect the symmetric monoidal structure.

Definition III.2.4. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, S_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, S_{\mathcal{D}})$ be two closed symmetric monoidal categories. We say $G : \mathcal{C} \to \mathcal{D}$ is symmetric monoidal if

$$G(S_{\mathcal{C}}) \cong S_{\mathcal{D}}, \text{ and}$$

$$G(X \otimes_{\mathcal{C}} Y) \cong G(X) \otimes_{\mathcal{D}} G(Y),$$

where the last isomorphism is natural in both variables.

III.3. Axiomatic stable homotopy theory

The purpose of this section is to set up the abstract framework in which we will prove all our results. Almost all of this material can be found in [**HPS97**], but we will present it here for the convenience of the reader.

We begin with the definition of certain types of subcategories which we will need to consider frequently.

Definition III.3.1 (Type of subcategories of $\mathcal{C}$). Let $\mathcal{C}$ be a closed symmetric monoidal triangulated category that is cocomplete (i.e., it has arbitrary coproducts).

(1) A subcategory $\mathcal{D}$ of $\mathcal{C}$ is said to be *thick* if it is closed under cofibrations and retracts. More explicitly, if

$$X \to Y \to Z \to \Sigma X$$

is a distinguished triangle and two of X, Y, Z are in $\mathcal{D}$, so is the third. And if we have $Y \in \mathcal{D}$ fitting in a commutative diagram

$$\begin{tikzcd} & Y \arrow[dr, "p"] & \\ X \arrow[ur, "i"] \arrow[rr, "1_X"] & & X \end{tikzcd}$$

then X is also in $\mathcal{D}$.

(2) A thick subcategory $\mathcal{D}$ is *localizing* if $\mathcal{D}$ is closed under coproducts, i.e., for any collection of objects $\{X_\alpha\}$ of $\mathcal{D}$, $\coprod_\alpha X_\alpha$ is also in $\mathcal{D}$. Given a collection of objects $\mathcal{A}$, $\mathsf{Loc}(\mathcal{A})$ will denote the localizing subcategory generated by $\mathcal{A}$.

(3) A *localizing ideal* $\mathcal{D}$ is a localizing subcategory satisfying $X \otimes Z \in \mathcal{D}$ whenever $Z \in \mathcal{D}$ and $X \in \mathcal{C}$. The localizing ideal generated by a collection of objects $\mathcal{A}$ will be denoted by $\mathsf{Locid}(\mathcal{A})$.

We will also need to consider invariants that capture the same formal properties that cohomology has on the derived category of a ring.

Definition III.3.2. A *cohomology functor* $H : \mathcal{C} \to \mathsf{Ab}$ is an additive contravariant functor from $\mathcal{C}$ into the category of Abelian groups Ab so that

(1) It is *exact*, i.e., for any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

the sequence

$$H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$$

is exact.

(2) H takes coproducts into products.

Finitely generated modules are fundamental to module theory as one can often reduce the proof of complicated results about a large class of modules to showing the statement holds for finitely generated ones. This is a concept that does not necessarily make sense in this level of generality, but we can instead consider the following categorical analog.

Definition III.3.3. An object Z of an additive category $\mathcal{C}$ is said to be *compact* if for any collection of objects $\{X_\alpha\}$, the natural map

$$\bigoplus_\alpha \mathrm{Hom}(Z, X_\alpha) \to \mathrm{Hom}(Z, \coprod_\alpha X_\alpha)$$

is an isomorphism.

Finally, we have the relevant notion of duality in the context of symmetric monoidal categories.

Definition III.3.4. Suppose that $\mathcal{C}$ is a closed symmetric monoidal category. We say that $Z \in \mathcal{C}$ is *strongly dualizable* if the natural map

$$F(Z,S) \otimes X \to F(Z,X)$$

is an isomorphism. The object $DZ = F(Z,S)$ is called the *Spanier-Whitehead dual of* Z.

We are now ready to introduce stable homotopy categories.

Definition III.3.5 (Stable homotopy category). A stable homotopy category $\mathcal{C}$ is a category with the following structure:

(1) A triangulation.
(2) A closed symmetric monoidal structure, compatible with the triangulation.
(3) A set $\mathcal{G}$ of strongly dualizable objects of $\mathcal{C}$, such that the only localizing subcategory of $\mathcal{C}$ containing $\mathcal{G}$ is $\mathcal{C}$ itself.

We also require that $\mathcal{C}$ satisfies the following:

(1) $\mathcal{C}$ has arbitrary coproducts.

(2) Every cohomology functor on $\mathcal{C}$ is representable. This is, for every cohomology functor H, there exists an object K in $\mathcal{C}$ such that

$$H(X) = \mathrm{Hom}(X, K)$$

for all $X \in \mathcal{C}$.

If in addition the objects of $\mathcal{G}$ are compact, we say that $\mathcal{C}$ is *algebraic*. If $\mathcal{C}$ is algebraic and the unit S is compact, we say that $\mathcal{C}$ is *algebraic unital*. Finally, if $\mathcal{C}$ is algebraic and $\mathcal{G} = \{S\}$, $\mathcal{C}$ is said to be *monogenic*.

Remark III.3.6. Condition (3) says that every object of $\mathcal{C}$ can be constructed from elements of $\mathcal{G}$ via coproducts and cofibers. This is similar to the fact that, for a fixed commutative ring R, every R-module can be written as a quotient of a direct sum of R's. We thus refer to the elements of $\mathcal{G}$ as *generators*.

We already know that $\mathrm{D}(R)$ is a triangulated closed symmetric monoidal category in a compatible way, and the next theorem shows that it is in fact a stable homotopy category generated by R as a complex concentrated in degree zero.

THEOREM III.3.7. [**HPS97**, Theorem 9.3.1] *Let R be a commutative ring. Then* $\mathrm{D}(R)$ *is a monogenic stable homotopy category.*

Other known examples of stable homotopy categories are the stable homotopy category of spectra and the stable category of comodules over a Hopf algebra [**HPS97**].

There are many results well known in the context of the examples above that can be proven for general stable homotopy categories. In the following, we will present the ones that are particularly useful to us.

Proposition III.3.8. *Let $\mathcal{C}$ be a stable homotopy category. Suppose that $f : X \to Y$ induces an isomorphism* $\mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ *for all $Z \in \{\Sigma^n W | W \in \mathcal{G}\}$. Then, f is an isomorphism.*

PROOF. This is Lemma 1.4.5 of [**HPS97**]. □

Proposition III.3.9. *Let $\mathcal{C}$ be an algebraic stable homotopy category. Then, Z is compact implies Z is strongly dualizable.*

PROOF. See Theorem 2.1.3 (c) of [**HPS97**]. □

III.4. Bousfield localization

An important concept in stable homotopy theory is the notion of localization, introduced by Bousfield in the context of spectra [**BK72**].

Definition III.4.1 (Bousfield (co)localization). We will say that an endofunctor L on $\mathcal{C}$ is a *localization functor*, if there is a natural transformation $i : 1 \to L$ so that $Li : L \to L^2$ is an equivalence, and the natural map

$$\mathrm{Hom}(LX, LY) \xrightarrow{i_X^*} \mathrm{Hom}(X, LY)$$

is an isomorphism for all $X, Y \in \mathcal{C}$. Dually, a *colocalization functor* is an endofunctor T on $\mathcal{C}$ with a natural transformation $q : T \to 1$ such that $Tq : T^2 \to T$ is an equivalence, and

$$\mathrm{Hom}(TX, TY) \xrightarrow{q_{Y*}} \mathrm{Hom}(TX, Y)$$

is an isomorphism for all $X, Y \in \mathcal{C}$.

To every localization functor L corresponds a colocalization functor T (and *vice versa*) such that $TX \to X \to LX$ is a cofiber sequence for all $X \in \mathcal{C}$.

Definition III.4.2 (Local and acyclic objects)**.** Let L be a localization. An object is said to be *L-local* if $X \to LX$ is an isomorphism. We say that X is *L-acyclic* when $\mathrm{Hom}(X, Z) = 0$ for all L-local objects Z.

The following proposition follows easily from the definition. These statements are part of Lemma 3.1.6 of [**HPS97**].

Proposition III.4.3. *Let L be a localization on $\mathcal{C}$.*

(1) *An object is L-local if and only if* $\mathrm{Hom}(Z, X) = 0$ *(or* $F(Z, X) = 0$*) for all L-acyclic objects Z.*
(2) *The subcategory of L-acyclic objects forms a localizing ideal.*

Another property that is nice about localizations is that they interact well with the symmetric monoidal product.

Proposition III.4.4. *For any localization functor L, the map*

$$L(i_x \otimes i_Y) : L(X \otimes Y) \to L(LX \otimes LY)$$

is an isomorphism.

PROOF. By the exactness of the symmetric monoidal product, we have distinguished triangles

$$X \otimes CY \to X \otimes Y \to X \otimes LY, \text{ and}$$

$$CX \otimes LY \to X \otimes LY \to LX \otimes LY.$$

But since the acyclic objects form a localizing ideal, the fact that CX is acyclic implies that $X \otimes CY$ and $CX \otimes LY$ are acyclic as well. Hence, applying L to the triangles above, we get isomorphisms

$$L(X \otimes Y) \xrightarrow{\cong} L(X \otimes LY) \xrightarrow{\cong} L(LX \otimes LY),$$

and the conclusion follows. □

The following is a particular type of localization that will be important for us.

THEOREM III.4.5 (**Existence of finite localization**). *Let $\{X_\alpha\}$ be a set of compact objects of $\mathcal{C}$. Let $\mathcal{D}$ be the localizing ideal generated by*

$$\mathcal{A} = \{Z_1 \otimes \cdots \otimes Z_r \otimes X_\alpha | Z_i \in \mathcal{G}\ 1 \leq i \leq r, \alpha\}.$$

Then, there is a unique localization whose acyclics are precisely $\mathcal{D}$. We refer to this localization as finite localization away from $\{X_\alpha\}$.

PROOF. This is Theorem 3.3.3 in [**HPS97**]. □

Notice that $\mathcal{A}$ is made out of compact objects. Indeed, if Z is strongly dualizable, W is compact, and $\{Y_\beta\}$ is some collection of objects, then

$$\begin{aligned}
\mathrm{Hom}(Z \otimes W, \coprod_\beta Y_\beta) &\cong \mathrm{Hom}(W, DZ \otimes \coprod_\beta Y_\beta) \\
&\cong \mathrm{Hom}(W, \coprod_\beta (DZ \otimes Y_\beta)) \\
&\cong \bigoplus_\beta \mathrm{Hom}(W, DZ \otimes Y_\beta) \\
&\cong \bigoplus_\beta \mathrm{Hom}(Z \otimes W, Y_\beta)
\end{aligned}$$

so $Z \otimes W$ is compact as well.

A localization L is *smashing* if $LX = LS \otimes X$ for every object $X \in \mathcal{C}$. The following proposition is a characterization of smashing localizations that will allow us to show that all finite localizations are smashing.

Proposition III.4.6. *A localization is smashing if and only if its category of L-local objects is localizing.*

PROOF. See Definition/Proposition 3.3.2 in [HPS97]. □

Proposition III.4.7. *Let L be a finite localization away from $\{X_\alpha\}$, some collection of compact objects. Then, L is smashing.*

PROOF. By Proposition III.4.6, it suffices to show that the category of local objects is closed under coproducts. Let Y_β be a collection of L-locals. Since $\mathcal{D} = \mathsf{Locid}(\mathcal{A})$, by Proposition III.4.3, X is local if and only if $\mathrm{Hom}(Z, X) = 0$ for all $Z \in \mathcal{A}$. But the objects of $\mathcal{A}$ are compact, hence

$$\mathrm{Hom}(Z, \coprod_\beta Y_\beta) \cong \bigoplus_\beta \mathrm{Hom}(Z, Y_\beta) = 0$$

for all $Z \in \mathcal{A}$. This completes the proof. □

Thus, the class of L-local objects of a finite localization forms a localizing subcategory. One can therefore expect to find another localization functor whose acyclic category is precisely the subcategory of L-local objects. This situation is completely characterized by the following theorem.

THEOREM III.4.8 ([HPS97, Theorem 3.3.5]). *Let $\mathcal{C}$ be an algebraic stable homotopy category and $\{X_\alpha\}$ a set of compact objects. Let L be the finite localization away* *from* $\{X_\alpha\}$ *and* $\mathcal{A}$ *as above. Then, there are (co)localization functors*

$$TX \to X \to LX$$

$$AX \to X \to CX$$

with the following properties:

(1) $LX = LS \otimes X$ *and* $TX = TS \otimes X$.

(2) $CX = F(TS, X)$ *and* $AX = F(LS, X)$.

(3) *The category of* C*-acyclics is equal to the category of* L*-locals,* $\mathcal{C}_L$*, both of which are equal to*

$$\{Y \in \mathcal{C} \,|\, \mathrm{Hom}(Z, Y) = 0, \forall\, Z \in \mathcal{A}\}.$$

(4) $\mathcal{C}_T = \mathsf{Loc}(\mathcal{A})$ *is the category of* L*- acyclics.*

(5) *We have isomorphisms* $CT = L$ *and* $TC = T$.

(6) *The functors* $C : \mathcal{C}_T \to \mathcal{C}_C$ *and* $T : \mathcal{C}_C \to \mathcal{C}_T$ *are mutually inverse equivalences.*

Remark III.4.9. Theorem 3.3.5 of [HPS97] holds under milder hypotheses on $\mathcal{C}$, but we will only use it as stated above.

It was suggested to the author that the equivalence of categories in part (6) of the theorem above might be an equivalence of symmetric monoidal categories. While it is known that the essential image of a localization (i.e. the category of local objects) inherits the structure of a stable homotopy category

[HPS97, Theorem 3.5.1], we were unable to find a similar statement for the essential image of a colocalization. The next result addresses the closed symmetric monoidal structure in the latter situation.

Proposition III.4.10. *Let $\mathcal{C}$ be a stable homotopy category and T a smashing colocalization functor on $\mathcal{C}$. Then, $\mathcal{C}_T$ inherits a closed symmetric monoidal structure from $\mathcal{C}$. The symmtric monoidal product is given by $X \otimes_T Y = X \otimes Y$ with unit $S_T = TS$, and the function objects are given by $F_T(X,Y) = TF(X,Y)$.*

PROOF. First, if $X, Y \in \mathcal{C}_T$, then $T(X \otimes Y) \cong (TS \otimes X) \otimes Y \cong X \otimes Y$ as T is smashing. Therefore, $X \otimes_T Y = X \otimes Y$ defines a symmetric monoidal product on $\mathcal{C}_T$. For the same reason, $TS \otimes X = TX = X$ for all objects X in $\mathcal{C}_T$, thus $S_T = TS$ is the unit for $\otimes_T$. Finally, for all $X, Y, Z \in \mathcal{C}_T$, we have natural isomorphisms

$$\mathrm{Hom}(X \otimes_T Y, Z) \cong \mathrm{Hom}(X, F(Y,Z)) \cong \mathrm{Hom}(X, TF(Y,Z)).$$

It follows that the function objects in $\mathcal{C}_T$ are given by $F_T(X,Y) = TF(X,Y)$ as claimed. This completes the proof. $\square$

Corollary III.4.11. *The functors $C : \mathcal{C}_T \to \mathcal{C}_C$ and $T : \mathcal{C}_C \to \mathcal{C}_T$ of Theorem III.4.8 are mutually inverse symmetric monoidal equivalences.*

PROOF. We want to show that C and T are symmetric monoidal functors. We have natural isomorphisms

$$CS_T = CTS \cong CS = S_C$$

$$TS_C = TCS \cong TS = S_T.$$

Moreover,

$$CX \otimes_C CY = C(CX \otimes CY) \cong C(X \otimes Y) = C(X \otimes_T Y),$$

where we are using Proposition III.4.4 to obtain the first equivalence above. Similarly,

$$TX \otimes_T TY = TX \otimes TY \cong T(X \otimes Y) \cong TC(X \otimes Y) = T(X \otimes_C Y).$$

□

As pointed out in [**HPS97**], the subcategory of colocal objects will not in general form a stable homotopy category. The issue is that it is unrealistic to expect that colocalization preserves generators. However, if T is a finite colocalization, then T preserves strongly dualizable objects. This is the key observation that makes the following theorem possible.

THEOREM III.4.12. *Let T be the colocalization corresponding to a finite localization L as in Theorem III.4.8. Then, $\mathcal{C}_T$ inherits the structure of a stable homotopy category.*

PROOF. Our proof closely follows the one of [HPS97, Theorem 3.5.1]. We repeat every step for the sake of completeness.

First, we declare $X \to Y \to Z \to \Sigma X$ to be a triangle in $\mathcal{C}_T$ if it is one in $\mathcal{C}$. Since $\mathcal{C}$ is thick, this forms a triangulation of $\mathcal{C}_T$.

The closed symmetric monoidal structure follows from Proposition III.4.10 as T is smashing.

$T \coprod X_\alpha = \coprod TX_\alpha$, hence $\coprod_T = \coprod$ is the coproduct in $\mathcal{C}_T$.

Let $\mathcal{G}_T = \{TZ | Z \in \mathcal{G}\}$, where $\mathcal{G}$ is a set of strongly dualizable generators of $\mathcal{C}$. Using part (2) of Theorem III.4.8, we can write natural isomorphisms

$$TF(TZ, X) \cong TF(TS, F(Z, X)) \cong TCF(Z, X) \cong TF(Z, X),$$

for all $Z \in \mathcal{C}$ and all $X \in \mathcal{C}_T$. Therefore, for Z strongly dualizable in $\mathcal{C}$ and X in $\mathcal{C}_T$, we have

$$TF(TZ, X) \cong TF(Z, X) \cong TF(Z, S) \otimes X \cong TF(TZ, S) \otimes X \cong TF(TZ, TS) \otimes X,$$

where the last isomorphism is owed to the fact that T is the right adjoint to the inclusion of $\mathcal{C}_T$ into $\mathcal{C}$. It follows that $\mathcal{G}_T$ is a set of strongly dualizable objects of $\mathcal{C}_T$. Let $\mathcal{D}$ be a localizing subcategory of $\mathcal{C}_T$ containing $\mathcal{G}_T$. Then $\{X \in \mathcal{C} | TX \in \mathcal{D}\}$ is a localizing subcategory of $\mathcal{C}$ that contains $\mathcal{G}$, hence all of $\mathcal{C}$. This shows that $\mathcal{G}_T$ generates $\mathcal{C}_T$.

Let $H : \mathcal{C}_T \to \mathsf{Ab}$ be a cohomology functor. The functor $\widehat{H}(X) := H(TX)$ is a cohomology functor on $\mathcal{C}$ because T is smashing. Hence, there exists an object

$K \in \mathcal{C}$ such that $\widehat{H}(X) \cong \mathrm{Hom}(X, K)$. If $X \in \mathcal{C}_T$, we have natural isomorphisms

$$\widehat{H}(X) = H(TX) \cong H(X) \cong \mathrm{Hom}(X, K) \cong \mathrm{Hom}(X, TK).$$

We conclude that TK represents the cohomology functor H.

We have thus shown that $\mathcal{C}_T$ is a stable homotopy category. □

Corollary III.4.13. *The functors $C : \mathcal{C}_T \to \mathcal{C}_C$ and $T : \mathcal{C}_C \to \mathcal{C}_T$ of Theorem III.4.8 are mutually inverse equivalences of stable homotopy categories.*

PROOF. By Theorem III.4.8 part (5), we have $C\mathcal{G}_T = \mathcal{G}_C = \{CZ | Z \in \mathcal{G}\}$ and $T\mathcal{G}_C = \mathcal{G}_T$. □

III.5. Bousfield classes

We now recall the theory of Bousfield classes. These were introduced in [BK72] for the stable homotopy category, but they can be defined in any triangulated symmetric monoidal category.

Definition III.5.1 (Bousfield class). Let $\mathcal{C}$ be an algebraic stable homotopy category. We say that W is *X-acyclic* if $W \otimes X = 0$. The collection of X-acyclics denoted by $\langle X \rangle$ is called the *Bousfield class* of X.

Notice that every Bousfield class is a localizing ideal.

The name X-acyclics is justified by the following proposition.

Proposition III.5.2. *Let $\mathcal{C}$ be an algebraic stable homotopy category. For any object X, there exists a localization L_X such that its class of acyclic objects is given by $\langle X\rangle$. Moreover, L_X is determined uniquely by the Bousfield class of X; $L_X \cong L_Y$ if and only if $\langle X\rangle = \langle Y\rangle$.*

PROOF. This is Lemma 3.6.6 together with Proposition 3.6.9 (e) of [**HPS97**]. □

There is a partial order on Bousfield classes given by reverse inclusion

$$\langle X\rangle \leq \langle Y\rangle \iff (W \otimes Y = 0 \implies W \otimes X = 0, \text{ for all } W \in \langle Y\rangle).$$

Note that $\langle 0\rangle$ is the smallest Bousfield class while $\langle S\rangle$ is the largest.

Given $X, Y \in \mathcal{C}$, we define the *join* of their corresponding Bousfield classes as $\langle X\rangle \vee \langle Y\rangle = \langle X \sqcup Y\rangle$. We get another operation of Bousfield classes by using the symmetric monoidal product, $\langle X\rangle \wedge \langle Y\rangle = \langle X \otimes Y\rangle$. Finally, if we assume that $\mathcal{C}$ has a set of Bousfield classes, we can define the *complementation operator* to be

$$a\langle X\rangle = \bigvee_{\langle Y \otimes X\rangle = \langle 0\rangle} \langle Y\rangle \,.$$

This is not a restrictive assumption; every well generated triangulated category has a set of Bousfield classes [**IK13**]. Moreover, the latter is also true for homotopy categories of pointed combinatorial monoidal model cateogories [**CGR14**]. For example, the stable homotopy category of spectra and the derived category of any commutative ring have a set of Bousfield classes.

We call a class $\langle X\rangle$ *complemented* if there exists another class $\langle Y\rangle$ such that $\langle X\rangle \wedge \langle Y\rangle = \langle 0\rangle$ and $\langle X\rangle \vee \langle Y\rangle = \langle S\rangle$. It is easy to check that if $\langle X\rangle$ is complemented, then its complement is given by $a\langle X\rangle$, thus its name. However, not every class needs to be complemented.

Another useful result is the fact that we can get a decomposition of any Bousfield class $\langle X\rangle$ in terms of the cofiber of a self map $f : \Sigma^d X \to X$ and the *telescope*, $f^{-1}X$, defined as the colimit of the sequence

$$\Sigma^d X \xrightarrow{f} X \xrightarrow{\Sigma^{-d} f} \Sigma^{-d} X \xrightarrow{\Sigma^{-2d} f} \cdots .$$

The notation suggests that f is becoming invertible at the limit, just as the direct limit of the sequence of maps $\mathbb{Z} \xrightarrow{p} \mathbb{Z}$ is $\mathbb{Z}[p^{-1}]$.

Proposition III.5.3. *Let $f : \Sigma^d X \to X$ be a self map in an algebraic stable homotopy category $\mathcal{C}$. Then, $\langle X\rangle = \langle X/f\rangle \vee \langle f^{-1}X\rangle$, where X/f denotes the cofiber of f and $f^{-1}X$ its telescope.*

The following result allows us to write a Bousfield class in terms of any representative of its complement. The proof of it depends on a result of Benson, Iyengar, and Krause proved in the context of symmetric monoidal triangulated categories stratified by the action of a Noetherian ring [**BIK11**].

Let $\mathcal{C}$ be an algebraic unital stable homotopy category. We say that *a Noetherian ring R acts on $\mathcal{C}$* if there is a ring homomorphism $R \to Z(\mathrm{Hom}(S, S))$

to the center of the ring $\mathrm{Hom}(S, S)$. In this way, $\mathrm{Hom}(X, Y)$ becomes an R-module for all $X, Y \in \mathcal{C}$. For example, $\mathbb{Z}$ acts on every algebraic unital stable homotopy category (as $\mathbb{Z}$ acts canonically on every ring), and R acts on $\mathrm{D}(R)$ since $\mathrm{Hom}(R, R) = R$. The action of R on $\mathcal{C}$ allows us to define the support of an object of $\mathcal{C}$. For this, we first recall the notions of support and minimal primes for modules.

The *injective hull* of an R-module M, is an injective module $E(M)$ containing M such that for every submodule $K \subseteq E(M)$, $K \cap M = 0$ implies $K = 0$. Let $\mathfrak{p}$ be a prime ideal of R. We say that $\mathfrak{p}$ *occurs in a minimal injective resolution I of M*, if for some k the module I^k has a direct summand isomorphic to $E(R/\mathfrak{p})$. We can then define the *support* of a module M as

$$\mathrm{supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) | \ \mathfrak{p} \text{ occurs in a minimal resolution of } M\}.$$

Finally, recall that a prime $\mathfrak{p}$ is *minimal prime ideal* if it is minimal under inclusion among the primes that contain 0. A Noetherian ring has only finitely many minimal prime ideals [**Eis95**]. Let us write $\min_R(M)$ for the set of minimal primes in $\mathrm{supp}_R(M)$

We now define the *support* of an object $X \in \mathcal{C}$ as

$$\mathrm{supp}_R X = \bigcup_{Z \text{ compact}} \min_R \mathrm{Hom}(Z, X).$$

$\mathcal{C}$ is said to be *stratified by* R, if there is inclusion-preserving one-to-one correspondence

$$\{\text{Localizing subcategories of } \mathcal{C}\} \leftrightarrow \{\mathrm{supp}_R(X) | X \in \mathcal{C}\}.$$

A natural example of a category of this type is the derived category of a Noetherian ring.

Proposition III.5.4. *Let $\mathcal{C}$ be an algebraic unital stable homotopy category stratified by the action of a Noetherian ring R. If $\langle X\rangle$ is complemented with complement $\langle Y\rangle$, then $\langle X\rangle = \mathsf{Locid}(Y)$.*

PROOF. For each $X \in \mathcal{C}$, there exists an object $aX \in \mathcal{C}$ such that $\langle X\rangle = \mathsf{Locid}(aX)$, where $\langle aX\rangle = a\langle X\rangle$ (see [**Wol14**], [**HP99**]). But $\langle X\rangle$ is complemented with complement $\langle Y\rangle$, hence $a\langle X\rangle = \langle Y\rangle$. The conclusion follows from the fact that in a category stratified by the action of a Noetherian ring, $\langle Z\rangle = \langle Y\rangle$ implies that $\mathsf{Locid}(Z) = \mathsf{Locid}(Y)$ [**IK13**]. □

Remark III.5.5. The statement $\langle X\rangle = \langle Y\rangle$ implies $\mathsf{Locid}(X) = \mathsf{Locid}(Y)$ for all $X, Y \in \mathcal{C}$ is equivalent to having every localizing subcategory of $\mathcal{C}$ being equal to the Bousfield class of some object [**HP99**]. This condition does not always hold. For example, Stevenson shows that the derived category of an absolute flat ring that is not semi-Artinian fails to have this property [**Ste14**, Corollary 4.10]. The same is true for the category of $H\mathbb{F}_p$-local spectra [**Wol13**, Proposition 6.4].

CHAPTER IV

Completion and torsion on the derived category

The purpose of this chapter is to show that the natural extension of the L-completion and I-torsion functors to the derived category can be interpreted in terms of Bousfield localization. This is an important step towards our goal because it allows us to introduce the techniques of axiomatic homotopy theory into the picture. Most of the results here are not new; the connection between completion and Bousfield localization was introduced in [**GM95**], and further developed in [**Gre01**] and [**DG02**]. Nevertheless, the homotopical tools are the key to our new criterion for L-completeness, thus showing the power of this approach.

IV.1. Identification of functors on the derived category

Consider the complex $K^\bullet(I) \otimes M$ used to define the local cohomology of M. If we replace M by an arbitrary complex $X \in \mathrm{Ch}(R)$, and use the tensor product of chain complexes instead of the one of modules, we obtain an additive endofunctor on $\mathrm{Ch}(R)$. Moreover, our endofunctor descends to the derived category

$$\mathrm{D}(R) \to \mathrm{D}(R)$$

$$X \mapsto K^{\bullet}(I) \otimes X,$$

where now the tensor product is the total left derived tensor product.

One can obtain similar functors if we extend the formulas defining the rest of the (co)homology functors on Mod_R. In [**GM95**], Greenlees gives an interpretation of these functors in terms of Bousfield (co)localizations determined uniquely by R/I. Let L be the finite localization away from R/I and T its corresponding colocalization functor. The key observation is that since R/I is a compact object in $\mathrm{D}(R)$, it follows from Theorem III.4.8 that it suffices to identify L (or equivalently T) to get a characterization of the rest of the functors. Moreover, since finite localization and colocalization are smashing, they are completely determined by their corresponding value at the unit R. In the next paragraphs, we will show that the cofiber sequence

$$K^{\bullet}(I) \to R \to C^{\bullet}(I)$$

is indeed $TR \to R \to LR$. This is equivalent to showing that $C^{\bullet}(I)$ is L-local while $K^{\bullet}(I)$ is L-acyclic.

For the rest of this section we will assume that I is generated by a regular sequence $x_0, \cdots, x_{n-1}$. Recall that $x_0, \cdots, x_{n-1}$ is a regular sequence if x_i is not a zero divisor of $R/(x_0,\cdots,x_{i-1})$, for $i=0,\cdots,n-1$. Our proofs closely follows the ones found in [**GM95**], however, they are much more straightforward given our assumption on I.

Proposition IV.1.1. *$K^\bullet(I)$ is an L-acyclic object.*

PROOF. We need to show that $K^\bullet(m)$ is in the localizing subcategory generated by R/I, $\mathsf{Loc}(R/I)$. Let $K^\bullet_s(x) = R \xrightarrow{x^s} R$, and recall that $K^\bullet(x) = \operatorname{colim}_s K^\bullet_s(x)$, where the maps are given by

$$\begin{array}{ccc} R & \xrightarrow{x^s} & R \\ \| & & \downarrow{\scriptstyle x} \\ R & \xrightarrow{x^{s+1}} & R \end{array}$$

It is easy to see that the diagram above gives rise to cofibrations $K^\bullet_s(x) \to K^\bullet_{s+1}(x) \to K^\bullet_1(x)$. As with the Koszul complex, we have $K^\bullet_s(I) = K^\bullet_s(x_0)\otimes \cdots \otimes K^\bullet_s(x_{n-1})$. Because each of the $K^\bullet_s(x_i)$'s is a complex of free R-modules, we still have cofibrations $K^\bullet_s(I)\to K^\bullet_{s+1}(I)\to K^\bullet_1(I)$. It follows by induction that if $K^\bullet_1(I)\in \mathsf{Loc}(R/I)$, then $K^\bullet_s(I)\in \mathsf{Loc}(R/I)$ for all $s\geq 1$. Furthermore, since the tensor product commutes with colimits, we have $K^\bullet(I) = \operatorname{colim}_s K^\bullet_s(I)$. Thus, it suffices to show that $K^\bullet_1(I)$ is in the localizing subcategory generated by R/I. But this is easy because $H^k(K^\bullet_1(I)) = R/I$ if $k=n$ or zero otherwise, so $K^\bullet_1(I)$ is isomorphic to $\Sigma^n R/I$ in $\mathrm{D}(R)$. ∎

Proposition IV.1.2. *$C^\bullet(I)$ is an L-local object.*

PROOF. The following is exactly the proof in [Gre01]. We need to show that $\mathrm{Hom}(R/I, C^{\bullet}(I)) = 0$. It is not hard to construct a finite filtration of $C^{\bullet}(I)$ with successive quotients of the form $R[x^{-1}]$ for some $x \in I$. Therefore, it suffices to show that $\mathrm{Hom}(R/I, R[x^{-1}]) = 0$ for any $x \in I$. But multiplication by x is nilpotent on R/I, while it is invertible on $R[x^{-1}]$, so the conclusion follows. □

Thus, our identification theorem follows directly by applying III.4.8 to the finite localization away from R/I.

THEOREM IV.1.3. *Let L be the finite localization away from R/I. Then, there are (co)localizations*

(1) $LX = C^{\bullet}(I) \otimes X$ *and* $TX = K^{\bullet}(I) \otimes X$.
(2) $CX = F(PK^{\bullet}(I), X)$ *and* $AX = F(PC^{\bullet}(I), X)$.
(3) *The category of C-acyclics is equal to the category of L-locals, $\mathrm{D}(R)_L$, both of which are equal to*

$$\{Y \in \mathrm{D}(R) | \mathrm{Hom}(R/I, Y) = 0\}.$$

(4) $\mathrm{D}(R)_T = \mathsf{Loc}(R/I)$ *is the class of L-acyclics.*
(5) *There are isomorphism $CT = C$ and $TC = T$.*
(6) *The functors $C : \mathrm{D}(R)_T \to \mathrm{D}(R)_C$ and $T : \mathrm{D}(R)_C \to \mathrm{D}(R)_T$ are mutually inverse equivalences.*

In particular, using the definitions of local (co)homology and Čech (co)homolgy, we get:

Corollary IV.1.4. *Let M be an R-module. Then,*

(1) $H^k(TM) = H^k_I(M)$ *and* $H^k(LM) = \check{C}H^k_I(M)$,

(2) $H_k(CM) = L^I_k(M)$ *and* $H_k(AM) = \check{C}H^I_k(M)$,

where we think of M as the complex of R-modules concentrated in degree zero. In particular, M is L-complete if and only if it is C-local, and M is I-torsion if and only if it is T-colocal.

The compact objects of $\mathrm{D}(R)$ can be characterized in terms of *perfect complexes*. $X \in \mathrm{D}(R)$ is called perfect if it is quasi-isomorphic to a bounded complex of finitely generated projective R-modules. It is well known that the perfect complexes are precisely the compact objects in $\mathrm{D}(R)$ (see [**BIK12**] for example). For instance, R/I is a compact object when I is generated by a regular sequence. Indeed, we know $\Sigma^{-n}K_1^\bullet(I)$ is quasi-isomorphic to R/I and hence we have a bounded free resolution of R/I.

Note that since R/I is a compact object and $\mathrm{D}(R)$ is algebraic, R/I is strongly dualizable (Proposition III.3.9). Moreover, if I is generated by a regular sequence of length n, then

$$\begin{aligned} H_k(D(R/I)) = H_k(F(R/I,R)) &\cong H_k(F(\Sigma^{-n}K_1^\bullet(I),R)) \\ &\cong H^{n-k}(\Sigma^{-n}K_1^\bullet(I)) \\ &= R/I \end{aligned}$$

if $k = n$, and zero otherwise (e.g. [**Wei94**, Section 4.5]). We have thus proved:

Lemma IV.1.5. *$D(R/I) = \Sigma^n(R/I)$ in $\mathrm{D}(R)$, i.e., R/I is self-dual up to suspension.*

Using part (3) of Theorem IV.1.3, we can now get a different description of C.

Proposition IV.1.6. *The functor C is the localization associated to the Bousfield class of R/I.*

PROOF. We need to show that the category of C-acyclics is equal to $\langle R/I \rangle$. Using Spanier-Whitehead duality, we can write

$$\mathrm{Hom}(R/I, Y) \cong \mathrm{Hom}(R, D(R/I) \otimes Y) \cong \mathrm{Hom}(R, \Sigma^{-n}(R/I \otimes Y)).$$

Since $\mathrm{D}(R)$ is monogenic, Proposition III.3.8 implies that

$$\mathrm{Hom}(R, \Sigma^{-n}(R/I \otimes Y)) = 0 \iff \Sigma^{-n}(R/I \otimes Y) = 0.$$

We conclude from part (3) of IV.1.3 that Y is C-acyclic if and only if it is R/I-acyclic. □

Next, we use the algebraic formulas in IV.1.3 to derive spectral sequences that compute the (co)homology of the various (co)localizations in consideration. These have appeared in [**GM95**]. When we write the local (co)homology or Čech (co)homology applied to a complex X, we mean the complex obtained by applying the corresponding functor degreewise, together with the obvious (co)chain maps.

Proposition IV.1.7. *There are strongly convergent spectral sequences*

(1) ${}^{I}E_2^{pq} = \check{C}H_I^p(H^q(X))$ *and* ${}^{II}E_2^{pq} = H^p(\check{C}H_I^q(X))$ *converging to* $H^{p+q}(LX)$,

(2) ${}^{I}E_2^{pq} = H_I^p(H^q(X))$ *and* ${}^{II}E_2^{pq} = H^p(H_I^q(X))$ *converging to* $H^{p+q}(TX)$,

(3) ${}^{I}E^2_{pq} = L_p^I(H_q(X))$ *and* ${}^{II}E^2_{pq} = H_p(L_q^I(X))$ *converging to* $H_{p+q}(CX)$,

(4) ${}^{I}E^2_{pq} = \check{C}H_p^I(H_q(X))$ *and* ${}^{II}E^2_{pq} = H_p(\check{C}H_q^I(X))$ *converging to* $H_{p+q}(AX)$.

PROOF. These are all spectral sequences arising from the relevant double complexes. For example, consider $C^{p,q} = K^p(I) \otimes X^q$. The ${}^{II}E_2^{pq}$ term follows by the definition of local cohomology. For the ${}^{I}E_2^{pq}$ term, one just needs to observe that $K^\bullet(I)$ is a complex of flat modules to obtain the desired formula. Finally, since $K^\bullet(I)$ is a bounded complex, we have $\mathrm{Tot}^{\oplus}(C^{\bullet,\bullet}) = \mathrm{Tot}^{\Pi}(C^{\bullet,\bullet}) = K^\bullet(I) \otimes X$, and thus both spectral sequences have the desired target. All the spectral sequences in the statement are strongly convergent because both $K^\bullet(I)$ and $C^\bullet(I)$ are bounded complexes. □

We obtain an immediate corollary after observing that the relevant spectral sequences collapse.

Corollary IV.1.8.

(1) *If* $H^*(X)$ *is* I*-torsion, then* $TX \to X$ *is an equivalence.*

(2) *If* $H_*(X)$ *is* L*-complete, then* $X \to CX$ *is an equivalence.*

As an application of Corollary IV.1.8, we present a formula for C that, to the best of our knowledge, is not present in the current literature.

Proposition IV.1.9. *Let $X \in \mathrm{D}(R)$. Then, $CX \cong \lim_k(R/I^k \otimes X)$.*

PROOF. First, we want to prove that $\lim_k(R/I^k \otimes X)$ is C-local. By the corollary above, it suffices to show that $H_*(\lim_k(R/I^k \otimes X))$ is L-complete. We have

$$H_*(R/I^k \otimes X) = H_*(K_k \otimes X),$$

where K_k is a projective approximation of of R/I^k. We get a Milnor exact sequence

$$0 \to \lim_k{}^1 H_{q+1}(K_k \otimes X) \to H_q(CX) \to \lim_k H_q(K_k \otimes X) \to 0,$$

and since L-complete modules are closed under taking extensions, lim and $\lim^1$ (see Theorem II.3.13), it suffices to show that $H_q(K_k \otimes X)$ is L-complete. Because K_k is a complex of flat modules, we can compute $H_q(K_k \otimes X)$ using the Künneth spectral sequence

$$E_2^{pq} = \bigoplus_{q=s+t} \mathrm{Tor}_p^R(H_s(K_k), H_t(X)) \Longrightarrow H_{p+q}(K_k \otimes X).$$

By definition, $H_0(K_k) = R/I^k$ and $H_s(K_k) = 0$ otherwise, so we have $E_2^{pq} = \mathrm{Tor}_p^R(R/I^k, H_q(X))$. Moreover, $I^k \,\mathrm{Tor}_p^R(R/I^k, H_q(X)) = 0$ so $\mathrm{Tor}_p^R(R/I^k, H_q(X))$ is I-adically complete and hence L-complete. Finally, $H_*(K_k \otimes X)$ is recovered by taking extensions of quotients of the E_2 page, all operations that have results within the category of L-complete modules (Theorem II.3.13).

Next, we need to show that CX and $\lim_k(R/I^k \otimes X)$ are quasi-isomorphic on R/I-homology, i.e., $\mathrm{Hom}(R, R/I \otimes CX) \cong \mathrm{Hom}(R, R/I \otimes \lim_k(R/I^k \otimes X))$.

This is equivalent to $\mathrm{Hom}(R/I, CX) \cong \mathrm{Hom}(R/I, \lim_k(R/I^k \otimes X))$, and using formula in (3) Theorem IV.1.3, we get

$$\mathrm{Hom}(R/I, CX) = \mathrm{Hom}(R/I, F(K^\bullet(I), X)) \cong \mathrm{Hom}(R/I \otimes K^\bullet(I), X),$$

but the map $T(R/I) \cong R/I \otimes K^\bullet(I) \to R/I$ induces an isomorphism $H^*(R/I \otimes K^\bullet(I)) = H_I^*(R/I) = R/I$. Hence, $\mathrm{Hom}(R/I, CX) \cong \mathrm{Hom}(R/I, X)$. Similarly,

$$\begin{aligned}
\mathrm{Hom}(R/I, R/I^k \otimes X) &\cong \mathrm{Hom}(R/I \otimes D(R/I^k), X) \\
&\cong \mathrm{Hom}(R/I \otimes \Sigma^n(R/I^k), X) \\
&\cong \mathrm{Hom}(R/I \otimes K_k^\bullet(I), X) \\
&\cong \mathrm{Hom}(R/I, X).
\end{aligned}$$

Thus, the tower in the Milnor exact sequence

$$\lim_k{}^1 \mathrm{Hom}(R/I, R/I^k \otimes X) \hookrightarrow \mathrm{Hom}(R/I, \lim_k(R/I^k \otimes X)) \twoheadrightarrow \lim \mathrm{Hom}(R/I, R/I^k \otimes X)$$

is constant, and $\mathrm{Hom}(R/I, \lim_k(R/I^k \otimes X) \cong \mathrm{Hom}(R/I, X) = \mathrm{Hom}(R/I, CX)$ as we wanted to show. $\square$

Remark IV.1.10. The result above is a general algebraic analog of the following. Let X be an arbitrary spectrum. Then, the map

$$X \to \mathrm{holim}(X \wedge M(p^{i_0}, v_1^{i_1}, \cdots, v_{n-1}^{i_{n-1}}),$$

is $F(n)$-localization, where $M(p^{i_0}, v_1^{i_1}, \cdots, v_{n-1}^{i_{n-1}})$ is a generalized Moore spectrum, and $F(n)$ is a finite spectrum of type n. For more details, please refer to [**Hov95**].

IV.2. The Ext $-I$ completeness criterion

The fact that we are assuming I is generated by a regular sequence allows us to get an explicit characterization of $\langle R/I\rangle$, and also of L. We begin by giving a decomposition of the maximal Bousfield class.

Proposition IV.2.1. *Let R be a commutative Noetherian ring and I an ideal generated by the regular sequence $(x_0, \ldots, x_{n-1})$. Then, there is a tensor complemented decomposition of Bousfield classes in* $\mathrm{D}(R)$

$$\langle R\rangle = \langle R/I\rangle \sqcup \bigsqcup_{i=0}^{n-1} \left\langle x_i^{-1}R/(x_0, \ldots, x_{i-1})\right\rangle .$$

PROOF. Proposition III.5.3 says that if $X \xrightarrow{f} X$ is a self map with cofiber X/f and telescope $f^{-1}X$, then $\langle X\rangle = \langle X/f\rangle \sqcup \langle f^{-1}X\rangle$. Applying this to the self map $R \xrightarrow{x_0} R$, we get

$$\langle R\rangle = \langle R/x_0\rangle \sqcup \left\langle x_0^{-1}R\right\rangle .$$

Note that it is important that the map $R \xrightarrow{x_0} R$ is injective because then by Proposition III.1.4, $R/(x_0)$ is indeed the cofiber of $R \xrightarrow{x_0} R$, where we all modules are interpreted as complexes concentrated in degree zero. Since $(x_0, \ldots, x_{n-1})$

is a regular sequence, we can do this again now with the injective map $R/x_0 \xrightarrow{x_1} R/x_0$ to get

$$\langle R/x_0\rangle = \langle R/(x_0,x_1)\rangle \sqcup \left\langle x_1^{-1}R/x_0\right\rangle.$$

We get the desired formula by repeating this process until we obtain $\langle R/(x_0,\dots,x_{n-2})\rangle = \langle R/I\rangle \sqcup \left\langle x_{n-1}^{-1}R/(x_0,\dots,x_{n-2})\right\rangle$.

It remains to show that any two Bousfield classes in this decomposition are tensor complemented. Consider the self map

$$R/I\otimes^L x_i^{-1}R/(x_0,\dots,x_{i-1}) \xrightarrow{x_i} R/I\otimes^L x_i^{-1}R/(x_0,\dots,x_{i-1}).$$

On the first factor, x_i acts trivially while on the second it acts as an isomorphism. The only way this is possible is that $R/I\otimes^L x_i^{-1}R/(x_0,\dots,x_{i-1}) = 0$. The same argument shows that $x_i^{-1}R/(x_0,\dots,x_{i-1})\otimes^L x_j^{-1}R/(x_0,\dots,x_{j-1}) = 0$ for $i\neq j$. □

We can now use these results to obtain the desired characterizations.

Proposition IV.2.2. *Let R be a Noetherian regular local commutative ring with the maximal ideal generated by the regular sequence $(x_0,\dots,x_{n-1})$. Then,*

(1) $\langle R/I\rangle = \mathsf{Loc}\left(\bigsqcup_{i=0}^{n-1} x_i^{-1}R/(x_0,\dots,x_{i-1})\right)$.

(2) $L = L_{\bigsqcup_{i=0}^{n-1} x_i^{-1}R/(x_0,\dots,x_{i-1})}$.

PROOF. For (1), apply III.5.4 with $X = R/I$. By the Bousfield decomposition of IV.2.1, the smash-complement of R/I is $\bigsqcup_{i=0}^{n-1} x_i^{-1}R/(x_0,\dots,x_{i-1})$ and we are done. Similarly, it follows that $\left\langle \bigsqcup_{i=0}^{n-1} x_i^{-1}R/(x_0,\ldots,x_{i-1}) \right\rangle = \mathsf{Loc}(R/I)$. But the class of L-acyclic objects is precisely the localizing subcategory generated by R/I. $\square$

We finish this section by combining the main results above to obtain a new criterion for when an R-module is L-complete.

THEOREM IV.2.3 (Ext $-I$ **completeness criterion**). *Let M be an R-module. Then,*

$$M \textit{ is } L\textit{-complete} \iff \mathrm{Ext}_R^q(x_i^{-1}R/(x_0,\ldots,x_{i-1}),M) = 0,\ \forall q \geq 0,\ i = 1,\ldots,n-1.$$

PROOF. First, by Corollary IV.1.4, M is L-complete if and only if M is C-local as a complex concentrated in degree zero. Thus, M is L-complete if and only if $F(Z,M) = 0$, for all C-acyclic objects Z. But by Proposition IV.2.2, it is equivalent to check that

$$F(\bigsqcup_{i=0}^{n-1} x_i^{-1}R/(x_0,\ldots,x_{i-1}),M)) \cong \prod_{i=0}^{n-1} F(\, x_i^{-1}R/(x_0,\ldots,x_{i-1}),M)) = 0.$$

But the product above is zero if and only if $F(\, x_i^{-1}R/(x_0,\ldots,x_{i-1}),M)) = 0$ for all $i = 1,\ldots,n-1$. The conclusion follows by applying homology in the latter statement. $\square$

The explanation for the name of our criterion comes from Bousfield in [BK72], where he defines an Abelian group A to be Ext $-p$ complete if the map $A \to \mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, A)$ is an isomorphism. It is easy to see that this is equivalent to A

having $\mathrm{Hom}(\mathbb{Z}[p^{-1}], A) = 0 = \mathrm{Ext}(\mathbb{Z}[p^{-1}], A)$, which is in fact the definition for Ext $-p$ completness given in [**Mit**]. In our language, these are just equivalent ways to say that A is L-complete in the category of $\mathbb{Z}$-modules. Thus, Theorem IV.2.3 shows that our condition is the correct generalization of the notion of Ext completeness for any Noetherian ring as long as I is generated by a finite regular sequence.

CHAPTER V

The derived category of $\widehat{\mathrm{Mod}}_R$

V.1. The category of L-complete modules

As in the previous Chapter, we let R be a Noetherian ring and I an ideal generated by a regular sequence $x_0, x_1, \ldots, x_{n-1}$.

One consequence of the I-adic completion not being left or right exact is that the full subcategory of I-adic complete modules is not in general Abelian. While kernels and cokernels exist, it is not true that the image and coimage agree in general. Here is an example from [**Van12**].

Example V.1.1. Let $R = \mathbb{Z}_p$ and $I = (p)$. Let f be the map

$$\left(\bigoplus_{k\geq 0} \mathbb{Z}_p\right)^{\wedge}_p \xrightarrow{f} \prod_{k\geq 0} \mathbb{Z}_p,$$

sending a compatible sequence $(x_1, x_2, x_3 \ldots)$ to $(px_1, p^2x_2, p^3x_3, \ldots)$. Recall that for a map $f : X \to Y$, $\operatorname{im} f = \ker(Y \to \operatorname{coker} f)$ while $\operatorname{coim} f = \operatorname{coker}(\ker f \to X)$. We have $\operatorname{coim} f = \operatorname{coker}(\ker f \to (\bigoplus_{k\geq 0} \mathbb{Z}_p)^{\wedge}_p = \operatorname{coker}(0 \to (\bigoplus_{k\geq 0} \mathbb{Z}_p)^{\wedge}_p = (\bigoplus_{k\geq 0} \mathbb{Z}_p)^{\wedge}_p$. However, it can be shown that the image of f in the category of p-complete Abelian groups is the closure of the image of f as Abelian groups

$$\operatorname{im} f = \overline{\{(px_1, p^2x_2, \ldots) | (x_i) \in \left(\bigoplus_{k\geq 0} \mathbb{Z}_p\right)^{\wedge}_p \}} = \prod_{k\geq 0} \mathbb{Z}_p.$$

Thus, the category of p-complete Abelian groups is not Abelian.

However, the category of L-complete modules is indeed Abelian, and by Theorem II.3.13, it contains the category of I-adically complete modules as a full subcategory. Furthermore, it is the best reflexive subcategory of R-modules containing the latter (see [**Sal10**]), thus confirming the choice of L-complete modules as the optimal setting where to study the homological algebra of I-adically complete modules.

In the following, we will use the symbol $\widehat{\mathrm{Mod}}_R$ to denote the category of L-complete modules. We collect the main features of this category in the following theorem. All of these facts follow from the results in Appendix A of [**HS99**] with word-by-word proofs.

THEOREM V.1.2. *$\widehat{\mathrm{Mod}}_R$ is a symmetric monoidal bicomplete Abelian full subcategory of the category of R-modules. Furthermore, the functor L_0 is left adjoint to the inclusion of $\widehat{\mathrm{Mod}}_R$ into all R-modules,*

$$\mathrm{Mod}_R \underset{i}{\overset{L_0}{\rightleftarrows}} \widehat{\mathrm{Mod}}_R$$

and thus the limits of objects in $\widehat{\mathrm{Mod}}_R$ coincide with the limit as R-modules, while the filtered colimits in $\widehat{\mathrm{Mod}}_R$ are the L-completion of the corresponding colimits in R-modules.

As we can see, the category $\widehat{\mathrm{Mod}}_R$ has many desirable properties. However, it follows from the description above that colimits in $\widehat{\mathrm{Mod}}_R$ may have higher derived functors because L_0 does. In fact, Hovey gives the following example in [**Hov08**].

Example V.1.3. Let $R = \mathbb{Z}_p$ and $I = (p)$. Consider the following tower of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_p & \xrightarrow{p^k} & \mathbb{Z}_p & \longrightarrow & \mathbb{Z}/p^k & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle p} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_p & \xrightarrow{p^{k+1}} & \mathbb{Z}_p & \longrightarrow & \mathbb{Z}/p^{k+1} & \longrightarrow & 0.
\end{array}$$

Taking the usual colimit, we get the short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Z}/p^\infty \to 0,$$

where $\mathbb{Z}/p^\infty$ is the Prüfer group, the union of all p-power roots of unity. However, multiplying by p is invertible in the last two groups so applying L_0 (or equivalently p-completing), we get

$$0 \to \mathbb{Z}_p \to 0 \to 0 \to 0,$$

which is obviously not exact.

Thus, $\widehat{\mathrm{Mod}}_R$ is not Grothendieck and the existence of its derived category is not trivial, as illustrated by Example II.2.6.

V.2. The derived category

To show that $\mathrm{D}(\widehat{\mathrm{Mod}}_R)$ does indeed exist, we will use the following strategy. As before, we extend L_0 to $\mathrm{Ch}(R)$. Consider the following diagram of functors

$$\begin{tikzcd} \mathrm{Ch}(R) \arrow[r, shift left, "L_0"] \arrow[d, "Q"] & \mathrm{Ch}(\widehat{\mathrm{Mod}}_R) \arrow[l, shift left, "i"] \arrow[d, "S"] \\ \mathrm{D}(R) \arrow[r, "C"] & \mathrm{D}(R)_C \end{tikzcd}$$

where i is the inclusion functor, the functor S is the composition $C \circ Q \circ i$, and $\mathrm{D}(R)_C$ is the category of C-locals (the essential image of C). First, we claim S factors through $\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)$ with the quasi-isomorphisms inverted.

Proposition V.2.1. *Suppose that $f : X \to Y$ is a quasi-isomorphism in $\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)$. Then, $S(f)$ is an isomorphism in $\mathrm{D}(R)_C$.*

PROOF. Since $i(f)$ is also a quasi-isomorphism in $\mathrm{Ch}(R)$, it follows that $(Q \circ i)(f)$ is an isomorphism in $\mathrm{D}(R)$. Since C is a functor, $C \circ Q \circ i$ is also an isomorphism. $\square$

Let us denote the resulting functor

$$\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)[\simeq]^{-1} \to \mathrm{D}(R)_C$$

by C. We will show that this is in fact an equivalence of categories.

We start by showing that C is essentially the inclusion functor when restricted to $\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)[\simeq]^{-1}$.

Proposition V.2.2. *If X is a complex of L-complete modules, then $X \to CX$ is an equivalence in* $\mathrm{D}(R)$.

PROOF. The result follows immediately from Proposition IV.1.8 once we observe that for any X in $\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)$, $H_*(X)$ is L-complete. □

The next proposition is an easy consequence of the properties of $\widehat{\mathrm{Mod}}_R$.

Proposition V.2.3. *Let $X \in \mathrm{D}(R)$. Then, CX has L-complete homology.*

PROOF. Recall one of the spectral sequences from IV.1.7

$$E^2_{pq} = L_p(H_q(X)) \implies H_{p+q}(CX).$$

By Theorem A.6 (a) in [**HS99**], the E^2 term of this spectral sequence is L-complete. Since $\widehat{\mathrm{Mod}}_R$ is an Abelian category, we can conclude that the target of the spectral sequence is also an L-complete module. □

Remark V.2.4. Combining Proposition V.2.3 with Proposition IV.1.8, we get a characterization of the C-local objects of $\mathrm{D}(R)$:

$$\mathrm{D}(R)_C = \{X \in \mathrm{D}(R) | H_*(X) \text{ is } L\text{-complete}\}.$$

Next, we can specialize one of the spectral sequences in Proposition IV.1.7, to obtain one computing the homology of $L_0(X)$.

Proposition V.2.5. *Let X be a complex of flat modules. There is a strongly convergent spectral sequence*

$$E^2_{pq} = L_p(H_q(X)) \Longrightarrow H_{p+q}(L_0(X)).$$

PROOF. If M is flat, then $L_p(M) = 0$ for all $p \geq 1$ (Theorem II.3.13). Thus, the spectral sequence with E^2 term ${}^{II}E^2_{pq} = H_p(L_q(X))$ collapses giving $H_*(CX) \cong H_*(L_0 X)$. Hence, the desired spectral sequence is just the first spectral sequence in (3) of Proposition IV.1.7 with the updated target. □

Corollary V.2.6. *Suppose that X in $\mathrm{Ch}(R)$ is a complex of flat R-modules and $H_*(X)$ is L-complete. Then, $X \to L_0(X)$ is a quasi-isomorphism.*

PROOF. Apply the spectral sequence above to X and note that it collapses with $E^2_{pq} \cong H_q(X)$ if $p = 0$, or zero otherwise. □

We can now prove a proposition that allows us to construct the inverse of $\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)[\simeq]^{-1} \xrightarrow{C} \mathrm{D}(R)_C$.

Proposition V.2.7. *Let $X \in \mathrm{D}(R)$ be C-local. Then, the following is a diagram of natural quasi-isomorphisms*

$$X \leftarrow QX \to L_0(QX),$$

where QX is the projective approximation of X.

PROOF. Let QX be the projective approximation of X, with the natural quasi-isomorphism $QX \to X$ (i.e. QX is the cofiber replacement in the projective model structure of $\mathrm{Ch}(R)$). By Proposition V.2.3, $H_*(QX)$ is L-complete as X is C-local. Furthermore, by Corollary VI.2.6, it follows that $QX \to L_0(QX)$ is a quasi-isomorphism. $\square$

We are now ready to prove the main theorem of this chapter. The proof of the existence of the derived category of L-complete modules can also be obtained by putting a model structure on $\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)$ induced by the projective model structure on $\mathrm{Ch}(R)$ via the adjunction between L_0 and the inclusion functor (see [**Rez13**]). The key result here is Lemma 9.3 in [**Rez13**], which can be easily proven using the spectral sequence from Proposition V.2.5. Furthermore, this model structure is cofibrantly generated and monoidal, as shown by Barthel and Frankland (see the appendix of [**BF13**]). However, we believe that our direct approach has the advantage of providing explicit characterizations which give a better understanding of the derived category of $\widehat{\mathrm{Mod}}_R$. For example, Theorem IV.2.3 is an easy consequence when working from our point of view. Moreover, the connection with I-torsion modules appears naturally from our framework, and this results in the derived equivalence of the category of L

THEOREM V.2.8 (**Derived category of $\widehat{\mathrm{Mod}}_R$**). *The functor*

$$\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)[\simeq]^{-1} \xrightarrow{C} \mathrm{D}(R)_C$$

is an equivalence of categories. Hence, the derived category of $\widehat{\mathrm{Mod}}_R$ *exists, and*

$$\mathrm{D}(\widehat{\mathrm{Mod}}_R) = \mathrm{D}(R)_C = \{X \in \mathrm{D}(R) | H_*(X) \textit{ is L-complete}\}.$$

Moreover, $\mathrm{D}(\widehat{\mathrm{Mod}}_R)$ *is a monogenic stable homotopy category.*

PROOF. By Proposition VI.2.7, we can define a functor $J : \mathrm{D}(R)_C \to \mathrm{Ch}(\widehat{\mathrm{Mod}}_R)[\simeq]^{-1}$ by the correspondence $X \mapsto L_0(QX)$. Note that the latter map is always an isomorphism in $\mathrm{D}(R)$. Moreover, $X \mapsto L_0(QX)$ is the composition of two natural transformations on $\mathrm{D}(R)$. Therefore,

$$X \to CX \to L_0(Q(CX))$$

is a natural isomorphism from $\mathbf{1}_{\mathrm{Ch}(\widehat{\mathrm{Mod}}_R)[\simeq]^{-1}}$ to $J \circ C$ as IV.1.8 implies that $X \to CX$ is a natural isomorphism for all $X \in \mathrm{Ch}(\widehat{\mathrm{Mod}}_R)$.

Likewise, it follows from VI.2.7 and IV.1.8 that

$$C(L_0(QX)) \leftarrow L_0(QX) \leftarrow QX \to X$$

is a diagram of natural quasi-isomorphisms whenever $X \in \mathrm{D}(R)_C$. Thus, they give a natural isomorphism from $C \circ J$ to $\mathbf{1}_{\mathrm{D}(R)_C}$.

Since $\mathrm{D}(\widehat{\mathrm{Mod}}_R)$ is the image of a localization functor, it follows that it inherits a stable homotopy category structure from $\mathrm{D}(R)$ with symmetric monoidal product $X \otimes_C Y = C(X \otimes Y)$, and (not compact) generator $R_C = CR$ (see [**HPS97**, Theorem 3.5.1]). □

CHAPTER VI

A derived version of Grothendieck's local duality

VI.1. The category of I-power torsion modules

We begin with a short discussion about the full Abelian subcategory of I-power torsion R-modules, which we will denote by $\mathrm{Mod}_R^{\mathrm{tor}}$. The results that we present here are surely know to commutative algebraist or algebraic geometers, however, we could not find a source containing all this material.

First, it is clear that kernels and cokernels are the same in $\mathrm{Mod}_R^{\mathrm{tor}}$ as in Mod_R. Thus, $\mathrm{Mod}_R^{\mathrm{tor}}$ is an Abelian subcategory of Mod_R. Furthermore, we have the following relationship between $\mathrm{Mod}_R^{\mathrm{tor}}$ and Mod_R.

Proposition VI.1.1. *The I-power torsion is right adjoint to the inclusion functor*

$$\mathrm{Mod}_R \underset{\Gamma_I}{\overset{i}{\leftrightarrows}} \mathrm{Mod}_R^{\mathrm{tor}}$$

PROOF. We need to show that there is a natural isomorphism

$$\mathrm{Hom}(M, N) \to \mathrm{Hom}(M, \Gamma_I(N)),$$

for all $M \in \mathrm{Mod}_R^{\mathrm{tor}}$ and all $N \in \mathrm{Mod}_R$. Observe that given a homomorphism $f : M \to N$ and $x \in M$, we have $I^k f(x) = 0$ for some positive integer k.

Therefore, f factors through $\Gamma_I N$

$$f : M \to \Gamma_I(N) \hookrightarrow N,$$

where the last map is the natural inclusion of $\Gamma_I(N)$ in N. The conclusion is then clear. $\square$

Since the inclusion is a left adjoint and left adjoints preserve filtered colimits, it follows that the filtered colimits in $\mathrm{Mod}_R^{\mathrm{tor}}$ are just the filtered colimits in the category of R-modules.

Corollary VI.1.2. *The category of I-power torsion has arbitrary filtered colimits given by the the usual colimit of R-modules. In particular,* $\mathrm{Mod}_R^{\mathrm{tor}}$ *has arbitrary direct sums.*

Notice that filtered colimits are then exact in $\mathrm{Mod}_R^{\mathrm{tor}}$ because their usual R-modules counterparts are so. Additionally, $\bigoplus_k R/I^k$ is a generator; if M is an I-torsion module, then we have an epimorphism $\bigoplus_{x \in M} R/I^{k(x)} \to M$, where $k(x)$ is an integer so that $I^k x = 0$. Hence, one can extend the previous map to an epimorphism $\bigoplus_{x \in M} \bigoplus_k R/I^k \to M$ by sending all the extra copies of R/I^k to zero. We have thus proved:

Proposition VI.1.3. $\mathrm{Mod}_R^{\mathrm{tor}}$ *is a Grothendieck category.*

In consequence, the derived category $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$ exists. Furthermore, as stated in Theorem 2.1.1 [**Hov04a**], there is a model structure for $\mathrm{Ch}(\mathrm{Mod}_R^{\mathrm{tor}})$, whose homotopy category is the derived category. However, we can run a similar proof to the one used to prove the existence of $\mathrm{D}(\widehat{\mathrm{Mod}}_R)$ to obtain an explicit characterization of $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$.

Remark VI.1.4. Note that if either M or N is I-power torsion, then so is $M \otimes N$. Therefore, the usual tensor product gives a symmetric monoidal product in $\mathrm{Mod}_R^{\mathrm{tor}}$. However, R does not need to be I-torsion and hence the tensor product in $\mathrm{Mod}_R^{\mathrm{tor}}$ has no unit. Thus, $\mathrm{Mod}_R^{\mathrm{tor}}$ *is not* a closed symmetric monoidal category, in contrast with $\widehat{\mathrm{Mod}}_R$.

VI.2. The derived category

In this section, we will show that $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$ is equivalent to the full subcategory of T-colocal objects. This has a surprising consequence: since by Theorem IV.1.3 (7) the category of T-colocal objects is equivalent to the C-local objects, it follows that $\mathrm{D}(\widehat{\mathrm{Mod}}_R)$ and $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$ are equivalent. However, the Abelian subcategories $\widehat{\mathrm{Mod}}_R$ and $\mathrm{Mod}_R^{\mathrm{tor}}$ are obviously far from being equivalent. We will state this result and its consequences in the next section.

Consider the diagram of functors

$$\begin{tikzcd}
\mathrm{Ch}(R) \arrow[r, shift right, "\Gamma_I"'] \arrow[d, "Q"] & \mathrm{Ch}(\mathrm{Mod}_R^{\mathrm{tor}}) \arrow[l, shift right, "i"'] \arrow[d, "S"] \\
\mathrm{D}(R) \arrow[r, "T"] & \mathrm{D}(R)_T
\end{tikzcd}$$

where i is the inclusion functor, the functor S is the composition $T \circ Q \circ i$, and Γ_I is applied degreewise to get a functor on $\mathrm{Ch}(\mathrm{Mod}_R^{\mathrm{tor}})$. As before, S factors through $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$.

Proposition VI.2.1. *Suppose that $f : X \to Y$ is a quasi-isomorphism in $\mathrm{Ch}(\mathrm{Mod}_R^{\mathrm{tor}})$. Then, $S(f)$ is an isomorphism in $\mathrm{D}(R)_T$.*

PROOF. Since $i(f)$ is also a quasi-isomorphism in $\mathrm{Ch}(R)$, it follows that $(Q \circ i)(f)$ is an isomorphism in $\mathrm{D}(R)$. Since T is a functor, $T(Q \circ i)$ is also an isomorphism. $\square$

Let us also denote the resulting functor

$$\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}}) \to \mathrm{D}(R)_T$$

by T. Then, T is an equivalence when restricted to $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$.

Proposition VI.2.2. *If X is a complex of I-torsion modules, then $TX \to X$ is an equivalence in $\mathrm{D}(R)$.*

PROOF. The result follows immediately from Proposition IV.1.8 as $H^*(X)$ is I-torsion for any X in $\mathrm{Ch}(\mathrm{Mod}_R^{\mathrm{tor}})$. $\square$

The next proposition is an easy consequence of the properties of $\mathrm{Mod}_R^{\mathrm{tor}}$.

Proposition VI.2.3. *Let $X \in \mathrm{D}(R)$. Then, TX has I-torsion cohomology.*

PROOF. Recall one of the spectral sequences from Proposition IV.1.7

$$E_2^{pq} = H_I^p(H^q(X)) \implies H^{p+q}(TX).$$

Since $H_I^p(M)$ is a quotient of I-torsion modules and $\mathrm{Mod}_R^{\mathrm{tor}}$ is Abelian, it follows that $H_I^p(M)$ is I-torsion for all $p \geq 0$. Thus, the E_2 term of this spectral sequence is I-torsion and we can conclude that the target of the spectral sequence is an I-torsion module as well. □

Remark VI.2.4. Combining Proposition VI.2.3 with Proposition IV.1.8, we get a characterization of the T-colocal objects of $\mathrm{D}(R)$:

$$\mathrm{D}(R)_T = \{X \in \mathrm{D}(R) | H^*(X) \text{ is } I\text{-torsion}\}.$$

Next, we specialize one of the spectral sequences in Proposition IV.1.7, to obtain one converging to the homology of $\Gamma_I X$.

Proposition VI.2.5. *Let X be a complex of injective modules. There is a strongly convergent spectral sequence*

$$E_2^{pq} = H_I^p(H^q(X)) \Longrightarrow H^{p+q}(\Gamma_I(X)).$$

PROOF. First, using the fact that $H_I^p(M) = \mathrm{colim}_k \operatorname{Ext}_R^p(R/I^k, M)$, we see that the local cohomology of an injective module vanishes for $p \geq 1$. Therefore, the spectral sequence ${}^{II}E_2^{pq} = H^p(H_I^q(X))$ collapses to give $H^*(TX) \cong H^*(\Gamma_I X)$. The desired spectral sequence is just the first spectral sequence in (2) of Proposition IV.1.7. □

Corollary VI.2.6. *Suppose that X in $\mathrm{Ch}(R)$ is a complex of injective R-modules and $H^*(X)$ is I-torsion. Then, the natural map $\Gamma_I X \to X$ is a quasi-isomorphism.*

PROOF. Apply the spectral sequence above to X and note that it collapses with $E_2^{pq} \cong H^q(X)$ if $p = 0$, or zero otherwise. $\square$

We can now prove a proposition that will allow us to construct the inverse of $\mathrm{Ch}(\mathrm{Mod}_R^{\mathrm{tor}})[\simeq]^{-1} \xrightarrow{T} \mathrm{D}(R)_T$.

Proposition VI.2.7. *Let $X \in \mathrm{D}(R)$ be T-colocal. Then, the following is a diagram of natural quasi-isomorphisms*

$$X \to RX \leftarrow \Gamma_I(RX),$$

where RX is the injective approximation of X.

PROOF. Let RX be the injective approximation of X, with the natural quasi-isomorphism $X \to RX$ (i.e. RX is the fiber replacement in the injective model structure of $\mathrm{Ch}(R)$). By Proposition V.2.3, $H^*(RX)$ is I-torsion as X is T-colocal. Moreover, by Corollary VI.2.6 it follows that $\Gamma_I(RX) \to RX$ is a quasi-isomorphism.

$\square$

We are now ready to prove our characterization of $\mathrm{D}(tM)$.

THEOREM VI.2.8 (**The derived category of** $\mathrm{Mod}_R^{\mathrm{tor}}$). *The functor*

$$\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}}) \xrightarrow{T} \mathrm{D}(R)_T$$

is an equivalence of categories, and

$$\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}}) = \mathrm{D}(R)_T = \{X \in \mathrm{D}(R) | H_*(X) \textit{ is } I\textit{-torsion}\}.$$

Moreover, $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$ *is a monogenic stable homotopy category.*

PROOF. By VI.2.7, we can define a functor $J : \mathrm{D}(R)_T \to \mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$ by the correspondence $X \mapsto \Gamma_I(RX)$. Note that the latter map is always an isomorphism in $\mathrm{D}(R)$. Moreover, $X \mapsto \Gamma_I(RX)$ is the composition of two natural transformations on $\mathrm{D}(R)$. Therefore,

$$X \to TX \to \Gamma_I(R(TX))$$

is a natural isomorphism from $\mathbf{1}_{\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})}$ to $J \circ T$ as IV.1.8 implies that $X \to TX$ is a natural isomorphism for all $X \in \mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$.

Likewise, it follows from Propositions VI.2.7 and IV.1.8 that

$$T(\Gamma_I(QX)) \to \Gamma_I(RX) \to RX \leftarrow X$$

is a diagram of natural quasi-isomorphisms whenever $X \in \mathrm{D}(R)_T$. Thus, they give a natural isomorphism from $T \circ J$ to $\mathbf{1}_{\mathrm{D}(R)_T}$.

Finally, $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}})$ is a monogenic stable homotopy category, as a consequence of Corollary III.4.12. $\square$

VI.3. Local duality

Duality theorems abound in all areas of mathematics, and they are often powerful tools to express a problem in an equivalent way that is often more approachable than the original formulation. In this section, we will show that the category of L-complete modules is derived equivalent to the category of I-power torsion modules, i.e., their derived categories are equivalent. This is a concrete example of two very different Abelian categories ($\mathrm{Mod}_R^{\mathrm{tor}}$ is Grothendieck but not symmetric monoidal, while $\widehat{\mathrm{Mod}}_R$ is exactly the opposite) whose derived category agrees, which indicates the lack of rigidity of the derived category in general.

Moreover, we will see how this equivalence implies a generalized version of Grothendieck's local duality that unifies several well known duality theorems. This result is not new; it is for example stated in [**LAL99**] where the authors prove a more general version of our theorem in a geometric context. However, the point here is to emphasize how the power of the homotopy theoretical approach makes the proof of such a result almost trivial, and furthermore, how we could use these techniques to prove similar results in completely different contexts (i.e., in other stable homotopy categories).

THEOREM VI.3.1 (**Completion-torsion derived equivalence**). *Let R be a Noetherian ring and I an ideal generated by a regular sequence. Then, the category of L-complete R-modules is derived equivalent to the category of I-torsion R-modules.*

Moreover, the equivalence is given by

$$\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}}) \underset{T}{\overset{C}{\rightleftarrows}} \mathrm{D}(\widehat{\mathrm{Mod}}_R),$$

and this is an equivalence of monogenic stable homotopy categories.

PROOF. By Theorems V.2.8 and VI.2.8, we have equivalences of categories $\mathrm{D}(\widehat{\mathrm{Mod}}_R) = \mathrm{D}(R)_C$ and $\mathrm{D}(\mathrm{Mod}_R^{\mathrm{tor}}) = \mathrm{D}(R)_T$, respectively. The conclusion is then immediate by Corollary III.4.13. ☐

We remark that the abstract equivalence of the categories $\mathrm{D}(R)_C$ and $\mathrm{D}(R)_T$ (which is one of the statements in Theorem III.4.8) was also proved in [**DG02**] by comparing them to a third equivalent derived category over a certain ring of endomorphisms. In addition, the recent paper of Porta, Shaul, and Yekutieli [**PSY12**] generalizes this abstract equivalence to the case of ideals generated by a *weakly proregular sequence*, which is a weakening of the regularity condition. We observe that our results hold for any ideal provided that R/I is compact in $\mathrm{D}(R)$, but the author does not know whether this is the case when I is generated by a proregular sequence. Non of these results make the connection with derived categories, and non of their equivalences are stable homotopy equivalences (or closed symmetric monoidal equivalences for that matter).

Next, we state our generalized version of Grothendieck's duality, and proceed to explain how this recovers several notions of duality in the literature.

THEOREM VI.3.2 (**Local duality**). *T is left adjoint to C, i.e., we have a natural isomorphism*

$$\mathrm{Hom}(TX, Y) \xrightarrow{\cong} \mathrm{Hom}(X, CY)$$

for all X, Y in $\mathrm{D}(R)$.

PROOF. The conclusion follows from the commutative diagram of natural equivalences

$$\begin{array}{ccc} \mathrm{Hom}(TX, TY) & \longrightarrow & \mathrm{Hom}(CX, CY) \\ \uparrow & & \downarrow \\ \mathrm{Hom}(TX, Y) & \longrightarrow & \mathrm{Hom}(X, CY) \end{array}$$

that we now explain. The top horizontal arrow is a consequence of the equivalence of $D(R)_T$ and $D(R)_C$, where we are identifying CT with C using Theorem IV.1.3. Both vertical arrows are the corresponding natural isomorphisms included in the definition of (co)localization. Thus, the composition of the three natural maps gives the desired natural isomorphism. □

We can also express local duality in terms of internal hom-objects, which is the way the theorem is stated in [**LAL99**].

Corollary VI.3.3. *For all $X, Y \in \mathrm{D}(R)$, we have a natural isomorphism*

$$F(TX, Y) \xrightarrow{\cong} F(X, CY).$$

PROOF. We have a chain of natural isomorphisms

$$\mathrm{Hom}(T\Sigma^n X, Y) \cong \mathrm{Hom}(\Sigma^n R \otimes TX, Y) \cong \mathrm{Hom}(\Sigma^n R, F(TX, Y)), \text{ and}$$

$$\mathrm{Hom}(\Sigma^n X, CY) \cong \mathrm{Hom}(\Sigma^n R \otimes X, CY) \cong \mathrm{Hom}(\Sigma^n R, F(X, CY)),$$

for all integers n. The conclusion follows from Proposition III.3.8. □

Recall that the *Krull dimension* of a ring R, $\dim(R)$, is the length of its longest strictly ascending sequence of prime ideals. R is a *regular local ring* if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$, where $\mathfrak{m}$ is the maximal ideal and $k = R/\mathfrak{m}$ is the residue field; in this case, the maximal ideal is generated by a regular sequence of length $d = \dim R$. Thus, we can recover a particular case of Greenlees-May duality from our local duality theorem.

THEOREM VI.3.4 (**Greenlees-May duality**, [**GM95**]). *If R is a Noetherian regular local ring and* $\dim R = d$*, then*

$$L_k^{\mathfrak{m}}(M) \cong \mathrm{Ext}^{d-k}(H_{\mathfrak{m}}^d(R), M),$$

for every R-module M.

PROOF. First, note that taking $X = R$ and $Y = M$ for an R-module M in Theorem VI.3.3, we get the formula $F(R, CM) = CM = F(TR, M) = F(K^\bullet(\mathfrak{m}), M)$ (which we already had from Theorem IV.1.3). On the other hand, for the case of a Noetherian regular local ring, we know that $K^\bullet(\mathfrak{m})$ is quasi-isomorphic to

$\Sigma^d H^d_{\mathfrak{m}}(R)$. Thus, the conclusion follows from Proposition IV.1.4 after applying homology to the isomorphism $CM = F(K^\bullet(\mathfrak{m}), M)$. $\square$

Example VI.3.5. If we take $R = \mathbb{Z}_p$ and $\mathfrak{m} = p\mathbb{Z}_p$, then R is a regular local ring with $\dim R = 1$, and therefore

$$L_0^{\mathfrak{m}}(M) = \mathrm{Ext}^1(Z/p^\infty, M) \text{ and } L_1^{\mathfrak{m}}(M) = \mathrm{Hom}(\mathbb{Z}/p^\infty, M).$$

The original local duality theorem of Grothendieck appeared in [**Har67**] by Hartshorne as lecture notes from a seminar given by Grothendieck at Harvard in 1962. As we see in the following, this result is another particular case of Corollary VI.3.3, if we assume that R is Noetherian regular local.

THEOREM VI.3.6 (**Grothendieck local duality**, [**Har67**]). *Suppose that R is a Noetherian regular local ring of dimension d, and let $E(k)$ be the injective hull of the residue field $k = R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of R. Then, for any R-module M there is an isomorphism of modules*

$$\mathrm{Ext}^k(M, \widehat{R}) \cong \mathrm{Hom}(H_{\mathfrak{m}}^{d-k}(M), E(k)).$$

PROOF. Apply Corollary VI.3.3 to X equal an R-module M concentrated in degree zero, and $Y = R$ to get $F(TM, R) \cong F(M, CR)$. Moreover, since T is right adjoint to the inclusion we have $F(TM, TR) = F(TM, K^\bullet(\mathfrak{m})) \cong F(TM, R)$. On the other hand, $K^\bullet(\mathfrak{m}) \cong \Sigma^d H^d_{\mathfrak{m}}(R)$, and $H^d_{\mathfrak{m}}(R)$ is equal to the injective hull of $k = R/\mathfrak{m}$. Additionally, combining Theorem II.3.13 and Proposition IV.1.4, we see that CM is quasi-isomorphic to $L_0^{\mathfrak{m}}(R) = \widehat{R}$. The result then follows from applying homology to the isomorphism $F(TM, \Sigma^d E(k)) \cong F(M, \widehat{R})$. □

Finally, we use the properties of $\mathrm{D}(R)$ as a monogenic stable homotopy category to deduce a duality theorem due to Hartshorne [**Har69**]. While this is not a consequence of our local duality theorem, we again want to highlight the way our formalism can considerably simplify the proof of such results.

THEOREM VI.3.7 (**Affine duality**, [**Har69**]). *Let R be a Noetherian regular local ring with maximal ideal $\mathfrak{m}$. Then, there is a natural isomorphism*

$$TX \xrightarrow{\cong} F(F(X,R), TR),$$

for all $X \in \mathrm{D}(R)$ compact.

PROOF. Recall $DX = F(X,R)$ denotes the Spanier-Whitehead dual of X and that X is strongly dualizable if $F(X,R) \otimes Y \cong F(X,Y)$ for all objects Y. Also, recall that if X is strongly dualizable so is DX, and we have an isomorphism $X \cong D^2X$.

Suppose that X is compact. Since $\mathrm{D}(R)$ is monogenic, X is strongly dualizable (Proposition III.3.9), and we have a chain of natural isomorphisms

$$\begin{aligned} TX \cong F(R, TX) &\cong F(R, X \otimes TR) \\ &\cong F(R, D^2X \otimes TR) \\ &\cong F(R, F(DX, R) \otimes TR) \\ &\cong F(R, F(DX, TR)) \\ &\cong F(R \otimes DX, TR) \\ &\cong F(F(X, R), TR). \end{aligned}$$

□

As a corollary, we get another form of local duality that was introduced in [**Har67**].

THEOREM VI.3.8 (Grothendieck, [**Har67**]). *Let R be a Noetherian regular local ring of dimension d with maximal ideal $\mathfrak{m}$. Let M be a finitely generated R-module. Then, we have a natural isomorphism*

$$H^k_{\mathfrak{m}}(M) \cong D(\mathrm{Ext}^{d-k}(M, R)),$$

where $D = \mathrm{Hom}(-, E(k))$.

PROOF. Recall that if M is finitely generated, then it has a bounded resolution of finitely generated projective modules, i.e., it is a compact object in $\mathrm{D}(R)$.

Thus, the result is a consequence of Theorem VI.3.7 with $X = M$ after applying homology to the corresponding isomorphism of complexes. $\square$

CHAPTER VII

Future directions

VII.1. Abstract local duality

As we have seen in the last section of the previous Chapter, Theorem VI.3.1 is an easy consequence of Theorem IV.1.3 where the existence of the key localizations is provided, together with algebraic formulae and characterization of the relevant subcategories. However, the latter is a particular case of Theorem III.4.8 from Hovey-Palmieri-Strickland's Axiomatic stable homotopy theory [**HPS97**], which we recall holds for any algebraic stable homotopy category. This observation has led Tobias Barthel, Drew Heard (from the Max-Planck-Institute für Mathematik in Bonn) and the author to consider other contexts where this theory could potential give new insights.

The first observation is that the proof of the duality theorem VI.3.2 from the equivalence of the derived categories of I-torsion and L-complete modules is completely formal. Let $\mathcal{C}$ be an algebraic stable homotopy category and let B be a compact object in $\mathcal{C}$. Let T be the finite localization away from B. Then, according to Theorem III.4.8, we have (co)localizations

$$TX \to X \to LX$$

$$AX \to X \to CX$$

satisfying several properties including the equivalence of the category of C-local and T-colocal objects, $\mathcal{C}_C$ and $\mathcal{C}_T$, respectively. Repeating the proof of our local duality theorem, we get

THEOREM VII.1.1 (**Abstract local duality**, [BHV15]). *T and C are an adjoint pair, this is, there is a natural isomorphism*

$$\mathrm{Hom}(TX, Y) \xrightarrow{\cong} \mathrm{Hom}(X, CX)$$

for all $X, Y \in \mathcal{C}$.

Now, this result by itself does not add much to the theory of stable homotopy categories. The idea is to find meaningful localization functors in categories of interest so that this result turns into some kind of duality that hopefully sheds new light in the corresponding context.

One of this categories that has interested the author since the beginning of this project is the category of comodules over a Hopf algebroid, which we discuss in the sequel.

VII.2. Comodules over a Hopf algebroid

We will first review the category of comodules over a Hopf algebroid (A, Γ); we refer the reader to [**Hov04c**, Section 1] and [**Rav86**, Appendix A1] for further details. The homology theories in common use in algebraic topology take values in comodules over certain Hopf algebroids.

We recall that a *Hopf algebroid* (A, Γ) over a commutative ring K is a cogroupoid object in the category of commutative K-algebras. Concretely, it consists of a pair (A, Γ) of commutative K-algebras, along with K-algebra morphisms

$$\eta_L : A \to \Gamma$$

$$\eta_R : A \to \Gamma$$

$$\Delta : \Gamma \to \Gamma \otimes_A \Gamma$$

$$\epsilon : \Gamma \to A$$

$$c : \Gamma \to \Gamma,$$

where we note that Γ is a left A-module via η_L, a right A-module via η_R and $\Gamma \otimes_A \Gamma$ refers to the bimodule tensor product. This maps must satisfy

(1) $\epsilon\eta_L = \epsilon\eta_R = 1_A$, the identity on A.

(2) $(\Gamma \otimes \epsilon)\Delta = (\epsilon \otimes \Gamma)\Delta = 1_\Gamma$.

(3) $(\Gamma \otimes \Delta)\Delta = (\Delta \otimes \Gamma)\Delta$.

(4) $c\eta_R = \eta_L$ and $c\eta_L = \eta_R$.

(5) $cc = 1_\Gamma$.

(6) Maps exist which make the following diagram commute

$$\begin{tikzcd}
\Gamma & \Gamma \otimes_K \Gamma \arrow[l, "c\cdot\Gamma"'] \arrow[r, "\Gamma\cdot c"] & \Gamma \\
 & \Gamma \otimes_A \Gamma \arrow[ul, dashed] \arrow[ur, dashed] & \\
A \arrow[uu, "\eta_R"] & \Gamma \arrow[l, "\epsilon"'] \arrow[r, "\epsilon"] \arrow[u, "\Delta"] & A \arrow[uu, "\eta_L"']
\end{tikzcd}$$

where $c \cdot \Gamma(\gamma_1 \otimes \gamma_2) = c(\gamma_1)\gamma_2$ and $\Gamma \cdot c(\gamma_1 \otimes \gamma_2) = \gamma_1 c(\gamma_2)$.

A *left* Γ*-comodule* M is a left A-module M with a counitary and coassociative left A-linear morphism $\psi : M \to \Gamma \otimes_A M$, i.e., $(\epsilon \otimes 1_M)\psi = 1_M$ and $(\Delta \otimes 1_M)\psi = (1_\Gamma \otimes \psi)\psi$. A comodule homomorphism $f : M \to N$, is an A-linear map such that the diagram

$$\begin{tikzcd}
M \arrow[r, "\psi_M"] \arrow[d, "f"] & \Gamma \otimes M \arrow[d, "1\otimes f"] \\
N \arrow[r, "\psi_N"] & \Gamma \otimes N
\end{tikzcd}$$

commutes.

We write Comod_Γ for the category of Γ-comodules; when Γ is a flat A-module, then Comod_Γ is a cocomplete Abelian subcategory of Mod_A [**Hov04c**, Lemma 1.1.1]. One could then consider the derived category of Comod_Γ as the framework in which to apply our abstract duality result, but this is not the right category. For instance, it would be natural to consider the finite localization away from A/I, where I is an ideal generated by a finite regular sequence as in the case of modules, however, A is not a compact object in $\mathrm{D}(\mathrm{Comod}_\Gamma)$, as discussed at the beginning of Section 3 in [Hov07]. Instead, we consider Hovey's construction of a variation of the usual derived category for comodules over an Adams Hopf algebroid, denoted Stable_Γ [Hov04c]. The main features of Stable_Γ are that it is a triangulated category with a compatible closed symmetric monoidal structure and a set of strongly dualizible generators, that is, it is in fact a stable homotopy category. Moreover,

$$\mathrm{Hom}_{\mathrm{Stable}_\Gamma}(A, M)_* \cong \mathrm{Ext}^*_\Gamma(A, M),$$

where A, M are thought as complexes of comodules concentrated in degree zero, and $\mathrm{Ext}^*_\Gamma(A, -)$ are the right derived functors of $\mathrm{Hom}_\Gamma(A, -)$. The latter is a desired property because important computational tools in stable homotopy theory can be expressed in terms of Ext for certain Hopf algebroids, and thus Stable_Γ gives a framework for this homological algebra of comodules.

We now present our local duality for comodules, as stated in [BHV15] at the moment of writing. For the sake of consistency, we change some of the notation of [BHV15] to match the rest of this document. First, it turns out that A is compact in Stable_Γ, from which we show that A/I is compact when I is generated by a finite regular sequence. We can then define the category $\mathrm{Stable}_\Gamma^{I-\mathrm{tors}}$ of I-torsion Γ-comodules as

$$\mathrm{Stable}_\Gamma^{I-\mathrm{tors}} = \mathsf{Loc}_{\mathrm{Stable}_\Gamma}(A/I),$$

the localizing subcategory of Stable_Γ generated by the compact object A/I. The inclusion of the category $\mathrm{Stable}_\Gamma^{I-\mathrm{tors}}$ of I-torsion Γ-comodules into Stable_Γ will be denoted ι_{tors}.

The category $\mathrm{Stable}_\Gamma^{I-\mathrm{tors}}$ is compactly generated by A/I, so ι_{tors} admits a right adjoint T,

$$T\colon \mathrm{Stable}_\Gamma \rightleftarrows \mathrm{Stable}_\Gamma^{I-\mathrm{tors}} \colon \iota_{\mathrm{tors}},$$

that is, T is the finite localization away from A/I. Therefore, we can apply Theorem III.4.8 to obtain localization and completion adjunctions

$$\mathrm{Stable}_\Gamma^{I-\mathrm{loc}} \underset{\iota_{\mathrm{loc}}}{\overset{L}{\rightleftarrows}} \mathrm{Stable}_\Gamma \underset{\iota_{\mathrm{comp}}}{\overset{C}{\rightleftarrows}} \mathrm{Stable}_\Gamma^{I-\mathrm{comp}}$$

with respect to the ideal I.

We are now ready to state our version of local duality for comodules, which follows from Theorem VII.1.1.

THEOREM VII.2.1 (**Local duality for comodules**, [**BHV15**]). *Let (A, Γ) be a Hopf algeboid satisfying Adams' condition and let $I \subset A$ be a finitely-generated ideal.*

(1) *The functors T and L are smashing localization and colocalization functors and C is the localization functor with category of acyclics given by* $\mathrm{Stable}_\Gamma^{I-\mathrm{loc}}$. *Moreover, T and C satisfy $C \circ T \simeq C$ and $T \circ C \simeq T$ and they induce mutually inverse equivalences*

$$\mathrm{Stable}_\Gamma^{I-\mathrm{tors}} \underset{C}{\overset{T}{\rightleftarrows}} \mathrm{Stable}_\Gamma^{I-\mathrm{comp}}$$

(2) *For* $X, Y \in \mathrm{Stable}_\Gamma$*, there is an equivalence*

$$F(TX, Y) \xrightarrow{\cong} F(X, CY),$$

natural in each variable.

Remark VII.2.2. Since there is an equivalence $\mathrm{Stable}_A \simeq \mathrm{D}(A)$ [**Hov04c**], it follows that $\mathrm{Stable}_A^{I-\mathrm{tors}} \simeq \mathrm{D}(\mathrm{Mod}_A^{\mathrm{tor}})$ and $\mathrm{Stable}_A^{I-\mathrm{comp}} \simeq \mathrm{D}(\widehat{\mathrm{Mod}_A})$ and so local duality for A-modules is a special case of local duality for comodules.

However, at the moment, the relationship of the subcategories of Stable_Γ that result from the theorem above with subcategories of Comod_Γ that can be defined in a similar fashion to the case of modules is not clear. A major difficulty is that while we can consider the Koszul object as a complex of A-modules, it does not need to be a complex of comodules. The issue is that the map $A \xrightarrow{x} A$ does not need not be a comodule map for a given $x \in I$ (see [**JY80**] for a counterexample). Nonetheless, the hope is that we can still draw some connection with completion and torsion of comodules at the Abelian level.

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