Monoidal Bousfield Localizations and Algebras over Operads

By

David White

Faculty Advisor: Mark Hovey,
Professor of Mathematics

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Abstract

We give conditions on a monoidal model category $\mathcal{M}$ and on a set of maps $\mathcal{C}$ so that the Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{C}$ preserves the structure of algebras over various operads. This problem was motivated by a step in the solution of the Kervaire Invariant One Theorem and by an example which demonstrates that for the model category of equivariant spectra, preservation does not come for free, even for cofibrant operads. We provide a general theorem regarding when localization preserves $P$-algebra structure for an arbitrary operad $P$. We then characterize the localizations which respect monoidal structure and prove that all such localizations preserve algebras over cofibrant operads. As a special case we recover numerous classical theorems about preservation of algebraic structure under localization, we improve upon known results regarding preservation for equivariant spectra, and we introduce a collection of operads which allow us to study the phenomenon of localization destroying some, but not all, of equivariant commutative structure.

To demonstrate our preservation result for non-cofibrant operads, we develop a theory for when the category of commutative monoids in $\mathcal{M}$ inherits a model structure from $\mathcal{M}$ in which a map is a weak equivalence or fibration if and only if it is so in $\mathcal{M}$. We then investigate properties of cofibrations of commutative monoids, functoriality of the passage from a commutative monoid $R$ to the category of commutative $R$-algebras, rectification between $E_\infty$-algebras and commutative monoids, and the relationship between
commutative monoids and monoidal Bousfield localization functors. We recover numerous known examples and a few new examples of model categories in which commutative monoids inherit a model structure. We then work out when localization preserves commutative monoids and the commutative monoid axiom. Finally, we provide conditions so that a left Bousfield localization satisfies the monoid axiom.
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## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Abstract</td>
<td>i</td>
</tr>
<tr>
<td>Acknowledgements</td>
<td>iii</td>
</tr>
<tr>
<td>Chapter 1. Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Chapter 2. Preliminaries</td>
<td>4</td>
</tr>
<tr>
<td>2.1. Categories and Localization</td>
<td>4</td>
</tr>
<tr>
<td>2.2. Model Categories</td>
<td>5</td>
</tr>
<tr>
<td>2.3. Bousfield Localization</td>
<td>16</td>
</tr>
<tr>
<td>2.4. Examples</td>
<td>18</td>
</tr>
<tr>
<td>2.5. Operads</td>
<td>28</td>
</tr>
<tr>
<td>2.6. Equivariant Homotopy Theory</td>
<td>32</td>
</tr>
<tr>
<td>Chapter 3. Preservation of Operad Algebras by Bousfield Localization</td>
<td>43</td>
</tr>
<tr>
<td>3.1. Definition of Preservation</td>
<td>43</td>
</tr>
<tr>
<td>3.2. General Preservation Result</td>
<td>44</td>
</tr>
<tr>
<td>3.3. Semi-Model Categories</td>
<td>47</td>
</tr>
<tr>
<td>Chapter 4. Monoidal Bousfield Localizations</td>
<td>52</td>
</tr>
<tr>
<td>4.1. A Non-Monoidal Bousfield Localization</td>
<td>52</td>
</tr>
<tr>
<td>4.2. Defining Monoidal Bousfield Localizations</td>
<td>54</td>
</tr>
</tbody>
</table>
CONTENTS

9.3. Equivariant Stable Homotopy Theory 175

9.4. Further Examples 177

Bibliography 180
CHAPTER 1

Introduction

In recent years, the importance of monoidal model categories has been demonstrated by a number of striking results related to structured (equivariant) ring spectra, c.f. [26], [41], [50], [66], [89]. Bousfield localization has played a crucial role in all of these advances. Bousfield localization allows for the passage from levelwise model structures to stable model structures (see [47]), allows for the construction of point-set models for numerous ring spectra, and provides a powerful computational tool.

Bousfield localization was originally introduced as a method to better understand the interplay between homology theories and the categories of spaces and spectra (see [17] and [18]). Thanks to the efforts of [27] and [42], Bousfield localization can now be understood as a process one may apply to general model categories, and the classes of maps which are inverted can be far more general than homology isomorphisms.

Monoids, commutative monoids, $A_\infty$-spectra (nowadays called structured ring spectra), and $E_\infty$-spectra have also played a key role in many recent applications, and the importance of model structures on commutative monoids (compatible with the monoidal model structure on the underlying category $\mathcal{M}$) has been conclusively demonstrated. Nowadays, $A_\infty$ and $E_\infty$ ring spectra are often thought of as algebras over the operads $A_\infty$ and $E_\infty$ acting in any of the monoidal model categories for spectra. It is therefore natural to ask the extent to which Bousfield localization preserves such algebraic structure. For Bousfield localizations at homology isomorphisms this question is answered
1. INTRODUCTION

in [26] and [66]. The case for spaces is subtle and is addressed in [20] and [27]. More general Bousfield localizations are considered in [19].

The preservation question may also be asked in the context of equivariant and motivic spectra, and it turns out the answer is far more subtle. Mike Hill found an example of a naturally occurring Bousfield localization of equivariant spectra which preserves the type of algebraic structure considered in [26] but which fails to preserve the equivariant commutativity needed for the landmark results in [41].

In this thesis we will answer the preservation question. In Chapter 2 we introduce the objects of study. In Chapter 3 we provide a general preservation result which will be used in all of our applications. In Chapter 4 we introduce the notion of a monoidal Bousfield localization and we characterize such localizations. We then use this characterization in Chapter 5 to give numerous examples of our preservation result to $\Sigma$-cofibrant operads such as $A_\infty$ and $E_\infty$ in model categories of spaces, spectra, and chain complexes.

In Chapter 5 we also provide an in-depth study of the case of equivariant spectra. We highlight precisely what is failing in Hill’s example and how to prohibit this behavior. We then discuss the connection between our work and the theorem of Hill and Hopkins presented in [40] which guarantees preservation of equivariant commutativity. We also provide preservation results for a collection of operads which interpolate between naive $E_\infty$ and genuine $E_\infty$. These operads are introduced in the last section of Chapter 2.

In the latter half of the thesis we turn to preservation of structure over non-cofibrant operads. An example is preservation of commutative monoids. For categories of spectra the phenomenon known as rectification means that preservation of strict commutativity is equivalent to preservation of $E_\infty$-structure, but in general there can be Bousfield
localizations which preserve the latter type of structure and not the former. In Chapter 6 we introduce a condition on a monoidal model category called the \textit{commutative monoid axiom}, which guarantees that the category of commutative monoids inherits a model structure.

In Chapter 7 we study the interplay between this new axiom and our preservation results. We provide conditions so that Bousfield localization preserves the commutative monoid axiom. We then apply our general preservation results to deduce preservation results for commutative monoids in numerous categories of interest. Finally, in Chapter 8 we provide conditions so that Bousfield localization preserves the monoid axiom.
CHAPTER 2

Preliminaries

2.1. Categories and Localization

Definition 2.1.1. A category is a class of objects together with a class of maps (a.k.a. morphisms) which contains identity maps for every object, is closed under composition, and such that between any two objects there is only a set worth of maps. An isomorphism is a map $f : A \rightarrow B$ such that there is some $g : B \rightarrow A$ satisfying $f \circ g = 1_B$ and $g \circ f = 1_A$. A functor $F : \mathcal{M} \rightarrow \mathcal{N}$ is a map of categories, i.e. $F$ takes objects to objects, takes morphisms to morphisms, and respects the category structure by satisfying $F(1_X) = 1_{F(X)}$ and $F(g \circ f) = F(g) \circ F(f)$.

Categories provide a general setting in which mathematics can be done. It is often useful for a mathematician to shift viewpoint and view a collection of objects as the same even if they were considered different previously. This process is known as localization, and proceeds by inverting a prescribed class of maps so that they become isomorphisms.

Definition 2.1.2. The localization of a category $\mathcal{M}$ at a class of maps $\mathcal{W}$ is a category $\mathcal{M}[\mathcal{W}^{-1}]$ together with a functor $\mathcal{M} \rightarrow \mathcal{M}[\mathcal{W}^{-1}]$ taking $\mathcal{W}$ into the isomorphisms and satisfying the universal property that any functor $\mathcal{M} \rightarrow \mathcal{N}$ taking $\mathcal{W}$ into the isomorphisms of $\mathcal{N}$ will factor through $\mathcal{M}[\mathcal{W}^{-1}]$. 

The field of topology provides an excellent example of localization. Point-set topology deals with spaces up to homeomorphism, whereas homotopy theory is less discriminating and views spaces $X$ and $Y$ as the same if they are homotopy equivalent. Localization comes into play in the passage from the category $\text{Top}$ whose objects are topological spaces and whose morphisms are continuous maps to the category $\text{HoTop}$ whose objects are spaces and whose morphisms are homotopy classes of maps. The functor $\text{Top} \to \text{HoTop}$ zooms in on homotopy theoretic information by formally setting the homotopy equivalences to be isomorphisms.

Unfortunately, when one applies the same formal process to pass from a general category $\mathcal{M}$ to some $\mathcal{M}[\mathcal{W}^{-1}]$, one loses control over the maps in $\mathcal{M}[\mathcal{W}^{-1}]$. Reestablishing control requires hypotheses on $\mathcal{M}$ which force it to behave enough like $\text{Top}$ so that $\mathcal{M}[\mathcal{W}^{-1}]$ behaves like $\text{HoTop}$. Such $\mathcal{M}$ are called model categories, and their classes $\mathcal{W}$ are called weak equivalences.

2.2. Model Categories

Model categories were invented in 1967 by Dan Quillen [74], whose key observation was that in $\text{Top}$ one cares not just about weak equivalences, but also about cofibrations and fibrations. Cofibrations are maps which are used to build more complicated spaces from simpler ones, e.g. maps which glue new cells to old spaces. Fibrations are the homotopy theorist’s version of projection, e.g. the map from the total space to the base space in a vector bundle, or projection from a Lie group onto its quotient by a subgroup. With these classes in mind, we now define the notion of a model category.
2.2. MODEL CATEGORIES

Definition 2.2.1. A model category is a category $\mathcal{M}$ with all limits and colimits, and with three distinguished classes of morphisms called weak equivalences $\mathcal{W}$, cofibrations $\mathcal{L}$, and fibrations $\mathcal{F}$ such that

- $\mathcal{W}$ satisfies the 2-out-of-3 property, i.e. if two out of the three morphisms $f, g, g \circ f$ are in $\mathcal{W}$, then so is the third.
- $\mathcal{W}, \mathcal{L}, \mathcal{F}$ are closed under retracts, where $f$ is a retract of $g$ if $f$ and $g$ fit into a commutative diagram where the horizontal composites are $id_A$ and $id_B$ respectively:

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longrightarrow & D
\end{array}
\begin{array}{cc}
\downarrow f \\
B & \longrightarrow & D
\end{array}
$$

- Trivial cofibrations (i.e. maps in $\mathcal{L} \cap \mathcal{W}$) satisfy the left lifting property with respect to fibrations, i.e. when $f \in \mathcal{L} \cap \mathcal{W}$ and $g \in \mathcal{F}$ then the lift below exists and makes both triangles commute.

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longrightarrow & D
\end{array}
$$

Similarly, cofibrations satisfy the left lifting property with respect to trivial fibrations. Dually, we say fibrations satisfy the right lifting property.

- Any morphism $f$ can be factored into two composites: either a cofibration followed by a trivial fibration or a trivial cofibration followed by a fibration. Furthermore, the assignment of $f$ to any of these factoring maps is functorial.
2.2. MODEL CATEGORIES

The benefit of model categories is that they have homotopy categories \( \text{Ho}(M) := \mathcal{M}[\mathbb{W}^{-1}] \) and the maps in these homotopy categories can be understood via the structure on \( \mathcal{M} \). Let \( \emptyset \) be the initial object of \( \mathcal{M} \) and \( * \) be the terminal object. If \( \emptyset = * \) then we say \( \mathcal{M} \) is \textit{pointed}.

**Definition 2.2.2.** An object \( A \) is \textit{cofibrant} if the natural map \( \emptyset \to A \) is a cofibration. Dually, an object \( B \) is \textit{fibrant} if \( B \to * \) is a fibration.

Let \( \mathcal{M}_{cf} \) denote the subcategory of cofibrant and fibrant objects of \( \mathcal{M} \) (a.k.a. bi-fibrant objects). It is a theorem (see [46]) that there is an equivalence of categories \( \mathcal{M}_{cf}/\sim \cong \text{Ho}(\mathcal{M}) \) where \( \sim \) is the homotopy relation.

Considering the motivation from topological spaces, one might naturally think that the only examples of model categories are topological in nature, e.g. \( \text{Top} \), simplicial sets, symmetric spectra (a setting for stable homotopy theory), and equivariant spectra (where a group \( G \) is allowed to act). Surprisingly, model categories also arise in algebraic situations, e.g. categories such as the category \( \text{Ch}(R) \) of chain complexes over a ring, the category of quasi-coherent sheaves over a scheme, and categories of motivic spaces and spectra (which allow for a homotopy theory of schemes and for a meaningful definition of Grothendieck’s motivic cohomology in algebraic geometry). These applications led to Fields Medals [73] for Quillen (1978) and [91] for Vladimir Voevodsky (2002, based on motivic homotopy theory). We will discuss several of these examples further in Section 2.4.
The classes $\mathcal{Q}$ and $\mathcal{F}$ have the added benefit of providing cofibrant and fibrant replacement functors on $\mathcal{M}$. These are methods of functorially replacing objects by nicer objects which are the same up to homotopy (i.e. isomorphic in $\mathcal{M}[^\mathcal{W}^{-1}]$).

**Definition 2.2.3.** For any object $X$ in a model category $\mathcal{M}$, the *cofibrant replacement* of $X$ is the cofibrant object $QX$ and the map $QX \to X$ obtained by applying the cofibration-trivial fibration factorization to the natural map $\emptyset \to X$. So $QX \to X$ is a trivial fibration with cofibrant domain, and any map $f : X \to Y$ yields a map $Qf : QX \to QY$ by functoriality.

Dually, the *fibrant replacement* of an object $A$ is a fibrant object $RA$ together with a trivial cofibration $A \to RA$. Fibrant replacement is obtained by applying the trivial cofibration-fibration factorization to the natural map $Y \to \ast$. A map $g : A \to B$ gives rise to a map $Rg : RA \to RB$ by functoriality.

In $\text{Top}$, cofibrant replacement is cellular approximation. In $\text{Ch}(\mathcal{R})$ cofibrant replacement can be projective resolution, or fibrant replacement can be injective resolution. The fact that $\text{Ch}(\mathcal{R})$ is a model category allows for hands-on constructions within the derived category $\mathcal{D}(\mathcal{R})$, leading to a better understanding of the maps between two objects, the triangulated structure, derived functors, and algebraic invariants such as André-Quillen cohomology.

**Remark 2.2.4.** We will always assume that cofibrant replacement takes a map $f$ to a cofibration $Qf$. This can always be arranged by taking $Qf$ to be the cofibration in the cofibration-trivial fibration factorization applied to the map $QX \to QY$ induced by $f$. 
Remark 2.2.5. Sometimes fibrant and cofibrant replacement exist even when $\mathcal{M}$ does not satisfy all the axioms of a model category. We shall see in Section 3.3 that a semi-model structure implies the existence of these functors (with the caveat that fibrant replacement only works after first applying cofibrant replacement).

Localization can now be thought of as a functor from a model category $\mathcal{M}$ to its homotopy category $\text{Ho}(\mathcal{M}) = \mathcal{M}[\mathcal{W}^{-1}]$. However, the question remains of how to invert a set of maps $\mathcal{C} \not\subseteq \mathcal{W}$. The answer is to find a new model category, denoted $L_{\mathcal{C}}(\mathcal{M})$ and called the (left) Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{C}$, such that both $\mathcal{C}$ and $\mathcal{W}$ are contained in the weak equivalences and hence sent to isomorphisms in $\text{Ho}(L_{\mathcal{C}}(\mathcal{M}))$. A general theory for when this can be done has been worked out in [42]. Before discussing the theory of Bousfield localization further, we must introduce more model category axioms.

Throughout this thesis we will assume $\mathcal{M}$ is a cofibrantly generated model category, so we now define this concept. Let $I$-cell denote the class of transfinite compositions of pushouts of maps in $I$, and let $I$-cof denote retracts of such. The small object argument (originally due to Dan Kan and expounded in Theorem 2.1.14 of [46]) is a transfinite construction which allows for the construction of the functorial factorizations required in the definition of a model category. In order to know that this process halts, we must introduce the notion of smallness.

Definition 2.2.6. An object $K$ is said to be $\kappa$-small relative to a class of maps $\mathcal{D}$ if given a regular cardinal $\lambda \geq \kappa$ and any $\lambda$-sequence $X_0 \to X_1 \to \ldots$ formed of maps
$X_\beta \to X_{\beta+1}$ in $\mathcal{D}$, then the map of sets $\text{colim}_{\beta<\lambda} \mathcal{M}(K, X_\beta) \to \mathcal{M}(K, \text{colim}_{\beta<\lambda} X_\beta)$ is a bijection.

An object is *small* relative to $\mathcal{D}$ if there is some $\kappa$ for which it is $\kappa$-small. An object $K$ is *finite* (a.k.a. *compact*) relative to $\mathcal{D}$ if for all limit ordinals $\lambda$, $\text{colim}_{\beta<\lambda} \mathcal{M}(K, X_\beta) \to \mathcal{M}(K, \text{colim}_{\beta<\lambda} X_\beta)$ is a bijection.

An object is small (resp. finite) if it is small (resp. finite) relative to the collection of all maps in $\mathcal{M}$.

See Chapter 10 of [42] for a more thorough treatment of this material.

**Definition 2.2.7.** A model category $\mathcal{M}$ is said to be *cofibrantly generated* if there is a set $I$ of cofibrations and a set $J$ of trivial cofibrations such that

1. The domains $K$ of the maps in $I$ (resp. $J$) are $\kappa$-small relative to $I$-cell (resp. $J$-cell) for some regular cardinal $\kappa$. When this holds we say $I$ and $J$ permit the small object argument.
2. A map is a fibration if and only if it satisfies the right lifting property with respect to all maps in $J$.
3. A map is a trivial fibration if and only if it satisfies the right lifting property with respect to all maps in $I$.

$\mathcal{M}$ is said to be *tractable* if it is cofibrantly generated and the domains (hence codomains) of the maps in $I$ and $J$ are cofibrant.

$\mathcal{M}$ is said to be *finitely generated* if the domains and codomains of $I$ and $J$ can be chosen to be finite relative to the cofibrations.
Remark 2.2.8. The author only knows one example where tractability of a cofibrantly generated model category fails, and that example was constructed precisely to demonstrate that tractability does not come for free. This example is discussed in Remark 4.4.4.

In some of our examples we will be dealing with stable model categories, i.e. \( \mathcal{M} \) for which \( \text{Ho}(\mathcal{M}) \) forms a triangulated category in the sense of Chapter 7 of [46]. This is the additional structure one sees when working in the stable homotopy category, and so many of our model categories of spectra are stable model categories.

The correct notion of a functor between model categories (i.e. one which respects the homotopy-theoretic structure) is that of a left Quillen functor.

Definition 2.2.9. Let \( \mathcal{M} \) and \( \mathcal{N} \) be model categories and let \( F : \mathcal{M} \rightleftarrows \mathcal{N} : U \) be an adjoint pair of functors (where \( F \) is left adjoint to \( U \)). We say \( F \) is a left Quillen functor, if it preserves cofibrations and trivial cofibrations. We say \( U \) is a right Quillen functor if it preserves fibrations and trivial fibrations.

We say the pair \( (F, U) \) is a Quillen equivalence if it descends to an adjoint equivalence of categories \( \text{Ho}(\mathcal{M}) \rightleftarrows \text{Ho}(\mathcal{N}) \).

Ken Brown’s Lemma (Lemma 1.1.12 in [46]) states that any functor which takes trivial cofibrations between cofibrant objects to weak equivalences (e.g. any left Quillen functor) must take all weak equivalences between cofibrant objects to weak equivalences.

2.2.1. Monoidal Model Categories. Recalling the core idea of algebraic topology, it is natural to consider model categories in which one can “do algebra.” These are
called monoidal model categories and are studied in [46]. Before introducing this notion we must first recall the notion of a monoidal category.

**Definition 2.2.10.** A *monoidal category* $\mathcal{M}$ is a category with a product bifunctor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, a unit object $S$, natural isomorphisms encoding the associativity of the product and the fact that $S \otimes E \cong E \cong E \otimes S$ for all $E$, and coherence diagrams for the unit and associator isomorphisms. We will always assume $\mathcal{M}$ is *symmetric monoidal*, i.e. there is a commutativity isomorphism $T : X \otimes Y \to Y \otimes X$ compatible with the monoidal structure. We say $\mathcal{M}$ is *closed symmetric monoidal* if for all $B$ the functor $A \mapsto A \otimes B$ has a right adjoint $A \mapsto \text{Hom}(B, A)$. The objects $\text{Hom}(B, A)$ are called *internal hom objects*.

**Definition 2.2.11.** In a monoidal category, a *monoid* is an object $E$ with structure maps $\mu : E \otimes E \to E$ and $\eta : S \to E$ satisfying associativity and unitality conditions as encoded by the following diagrams.

$$
\begin{align*}
E \otimes (E \otimes E) & \xrightarrow{\cong} (E \otimes E) \otimes E \\
1 \otimes \mu & \downarrow \\
E \otimes E & \xrightarrow{\mu \otimes 1} E \otimes E
\end{align*}
\quad
\begin{align*}
S \otimes E & \xrightarrow{\eta \otimes 1} E \otimes E \\
\cong & \downarrow \\
E & \xrightarrow{\mu} E
\end{align*}
\quad
\begin{align*}
E \otimes S & \xrightarrow{1 \otimes \eta} E \otimes E \\
\cong & \downarrow \\
E & \xrightarrow{\mu} E
\end{align*}
$$

A left *$E$-module* is an object $X$ with an action map $E \otimes X \to X$ with diagrams encoding compatibility with the associativity and unitality above.

A *commutative monoid* has a twist isomorphism $\tau : E \otimes E \to E \otimes E$ and diagrams forcing the twist to commute with multiplication. An *algebra* over a commutative monoid is an $E$-module which is also a monoid.
In a monoidal category, the pushout product of two maps \( f : A \to B \) and \( g : X \to Y \) is the map \( f \Box g : A \otimes Y \coprod_{A \otimes X} B \otimes X \to B \otimes Y \). We shall now make use of this product to provide the key coherence condition required to “do algebra” in model categories (Definition 4.2.6 in [46]).

**Definition 2.2.12.** A **monoidal model category** is a model category with a monoidal product \( \otimes \) satisfying

1. **Unit Axiom:** For any cofibrant \( X \), the map \( QS \otimes X \to S \otimes X \cong X \) is a weak equivalence, where \( QS \to S \) is cofibrant replacement.

2. **Pushout Product Axiom:** Given any \( f : X_0 \to X_1 \) and \( g : Y_0 \to Y_1 \) cofibrations, \( f \Box g : X_0 \otimes Y_1 \coprod_{X_0 \otimes Y_0} X_1 \otimes Y_0 \to X_1 \otimes Y_1 \) is a cofibration. Furthermore, if either \( f \) or \( g \) is trivial then \( f \Box g \) is trivial.

Note that the pushout product axiom is equivalent to the statement that \(- \otimes -\) is a Quillen bifunctor (see Definition 4.2.1 in [46]). Furthermore, it is sufficient to check the pushout product axiom on the generating (trivial) cofibrations \( I \) and \( J \), by Proposition 4.2.5 of [46].

From now on let \( \mathcal{M} \) be a monoidal model category. We will at times need to assume that cofibrant objects are flat in \( \mathcal{M} \), i.e. that whenever \( X \) is cofibrant and \( f \) is a weak equivalence then \( f \otimes X \) is a weak equivalence.

Another hypothesis which is valuable in the context of monoidal model categories is the **monoid axiom** which appeared as Definition 3.3 in [81].
Definition 2.2.13. Given a class of maps $\mathcal{C}$ in $\mathcal{M}$, let $\mathcal{C} \otimes \mathcal{M}$ denote the class of maps $f \otimes id_X$ where $f \in \mathcal{C}$ and $X \in \mathcal{M}$. A model category is said to satisfy the monoid axiom if all maps in $\text{(Trivial-Cofibrations} \otimes \mathcal{M})$-cell are weak equivalences.

By Remark 3.4 in [81], if all objects are cofibrant then the monoid axiom is a consequence of the pushout product axiom. By Lemma 3.5 in [81] it is sufficient to check the monoid axiom on the generating trivial cofibrations.

Let $A$ be any monoid and let $R$ be any commutative monoid. In [81] and the follow-up paper [45], it is proven that if $\mathcal{M}$ satisfies the monoid axiom and if the domains of $I$ (resp. $J$) are small relative to $(\mathcal{M} \otimes I)$-cell (resp. $(\mathcal{M} \otimes J)$-cell), then the categories of (left or right) $A$-modules and of $R$-algebras inherit model structures from $\mathcal{M}$. We will undertake a similar program in Chapter 6 for commutative monoids and algebras over them. We will require the same smallness hypothesis, which is satisfied automatically if $\mathcal{M}$ is a combinatorial model category.

2.2.2. Simplicial Model Categories. Another property which a model category can possess is a certain coherence with respect to the original motivation coming from spaces. In order to state this property we must introduce another model for spaces which is often more convenient to a model category theorist. It is the category $sSet$ of simplicial sets discussed in Chapter 3 of [46] and Example 3.2.6 in [50]. It has the same homotopy theory as $Top$ (i.e. the two model categories are Quillen equivalent), but the objects of $sSet$ are often easier to work with because they are fundamentally combinatorial (they are built from simplices).
Definition 2.2.14. The simplicial category $\Delta$ is the category whose objects are $[n] = \{0, 1, \ldots, n\}$ and whose morphisms $\Delta([n],[k])$ are maps $f : [n] \to [k]$ such that $x \leq y \Rightarrow f(x) \leq f(y)$.

The category $sSet$ of simplicial sets is the functor category $Set^{\Delta^{op}} = \text{Fun}(\Delta^{op}, Set)$. For a simplicial set $K$, the set $K_n := K[n]$ is the set of $n$-simplices of $K$.

The simplicial set $\Delta[n]$ is the functor $[k] \mapsto \Delta([k],[n])$. The boundary of $\Delta[n]$ is the subfunctor $\partial \Delta[n] \subset \Delta[n]$ of non-surjective maps. The $r$-horn on $\Delta[n]$ is the subfunctor $\Lambda^r[n]$ of maps whose image does not contain $r$.

There is an adjunction $|\cdot| : sSet \rightleftarrows Top : \text{Sing}$ where $|X|$ is the geometric realization of $X$ and $\text{Sing}$ is the singular functor. For a geometric intuition for the simplicial sets $\Delta[n], \partial \Delta[n],$ and $\Lambda^r[n]$ we refer the reader to Chapter 3 of [46]. It is proven in Lemma 3.1.8 and Lemma 3.2.4 of [46] that $|\cdot|$ preserves finite products and all finite limits.

We will discuss the model structure on $Top$ and $sSet$ in Section 2.4. For now, we state the coherence condition alluded to earlier.

Definition 2.2.15. A model category $\mathcal{M}$ is a simplicial model category if it is enriched, tensored, and cotensored over $sSet$ in a way compatible with the model structure on $\mathcal{M}$. This means the morphism sets in $\mathcal{M}$ have the structure of simplicial sets, there is a product bifunctor $\otimes : sSet \times \mathcal{M} \to \mathcal{M}$, and there is a way to exponentiate an object $M$ of $\mathcal{M}$ with respect to a simplicial set $K$ and get an object $M^K$ in $\mathcal{M}$. The compatibility condition means $\otimes$ is a Quillen bifunctor.

We remark that a topological model category satisfies the definition above with $Top$ instead of $sSet$. A more general definition of when a model category $\mathcal{M}$ is a $D$-model
category for some monoidal model category $\mathcal{D}$ is given as Definition 4.2.18 of [46]. It requires a version of the unit axiom regarding the cofibrant replacement for the unit in $\mathcal{D}$. This condition is unnecessary in the definition above because the units of $sSet$ and $Top$ are cofibrant as we shall see in Section 2.4 below.

2.3. Bousfield Localization

We are now prepared to discuss Bousfield localization. The goal is to invert a class of maps $\mathcal{C} \not\in \mathcal{W}$ by forming a new model category $L_{\mathcal{C}}(\mathcal{M})$ where $\mathcal{C}$ is contained in the weak equivalences. This was first accomplished by Bousfield in [17] (and later in [18]) for the model category of topological spaces (and later for spectra) with the goal of studying homology isomorphisms for some generalized homology theory $E$. In Bousfield’s setting the class $\mathcal{C}$ consisted of maps $f$ such that $E_*(f)$ is an isomorphism.

The theory of Bousfield localization was extended to general model categories in [42]. Note that $L_{\mathcal{C}}(\mathcal{M})$ does not exist for every choice of $\mathcal{M}$ and $\mathcal{C}$, but conditions are given in [42] under which existence is guaranteed in the case when $\mathcal{C}$ is a set.

**Definition 2.3.1.** The *Bousfield localization* of $\mathcal{M}$ with respect to a class of maps $\mathcal{C}$ is a model category $L_{\mathcal{C}}(\mathcal{M})$ with the same objects and morphisms as $\mathcal{M}$, with the same cofibrations as $\mathcal{M}$, but with a larger class of weak equivalences (containing $\mathcal{W}$ and $\mathcal{C}$). Furthermore, $L_{\mathcal{C}}(\mathcal{M})$ satisfies a universal property: the identity functor $\mathcal{M} \to L_{\mathcal{C}}(\mathcal{M})$ is a left Quillen functor and for any model category $\mathcal{N}$ and any left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ taking the maps $\mathcal{C}$ to weak equivalences, there is a unique left Quillen functor $L_{\mathcal{C}}(\mathcal{M}) \to \mathcal{N}$ through which $F$ factors.
All of our Bousfield localizations will be left Bousfield localizations, and this is why the cofibrations of \( L_C(M) \) and \( M \) match. There is a notion of right Bousfield localization considered in [42], but we will make no use of this notion.

Proving the existence of a Bousfield localization proceeds by first constructing the fibrant objects of \( L_C(M) \) and then constructing the weak equivalences. In both cases this is done via simplicial mapping spaces \( \text{map}(-,-) \). If \( M \) is a simplicial or topological model category then one can use the hom-object in \( sSet \) or \( Top \) to test locality. Otherwise a framing is required to construct the simplicial mapping space. We refer the reader to [42] or [46] for details on this process. We will not make use of framings in this document.

**Definition 2.3.2.** An object \( N \) is said to be \( C \)-local if it is fibrant in \( M \) and if for all \( g : X \to Y \) in \( C \), \( \text{map}(g,N) : \text{map}(Y,N) \to \text{map}(X,N) \) is a weak equivalence in \( sSet \). These objects are precisely the fibrant objects in \( L_C(M) \). A map \( f : A \to B \) is a \( C \)-local equivalence if for all \( N \) as above, \( \text{map}(f,N) : \text{map}(B,N) \to \text{map}(A,N) \) is a weak equivalence. These maps are precisely the weak equivalences in \( L_C(M) \).

Throughout this thesis we assume \( C \) is a set of cofibrations between cofibrant objects. This can always be guaranteed in the following way. For any map \( f \) let \( Qf \) denote the cofibrant replacement and recall that by Remark 2.2.4 we may assume \( Qf \) is a cofibration between cofibrant objects. Define \( \overline{C} = \{ Qf \mid f \in C \} \). Localization with respect to \( \overline{C} \) yields the same result as localization with respect to \( C \), so our assumption that the maps in \( C \) are cofibrations between cofibrant objects loses no generality.
We also assume everywhere that the model category $L_C(M)$ exists. This can be guaranteed by assuming $M$ is left proper and either combinatorial (as discussed in [6]) or cellular (as discussed in [42]).

**Definition 2.3.3.** A model category is *left proper* if pushouts of weak equivalences along cofibrations are again weak equivalences. A model category is *right proper* if pullbacks of weak equivalences along fibrations are again weak equivalences. A model structure is *proper* if it is both left and right proper.

We will often assume $M$ is left proper. However, as we have not needed the cellularity or combinatoriality assumptions for our work we have decided not to assume them. In this way if a Bousfield localization is known to exist for some reason other than the theory in [42] then our results will be applicable. We remark that a model category is *combinatorial* if it is cofibrantly generated and locally presentable as a category (it is fine to think of this second condition as saying all objects are small). The definition of cellularity is more complicated, but the idea is that $M$ is cofibrantly generated, the cofibrations behave like those in $Top$ as regards smallness, and the cofibrations are so-called effective monomorphisms. We shall not make much use of this notion.

### 2.4. Examples

In this section we give numerous examples of model categories.

**2.4.1. Spaces.** We have already remarked that $Top$ is the motivating example of a model category. We will now spell out the way in which it forms a model category. We must first be clear about what we mean by topological space. Because this thesis
is concerned with monoidal model categories, we must work in a monoidal setting. In general, the set of continuous maps between two topological spaces need not have a topology which is compatible with the Cartesian product (in the sense of the adjunction in Definition 2.2.10). For this reason, we restrict to a subcategory of spaces. The following appears in Definition 2.4.21 in [46].

**Definition 2.4.1.** Let \( X \) be a topological space.

1. A subset \( U \) of \( X \) is **compactly open** if for every continuous map \( f : K \to X \) where \( K \) is compact Hausdorff, \( f^{-1}(U) \) is open in \( K \).
2. \( X \) is a \( k \)-**space** if every compactly open subset is open.

In this thesis, \( Top \) will always be taken to mean the category of \( k \)-spaces. In this category, the set \( Top(X, Y) \) of continuous maps between any two spaces can be endowed with the compact-open topology. Furthermore, there is an adjunction \( Top(X, Top(Y, Z)) \cong Top(X \times Y, Z) \). We are now prepared to discuss the model structure on \( Top \). The following is Theorem 2.4.23 in [46].

**Theorem 2.4.2.** The category \( Top \) forms a finitely generated model category where the weak equivalences are the weak homotopy equivalences, the fibrations are the Serre fibrations, and the generating (trivial) cofibrations are the following sets of inclusions:

\[
I = \{ i_n : S^{n-1} \to D^n \mid n \in \mathbb{N} \} \text{ and } J = \{ j_n : D^n \to D^n \times [0,1] \mid n \in \mathbb{N} \}
\]

In this model structure all objects are fibrant, so \( Top \) is right proper. Lemma 2.4.1 in [46] proves all spaces are small relative to inclusions. It is clear that \( Top \) is tractable. Furthermore, \( Top \) is cellular and left proper (see 12.1.4 and 13.3.11 in [42]). Note that \( Top \) is not combinatorial.
We now describe the model structure on $sSet$. Recall the adjunction $|\cdot| : sSet \rightleftarrows \text{Top} : \text{Sing}$. This adjunction is used to construct a model structure on $sSet$ from the one on $\text{Top}$, and in doing so the adjunction becomes a Quillen equivalence.

**Theorem 2.4.3.** There is a finitely generated model structure on $sSet$ in which a map $f$ is a weak equivalences if and only if $|f|$ is a weak equivalence in $\text{Top}$, in which the cofibrations are precisely the injective maps, and in which the generating (trivial) cofibrations are given by the sets of inclusions:

$$I = \{ i_n : \partial \Delta[n] \to \Delta[n] \mid n > 0 \}$$

and

$$J = \{ j_n : \Lambda^r[n] \to \Delta[n] \mid n > 0, r \in [n] \}$$

In this model structure all objects are cofibrant (so $sSet$ is left proper and tractable) and all objects are small (so $sSet$ is combinatorial).

Chapter 4 of [46] proves that both $\text{Top}$ and $sSet$ are monoidal model categories with respect to the Cartesian product. There are also pointed versions of these categories, normally denoted $\text{Top}_*$ and $sSet_*$, in which objects are spaces (resp. simplicial sets) with a disjoint basepoint. For the pointed versions, the monoidal product is given by the smash product $X \wedge Y = X \times Y / X \wedge Y$ and again these model categories are monoidal model categories. In both the pointed and unpointed settings cofibrant objects are flat and the monoid axiom is satisfied.

Our default position will be to work with pointed spaces, so we shall use the symbols $\text{Top}$ and $sSet$ for what might normally be called $\text{Top}_*$ and $sSet_*$. To avoid confusion we use the notation $X_* := X \amalg *$ for the pointed space corresponding to an unpointed space $X$. All of our localization results will hold for the unpointed versions as well, but
because spectra and operads are built from pointed spaces we find it more convenient to assume all spaces are pointed.

2.4.2. Spectra. The category of spectra is where stable homotopy theory is done. To motivate this category we will remind the reader of the Freudenthal Suspension Theorem, which is the starting place for stable homotopy theory. Before stating this theorem we must fix some terminology. The reduced suspension functor is defined to be \( \Sigma(X) = X \land S^1 \). A space is \( n \)-connected if \( \pi_k(X) = 0 \) for all \( 1 \leq k \leq n \) (a space is 0-connected if it is path connected).

**Theorem 2.4.4 (Freudenthal Suspension Theorem (1937)).** Let \( X \) be an \( n \)-connected pointed CW-complex (or pointed simplicial set) and let \( n \geq 1 \). The inclusion \( X \to \Sigma X \) induces a map \( \pi_k(X) \to \pi_{k+1}(\Sigma X) \) which is an isomorphism if \( k < 2n + 1 \) and an epimorphism if \( k = 2n + 1 \).

We see immediately that if \( X \) is \( n \)-connected, then \( \Sigma X \) is \( n+1 \)-connected, since \( \pi_{i+1}(\Sigma X) \cong \pi_i(X) = 0 \) for all \( i \leq n \).

Now take \( X \) any path-connected space and fix \( k \). Starting with \( n = 1 \) and increasing \( n \) at each step gives:

\[
\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \pi_{k+3}(\Sigma^3 X) \ldots
\]

For \( n > k + 1 \) we have \( \pi_{k+n}(\Sigma^n X) \cong \pi_{k+n+1}(\Sigma^{n+1} X) \), i.e. this chain of maps stabilizes. We can thus define: \( \pi^s_k(X) = \lim_n \pi_{k+n}(\Sigma^n X) \). A similar process beginning with \( S^0 \) yields \( \pi^s_k(S^0) \), called the *stable k-stem*. Stable homotopy groups are easier to compute than the unstable groups because all that matters is the difference between \( n \) and \( k \).
Most of the modern theory looks at all the $\pi_k(S^0)$ simultaneously, because they form a graded ring under composition. The point of spectra is to provide a topological object whose homotopy is this algebraic object. Doing so requires moving from the category of topological spaces to a new category where the objects are “stable.” In particular, this requires erasing topological dimension. The following is originally due to Lima, in [58].

**Definition 2.4.5.** The category $Sp = Spectra$ has objects given by sequences of spaces $(X_n)_{n \in \mathbb{N}}$ with structure maps $\Sigma X_n \to X_{n+1}$. A morphism of spectra is a sequence $f = (f_n)$ commuting with the structure maps. The *sphere spectrum* $S = (S^n)$. Given a space $X$, define the *suspension spectrum* of $X$ to be $(\Sigma^\infty X)_n$ to be $\Sigma^n X$. Given a ring $R$ the Eilenberg-Mac Lane spectrum $HR = (K(R,n))$ has $n^{th}$ space $K(R,n)$ the Eilenberg-Mac Lane space with $\pi_n(K(R,n)) = R$ and $\pi_k(K(R,n)) = 0$ for all $k \neq n$.

As we are concerned with monoidal model categories, we need a notion of spectra with an appropriate product and internal hom. There are several options.

2.4.2.1. *Symmetric Spectra.* The following is taken from [50].

**Definition 2.4.6.** A *symmetric sequence* is a sequence of pointed simplicial sets $(X_n)$ together with an action of $\Sigma_n$ on $X_n$ for all $n$. The tensor product of two symmetric sequences is given by the formula

$$(X \otimes Y)_n = \bigvee_{p+q=n} (\Sigma_p)_+ \land_{\Sigma_p \times \Sigma_q} (X_p \land X_q)$$

The category $Sp^\Sigma$ of *symmetric spectra* is the category of left $S$-modules in the category of symmetric sequences. As $S$ is a commutative monoid with respect to the product just given, the category of symmetric spectra is a monoidal category where the
product $X \otimes_S Y$ is defined as the coequalizer of the natural maps $X \otimes S \otimes Y \to X \otimes Y$. The spectrum structure maps $\nu_{p,q} : S^p \wedge X_q \to X_{p+q}$ are now $\Sigma_p \times \Sigma_q$-equivariant, unital ($\nu_{0,q}$ is the identity), and associative (i.e. $\nu_{p,q+r} \circ \nu_{q,r} = \nu_{p+q+r} \circ \mu_{p,q}$ where $\mu_{p,q} : S^p \wedge S^q \to S^{p+q}$ is the natural homeomorphism).

For an understanding of the closed structure (i.e. of the internal hom in this category) we refer the reader to [50].

These monoidal categories of spectra may be given model structures. We first need a new piece of information. For any $n$ there is an evaluation functor $Ev_n : Sp \Sigma \to sSet$ taking $X$ to $X_n$. This functor has a left adjoint $F_n$. The following is part of Theorem 5.1.2 in [50].

**Theorem 2.4.7.** The projective model structure on $Sp \Sigma$ has weak equivalences (resp. fibrations) defined to be maps which are levelwise weak equivalences (resp. fibrations) in $sSet$, and cofibrations defined via the left lifting property. This model structure has generating (trivial) cofibrations given by $FI = \cup_{n \geq 0} F_n(I)$ and $FJ = \cup_{n \geq 0} F_n(J)$ where $I$ and $J$ are the generating (trivial) cofibrations of $sSet$.

There is also an injective model structure with cofibrations and weak equivalences defined levelwise, and fibrations defined via lifting. The stable model structure on $Sp \Sigma$ is obtained from the projective model structure via left Bousfield localization. We refer the reader to Definition 3.1.3 and Theorem 3.4.4 of [50].

We remark that there are also positive levelwise and positive stable model structures on $Sp \Sigma$ introduced in [66]. These model structures have the same weak equivalences as the non-positive versions, but have fewer cofibrations. In particular, a map $f =$
$(f_n)$ is a positive cofibration if and only if $f$ is a cofibration as above and $f_0$ is an isomorphism. The reason for this variant is so that commutative monoids may inherit a model structure. This will be made clear in Chapter 6.

Lastly, we remark that there are flat and flat stable model structures on $Sp^\Sigma$ (introduced in [50] and [83]) in which cofibrations are enlarged to include all monomorphisms, and there are positive flat and positive flat stable model structures in which after this enlargement is made the cofibrations are required to be isomorphisms in level 0. Again, the reason for these model structures will be made clear in Chapter 6.

All of the model structures on symmetric spectra are cofibrantly generated, tractable, left proper, combinatorial, monoidal, simplicial, have cofibrant objects flat, and satisfy the monoid axiom.

2.4.2.2. Orthogonal Spectra and Diagram Spectra. For the next definition (introduced first in [66]), let $O(n)$ be the group of $n \times n$ orthogonal matrices and say $X$ is an $O(n)$-space if $O(n)$ acts continuously on $X$.

**Definition 2.4.8.** An orthogonal spectrum is a sequence of pointed $O(n)$-spaces $X_n$ for $n \geq 0$, together with $O(p) \times O(q)$-equivariant spectrum structure maps $\nu_{p,q} : Sp^p \wedge X_q \to X_{p+q}$ that are associative and unital. Here $O(p)$ acts on $Sp$ via the action on $\mathbb{R}^p$, since $Sp$ is the one-point compactification of $\mathbb{R}^p$. The category of orthogonal spectra will be denoted $Sp^O$.

The monoidal product on the category of orthogonal sequences is given by

$$(X \otimes Y)_n = \bigvee_{p+q=n} O(n)_+ \wedge_{O(p) \times O(q)} (X_p \wedge Y_q)$$
This extends to a product on the category of orthogonal spectra (i.e. left $S$-modules) in the same way as for symmetric spectra.

As for symmetric spectra, there are many model structures on $Sp^O$. It is clear from the treatment above that there are again evaluation functors from $Sp^O$ to $Top$ and left adjoints $F_n$. There is again a projective model structure (with generators given just as in Theorem 2.4.7), a stable model structure (obtained as a left Bousfield localization), and positive, flat, positive flat, positive stable, flat stable, and positive flat stable variants. These are considered in [66] (Sections 6 and 9) and [87] (Chapter 1). All of these model structures are cofibrantly generated, tractable, proper, cellular, monoidal, topological, have cofibrant objects flat, and satisfy the monoid axiom. We will prove in Chapter 8 that there are additionally injective model structures.

The third primary model for monoidal categories of spectra is the category of $S$-modules of [26]. The idea here is to define so-called ‘pre-spectra’ in a coordinate-free way, to encode the notion of the smash product of a space with a spectrum via the so-called ‘twisted half-smash product’ (Section I.2 of [26]), and to encode the monoidal product of two spectra via the linear isometries operad (making use of the twisted half-smash product for the operad-algebra structure maps, c.f. Section I.3 of [26]). The passage to spectra then proceeds as above via consideration of left $S$-modules. We will not have much use for $S$-modules, so we shall not say any more about them here.

Other categories of diagram spectra include $\Gamma$-spaces, $W$-spaces, and simplicial functors. The first two are considered in [66] and may be defined in a completely analogous way to orthogonal spectra, but with different indexing categories in place of the category of orthogonal groups. We will discuss the first two in Chapter 6 but as that is the
2.4. Examples

only place we discuss them we have delayed introducing them until then. The third was introduced in [64] and we do not study it at all here.

2.4.3. Stable Module Category. Let $R$ be a quasi-Frobenius ring, i.e. one for which any module is projective if and only if it is injective. We are implicitly assuming $R$ is commutative, but this theory works equally well if $R$ is not commutative. The interested reader is encouraged to consult Section 2.2 in [46], where all the material in this section can be found.

Definition 2.4.9. The stable category of $R$-modules $\text{StMod}(R)$ is a category whose objects are $R$-modules and whose morphisms are equivalence classes of $R$-module homomorphisms where $f \sim g$ if $f - g$ factors through a projective $R$-module. A map $f$ is said to be a stable equivalence if it is an isomorphism in $\text{StMod}(R)$.

Theorem 2.4.10. There is a cofibrantly generated stable model structure $\mathcal{M}$ on the category of $R$-modules in which the weak equivalences are the stable equivalences, the fibrations are surjective maps, the trivial fibrations are surjections with projective cokernel, and where the cofibrations are injections. The generating cofibrations $I$ are given by the set of inclusions $\{a \to R\}$ where $a$ is an ideal of $R$. The generating trivial cofibrations $J$ are given by the one-element set $\{0 \to R\}$.

Because the cofibrations are inclusions and the fibrations are surjections, all objects are cofibrant and fibrant in $\mathcal{M}$ (so $\mathcal{M}$ is tractable and proper). If $R$ is finitely generated then every object is finite. For any $R$, all objects are small (so $\mathcal{M}$ is combinatorial). We discuss the monoidal structure on $\mathcal{M}$ in Example 4.1.1.
2.4. Chain Complexes. Let $R$ be a commutative ring and let $Ch(R)$ denote the category whose objects are chain complexes of $R$-modules and whose morphisms are chain maps (i.e. maps commuting with the differential). Let $M$ be an $R$-module and let $S^n(M)$ denote the chain complex with $M$ in degree $n$ and 0 in all other degrees. Let $D^n(M)$ denote the chain complex with $M$ in degrees $n$ and $n-1$, with the differential $d_n = id_M$, and with 0 in all other degrees.

Theorem 2.4.11. There is a cofibrantly generated model structure on $Ch(R)$ in which the weak equivalences are the quasi-isomorphisms, the fibrations are surjections, and the cofibrations are the dimensionwise split injections with cofibrant cokernel. The set of generating cofibrations is defined to be the set $I = \{ S^{n-1}(R) \to D^n(R) \mid n \in \mathbb{Z} \}$ where the map is the identity in degree $n$. The set of generating trivial cofibrations is the set $J = \{ 0 \to D^n(R) \mid n \in \mathbb{Z} \}$.

All objects in $Ch(R)$ are small, so $Ch(R)$ is combinatorial. This model structure is called the projective model structure. There is also an injective model structure but we will not make use of it.

One could also restrict attention to bounded chain complexes, which we shall take to mean $X_n = 0$ for all $n < 0$. There is again a (projective) model structure. Bounded complexes of finitely presented $R$-modules are finite so this model structure is finitely generated.

Let $\mathcal{M}$ denote the projective model structure on either bounded or unbounded chain complexes. Then $\mathcal{M}$ is tractable, proper, combinatorial, monoidal, has cofibrant objects flat, has all objects fibrant, and satisfies the monoid axiom.
2.5. Operads

One of the main goals of this thesis is to discuss preservation of algebraic structure as encoded by an operad. We have already mentioned the many striking applications of studying ring spectra and commutative ring spectra. Encoding algebraic structure via diagrams as in Definition 2.2.11 is a powerful tool which allows one to do algebra in any monoidal category. One could also use diagrams to encode the notion of field, vector space, Lie algebra, etc.

Operads are bookkeeping tools which allow one to consider algebraic structures in general monoidal categories (i.e. to do ‘universal algebra’) without having to keep track of all these diagrams. By operad we will always mean symmetric operad. The formal definition of an operad is somewhat technical, but we include it for completeness.

In the following, the reader should think of the objects $P(n)$ as parameterizing functions of arity $n$ (i.e. with $n$ inputs and 1 output). The $\Sigma_n$-action is then a statement about permuting the inputs and the composition rule is a statement about plugging the outputs of $n$ functions (elements of $P(k_1), \ldots, P(k_n)$) into a function in $P(n)$.

**Definition 2.5.1.** An operad valued in $\mathcal{M}$ is a symmetric sequence $(P(n))_{n \in \mathbb{N}}$ of objects in $\mathcal{M}$, with an identity element $1 \in P(1)$ and compositions maps $\circ$ defined by

$$\circ : P(n) \times P(k_1) \times \cdots \times P(k_n) \rightarrow P(k_1 + \cdots + k_n)$$

$$(f, f_1, \ldots, f_n) \mapsto f \circ (f_1, \ldots, f_n),$$

satisfying the following identity, associativity, and equivariance axioms

**Identity:** $f \circ (1, \ldots, 1) = f = 1 \circ f$

**Associativity:** $f \circ (f_1 \circ (f_{1,1}, \ldots, f_{1,k_1}), \ldots, f_n \circ (f_{n,1}, \ldots, f_{n,k_n}))$

is equal to $(f \circ (f_1, \ldots, f_n)) \circ (f_{1,1}, \ldots, f_{1,k_1}, \ldots, f_{n,1}, \ldots, f_{n,k_n})$
Equivariance: given \( \sigma \in \Sigma_n \), \( (\sigma \cdot f)(f_{\sigma(1)}, \ldots, f_{\sigma(n)}) = \sigma \cdot (f \circ (f_1, \ldots, f_n)) \) and given \( s_i \in \Sigma_{k_i}, f \circ (s_1 \cdot f_1, \ldots, s_n \cdot f_n) = (s_1, \ldots, s_n) \cdot (f \circ (f_1, \ldots, f_n)) \).

A morphism of operads is a morphism \( g \) of symmetric sequences such that \( g(1) = 1 \) and \( g \) respects the composition maps, i.e. \( g(f \circ (f_1, \ldots, f_n)) = g(f) \circ (g(f_1), \ldots, g(f_n)) \).

We care about operads mostly for their algebras.

**Definition 2.5.2.** An algebra over the operad \( P \) is an object \( A \in \mathcal{M} \) equipped with coherent maps \( P(n) \otimes A^\otimes n \to A \). A morphism of \( P \)-algebras is a map respecting this action. Let \( P\text{-alg}(\mathcal{M}) \) denote the category of algebras over \( P \).

More compactly, an algebra over an operad is a map of operads from \( P \) to the endomorphism operad \( \text{End}_A = (\mathcal{M}(A^\otimes n, A))_{n \in \mathbb{N}} \). Given an operad \( P \) and an algebra \( A \), one may form the enveloping operad \( P_A \) whose algebras are \( A \)-modules in \( \mathcal{M} \). We refer the reader to [12] for details on how this is done. Monoids are algebras over the associative operad \( \text{Ass} \) and \( R \)-modules are algebras over \( \text{Ass}_R \). Commutative monoids are algebras over the commutative operad \( \text{Com} \), whose \( n \)th space is the monoidal unit (with the trivial \( \Sigma_n \)-action) for all \( n \). Commutative \( R \)-algebras are algebras over \( \text{Com}_R \).

The free \( P \)-algebra functor from \( \mathcal{M} \) to \( P\text{-alg}(\mathcal{M}) \) is left adjoint to the forgetful functor. A formula may be given by

\[
P(X) = \coprod_{n \geq 0} P(n) \otimes_{\Sigma_n} A^\otimes n.
\]

When \( P \) is \( \text{Ass} \), the free monoid functor \( X \mapsto S \coprod X \coprod X^\otimes 2 \coprod \ldots \) has been known to topologists for years as the James construction. When \( P \) is \( \text{Com} \), the free commutative
monoid functor $X \mapsto S \coprod X \coprod X^\otimes 2 / \Sigma 2 \coprod \ldots$ is sometimes called the $SP^\infty$ functor, or the Dold-Thom functor.

**Definition 2.5.3.** For a cofibrantly generated monoidal model category $\mathcal{M}$ and for a group $G$, let $\mathcal{M}^G$ denote the *projective model structure* on objects which have a $G$ action. In this model structure a map is a weak equivalence (resp. fibration) if and only if it is so when we forget the $G$-action.

**Definition 2.5.4.** The category $P\text{-alg}(\mathcal{M})$ *inherits* a model structure from $\mathcal{M}$ if the model structure is transferred across the free-forgetful adjunction, i.e. if a $P$-algebra homomorphism is a weak equivalence (resp. fibration) if and only if it is so in $\mathcal{M}$. An operad $P$ is said to be *admissible* if $P\text{-alg}(\mathcal{M})$ inherits a model structure in this way.

Proving the existence of this model structure often comes down to Lemma 2.3 in [81]:

**Lemma 2.5.5.** Suppose $\mathcal{M}$ is cofibrantly generated and $T$ is a monad which commutes with filtered direct limits. If the domains of $T(I)$ and $T(J)$ are small relative to $T(I)$-cell and $T(J)$-cell respectively and either

1. $T(J)$-cell $\in \mathcal{W}$, or
2. All objects are fibrant and every $T$-algebra has a path object (factoring $\delta : X \to X \otimes X$ into a trivial cofibration followed by a fibration)

then $T\text{-alg}$ inherits a cofibrantly generated model structure from $\mathcal{M}$.

When the conditions of this lemma are satisfied, $P\text{-alg}$ inherits a cofibrantly generated model structure in which $P(I)$ and $P(J)$ are the generating (trivial) cofibrations. The
case $P = \text{Ass}$ was treated in [81] and checking the first condition of the lemma led to the introduction of the monoid axiom. The monoid axiom therefore implies monoids inherit a model structure. We will develop the commutative analogue of this result in Chapter 6.

Admissibility will be a central feature for our general result regarding preservation of operad-algebras by Bousfield localization (see Chapter 3). A result due to Markus Spitzweck [32] will allow us to deduce a weaker form of admissibility for all $\Sigma$-cofibrant operads, and this will be used to obtain many examples of our main result in Chapter 5.

We recall now the method by which cofibrancy and $\Sigma$-cofibrancy is defined for an operad. The category of symmetric sequences may be endowed with a \textit{product model structure} (Example 1.1.6 of [46]) coming from the projective model structures $\mathcal{M}^{\Sigma_n}$ of Definition 2.5.3:

$$\text{SymSeq}(\mathcal{M}) = \prod_{n \geq 1} \mathcal{M}^{\Sigma_n}.$$  

A map $f$ is a weak equivalence (resp. fibration) if and only if $f_n$ is a weak equivalence (resp. fibration) in $\mathcal{M}^{\Sigma_n}$ for all $n$. This occurs if and only if $f_n$ is a weak equivalence (resp. fibration) in $\mathcal{M}$ for all $n$.

Berger and Moerdijk [9] constructed a free-operad functor from the category of symmetric sequences to the category of operads. They considered the passage of a model structure from the category of symmetric sequences to the category of operads via the free-forgetful adjunction. Under certain hypotheses on $\mathcal{M}$, this transfer endows the category of operads with a model structure. In particular, they verify that when $\mathcal{M} = \text{Top}$, the category of operads is a cofibrantly generated model category.
2.6. EQUIVARIANT HOMOTOPY THEORY

Even if the category of operads does not form a model category, one can still discuss cofibrancy. An operad $P$ is said to be $\Sigma$-cofibrant if, after applying the forgetful functor and viewing $P$ as a symmetric sequence, the map from the initial operad $\emptyset$ to $P$ satisfies the left lifting property with respect to all trivial fibrations of symmetric sequences. We say $P$ is cofibrant if it satisfies the left lifting property in the category of operads. By Proposition 4.3 in [9], a cofibrant operad is $\Sigma$-cofibrant.

The primary examples of $\Sigma$-cofibrant operads are $A_\infty$ and $E_\infty$ operads considered in [67], which are $\Sigma$-cofibrant and weakly equivalent to $Ass$ and $Com$ respectively. When $\mathcal{M}$ is a category of spectra we are free to work with operads valued in spaces because the $\Sigma^\infty$ functor will take a ($\Sigma$-cofibrant) space-valued operad to a ($\Sigma$-cofibrant) spectrum-valued operad with the same algebras. So for both spaces and spectra, the $n^{th}$ space of an $E_\infty$ operad will be an $E\Sigma_n$ space, i.e. the cofibrant replacement for the point in $Top^{\Sigma_n}$.

2.6. Equivariant Homotopy Theory

In equivariant homotopy theory there is a group acting on everything in sight. This additional structure was pivotal in the computations used in the Kervaire Invariant One Theorem [41] and has been very useful in motivic homotopy theory when $G$ is the Galois group.

As we are interested in topological situations, we will insist that this group be a topological object. In particular, assume $G$ is a compact Lie group. At times we will restrict attention to a finite group $G$ with the discrete topology, but we will always state when such a restriction is being made.
2.6.1. Equivariant Spaces. Let $GTop$ denote the category whose objects are $k$-spaces with a continuous $G$-action and whose morphisms are $G$-equivariant maps. As we always work with pointed spaces, we will additionally assume the $G$-action fixes the distinguished basepoint. Classically, a family of subgroups of a finite group $G$ is taken to mean a non-empty set of subgroups closed under conjugation and taking subgroups.

Families of subgroups are crucial to the study of equivariant homotopy theory; they are necessary for the definition of the geometric fixed points functor, they come up several places in [56] in constructions of free spectra, and they are related to the Baum-Connes and Farrell-Jones Conjectures. We refer the reader to the excellent survey articles [59] and [10] for more information on the importance of families. Fixed-point model structures allow for the homotopical study of the information which can be "seen" by a family.

Families will also be useful to us (in the context of compact Lie groups) in Chapter 5. Given a $G$-space $X$ and a closed subgroup $H$ of $G$, we can consider the space of fixed points $X^H$. We need $H$ to be closed so that the quotient topology on $G/H$ is weak Hausdorff. For this reason, whenever we consider subgroups we will assume they are closed.

The following model structure is considered in Theorem IV.6.3 in [65]:

**Definition 2.6.1.** Let $\mathcal{F}$ be a non-empty set of closed subgroups of $G$ which is closed under conjugation and taking subgroups. Then the $\mathcal{F}$-fixed point model structure on pointed $G$-spaces is a cofibrantly generated model structure in which a map $f$ is a weak equivalence (resp. fibration) if and only if $f^H$ is a weak equivalence (resp. fibration)
in $\text{Top}$ for all $H \in \mathcal{F}$. The generating (trivial) cofibrations are $(G/H \times g)_+$, where $g$ is a generating (trivial) cofibration of topological spaces, and $H \in \mathcal{F}$.

We will denote this model structure by $\text{Top}^\mathcal{F}$, or by $\text{Top}^G$ if $\mathcal{F}$ is the set of all closed subgroups of $G$.

Turning now to monoidal structure, recall that $G\text{Top}$ is closed symmetric monoidal, where we use the diagonal action of $G$ on $X \times Y$, and the conjugation action of $G$ on the equivariant mapping space $\text{Map}_G(X, Y)$. That is, $(g \cdot f)(x) = g \cdot f(g^{-1} \cdot x)$. The category $G\text{Top}$ is enriched, tensored, and cotensored over $\text{Top}$. The left adjoint to the forgetful functor $G\text{Top} \to \text{Top}$ takes a space to itself, endowed with the trivial $G$-action.

The pushout product axiom on $\text{Top}^\mathcal{F}$ requires $(G/H)_+ \wedge (G/K)_+$ with the diagonal action to be cofibrant. If $G$ is finite then the assumption that $\mathcal{F}$ is closed under intersections implies this. In the case where $G$ is a compact Lie group rather than a finite group, additional care must be taken to ensure the model structure on $\text{Top}^\mathcal{F}$ satisfies the pushout product axiom. As noticed in Lemma 2.9 of [28], the key condition to assume on $\mathcal{F}$ is that it forms an Illman collection. This means that $(G/H \times G/K)_+$ is an $\mathcal{F}I$-cell complex for any $H, K \in \mathcal{F}$.

Remark 2.6.2. Because we do our work in the setting of monoidal model categories, when we say family of subgroups we will always mean an Illman collection (or a family in the classical sense if $G$ is finite).

We summarize the considerations above as:
Proposition 2.6.3. Let $G$ be a compact Lie group and $\mathcal{F}$ a family of subgroups as in Remark 2.6.2. Then $\text{Top}^\mathcal{F}$ is a cofibrantly generated, tractable, proper, topological, monoidal model category satisfying the hypothesis that cofibrant objects are flat.

2.6.2. Equivariant Spectra. We follow [51] and define a $G$-spectrum to be an orthogonal spectrum with a $G$-action, i.e. a sequence $X$ of pointed $G \times O(n)$-spaces $X_n$ for $n \geq 0$ (such $X$ are called $G$-orthogonal sequences) with associative and unital $G \times O(n) \times O(m)$-equivariant structure maps $S^n \times X_m \to X_{n+m}$. When $G = \{e\}$, a $G$-spectrum is an orthogonal spectrum.

The category of $G$-spectra is closed symmetric monoidal, because a $G$-spectrum $X$ is an $S$-module in the category of $G$-orthogonal sequences (here $S$ is the sphere spectrum, and a commutative monoid in the usual way). The monoidal product on $G$-orthogonal sequences is given by

$$(X \otimes Y)_n = \bigvee_{p+q=n} O(n) \wedge_{O(p) \times O(q)} (X_p \wedge Y_q)$$

with diagonal $G$-action. The closed structure is given by

$$\text{Hom}(X,Y)_n = \prod_{m \geq n} \text{Map}_{O(m-n)}(X_{m-n}, Y_m)$$

where $g \in O(n)$ acts on a map $f$ by acting on $f(x) \in Y_m$ using the inclusion

$$O(n) \subseteq O(m-n) \times O(n) \to O(m).$$
The enrichment over topological spaces is given by

$$\text{Map}(X,Y) = \prod_n \text{Map}_{O(n)}(X_n,Y_n).$$

Following [51], let $\mathcal{U}$ denote a complete $G$-universe, let $V$ be an $n$-dimension $G$-representation in $\mathcal{U}$, and let $\text{Ev}_V$ be the functor from $G$-spectra to $\text{Top}^G$ which takes a spectrum $X$ to the space

$$X(V) = O(\mathbb{R}^n, V)_+ \wedge_{O(n)} X_n$$

The left adjoint to $\text{Ev}_V$ is $F_V$, defined as

$$(F_V K)_{n+k} = O(n+k)_+ \wedge_{O(k) \times O(n)} (S^k \wedge (O(V, \mathbb{R}^n)_+ \wedge K)),$$

Just as family model structures may be considered for spaces, so may they be considered for spectra. Theorem IV.6.3 in [65] provides:

**Definition 2.6.4.** Let $\mathcal{F}$ be a family of subgroups of $G$. The $\mathcal{F}$-fixed point model structure on $Sp^G$ has weak equivalences (resp. fibrations) maps $f$ such that $f^H$ is a weak equivalence (resp. fibration) of orthogonal spectra for all $H \in \mathcal{F}$. The generating (trivial) cofibrations are $F_W((G/H)_+ \wedge g)$ as $H$ runs through $\mathcal{F}$, $g$ runs through the generating (trivial) cofibrations of spaces, and $W$ runs through some $G$-universe $\mathcal{U}$.

We denote this model structure by $Sp^\mathcal{F}$. We remark that there are many variants on this model structure, including positive, flat, positive flat, stable, positive stable, flat stable, and positive flat stable. See [87] for a thorough treatment, especially 2.3.19 and
2.3.30. These model structures are tractable, right proper, monoidal, have cofibrant objects flat, and satisfy the monoid axiom. The stable model structures are left proper. All of these model structures are $\text{Top}^G$-model structures in the sense of Definition 4.2.18 of [46].

### 2.6.3. Equivariant Operads.

The following definitions appeared in [56]:

**Definition 2.6.5.** A $G$-operad is an operad $P$ valued in $\text{Top}^G$, i.e. $P(n)$ has an action of $G \times \Sigma_n$ for all $n$ and the structure maps of the operad are $G$-equivariant. Furthermore, $G$ must fix the unit $1 \in P(1)$. All our operads will be reduced, i.e. $P(0)$ is a point and is fixed by $G$.

A morphism $f : P \to Q$ of $G$-operads is a sequence $(f(n) : P(n) \to Q(n))$ of $G \times \Sigma_n$-equivariant maps such that $f(1) = 1$ and $f$ commutes with composition. We denote by $G-\text{Oper}$ the category of $G$-operads.

Let $\text{Oper}^G$ denote the category of operads with a $G$-action, i.e. the category of functors from the category $G$ to the category of operads valued in $\text{Top}$. Morphisms are natural transformations.

**Proposition 2.6.6.** The category of (reduced) $G$-operads is identical to the category of (reduced) operads with a $G$-action.

**Proof.** An object in $\text{Oper}^G$ is a sequence of spaces $(P(n))$ with a $G$-action on the sequence, such that each $P(n)$ has an action of $\Sigma_n$. An action of $G$ on a symmetric sequence is a levelwise action, so in fact each $P(n)$ has a $G \times \Sigma_n$ action and there is no interplay between the $G$-action and the $\Sigma_n$-action. Thus, the structure maps in...
$\text{Oper}^G$ are $G \times \Sigma_n$-equivariant, because they are $G$-equivariant and $\Sigma_n$-equivariant with no interplay between the two group actions.

If the operad $(P(n))$ is reduced then $P(0)$ is a point, and so the $G$ action on $P(0)$ has no choice but to fix this point. For the unit condition, observe that en element $1 \in P(1)$ is the same thing as a map $\ast \to P(1)$ which picks out the unit element. When we pass from $\text{Oper}$ to $\text{Oper}^G$ this map becomes $G$-equivariant. The domain is $G$-fixed, so the image of the map must also be $G$-fixed, by equivariance. So $1$ is $G$-fixed.

Lastly, consider a morphism in $\text{Oper}^G$. It is in particular a morphism in $\text{Oper}$ and so takes the form $(f(n) : P(n) \to Q(n))$ where $f(1) = 1$ and $f$ commutes with composition. The $G$-action makes each map $f(n) G \times \Sigma_n$-equivariant.

We could just as well begin with an object of $G - \text{Oper}$ and recover a functor $G \to \text{Oper}$. We leave the details as an exercise for the reader. \hfill \Box

We follow [14] rather than [56] in our definition of an algebra over an equivariant operad. Rather than requiring the twisted half-smash product, we let the operad act in the way described in Section 2.5 using that $G$-spaces and $G$-spectra are both tensored over $G$-spaces. So a $P$-algebra is a based $G$-space $X$ together with $G$-maps $P(n) \times X^n \to X$ compatible with the $\Sigma_n$-action and the operad structure maps. A map of $P$-algebras is a map of $G$-spaces which is compatible with the $P$-action. We denote by $P$-alg the category of $P$-algebras.

We are now ready to discuss family model structures on the category of $G$-operads.
Theorem 2.6.7. For any family $\mathcal{F}$ of subgroups of $G$, the category of operads inherits a model structure denoted $\text{Oper}^\mathcal{F}$ in which weak equivalences and fibrations are maps which are levelwise weak equivalences and fibrations in $\text{Top}^\mathcal{F}$.

Proof. We will apply Proposition 3.11(b) from [86] to the cofibrantly generated model category $\text{Oper}$ (using the main result of [9]). We must demonstrate that in this model category all objects are fibrant and that the cofibrations are contained in the monomorphisms. Both of these properties hold for $\text{Top}$, because all cofibrations are built as retracts of transfinite compositions of pushouts of the closed inclusions $\partial \Delta[n]_+ \to \Delta[n]_+$, and the class of inclusions is closed under transfinite compositions, pushouts, and retracts in $k$-spaces (see Section 2.4 in [46]). The class of inclusions in contained in the class of monomorphisms.

Next, both of these properties pass to $\text{Top}^{\Sigma n}$ because a projective cofibration forgets to a cofibration (which forgets to a monomorphism) and because fibrations are the same in the projective and underlying model structures. Next, both of these properties pass to the product model structure on symmetric sequences, because monomorphisms in this category are defined to be levelwise monomorphisms (so cofibrations are contained in the class of levelwise monomorphisms) and because fibrations are levelwise.

Finally, both of these properties pass to the model category of operads because a cofibration of operads forgets to a cofibration of symmetric sequences (which is in particular a levelwise monomorphism) and fibrations are the same in operads and symmetric sequences. □
Consider the operad $Com$, whose algebras are strictly commutative monoids. Then $Com(n) = *$ for all $n$, and $Com$ is an object in $G - Oper$, where $G$ always acts trivially. Regardless of which family semi-model structure we place on $Oper^G$, $Com$ is not cofibrant. Indeed, it is not even $\Sigma$-cofibrant, because this would require the $n^{th}$ space to be a free $\Sigma_n$-space for all $n$ and this fails badly. We have remarked that we often desire to work with cofibrant operads. Thus, the following is a natural definition.

**Definition 2.6.8.** Let $E^\mathcal{F}_\infty$ be the cofibrant replacement for the operad $Com$ in the $\mathcal{F}$-fixed point model structure on $G$-operads.

The existence of $E^\mathcal{F}_\infty$ is guaranteed by the model structure. A more hands-on construction, which includes a formula for the $n^{th}$ space of the cofibrant replacement, is given in [10], generalizing the work of [15]. Another hands-on construction for cofibrant replacement of operads is given in [55].

**Example 2.6.9.** When $\mathcal{F} = \{e\}$, the $\mathcal{F}$-model structure on $G - Oper$ is equivalent to the usual model structure on (non-equivariant) operads because $Top^\mathcal{F}$ is the same as $Top$. Thus, the cofibrant replacement of $Com$ is an $E_\infty$-operad, i.e. has $P(n)$ contractible and $\Sigma_n$ acting freely. Recalling the universal property of the total space $E\Sigma_n$ of the principal $\Sigma_n$-bundle, we see that $P(n)$ is $\Sigma_n$-weakly equivalent to $E\Sigma_n$. Examples include the linear isometries operad and the little cubes operad.

When $\mathcal{F} = \{\text{all subgroups of } G\}$, the cofibrant replacement of $Com$ encodes genuine equivariant commutativity. As discussed in [40], the $n^{th}$ space $P(n)$ of a genuine $E_\infty^G$ operad is an $E_G\Sigma_n$, i.e. a space with a $G \times \Sigma_n$-action which is characterized up to $G \times \Sigma_n$-weak equivalence by the property that for all closed $\Lambda < G \times \Sigma_n$, we have
Similar considerations demonstrate that the $n^{th}$ space of the operad $E^G_{\infty}$ is an $E_{\mathcal{F}} \Sigma_n$ space in the language of [59]. This space is characterized up to weak equivalence in the model structure $(\text{Top}^{\Sigma_n})^{\mathcal{F}}$ by the property that for all closed $\Lambda < G \times \Sigma_n$, we have

$$(E_{\mathcal{F}} \Sigma_n)^\Lambda \simeq \begin{cases} * & \text{if } \Lambda \cap \Sigma_n = \{e\} \\ \emptyset & \text{if } \Lambda \cap \Sigma_n \neq \{e\} \end{cases}$$

Observe that equivariantly, the naive $E_{\infty}$ operad is not cofibrant, unless the family $\mathcal{F}$ consists of only the trivial group. To see this, notice that the $n^{th}$ level $E \Sigma_n$ is not cofibrant in $\text{Top}^{G \times \Sigma_n}$, for instance because it has a trivial $G$-action.

Forgetting the $G$-action provides a map $E^G_{\infty} \to E_{\infty}$. Any cofibrant replacement of $E_{\infty}$ in $\text{Oper}^{\mathcal{F}}$ will be weakly equivalent to $E^G_{\infty}$, and thus $E^G_{\infty}$ may be viewed as the cofibrant replacement for $E_{\infty}$ as well as for $\text{Com}$.

Remark 2.6.10. As $E_{\infty} = E_{\infty}^{(e)}$ and $E^G_{\infty} = E_{\infty}^{(All)}$, we may view the collection of operads $E^G_{\infty}$ as interpolating between these two extremes. In particular, along any chain of families subgroups (ordered by inclusion) beginning with $\{e\}$ and ending at $\{All\}$ we can iteratively take cofibrant replacements in the model structures $\text{Oper}^{\mathcal{F}}$ and obtain a tower of operads linked by forgetful functors: $E^G_{\infty} \to E^G_{\infty} \to E^G_{\infty} \to \cdots \to E_{\infty}$. More generally there is a lattice of operads between $E_{\infty}$ and $E^G_{\infty}$ in which each vertex is a $E^G_{\infty}$. 

These $E_\infty$ operads isolate the difference between norm, restriction, and transfer. An $E_\infty$-algebra $X$ has a multiplicative structure on $\text{res}_H(X)$ (compatible with the transfers) for every $H \in \mathcal{F}$. However, $N^G_H(\text{res}_H(X))$ need not have a multiplicative structure. We will not address $E_\infty$-algebras further here, but we will make use of these operads $E_\infty$ in Chapter 5.
CHAPTER 3

Preservation of Operad Algebras by Bousfield Localization

In this chapter we provide a general result regarding when Bousfield localization preserves \( P \)-algebras. We must first provide a precise definition for this concept. Throughout this chapter, let \( \mathcal{M} \) be a monoidal model category and let \( \mathcal{C} \) be a class of maps in \( \mathcal{M} \) such that Bousfield localization \( L_{\mathcal{C}}(\mathcal{M}) \) is a also monoidal model category.

3.1. Definition of Preservation

On the model category level the functor \( L_{\mathcal{C}} \) is the identity. So when we write \( L_{\mathcal{C}} \) as a functor we shall mean the composition of derived functors \( \text{Ho}(\mathcal{M}) \to \text{Ho}(L_{\mathcal{C}}(\mathcal{M})) \to \text{Ho}(\mathcal{M}) \), i.e. \( E \to L_{\mathcal{C}}(E) \) is the unit map of the adjunction \( \text{Ho}(\mathcal{M}) \cong \text{Ho}(L_{\mathcal{C}}(\mathcal{M})) \). In particular, for any \( E \) in \( \mathcal{M} \), \( L_{\mathcal{C}}(E) \) is weakly equivalent to \( R_{\mathcal{C}}QE \) where \( R_{\mathcal{C}} \) is a choice of fibrant replacement in \( L_{\mathcal{C}}(\mathcal{M}) \) and \( Q \) is a cofibrant replacement in \( \mathcal{M} \).

Let \( P \) be an operad valued in \( \mathcal{M} \). Because the objects of \( L_{\mathcal{C}}(\mathcal{M}) \) are the same as the objects of \( \mathcal{M} \), \( P \) is also valued in \( L_{\mathcal{C}}(\mathcal{M}) \). Thus, we may consider \( P \)-algebras in both categories and these classes of objects agree (because the \( P \)-algebra action is independent of the model structure). We denote the categories of \( P \)-algebras by \( P\text{-alg}(\mathcal{M}) \) and \( P\text{-alg}(L_{\mathcal{C}}(\mathcal{M})) \). These are identical as categories, but in a moment they will receive different model structures.

Definition 3.1.1. Assume that \( \mathcal{M} \) and \( L_{\mathcal{C}}(\mathcal{M}) \) are monoidal model categories.

Then \( L_{\mathcal{C}} \) is said to preserve \( P\text{-algebras} \) if
3.2. General Preservation Result

(1) When $E$ is a $P$-algebra there is some $P$-algebra $\tilde{E}$ which is weakly equivalent in $\mathcal{M}$ to $L_C(E)$.

(2) In addition, when $E$ is a cofibrant $P$-algebra, then there is a choice of $\tilde{E}$ and a lift of the localization map $E \rightarrow L_C(E)$ to a $P$-algebra homomorphism $E \rightarrow \tilde{E}$.

The notion of preservation was also considered in [19], but only for cofibrant $E$.

3.2. General Preservation Result

Recall that when we say $P$-$\text{alg}(\mathcal{M})$ inherits a model structure from $\mathcal{M}$ we mean that this model structure is transferred by the free-forgetful adjunction. In particular, a map of $P$-algebras $f$ is a weak equivalence (resp. fibration) if and only if $f$ is a weak equivalence (resp. fibration) in $\mathcal{M}$.

**Theorem 3.2.1.** Let $\mathcal{M}$ be a monoidal model category such that the Bousfield localization $L_C(\mathcal{M})$ exists and is a monoidal model category. Let $P$ be an operad valued in $\mathcal{M}$. If the categories of $P$-algebras in $\mathcal{M}$ and in $L_C(\mathcal{M})$ inherit model structures from $\mathcal{M}$ and $L_C(\mathcal{M})$ then $L_C$ preserves $P$-algebras.

**Proof.** Let $R_C$ denote fibrant replacement in $L_C(\mathcal{M})$, let $R_{C,P}$ denote fibrant replacement in $P$-$\text{alg}(L_C(\mathcal{M}))$, and let $Q_P$ denote cofibrant replacement in $P$-$\text{alg}(\mathcal{M})$. We will prove the first form of preservation and our method of proof will make it clear that in the special case where $E$ is a cofibrant $P$-algebra then in fact we may deduce the second form of preservation.

In our proof, $\tilde{E}$ will be $R_{C,P}Q_P(E)$. Because $Q$ is the left derived functor of the identity adjunction between $\mathcal{M}$ and $L_C(\mathcal{M})$, and $R_C$ is the right derived functor of
the identity, we know that $L_C(E) \simeq R_C Q(E)$. We must therefore show $R_C Q(E) \simeq R_{C,P} Q_P(E)$.

The map $Q_P E \to E$ is a trivial fibration in $P$-$\text{alg}(\mathcal{M})$, hence in $\mathcal{M}$. The map $Q E \to E$ is also a weak equivalence in $\mathcal{M}$. Consider the following lifting diagram in $\mathcal{M}$:

\[
\begin{array}{c}
\phi \\
\downarrow \\
Q E \\
\downarrow \\
E
\end{array} \quad \xrightarrow{\simeq} \quad \begin{array}{c}
\emptyset \\
\downarrow \\
Q P E \\
\downarrow \\
E
\end{array}
\]

(3.1)

The lifting axiom gives the map $Q E \to Q P E$ and it is necessarily a weak equivalence in $\mathcal{M}$ by the two out of three property.

Since $Q_P E$ is a $P$-algebra in $\mathcal{M}$ it must also be a $P$-algebra in $L_C(\mathcal{M})$, since the monoidal structure of the two categories is the same. We may therefore construct a lift:

\[
\begin{array}{c}
Q_P E \\
\downarrow \\
R_{C,P} Q_P E \\
\downarrow \\
R_C Q_P E
\end{array} \quad \xrightarrow{\simeq} \quad \begin{array}{c}
\emptyset \\
\downarrow \\
R_{C,P} Q_P E \\
\downarrow \\

* \)

In this diagram the left vertical map is a weak equivalence in $L_C(\mathcal{M})$ and the top horizontal map is a weak equivalence in $P$-$\text{alg}(L_C(\mathcal{M}))$. Because the model category $P$-$\text{alg}(L_C(\mathcal{M}))$ inherits weak equivalences from $L_C(\mathcal{M})$, this map is a weak equivalence in $L_C(\mathcal{M})$. Therefore, by the two out of three property, the lift is a weak equivalence in $L_C(\mathcal{M})$. We make use of this map as the horizontal map in the lower right corner of the diagram below.
The top horizontal map $QE \to Q_PE$ in the following diagram is the first map we constructed, which was proven to be a weak equivalence in $\mathcal{M}$. The square in the diagram below is then obtained by applying $R_C$ to that map. In particular, $R_CQE \to R_CQP_E$ is a weak equivalence in $L_C(\mathcal{M})$:

\[
\begin{array}{ccc}
QE & \longrightarrow & Q_PE \\
\downarrow & & \downarrow \\
R_CQE & \longrightarrow & R_CQP_E \longrightarrow R_C,PQP_E
\end{array}
\]

We have shown that both of the bottom horizontal maps are weak equivalences in $L_C(\mathcal{M})$. Thus, by the two out of three property, their composite $R_CQE \to R_C,PQP_E$ is a weak equivalence in $L_C(\mathcal{M})$. All the objects in the bottom row are fibrant in $L_C(\mathcal{M})$, so these $C$-local equivalences are weak equivalences in $\mathcal{M}$.

As $E$ was a $P$-algebra and $Q_P$ and $R_{C,P}$ are endofunctors on categories of $P$-algebras, it is clear that $R_{C,P}QP_E$ is a $P$-algebra. We have just shown that $L_C(E)$ is weakly equivalent to this $P$-algebra, so we are done.

We turn now to the case where $E$ is assumed to be a cofibrant $P$-algebra. We have seen that there is an $\mathcal{M}$-weak equivalence $R_CQE \to R_C,PQP_E$, and above we took $R_{C,P}QP_E$ in $\mathcal{M}$ as our representative for $L_C(E)$ in $\text{Ho}(\mathcal{M})$. Because $E$ is a cofibrant $\mathcal{M}$-algebra, there are weak equivalences $E \Leftrightarrow Q_P(E)$ in $P\text{-alg}(L_C(\mathcal{M}))$. This is because all cofibrant replacements of a given object are weakly equivalent, e.g. by diagram (3.1). So passage to $Q_P(E)$ is unnecessary when $E$ is cofibrant, and we take $R_{C,P}E$ as our representative for $L_C(E)$. We may then lift the localization map $E \to L_C(E)$ in $\text{Ho}(\mathcal{M})$.
to the fibrant replacement map $E \to R_{C,P}E$ in $\mathcal{M}$. As this fibrant replacement is taken in $P\text{-alg}(L_C(\mathcal{M}))$, this map is a $P$-algebra homomorphism, as desired.

□

This theorem alone would not be a satisfactory answer to the question of when $L_C$ preserves $P$-algebras, because there is no clear way to check the hypotheses. For this reason, in the coming chapters we will discuss conditions on $\mathcal{M}$ and $P$ so that $P$-algebras inherit model structures, and then we will discuss which localizations $L_C$ preserve these conditions (so that $P\text{-alg}(L_C(\mathcal{M}))$ inherits a model structure from $L_C(\mathcal{M})$). The most important condition is the pushout product axiom, and we will discuss in the next chapter when this axiom passes to $L_C(\mathcal{M})$.

One such condition on $\mathcal{M}$ is the monoid axiom. In Chapter 8 we discuss when this condition passes to $L_C(\mathcal{M})$. However, it will turn out that the monoid axiom is not necessary in order for our preservation results to apply. This is because the work in \[45\] and \[85\] produces semi-model structures on $P$-algebras and these will be enough for our proof above to go through.

### 3.3. Semi-Model Categories

Observe that in the proof of Theorem 3.2.1 we only used formal properties of fibrant and cofibrant replacement functors, and the fact that the model structures on $P$-algebras were inherited from $\mathcal{M}$ and $L_C(\mathcal{M})$. So it should not come as a surprise that the same proof works when $P$-algebras only form semi-model categories. The motivating example of a semi-model category is when $\mathcal{D}$ is obtained from $\mathcal{M}$ via the general transfer principle for transferring a model structure across an adjunction (see Lemma 2.3 in \[81\].
or Theorem 12.1.4 in [29] when not all the conditions needed to get a full model structure are satisfied.

In particular, the reader should imagine that weak equivalences and fibrations in $\mathcal{D}$ are maps which forget to weak equivalences and fibrations in $\mathcal{M}$, and that the generating (trivial) cofibrations of $\mathcal{D}$ are maps of the form $F(I)$ and $F(J)$ where $F : \mathcal{M} \to \mathcal{D}$ is the free algebra functor and $I$ and $J$ are the generating (trivial) cofibrations of $\mathcal{M}$. The following is Definition 1 from [85] and Definition 12.1.1 in [29]. Cofibrant should be taken to mean cofibrant in $\mathcal{D}$.

**Definition 3.3.1.** A *semi-model category* is a bicomplete category $\mathcal{D}$, an adjunction $F : \mathcal{M} \rightleftarrows \mathcal{D} : U$ where $\mathcal{M}$ is a model category, and subcategories of weak equivalences, fibrations, and cofibrations in $\mathcal{D}$ satisfying the following axioms:

1. $U$ preserves fibrations and trivial fibrations.
2. $\mathcal{D}$ satisfies the two out of three axiom and the retract axiom.
3. Cofibrations in $\mathcal{D}$ have the left lifting property with respect to trivial fibrations. Trivial cofibrations in $\mathcal{D}$ whose domain is cofibrant have the left lifting property with respect to fibrations.
4. Every map in $\mathcal{D}$ can be functorially factored into a cofibration followed by a trivial fibration. Every map in $\mathcal{D}$ whose domain is cofibrant can be functorially factored into a trivial cofibration followed by a fibration.
5. The initial object in $\mathcal{D}$ is cofibrant.
6. Fibrations and trivial fibrations are closed under pullback.
\(\mathcal{D}\) is said to be \textit{cofibrantly generated} if there are sets of morphisms \(I'\) and \(J'\) in \(\mathcal{D}\) such that \(I'\)-inj is the class of trivial fibrations and \(J'\)-inj the class of fibrations in \(\mathcal{D}\), if the domains of \(I'\) are small relative to \(I'\)-cell, and if the domains of \(J'\) are small relative to maps in \(J'\)-cell whose domain becomes cofibrant in \(\mathcal{M}\).

Note that the only difference between a semi-model structure and a model structure is that one of the lifting properties and one of the factorization properties requires the domain of the map in question to be cofibrant. Because fibrant and cofibrant replacements are constructed via factorization, (4) implies that every object has a cofibrant replacement and that objects with cofibrant domain have fibrant replacements. So one could construct a fibrant replacement functor which first does cofibrant replacement and then does fibrant replacement. These functors behave as they would in the presence of a full model structure.

We are now prepared to state our preservation result in the presence of only a semi-model structure on \(P\)-algebras. Again, when we say \(P\)-algebras inherit a semi-model structure we mean with weak equivalences and fibrations reflected and preserved by the forgetful functor.

**Corollary 3.3.2.** Let \(\mathcal{M}\) be a monoidal model category such that the Bousfield localization \(L_C(\mathcal{M})\) exists and is a monoidal model category. Let \(P\) be an operad valued in \(\mathcal{M}\). If the categories of \(P\)-algebras in \(\mathcal{M}\) and in \(L_C(\mathcal{M})\) inherit semi-model structures from \(\mathcal{M}\) and \(L_C(\mathcal{M})\) then \(L_C\) preserves \(P\)-algebras.

**Proof.** The proof proceeds exactly as the proof of the theorem above. We highlight where care must be taken in the presence of semi-model categories. As remarked above,
the cofibrant replacement $Q_P$ in the semi-model category $P$-alg$(\mathcal{M})$ exists and $Q_P E \to E$ is a weak equivalence in $P$-alg$(\mathcal{M})$, hence in $\mathcal{M}$. Diagram (3.1) is a lifting diagram in $\mathcal{M}$, so still yields a weak equivalence $Q E \to Q_P E$.

Next, the fibrant replacement $R_C Q_P E$ is a replacement in $L_C(\mathcal{M})$, which is a model category. The fibrant replacement $Q_P E \to R_{C,P} Q_P E$ is a fibrant replacement in the semi-model category $P$-alg$(L_C(\mathcal{M}))$, and exists because the domain of $Q_P E \to \ast$ is cofibrant in $P$-alg$(L_C(\mathcal{M}))$. The resulting object $R_{C,P} Q_P E$ is fibrant in $P$-alg$(L_C(\mathcal{M}))$ hence in $L_C(\mathcal{M})$. The lift in the next diagram is a lift in $L_C(\mathcal{M})$, and again by the two out of three property in $L_C(\mathcal{M})$ the diagonal map is a $C$-local equivalence:

$$
\begin{array}{ccc}
Q_P E & \to & R_{C,P} Q_P E \\
\downarrow & & \downarrow \\
R_C Q_P E & \to & \ast
\end{array}
$$

Finally, the last diagram is fibrant replacement in the model category $L_C(\mathcal{M})$, and so the argument that $R_C Q E \to R_{C,P} Q_P E$ is a $C$-local equivalence remains unchanged.

$$
\begin{array}{ccc}
Q E & \to & Q_P E \\
\downarrow & & \downarrow \\
R_C Q E & \to & R_{C,P} Q_P E & \to & R_{C,P} Q_P E
\end{array}
$$

The composite across the bottom $R_C Q E \to R_{C,P} Q_P E$ is a weak equivalence between fibrant objects in $L_C(\mathcal{M})$ and so is a weak equivalence in $\mathcal{M}$, as in the proof of the theorem.
Finally, for the case of $E$ cofibrant in the semi-model category $P\text{-alg}(\mathcal{M})$, note that the localization map $E \to L_C(E)$ is again fibrant replacement $E \to R_{C,P}E$ in $P\text{-alg}(L_C(\mathcal{M}))$. This exists because the domain is cofibrant by assumption. By construction, this map is a $P$-algebra homomorphism, as desired. \qed
CHAPTER 4

Monoidal Bousfield Localizations

In both Theorem 3.2.1 and Corollary 3.3.2 we assumed that $L_C(M)$ is a monoidal model category. In this chapter we provide conditions on $M$ and $C$ so that this occurs. First, we provide an example demonstrating that the pushout product axiom can fail for $L_C(M)$, even if it holds for $M$.

4.1. A Non-Monoidal Bousfield Localization

Example 4.1.1. It is not true that every Bousfield localization of a monoidal model category is a monoidal model category. Let $R = \mathbb{F}_2[\Sigma_3]$. An $R$ module is simply an $\mathbb{F}_2$ vector space with an action of the symmetric group $\Sigma_3$. Because $R$ is a Frobenius ring, we may pass from $R$-mod to the stable module category $StMod(R)$ by identifying any two morphisms whose difference factors through a projective module. Let $\mathcal{M}$ be the corresponding model category, discussed in Subsection 2.4.3.

Proposition 4.2.15 of [46] proves that for $R = \mathbb{F}_2[\Sigma_3]$, this model category is a monoidal model category because $R$ is a Hopf algebra over $\mathbb{F}_2$. The monoidal product of two $R$-modules is $M \otimes_{\mathbb{F}_2} N$ where $R$ acts via its diagonal $R \to R \otimes_{\mathbb{F}_2} R$.

We now check that cofibrant objects are flat in $\mathcal{M}$. By the pushout product axiom, $X \otimes -$ is left Quillen. Since all objects are cofibrant, all weak equivalences are weak equivalences between cofibrant objects. So Ken Brown’s lemma implies $X \otimes -$ preserves weak equivalences.
4.1. A NON-MONOIDAL BOUSFIELD LOCALIZATION

Let \( f : \emptyset \to \mathbb{F}_2 \), where the codomain has the trivial \( \Sigma_3 \) action. We'll show that the Bousfield localization with respect to \( f \) cannot be a monoidal Bousfield localization.

First observe that being \( f \)-locally trivial is equivalent to having no \( \Sigma_3 \)-fixed points, and this is equivalent to failing to admit \( \Sigma_3 \)-equivariant maps from \( \mathbb{F}_2 \) (the non-identity element would need to be taken to a \( \Sigma_3 \)-fixed point because the \( \Sigma_3 \)-action on \( \mathbb{F}_2 \) is trivial).

If the pushout product axiom held in \( L_f(M) \) then the pushout product of two \( f \)-locally trivial cofibrations \( g, h \) would have to be \( f \)-locally trivial. We will now demonstrate an \( f \)-locally trivial object \( N \) for which \( N \otimes_{\mathbb{F}_2} N \) is not \( f \)-locally trivial, so \((\emptyset \to N) \Box (\emptyset \to N) \) is not a trivial cofibration in \( L_f(M) \).

Define \( N \cong \mathbb{F}_2 \oplus \mathbb{F}_2 \) where the element \((12)\) sends \( a = (1, 0) \) to \( b = (0, 1) \) and the element \((123)\) sends \( a \) to \( b \) and \( b \) to \( c = a + b \). The reader can check that \((12)(123)\) acts the same as \((123)^2(12)\), so that this is a well-defined \( \Sigma_3 \)-action. This object \( N \) is \( f \)-locally trivial. It does not admit any maps from \( \mathbb{F}_2 \) because it has no \( \Sigma_3 \)-fixed points. However, \( N \otimes_{\mathbb{F}_2} N \) is not \( f \)-locally trivial because \( N \otimes_{\mathbb{F}_2} N \) does admit a map from \( \mathbb{F}_2 \) which takes the non-identity element of \( \mathbb{F}_2 \) to the \( \Sigma_3 \)-invariant element \( a \otimes a + b \otimes b + c \otimes c \). Thus, \( L_f(M) \) is not a monoidal model category.

In order to get around examples such as the above we must place hypotheses on the maps \( C \) which we are inverting. A similar program was conducted in [19], where localizations of stable model categories were assumed to commute with suspension. Similarly, a condition on a stable localization to ensure that it is additionally monoidal was given in Definition 5.2 of [2] and the same condition appeared in Theorem 4.46 of [6]. This condition states that \( C \Box I \) is contained in the \( C \)-local equivalences.
4.2. Defining Monoidal Bousfield Localizations

Remark 4.1.2. The counterexample above fails to satisfy the condition that $\mathcal{C} \square I$ is contained in the $\mathcal{C}$-local equivalences. If this condition were satisfied then $I$ would be contained in the $f$-local equivalences and this would imply all cofibrant objects (hence all objects) are $f$-locally trivial. But $0 \to N \otimes_{F_2} N$ is not $f$-locally trivial. Thus, this counterexample has no bearing on the work of [2] or [6].

Remark 4.1.3. The counterexample demonstrates a general principle which we now highlight. In any $G$-equivariant world, there are multiple spheres due to the different group actions. In the example above, one can suspend by representations of $\Sigma_n$, i.e. copies of $F_2$ on which $\Sigma_n$ acts. The 1-point compactification of such an object is a sphere $S^n$ on which $\Sigma_n$ acts. A localization which kills a representation sphere should not be expected to respect the monoidal structure, because not all acyclic cofibrant objects can be built from one of the representation spheres alone. In particular, $N \otimes N$ will not be in the smallest thick subcategory generated by $F_2$. The point is that the homotopy categories of stable model categories in an equivariant context are not monogenic axiomatic stable homotopy categories in the sense of [49].

Note that this example also demonstrates that the monoid axiom can fail on $L_C(M)$. The author does not know an example of a model category satisfying the pushout product axiom but failing the monoid axiom.

4.2. Defining Monoidal Bousfield Localizations

In our applications we will need to know that $L_C(M)$ satisfies the pushout product axiom, the unit axiom, and the axiom that cofibrant objects are flat. We therefore give a name to such localizations, and then we characterize them.
4.2. DEFINING MONOIDAL BOUSFIELD LOCALIZATIONS

**Definition 4.2.1.** A Bousfield localization $L_C$ is said to be a monoidal Bousfield localization if $L_C(M)$ satisfies the pushout product axiom, the unit axiom, and the axiom that cofibrant objects are flat (in particular, $L_C(M)$ is a monoidal model category).

**Theorem 4.2.2.** Suppose $M$ is a tractable monoidal model category in which cofibrant objects are flat. Let $I$ denote the generating cofibrations of $M$. Then $L_C$ is a monoidal Bousfield localization if and only if every map of the form $f \otimes \text{id}_K$, where $f$ is in $C$ and $K$ is a domain or codomain of a map in $I$, is a $C$-local equivalence.

**Theorem 4.2.3.** Suppose $M$ is a cofibrantly generated monoidal model category in which cofibrant objects are flat. Then $L_C$ is a monoidal Bousfield localization if and only if every map of the form $f \otimes \text{id}_K$, where $f$ is in $C$ and $K$ is cofibrant, is a $C$-local equivalence.

We shall prove Theorem 4.2.2 in Section 4.3 and we shall prove Theorem 4.2.3 in Section 4.4. These theorems demonstrate precisely what must be done if one wishes to invert a given set of morphisms $C$ and ensure that the resulting model structure is a monoidal model structure.

**Definition 4.2.4.** Suppose $M$ is tractable, left proper, and either cellular or combinatorial. The smallest monoidal Bousfield localization which inverts a given set of morphisms $C$ is the Bousfield localization with respect to the set $C' = \{C \otimes \text{id}_K\}$ where $K$ runs through the domains and codomains of the generating cofibrations $I$. 
This notion has already been used in [51]. The reason for the tractability hypothesis is to ensure that $\mathcal{C}'$ is a set. Requiring left properness and either cellularity or combinatoriality ensures that $L_{\mathcal{C}'}$ exists. The smallest Bousfield localization has a universal property, which we now highlight.

**Proposition 4.2.5.** Suppose $\mathcal{C}'$ is the smallest monoidal Bousfield localization inverting $\mathcal{C}$, and let $j : \mathcal{M} \to L_{\mathcal{C}'}(\mathcal{M})$ be the left Quillen functor realizing the localization. Suppose $\mathcal{N}$ is a monoidal model category with cofibrant objects flat. Suppose $F : \mathcal{M} \to \mathcal{N}$ is a monoidal left Quillen functor such that $\mathbb{L}F$ takes the images of $\mathcal{C}$ in $\text{Ho}(\mathcal{M})$ to isomorphisms in $\text{Ho}(\mathcal{N})$. Then there is a unique monoidal left Quillen functor $\delta : L_{\mathcal{C}'}\mathcal{M} \to \mathcal{N}$ such that $\delta j = F$.

**Proof.** Suppose $F : \mathcal{M} \to \mathcal{N}$ is a monoidal left Quillen functor, that $\mathcal{N}$ has cofibrant objects flat, and that $\mathbb{L}F$ takes the images of $\mathcal{C}$ in $\text{Ho}(\mathcal{M})$ to isomorphisms in $\text{Ho}(\mathcal{N})$. Then $F$ also takes the images of maps in $\mathcal{C}'$ to isomorphisms in $\text{Ho}(\mathcal{N})$, because for any $f \in \mathcal{C}$ and any cofibrant $K$, $F(f \otimes K) \simeq F(f) \otimes F(K)$ is a weak equivalence in $\mathcal{N}$. This is because $F(K)$ is cofibrant in $\mathcal{N}$ (as $F$ is left Quillen), cofibrant objects are flat in $\mathcal{N}$, and $F(f)$ is a weak equivalence in $\mathcal{N}$ by hypothesis.

The universal property of the localization $L_{\mathcal{C}'}$ then provides a unique left Quillen functor $\delta : L_{\mathcal{C}'}\mathcal{M} \to \mathcal{N}$ which is the same as $F$ on objects and morphisms (c.f. Theorem 3.3.18 and Theorem 3.3.19 in [42], which also provide uniqueness for $\delta$). In particular, $\delta$ is a monoidal functor and $\delta q = Fq : F(QS) \to F(S)$ is a weak equivalence in $\mathcal{N}$ because the cofibrant replacement $QS \to S$ is the same in $L_{\mathcal{C}'}(\mathcal{M})$ as in $\mathcal{M}$. So $\delta$ is a
unique monoidal left Quillen functor as required, and the commutativity $\delta j = F$ follows immediately from the definition of $\delta$. \hfill \Box

### 4.3. Proof for Tractable Model Categories

In this section we will prove Theorem [4.2.2](#). We first prove that under the hypotheses of Theorem [4.2.2](#), cofibrant objects are flat in $L_C(M)$.

**Proposition 4.3.1.** Let $\mathcal{M}$ be a tractable monoidal model category in which cofibrant objects are flat. Let $I$ denote the generating cofibrations of $\mathcal{M}$. Suppose that every map of the form $f \otimes \text{id}_K$, where $f$ is in $\mathcal{C}$ and $K$ is a domain or codomain of a map in $I$, is a $\mathcal{C}$-local equivalence. Then cofibrant objects are flat in $L_C(M)$.

**Proof.** We must prove that the class of maps $\{g \otimes X \mid g$ is a $\mathcal{C}$-local equivalence and $X$ is a cofibrant object$\}$ is contained in the $\mathcal{C}$-local equivalences. Let $X$ be a cofibrant object in $L_C(M)$ (equivalently, in $\mathcal{M}$). Let $g : A \to B$ be a $\mathcal{C}$-local equivalence. To prove $- \otimes X$ preserves $\mathcal{C}$-local equivalences, it suffices to show that it takes $L_C(M)$ trivial cofibrations between cofibrant objects to weak equivalences. This is because we can always do cofibrant replacement on $g$ to get $Qg : QA \to QB$, and by Remark [2.2.4](#) we may assume this is a cofibration between cofibrant objects. Smashing with $X$ gives:

$$
\begin{array}{ccc}
QA \otimes X & \to & QB \otimes X \\
\downarrow & & \downarrow \\
A \otimes X & \to & B \otimes X
\end{array}
$$

If we prove that $Qg \otimes X$ is a $\mathcal{C}$-local equivalence, then $g \otimes X$ must also be by the two out of three property, since the vertical maps are weak equivalences in $\mathcal{M}$ due to
X being cofibrant and cofibrant objects being flat in $M$. So we may assume that $g$ is an $L_C(M)$ trivial cofibration between cofibrant objects. Since $X$ is built as a transfinite composition of pushouts of maps in $I$, we proceed by transfinite induction. For the rest of the proof, let $K, K_1$, and $K_2$ denote domains/codomains of maps in $I$. These objects are cofibrant in $M$ by hypothesis, so they are also cofibrant in $L_C(M)$.

For the base case $X = K$ we appeal to Theorem 3.3.18 in [42]. The composition $F = \text{id} \circ K \otimes - : M \to M \to L_C(M)$ is left Quillen because $K$ is cofibrant. $F$ takes maps in $C$ to weak equivalences by hypothesis. So Theorem 3.3.18 implies $F$ induces a left Quillen functor $K \otimes - : L_C(M) \to L_C(M)$. Thus, $K \otimes -$ takes $C$-local equivalences between cofibrant objects to $C$-local equivalences and in particular takes $Qg$ to a $C$-local equivalence. Note that this is the key place in this proof where we use the hypothesis that $L_C$ is a monoidal Bousfield localization. This theorem is the primary tool when one wishes to get from a statement about $C$ to a statement about all $C$-local equivalences.

For the successor case, suppose $X_\alpha$ is built from $K$ as above and is flat in $L_C(M)$. Suppose $X_{\alpha+1}$ is built from $X_\alpha$ and a map in $I$ via a pushout diagram:

\[
\begin{array}{ccc}
K_1 & \xleftarrow{i} & K_2 \\
\downarrow & & \downarrow \\
X_\alpha & \rightarrow & X_{\alpha+1}
\end{array}
\]

We smash this diagram with $g : A \to B$ and note that smashing a pushout square with an object yields a pushout square.
Because $g$ is a cofibration of cofibrant objects, $A$ and $B$ are cofibrant. Because pushouts of cofibrations are cofibrations, $X_{\alpha} \to X_{\alpha+1}$ for all $\alpha$. Because $X_0$ is cofibrant, $X_\alpha$ is cofibrant for all $\alpha$. So all objects above are cofibrant. Furthermore, $g \otimes K_i = g \Box (0 \to K_i)$. Thus, by the Pushout Product axiom on $\mathcal{M}$ and the fact that cofibrations in $\mathcal{M}$ match those in $L_C(\mathcal{M})$, these maps are cofibrations.

Finally, the maps $g \otimes K_i$ are weak equivalences in $L_C(\mathcal{M})$ by the base case above, while $g \otimes X_\alpha$ is a weak equivalence in $L_C(\mathcal{M})$ by the inductive hypothesis. Thus, by Dan Kan’s Cube Lemma (Lemma 5.2.6 in [46]), the map $g \otimes X_{\alpha+1}$ is a weak equivalence in $L_C(\mathcal{M})$.

For the limit case, suppose we are given a cofibrant object $X = \text{colim}_{\alpha<\beta} X_\alpha$ where each $X_\alpha$ is cofibrant and flat in $L_C(\mathcal{M})$. Because each $X_\alpha$ is cofibrant, $g \otimes X_\alpha = g \Box (0 \to X_\alpha)$ is still a cofibration. By the inductive hypothesis, each $g \otimes X_\alpha$ is also a $C$-local equivalence, hence a trivial cofibration in $L_C(\mathcal{M})$. Since trivial cofibrations are always closed under transfinite composition, $g \otimes X = g \otimes \text{colim} X_\alpha = \text{colim}(g \otimes X_\alpha)$ is also a trivial cofibration in $L_C(\mathcal{M})$. □
4.3. PROOF FOR TRACTABLE MODEL CATEGORIES

We now pause for a moment to extract the key point in the proof above, where we applied the universal property of Bousfield localization. This is a reformulation Theorem 3.3.18 in [42] which will be helpful to us in the sequel.

**Lemma 4.3.2.** A left Quillen functor $F : \mathcal{M} \to \mathcal{M}$ induces a left Quillen functor $L_C F : L_C(\mathcal{M}) \to L_C(\mathcal{M})$ if and only if for all $f \in \mathcal{C}$, $F(f)$ is $\mathcal{C}$-local equivalence.

We turn now to the unit axiom.

**Proposition 4.3.3.** If $\mathcal{M}$ satisfies the unit axiom then any Bousfield localization $L_C(\mathcal{M})$ satisfies the unit axiom. If cofibrant objects are flat in $\mathcal{M}$ then the map $QS \otimes Y \to Y$ which is induced by cofibrant replacement $QS \to S$ is a weak equivalence for all $Y$, not just cofibrant $Y$. Furthermore, for any weak equivalence $f : K \to L$ between cofibrant objects, $f \otimes Y$ is a weak equivalence.

**Proof.** Since $L_C(\mathcal{M})$ has the same cofibrations as $\mathcal{M}$, it must also have the same trivial fibrations. Thus, it has the same cofibrant replacement functor and the same cofibrant objects. Thus, the unit axiom on $L_C(\mathcal{M})$ follows directly from the unit axiom on $\mathcal{M}$, because a weak equivalence in $\mathcal{M}$ is in particular a $\mathcal{C}$-local equivalence.

We now assume cofibrant objects are flat and that $Y$ is an object of $\mathcal{M}$. Consider the following diagram:

\[
\begin{array}{ccc}
QS \otimes QY & \longrightarrow & QY \\
\downarrow & & \downarrow \\
QS \otimes Y & \longrightarrow & Y
\end{array}
\]
4.3. PROOF FOR TRACTABLE MODEL CATEGORIES

The top map is a weak equivalence by the unit axiom for the cofibrant object \( QY \).
The left vertical map is a weak equivalence because cofibrant objects are flat and \( QS \)
is cofibrant. The right vertical is a weak equivalence by definition of \( QY \). Thus, the
bottom arrow is a weak equivalence by the two out of three property.

For the final statement we again apply cofibrant replacement to \( Y \) and we get

\[
\begin{array}{ccc}
K \otimes QY & \longrightarrow & L \otimes QY \\
\downarrow & & \downarrow \\
K \otimes Y & \longrightarrow & L \otimes Y
\end{array}
\]

Again the top horizontal map and the vertical maps are weak equivalences because
cofibrant objects are flat (for the first use that \( QX \) is cofibrant, for the second use that
\( K \) and \( L \) are cofibrant).

\[\Box\]

We turn now to proving Theorem 4.2.2. As mentioned in the proof of Proposition 4.3.1 if \( h \) and \( g \) are \( L_C(M) \)-cofibrations then they are \( M \)-cofibrations and so \( h \Box g \) is
a cofibration in \( M \) (hence in \( L_C(M) \)) by the pushout product axiom on \( M \). To verify
the rest of the pushout product axiom on \( L_C(M) \) we must prove that if \( h \) is a trivial
cofibration in \( L_C(M) \) and \( g \) is a cofibration in \( L_C(M) \) then \( h \Box g \) is a weak equivalence
in \( L_C(M) \).

Proposition 4.3.4. Let \( M \) be a tractable monoidal model category in which cofibrant
objects are flat. Let \( I \) denote the generating cofibrations of \( M \). Suppose that every map
of the form \( f \otimes \text{id}_K \), where \( f \) is in \( C \) and \( K \) is a domain or codomain of a map in \( I \), is
a \( C \)-local equivalence. Then \( L_C(M) \) satisfies the pushout product axiom.
Proof. We have already remarked that the cofibration part of the pushout product axiom on $L_C(M)$ follows from the pushout product axiom on $M$, since the two model categories have the same cofibrations. By Proposition 4.2.5 of [10], it is sufficient to check the pushout product axiom on generating (trivial) cofibrations. So suppose $h : X \to Y$ is an $L_C(M)$ trivial cofibration and $g : K \to L$ is a generating cofibration in $L_C(M)$ (equivalently, in $M$). Then we must show $h \Box g$ is an $L_C(M)$ trivial cofibration.

By hypothesis on $M$, $K$ and $L$ are cofibrant. Because $h$ is a cofibration, $K \otimes h$ and $L \otimes h$ are cofibrations by the pushout product axiom on $M$ (because $K \otimes h = (\emptyset \to K) \Box h$). By Proposition 4.3.1, cofibrant objects are flat in $L_C(M)$. So $K \otimes -$ and $L \otimes -$ are left Quillen functors. Consider the following diagram:

$$
\begin{array}{ccc}
K \otimes X & \xrightarrow{=} & K \otimes Y \\
\downarrow & & \downarrow \\
L \otimes X & \xrightarrow{=} & (K \otimes Y) \amalg_{K \otimes X} (L \otimes X)
\end{array}
$$

The map $L \otimes X \to (K \otimes Y) \amalg_{K \otimes X} (L \otimes X)$ is a trivial cofibration because it is the pushout of a trivial cofibration. Thus, by the two out of three property for the lower triangle, $h \Box g$ is a weak equivalence. Since we already knew it was a cofibration (because it is so in $M$), this means it is a trivial cofibration. $\square$

We are now ready to complete the proof of Theorem 4.2.2.
Proof of Theorem 4.2.2. We begin with the forwards direction. Suppose \( L_C(M) \) satisfies the pushout product axiom and has cofibrant objects flat. Let \( f \) be any map in \( C \). Note that in particular, \( f \) is a \( C \)-local equivalence. Because cofibrant objects are flat, the map \( f \otimes K \) is a \( C \)-local equivalence for any cofibrant \( K \). So the collection \( C \otimes K \) is contained in the \( C \)-local equivalences, where \( K \) runs through the class of cofibrant objects, i.e. \( L_C \) is a monoidal Bousfield localization.

For the converse, we apply our three previous propositions. That cofibrant objects are flat in \( L_C(M) \) is the content of Proposition 4.3.1. The unit axiom on \( L_C(M) \) follows from Proposition 4.3.3 applied to \( L_C(M) \). That the pushout product axiom holds on \( L_C(M) \) is Proposition 4.3.4. \( \square \)

4.4. Proof for Non-tractable Model Categories

We will now prove Theorem 4.2.3 following the outline above. The proof that cofibrant objects are flat in \( L_C(M) \) will proceed just as it did in Proposition 4.3.1. Proposition 4.3.3 again implies the unit axiom in \( L_C(M) \). Deducing the pushout product axiom on \( L_C(M) \) will be more complicated without the tractability hypothesis. For this reason, we need the following lemma. First, let \( I' \) be obtained from the generating cofibrations \( I \) by applying any cofibrant replacement \( Q \) to all \( i \in I \) and then taking the left factor in the cofibration-trivial fibration factorization of \( Q_i \). So \( I' \) consists of cofibrations between cofibrant objects.

Lemma 4.4.1. Suppose \( M \) is a left proper model category cofibrantly generated by sets \( I \) and \( J \) in which the domains of maps in \( J \) are small relative to \( I \)-cell. Then the sets \( I' \cup J \) and \( J \) cofibrantly generate \( M \).
Proof. We verify the conditions given in Definition 11.1.2 of [42]. We have not changed $J$, so the fibrations are still precisely the maps satisfying the right lifting property with respect to $J$ and the maps in $J$ still permit the small object argument because the domains are small relative to $J$-cell.

Any map which has the right lifting property with respect to all maps in $I$ is a trivial fibration, so will in particular have the right lifting property with respect to all cofibrations, hence with respect to maps in $I' \cup J$. Conversely, suppose $p$ has the right lifting property with respect to all maps in $I' \cup J$. We are faced with the following lifting problem:

\[
\begin{array}{ccc}
A' & \rightarrow & A \\
\downarrow i' & & \downarrow i \\
B' & \rightarrow & B
\end{array}
\quad
\begin{array}{ccc}
 & & X \\
 & \downarrow p & \\
 & & Y
\end{array}
\]

Because $p$ has lifting with respect to $I' \cup J$, it has the right lifting property with respect to $J$. This guarantees us that $p$ is a fibration. Now because $\mathcal{M}$ is left proper, Proposition 13.2.1 in [42] applies to solve the lifting diagram above. In particular, because $p$ has the right lifting property with respect to $I'$, $p$ must have the right lifting property with respect to $I$. Thus, $p$ is a trivial fibration as desired.

We now turn to smallness. Any domain of a map in $J$ is small relative to $J$-cell, but in general this would not imply smallness relative to $I$-cell. We have assumed the domains of maps in $J$ are small relative to $I$-cell, so they are small relative to $(J \cup I')$-cell because $J \cup I'$ is contained in $I$-cell.
Any domain of a map in $I'$ is of the form $QA$ for $A$ a domain of a map in $I$. We will show $QA$ is small relative to $I$-cell. As $J \cup I'$ is contained in $I$-cell this will show $QA$ is small relative to $J \cup I'$. Consider the construction of $QA$ as the left factor in

$$
\begin{array}{c}
\emptyset \\
\Rightarrow
\end{array}
\quad
\Rightarrow
\begin{array}{c}
QA \\
\Rightarrow
\end{array}
\quad
A
$$

The map $\emptyset \to QA$ is in $I$-cell, so $QA$ is a colimit of cells (let us say $\kappa A$ many), each of which is $\kappa$-small where $\kappa$ is the regular cardinal associated to $I$ by Proposition 11.2.5 of [42]. So for any $\lambda$ greater than the cofinality of $\max(\kappa, \kappa_A)$, a map from $QA$ to a $\lambda$-filtered colimit of maps in $I$-cell must factor through some stage of the colimit because all the cells making up $QA$ will factor in this way. One can find a uniform $\lambda$ for all objects $QA$ by an appeal to Lemma 10.4.6 of [42].

\[\square\]

Remark 4.4.2. In a combinatorial model category no smallness hypothesis needs to be made because all objects are small. In a cellular model category, the assumption that the domains of $J$ are small relative to cofibrations is included. As these hypotheses are standard when working with left Bousfield localization, we shall say no more about the additional smallness hypothesis placed on $J$ above.

Corollary 4.4.3. Suppose $\mathcal{M}$ is a left proper model category cofibrantly generated by sets $I$ and $J$ in which the domains of maps in $J$ are small relative to $I$-cell and are cofibrant. Then $\mathcal{M}$ can be made tractable.
Remark 4.4.4. Note that this corollary does not say that any left proper, cofibrantly generated model category can be made tractable. There is an example due to Carlos Simpson (found on page 199 of [84]) of a left proper, combinatorial model category which cannot be made tractable. In this example the cofibrations and trivial cofibrations are the same, so cannot be leveraged against one another in the way we have done above.

We are now prepared to prove Theorem 4.2.3.

Proof of Theorem 4.2.3. First, if $L_C$ is a monoidal Bousfield localization then every map of the form $f \otimes id_K$, where $f \in C$ and $K$ is cofibrant, is a $C$-local equivalence. This is because $f$ is a $C$-local equivalence and cofibrant objects are flat in $L_C(M)$. We turn now to the converse.

Assume every map of the form $f \otimes id_K$, where $f \in C$ and $K$ is cofibrant, is a $C$-local equivalence. Then cofibrant objects are flat in $L_C(M)$. To see this, let $X$ be cofibrant and define $F(-) = X \otimes -$. Then Lemma 4.3.2 implies $F$ is left Quillen when viewed as a functor from $L_C(M)$ to $L_C(M)$. So $F$ takes $C$-local equivalences between cofibrant objects to $C$-local equivalences. By the reduction at the beginning of the proof of Proposition 4.3.1 this implies $F$ takes all $C$-local equivalences to $C$-local equivalences.

Next, the unit axiom on $L_C(M)$ follows from the unit axiom on $M$, by Proposition 4.3.3. Finally, we must prove the pushout product axiom holds on $L_C(M)$. As in the proof of Proposition 4.3.4, Proposition 4.2.5 of [46] reduces the problem to checking the pushout product axiom on a set of generating (trivial) cofibrations. We apply Lemma 4.4.1 to $M$ and check the pushout product axiom with respect to this set of generating maps.
As in the tractable case, let $h : X \to Y$ be a trivial cofibration in $L_C(M)$ and let $g : K \to L$ be a generating cofibration. By the lemma, the map $g$ is either a cofibration between cofibrant objects or a trivial cofibration in $M$. If the former, then the proof of Proposition 4.3.4 goes through verbatim and proves that $h \Box g$ is an $L_C(M)$-trivial cofibration, since cofibrant objects are flat in $L_C(M)$. If the latter, then because $g$ is a trivial cofibration in $M$ and $h$ is a cofibration in $M$ we may apply the pushout product axiom on $M$ to see that $h \Box g$ is a trivial cofibration in $M$ (hence in $L_C(M)$ too). This completes the proof of the pushout product axiom on $L_C(M)$. \hfill \Box

**Remark 4.4.5.** The use of the lemma demonstrates that this proposition proves something slightly more general. Namely, if a model category is cofibrantly generated, left proper, has cofibrant objects flat, and the class of cofibrations is closed under pushout product then the pushout product axiom is satisfied.

Additionally, one could also prove the forwards direction in the theorem using only that $L_C(M)$ satisfies the pushout product axiom. For any cofibrant $K$ we have a cofibration $\phi_K : \emptyset \rightarrow K$. Note that for any $f \in C$, $f \otimes K = f \Box \phi_K \in C$-local equivalences, because $f$ is a trivial cofibration in $L_C(M)$.

We record this remark because in the future we hope to better understand the connection between monoidal Bousfield localizations and the closed localizations which appeared in [19], and this remark may be useful.
CHAPTER 5

Preservation of algebras over cofibrant operads

In this chapter we will provide several applications of the results in the previous two

chapters. We remind the reader that for operads valued in \( \mathcal{M} \), a map of operads \( A \to B \)

is said to be a trivial fibration if \( A_n \to B_n \) is a trivial fibration in \( \mathcal{M} \) for all \( n \). An operad

\( P \) is said to be \textit{cofibrant} if the map from the initial operad into \( P \) has the left lifting

property in the category of operads with respect to all trivial fibrations of operads. An

operad \( P \) is said to be \( \Sigma\text{-cofibrant} \) if it has this left lifting property only in the category

of symmetric sequences.

We begin with a theorem due to Markus Spitzweck, proven as Theorem 5 in [85] and as Theorem A.8 in [32], which makes it clear that the hypotheses of Corollary 3.3.2 are satisfied when \( L_C \) is a monoidal Bousfield localization and \( P \) is a cofibrant operad.

\textbf{Theorem 5.0.6.} Suppose \( P \) is a \( \Sigma\text{-cofibrant operad} \) and \( \mathcal{M} \) is a monoidal model

category. Then \( P\text{-alg} \) is a semi-model category.

This theorem, applied to both \( \mathcal{M} \) and \( L_C(\mathcal{M}) \) (if the localization is monoidal),

endows the categories of \( P\text{-algebras} \) in \( \mathcal{M} \) and \( L_C(\mathcal{M}) \) with inherited semi-model structures.

By Corollary 3.3.2, monoidal Bousfield localizations preserve algebras over \( \Sigma\text{-cofibrant operads} \). In particular, monoidal localizations preserve \( A_\infty \) and \( E_\infty \) algebras

in \( \mathcal{M} \), where \( A_\infty \) and \( E_\infty \) are any operads which are \( \Sigma\text{-cofibrant} \) and weakly equivalent.
to Ass and Com. We remark that Theorem 4 in [85] provides a full model structure under strengthened hypotheses.

**Theorem 5.0.7.** Suppose $P$ is a cofibrant operad and $\mathcal{M}$ is a monoidal model category satisfying the monoid axiom. Then $P\text{-alg}$ is a model category.

### 5.1. Spaces and Spectra

For topological spaces the situation is especially nice.

**Proposition 5.1.1.** Let $\mathcal{M}$ be the model category of (pointed) simplicial sets or $k$-spaces. Every Bousfield localization of $\mathcal{M}$ is a monoidal Bousfield localization.

**Proof.** For a review of the monoidal model structures on spaces and simplicial sets see Chapter 4 of [46]. Both are tractable, left proper, monoidal model categories with cofibrant objects flat.

For $\mathcal{M} = sSet$ we can simply rely on Theorem 4.1.1 of [42], which guarantees that $L_C(\mathcal{M})$ is a simplicial model category. The pushout product axiom is equivalent to the SM7 axiom for $sSet$, so this proves $L_C(\mathcal{M})$ is a monoidal model category and hence that $L_C$ is monoidal. There is also an elementary proof of this fact which is obtained from the proof below by replacing $F(-,-)$ everywhere by map($-,-$).

We turn now to $\mathcal{M} = Top$. By definition, any Bousfield localization $L_C$ will be a monoidal Bousfield localization as soon as we show $C \land S^n_+$ is contained in the $C$-local equivalences (the codomains of the generating cofibrations are contractible, so do not matter). As remarked in the discussion below Definition 4.1 in [51], for topological model categories Bousfield localization with respect to a set of cofibrations can be defined
using topological mapping spaces rather than simplicial mapping spaces (at least when all maps in \( \mathcal{C} \) are cofibrations). Let \( F(X,Y) \) denote the space of based maps \( X \to Y \).

We will make use of Proposition 3.2 in [47], a version of which states that because \( \text{Top} \) is left proper and cofibrantly generated, a map \( f \) is a weak local equivalence if and only if \( F(T,f) \) is a weak equivalence of topological spaces for all \( T \) in the (co)domains of the generating cofibrations \( I \) in \( \text{Top} \).

Now consider the following equivalent statements, where \( T \) runs through the domains and codomains of generating cofibrations.

- \( f \) is a \( \mathcal{C} \)-local equivalence
- iff \( F(f,Z) \) is a weak equivalence for all \( \mathcal{C} \)-local \( Z \)
- iff \( F(T,F(f,Z)) \) is a weak equivalence for all \( \mathcal{C} \)-local \( Z \) and all \( T \) (by Prop. 3.2)
- iff \( F(T \wedge f,Z) \) is a weak equivalence for all \( \mathcal{C} \)-local \( Z \) (by adjointness)
- iff \( T \wedge f \) is a \( \mathcal{C} \)-local equivalence

This proves that the class of \( \mathcal{C} \)-local equivalences is closed under smashing with a domain or codomain of a generating cofibration, so \( L_\mathcal{C} \) is a monoidal Bousfield localization.

\[ \square \]

The reader may wonder whether all Bousfield localizations preserve algebras over cofibrant operads in general model categories \( \mathcal{M} \), i.e. whether all Bousfield localizations are monoidal. This is false, as demonstrated by the following example, which can be found at the end of Section 6 in [19].

**Example 5.1.2.** Let \( \mathcal{M} \) be symmetric spectra, orthogonal spectra, or \( \mathbb{S} \)-modules.
Recall that in topological spaces, the $n^{th}$ Postnikov section functor $P_n$ is the Bousfield localization $L_f$ corresponding to the map $\Sigma f$ where $f: S^n \to \ast$. Applying $\Sigma^\infty$ gives a map of spectra and we again denote by $P_n$ the Bousfield localization with respect to this map.

The Bousfield localization $P_{-1}$ on $\mathcal{M}$ does not preserve $A_\infty$-algebras. If $R$ is a non-connective $A_\infty$-algebra then the unit map $\nu: S \to P_{-1}R$ is null because $\pi_0(P_{-1}R) = 0$. Thus, $P_{-1}R$ cannot admit a ring spectrum structure (not even up to homotopy) because $S \wedge P_{-1}R \to P_{-1}R \wedge P_{-1}R \to P_{-1}R$ is not a homotopy equivalence as it would have to be for $P_{-1}R$ to be a homotopy ring.

In [19], examples of the sort above are prohibited by assuming that $L$-equivalences are closed under the monoidal product. It is then shown in Theorem 6.5 that for symmetric spectra this property is implied if the localization is stable, i.e. $L \circ \Sigma \simeq \Sigma \circ L$. We now compare our requirement that $L_C$ be a monoidal Bousfield localization to existing results regarding preservation of monoidal structure.

**Proposition 5.1.3.** Every monoidal Bousfield localization is stable. In a monogenic setting such as spectra, every stable localization is monoidal.

This is clear, since suspending is the same as smashing with the suspension of the unit sphere.

The Postnikov section is clearly not stable, and indeed the counterexample above hinges on the fact that the section has truncated the spectrum by making trivial the degree in which the unit must live. Under the hypothesis that localization respects the monoidal product, Theorem 6.1 of [19] proves that cofibrant algebras over a cofibrant
colored operad valued in $sSet_*$ or $Top_*$ are preserved. Theorem 3.2.1 recovers this result in the case of operads, and improves on it by extending the class of operads so that they do not need to be valued in $sSet_*$ or $Top_*$, by discussing preservation of non-cofibrant algebras, by weakening the cofibrancy required of the operad to $\Sigma$-cofibrancy (using Theorem 5.0.6 above), and by potentially weakening the hypothesis on the localization. A different generalization of [19] has been given in [32].

Proposition 5.1.4. Every Bousfeld localization for which the local equivalences are closed under $\otimes$ is a monoidal Bousfeld localization. The converse is not true.

Proof. To see why this is true, consider the maps $id_K$ as $L$-equivalences when testing whether or not $id_K \otimes C$ is a $C$-local equivalence. To see that the converse is not true, take $C$ to be the generating trivial cofibrations of any cofibrantly generated model category in which the weak equivalences are not closed under $\otimes$. □

Thus, our hypothesis on a monoidal Bousfeld localization is strictly weaker than requiring $L$-equivalences to be closed under $\otimes$. Theorems 4.2.2 and 4.2.3 demonstrate that the hypothesis that $C \otimes id_K$ is contained in the $C$-local equivalences is best-possible, since it $L_C$ is a monoidal Bousfeld localization if and only if this property holds, and without the pushout product axiom on $L_C(M)$ the question of preservation of algebras under localization is not even well-posed.

Remark 5.1.5. In light of the Postnikov section example, the argument of Proposition 5.1.1 must break down for spectra. The precise place where the argument fails is the passage through $\text{map}(T, \text{map}(f, Z))$. In spectra, this expression has no meaning, because $T$ is a spectrum but $\text{map}(f, Z)$ is a space. So the argument of Proposition 5.1.1...
5.2. EQUIVARIANT SPECTRA

relies precisely on the fact that $\mathcal{M} = sSet$ (or $\mathcal{M} = Top$ in the topological case), so that the SM7 axiom for $\mathcal{M}$ is precisely the same as the pushout product axiom.

Theorem 5.0.6 combines with Theorem 3.2.1 to prove that any monoidal Bousfield localization of spectra preserves $A_\infty$ and $E_\infty$ algebras. In particular, $A_\infty$ and $E_\infty$ algebras are preserved by stable Bousfield localizations such as $L_E$ where $E$ is a homology theory. So our results recover Theorems VIII.2.1 and VIII.2.2 of [26]. We remark that not all $E_\infty$-operads in the sense of [67] are cofibrant, but all are $\Sigma$-cofibrant, precisely because the $n^{th}$ space is taken to be an $E\Sigma_n$-space.

5.2. Equivariant Spectra

We turn now to the example which originally motivated this work. The author learned this example from a talk given by Mike Hill at Oberwolfach (the proceedings can be found in [39]). A similar example appeared in [68]. Before presenting this motivating example, we must introduce some new notation.

Let $G$ be a compact Lie group and let $\mathcal{M} = \text{Sp}^G$ be the positive stable model structure on equivariant orthogonal spectra. Given a $G$-space $X$ and a closed subgroup $H$, one may restrict the $G$ action to $H$ and obtain an $H$-space denoted $\text{res}_H(X)$. This association is functorial and lifts to a functor $\text{res}_H : \text{Sp}^G \to \text{Sp}^H$. This restriction functor has a left adjoint $G^+ \wedge_H (-)$, the induction functor. We refer the reader to Section 2.2.4 of [41] for more details. If one shifts focus to commutative monoids $\text{Comm}_G$ in $\text{Sp}^G$ (equivalently to genuine $E_\infty$-algebras) then there is again a restriction functor $\text{res}_H : \text{Comm}_G \to \text{Comm}_H$ and it again has a left adjoint functor $N^G_H(-)$ called the norm. This functor is discussed in Section 2.3.2 of [41].
Example 5.2.1. There are localizations which destroy genuine commutative structure but which preserve naive $E_\infty$-algebra structure. For this example, let $G$ be a (non-trivial) finite group.

Consider the reduced real regular representation $\overline{\rho}$ obtained by taking the quotient of the real regular representation $\rho$ by the trivial representation, which is a summand. We write $\overline{\rho}_G = \rho_G - 1$ where 1 means the trivial representation $\mathbb{R}[e]$. Taking the one-point compactification of this representation yields a representation sphere $S^{\overline{\rho}}$. There is a natural inclusion $a_\overline{\rho} : S^0 \to S^{\overline{\rho}}$ induced by the inclusion of the trivial representation into $\overline{\rho}$. Consider the spectrum $E = S[a_\overline{\rho}^{-1}]$ obtained from the unit $S$ (certainly a commutative algebra in $Sp^G$) by localization with respect to $a_\overline{\rho}$. We will show that this spectrum does not admit maps from the norms of its restrictions, and hence cannot be commutative.

First, $\rho_G|_H = [G : H] \rho_H$, so $\overline{\rho}_G|_H = [G : H] \overline{\rho}_H + ([G : H] - 1)$. We will use this to prove that for all proper $H < G$, $\text{res}_H(E)$ is contractible. Because $[G : H] - 1$ is a number $k$ greater than 0 we have $\text{res}_H S^G = (S^{\overline{\rho}_H})^\# [G : H] \wedge S^k$. This means that as an $H$-spectrum it is contractible, because there is enough space in the $S^k$ part to deform it to a point. Note, however, that $E$ itself is not locally trivial. Thinking of $S^0$ as $\{0, \infty\}$ we see that the only fixed points of $a_\overline{\rho}$ are 0 and $\infty$, so the map $a_\overline{\rho}$ is not equivariantly trivial.

If $E$ were a commutative algebra in $Sp^G$ then the counit of the norm-restriction adjunction would provide a ring homomorphism $N^G_H \text{res}_H(E) \to E$. But the domain is contractible for every proper subgroup $H$ because $\text{res}_H(E)$ is contractible. This cannot be a ring map unless $E$ to be contractible, and we know $E$ is not contractible because $a_\overline{\rho}$ fixes 0 and $\infty$. 
This example would have been a hole in the proof of the Kervaire Invariant One Theorem (because the spectrum $\Omega = D^{-1}MU^{(4)}$ needed to be commutative) if not for the following theorem from [40].

**Theorem 5.2.2.** Let $G$ be a finite group. Let $L$ be a localization of equivariant spectra. If for all $L$-acyclics $Z$ and for all subgroups $H$, $N_H^G Z$ is $L$-acyclic, then for all commutative $G$-ring spectra $R$, $L(R)$ is a commutative $G$-ring spectrum.

The hypothesis in this theorem is designed so that the proof in [26] regarding preservation of $E_\infty$ structure under localization (i.e. via the skeletal filtration) may go through.

We wish to understand how our general preservation result relates to this example and theorem, so we now specialize Corollary 3.3.2 to the case of $\mathcal{M} = Sp^G$, where $G$ is a compact Lie group.

We must first understand the generating cofibrations. For $\text{Top}^G$, the (co)domains of maps in $I$ take the form $((G/H) \times S^{n-1})_+$ and $((G/H) \times D^n)_+$ for $H$ a closed subgroup of $G$, by Definition 1.1 in [65]. For $Sp^G$, we first need a new piece of notation. For any finite dimensional orthogonal $G$-representation $W$ there is an evaluation functor $Ev_W : Sp^G \to \text{Top}^G$. This functor has a left adjoint $F_W$ (see Proposition 3.1 in [51] for more details). The (co)domains of maps in $I$ take the form $F_W((G/H)_+ \wedge S^{n-1})_+$ and $F_W((G/H)_+ \wedge D^n)_+$ by Definition 1.11 in [65], where $W$ runs through some fixed $G$-universe $\mathcal{U}$. The latter are contractible, and so smashing with them does not make a difference. Observe that these are tractable model structures. That $Sp^G$ is a monoidal model category with cofibrant objects flat is verified in [65], and may also be deduced
from Corollary 4.4 in [51]. Thus, for $\mathcal{M} = \mathcal{S}p^G$, our preservation result (Corollary 3.3.2 together with Theorem 4.2.2) becomes:

**Theorem 5.2.3.** Let $G$ be a compact Lie group. In $\mathcal{S}p^G$, a Bousfield localizations $L_C$ is monoidal if and only if $C \wedge F_W((G/H)_+ \wedge S^{n-1})$ is a $C$-local equivalence for all closed subgroups $H$ of $G$, for all $W$ in the universe, and for all $n$. Furthermore, such localizations preserve genuine equivariant commutativity.

Ignoring suspensions, monoidal Bousfield localizations are precisely the ones for which $L_C$ respects smashing with $(G/H)_+$ for all subgroups $H$. We think of these localizations as the ones which can ‘see’ the information of all subgroups. We now discuss Hill’s example in more detail, in light of this theorem. First, it is clear that Hill’s example fails to be a monoidal Bousfield localization because $E \wedge (G/H)_+$ is contractible for all proper $H$ (see Section 2.3.2 in [41]), but as we have remarked $E$ itself is not contractible (not even locally).

Example 4.1.1 has already demonstrated that localizations that kill a representation sphere should not be expected to be monoidal. The presence of $S^F$ demonstrates that Hill’s example is analogous, but the example can also be viewed in another way. In Hill’s example, smashing with $G/H$ for a non-trivial proper $H$ is equivalent to suspending with respect to the representation sphere corresponding to $H$. In this light, Hill’s example is demonstrating that a monoidal Bousfield localization of spectra must be stable with respect to all representation spheres, and it can be seen as an equivariant analogue of Example 5.1.2.
Hill’s example is stable with respect to the monoidal unit, so naive $E_\infty$ algebras are preserved. The failure only manifests when a non-trivial, proper $H$ is considered. However, because any such $H$ will lead to a failure of $L(S) = E$ to be commutative, Hill’s example is in some sense maximally bad. In light of this, it is natural to ask what happens when the localization respects some, but not all, of the subgroups of $G$. It is to answer this question that we introduced the lattice of operads $E_\infty^\mathcal{F}$ in Definition 2.6.8 which interpolate between naive and genuine $E_\infty$ operads.

Let $\mathcal{F}$ be a family of subgroups, and recall from Definition 2.6.4 that the generating cofibrations of the $\mathcal{F}$-fixed point model structure $Sp^\mathcal{F}$ on $G$-spectra are maps of the form $F_{W}((G/H)_+ \wedge i_n)$ where $i_n$ is a generating cofibration in $Top$. With the generating cofibrations in hand, Theorem 4.2.2 implies that monoidal Bousfield localizations in $Sp^\mathcal{F}$ are characterized by the property that $C \wedge (G/H)_+$ is a $C$-local equivalence for all $H \in \mathcal{F}$ (again, ignoring suspensions). We now wish to understand how $E_\infty^\mathcal{F}$-algebra structure interacts with Bousfield localization. We first re-formulate Example 5.2.1. This is the form of Hill’s example which was presented in [39] for $\mathcal{P}$ the family of proper subgroups.

**Example 5.2.4.** If $X$ is an $E_\infty^\mathcal{F}$-algebra then there is a localization $L$ sending $X$ to a naive $E_\infty$-algebra. Consider the cofiber sequence $E\mathcal{P}_+ \to S^0 \to \bar{E}\mathcal{P}$ for any family $\mathcal{P} \supseteq \mathcal{F}$ which does not contain $G$. Recall the fixed-point property of the space $E\mathcal{P}$ (discussed very nicely in Section 7 of [78]) and deduce:

$$(E\mathcal{P}_+)^H \simeq \begin{cases} *_+ = S^0 & \text{if } H \in \mathcal{P} \\ \emptyset_+ = * & \text{if } H \notin \mathcal{P} \end{cases}$$
For all $H$, the $H$-fixed points of $S^0$ are $S^0$. So that the cofiber obtained by mapping this space into $S^0$ satisfies the following fixed-point property

$$\left( \overset{H}{E\mathcal{P}} \right)^H = \begin{cases} * & \text{if } H \in \mathcal{P} \\ S^0 & \text{if } H \notin \mathcal{P} \end{cases}$$

Now apply $\Sigma_+^\infty$ to the map $S^0 \to \overset{E}{E\mathcal{P}}$. If $G$ is a finite group then the resulting map $S \to E$ is the same localization map considered in Example 5.2.1 (see Section 7 of [78]). This $E$ is not contractible because $E\mathcal{P}_+$ is not homotopy equivalent to $S^0$ (since $\mathcal{P}$ doesn’t contain $G$), though $\text{res}_H(E\mathcal{P}_+)$ is homotopy equivalent to $\text{res}_H(S^0)$ for any $H \in \mathcal{P}$.

In this formulation it is clear that the map $S \to E$ is a nullification which kills all maps out of the induced cells $G_+ \wedge_H (H/K)_+ = (G/H)_+$ for all $H \in \mathcal{P}$. With the characterization of monoidal Bousfield localizations in $Sp^{\mathcal{P}}$, we can see that in order to produce a localization which sends $E^{\mathcal{P}}$-algebras to naive $E_\infty$-algebras one need only apply the localization $S^0 \to \overset{E}{E\mathcal{P}}$ rather than the localization $S^0 \to \overset{E}{E\mathcal{P}}$ for the full family $\mathcal{P}$ of proper subgroups of $G$.

The presentation in Example 5.2.5 for finite $G$ makes it clear that this localization is simply killing a homotopy element (namely: the Euler class $a_\mathcal{P}$ discussed in Section 2.6.3 of [41]). The presentation in Example 5.2.4 has several benefits of its own: it generalizes to compact Lie groups $G$, it demonstrates that a smaller localization is needed to destroy $E^{\mathcal{P}}_\infty$-algebra structure rather than $E^G_\infty$-algebra structure, and it provides a generalization
of Hill’s example in which localization can reduce one’s place in the lattice $E_\infty^\mathcal{F}$ without reducing it all the way down to naive $E_\infty$.

**Example 5.2.5.** Localization can take an $E_\infty^\mathcal{F}$-algebra $E$ to a $E_\infty^\mathcal{K}$-algebra for $\mathcal{K} \subsetneq \mathcal{F}$. To define such a localization $L$ we need to kill some, but not all, maps from induced cells corresponding to $H \in \mathcal{F}$. This can be done by inverting a wedge of maps which kills whichever induced cells one desires to kill, as long as this localization does not kill any induced cells for $K \in \mathcal{K}$. This can be accomplished by inverting only cells corresponding to $H$ which intersect $\mathcal{K}$ in the identity subgroup. Then because maps from induced cells corresponding to $K \in \mathcal{K}$ have not been killed, the resulting object $LE$ has $E_\infty^\mathcal{K}$-algebra structure inherited from $E$.

This example demonstrates once again that the key property of a localization $L_C$ so that it preserves $E_\infty^\mathcal{F}$-algebra structure is a compatibility condition governing the behavior of the maps $C$ after the functor $- \wedge (G/H)_+$ is applied (as $H$ runs through the family $\mathcal{F}$). We formalize this by another application of Corollary 3.3.2 and Theorem 4.2.2. First, observe that both $\text{Oper}^\mathcal{F}$ and $\text{Sp}^\mathcal{F}$ are $\text{Top}^\mathcal{F}$-model structures (in the sense of Definition 4.2.18 in [46]) and the cofibrancy of $E_\infty^\mathcal{F}$ is relative to the $\mathcal{F}$-model structure. Thus, from a model category theoretic standpoint, $E_\infty^\mathcal{F}$-algebras are best viewed in $\text{Sp}^\mathcal{F}$.

**Theorem 5.2.6.** Let $\mathcal{M} = \text{Sp}^G$ and let $\mathcal{F}$ be a family of closed subgroups of $G$. Assume $F_W((G/H)_+ \wedge S_{+}^{n-1}) \wedge C$ is contained in the $C$-local equivalences for all $H \in \mathcal{F}$, for all $n$, and for all $W$ in the universe. Then $L_C$ takes any $E_\infty^G$-algebra to an $E_\infty^\mathcal{F}$-algebra.
Localizations of the form above are $\mathcal{F}$-monoidal but not necessarily $G$-monoidal. This is why $L_C(X)$ for $X \in E^G_{\infty}$-alg has $E^F_{\infty}$-algebra structure but may not have $E^G_{\infty}$-algebra structure, as demonstrated by Example 5.2.5. More generally, we have the following result, which encodes the fact that if we work in $Sp^\mathcal{X}$ rather than $Sp^G$ then localizations should be compatible with both $\mathcal{X}$ and $\mathcal{F}$. Because there are now two families involved, the localization will preserve algebraic structure corresponding to the meet of these two families in the lattice of families.

Theorem 5.2.7. Let $M$ be the $\mathcal{X}$-fixed point model structure on $G$-spectra and let $\mathcal{X}'$ be a subfamily of $\mathcal{X}$. Assume $F_W((G/H)_+ \wedge S^n) \wedge C$ is contained in the $C$-local equivalences for all $H \in \mathcal{X}'$, for all $n$, and for all $W$ in the universe. Then $L_C$ takes any $E^\mathcal{F}_{\infty}$-algebra to an $E^\mathcal{F}_{\infty} \cap \mathcal{X}'_{\infty}$-algebra.

Proof. In order to apply Corollary 3.3.2, first forget to the model structure $Sp^\mathcal{F} \cap \mathcal{X}'$ and observe that any $E^\mathcal{F}_{\infty}$-algebra is sent to a $E^\mathcal{F} \cap \mathcal{X}'_{\infty}$-algebra. The hypothesis on $L_C$ guarantees that $L_C$ is a monoidal Bousfield localization with respect to the $\mathcal{F} \cap \mathcal{X}'$ model structure, and so $E^\mathcal{F} \cap \mathcal{X}'_{\infty}$ is preserved.

This theorem also explains why $LE$ has $E^\mathcal{F}_{\infty}$-algebra structure in Example 5.2.5. The localization $L$ described in Example 5.2.5 is a monoidal Bousfield localization with respect to the $Sp^\mathcal{X}$ model structure.

Together, Theorem 5.2.7 and Example 5.2.5 resolve the question of preservation for the lattice $E^\mathcal{F}_{\infty}$. As expected, preservation of lesser algebraic structure comes down to requiring a less stringent condition on the Bousfield localization. The least stringent condition is for $\mathcal{F} = \{e\}$ and recovers the notion of a stable localization (i.e. one which
is monoidal on the category of spectra after forgetting the $G$-action). Thus, our preservation theorem is a generalization of the result in \cite{40} that any such localization takes commutative equivariant ring spectra to spectra with an action of an $E_\infty$ operad. We will discuss Theorem \ref{thm:preservation} more in Chapter \ref{chap:theory} after developing the theory of preservation for commutative monoids.
CHAPTER 6

General Model Categories of Commutative Monoids

In this chapter we will provide conditions on a monoidal model category $\mathcal{M}$ so that commutative monoids in $\mathcal{M}$ inherit a model structure. In [81], the authors refer to the commutative situation as “intrinsically more complicated” and indeed there are several known cases where commutative monoids cannot inherit a model structure in the way above, e.g. commutative differential graded algebras over a field of nonzero characteristic, $\Gamma$-spaces, and non-positive model structures on symmetric or orthogonal spectra (due to an example of Gaunce Lewis in [57]). Side-stepping Lewis’s example required the introduction of positive variants on diagram spectra in [66], and the convenient model structure on symmetric spectra introduced in [83] (nowadays referred to as the positive flat model structure). We discuss these examples in Section 6.3.

One way to get around these obstacles is to work with $E_\infty$-algebras everywhere and never ask for strict commutativity. It is much easier to place a model structure on $E_\infty$-algebras because $E_\infty$ is a cofibrant operad, while $\text{Com}$ is not. We feel it is important to also be able to treat the strict commutative case, in particular because outside of categories of structured ring spectra one does not know that there is a Quillen equivalence between $E_\infty$-algebras and strictly commutative monoids (because $\text{Com}$ is not even $\Sigma$-cofibrant, one cannot use the general rectification results in [9]). The crucial hypothesis which allows such a Quillen equivalence in the case of structured ring spectra is that for all cofibrant $X$, the map $(E\Sigma_n)_+ \wedge X^{\wedge n} \to X^{\wedge n}/\Sigma_n$ is a weak equivalence.
It is important to note that this hypothesis is not necessary for strictly commutative monoids to inherit a model structure (in particular, it fails for simplicial sets). This hypothesis appears to be more related to the rectification question than to the question of existence of model structures. We address the point further in Subsection 6.2.2.

Due to the difficulties associated with passing model structures to categories of commutative monoids, several important papers have folded the existence of a model structure on commutative monoids into their hypotheses. This is done in Assumption 1.1.0.4 in [89] and in Hypothesis 5.5 in [82], among other places. The results in Section 6.1 provide check-able conditions on \( \mathcal{M} \) so that those hypotheses are satisfied.

We remark that a different axiom on \( \mathcal{M} \) which guarantees commutative monoids inherit a model structure has appeared as Proposition 4.3.21 in [61]. However, it is pointed out in [63] that this work contains some errors and as written does not apply to the positive model structure on symmetric spectra. Furthermore, we will demonstrate that it does not apply to topological spaces, though it does apply to chain complexes over a field of characteristic zero. Our commutative monoid axiom is more general, and does apply in these situations.

After stating the main results in Section 6.1 we will highlight the differences from the situation of [61]. We additionally discuss when a cofibration of commutative monoids forgets to a cofibration in \( \mathcal{M} \), and we introduce the strong commutative monoid axiom to guarantee this occurs. Following [81], we place the details of the proofs of these main results at the end of the chapter in Section 6.5 and we also prove in Section 6.4 that it is sufficient to check the strong commutative monoid axiom on the generating (trivial) cofibrations. Using this, we collect examples in Section 6.3. We include additional results
regarding functoriality of the passage from $R$ to commutative $R$-algebras, regarding rectification between $\text{Com}$ and $E_{\infty}$, and remarks regarding the interplay between the strong commutative monoid axiom and Bousfield localization in Section 6.2.

### 6.1. The Commutative Monoid Axiom

For any map $h$ we may consider the $n^{th}$ iterated pushout product of $h$ with itself, and we denote this $h^{\boxtimes n} = h \Box \ldots \Box h$. This map has a natural axiom of $\Sigma_n$ coming from permuting the inputs and outputs. The action will be spelled out in the proof of Theorem 6.1.2

**Definition 6.1.1.** A monoidal model category $\mathcal{M}$ is said to satisfy the *commutative monoid axiom* if whenever $h$ is a trivial cofibration in $\mathcal{M}$ then $h^{\boxtimes n}/\Sigma_n$ is a trivial cofibration in $\mathcal{M}$ for all $n > 0$.

Under this hypothesis, we state our main theorem:

**Theorem 6.1.2.** Let $\mathcal{M}$ be a cofibrantly generated monoidal model category satisfying the commutative monoid axiom and the monoid axiom, and assume that the domains of the generating maps $I$ (resp. $J$) are small relative to $(I \otimes \mathcal{M})$-cell (resp. $(J \otimes \mathcal{M})$-cell). Let $R$ be a commutative monoid in $\mathcal{M}$. Then the category $\text{CAlg}(R)$ of commutative $R$-algebras is a cofibrantly generated model category in which a map is a weak equivalence or fibration if and only if it is so in $\mathcal{M}$. In particular, when $R = S$ this gives a model structure on the category of commutative monoids in $\mathcal{M}$.

It is clear from this description of $\text{CAlg}(R)$ that if $\mathcal{M}$ is simplicial then $\text{CAlg}(R)$ is simplicial. Simply use that (trivial) fibrations are created in $\mathcal{M}$ and use the pullback
formulation of the SM7 axiom. Furthermore, if $\mathcal{M}$ is combinatorial then $CAlg(R)$ is combinatorial.

As the generating (trivial) cofibrations of $CAlg(R)$ are of the form $R \otimes \text{Sym}(I)$ (resp. $R \otimes \text{Sym}(J)$), these are cofibrant in $CAlg(R)$ if $\mathcal{M}$ is tractable and satisfies the commutative monoid axiom. Hence, tractability also passes from $\mathcal{M}$ to $CAlg(R)$.

**Proof sketch.** We will focus first on the case where $R$ is the monoidal unit $S$, and discuss general $R$ at the end. As commutative $S$-algebras are simply commutative monoids, we denote the category of commutative monoids $CMon(\mathcal{M})$ rather than $CAlg(S)$. We will verify condition (1) of Lemma 2.5.5 for the monad coming from the $(\text{Sym}, U)$ adjunction between $\mathcal{M}$ and $CMon(\mathcal{M})$. Let $J$ denote the generating trivial cofibrations of $\mathcal{M}$. We must prove that maps in $\text{Sym}(J)$-cell are weak equivalences. Given a trivial cofibration $h : K \to L$ in $\mathcal{M}$, we form the following pushout square in $CMon(\mathcal{M})$ and must prove that the bottom map is a map of the sort considered by the monoid axiom, so that transfinite compositions of such maps are weak equivalences in $\mathcal{M}$ (hence weak equivalences of commutative monoids):

$$
\begin{array}{ccc}
\text{Sym}(K) & \longrightarrow & \text{Sym}(L) \\
\downarrow & & \downarrow \\
X & \longrightarrow & P
\end{array}
$$

Of course, in $CMon(\mathcal{M})$, the pushout is simply the tensor product, so $P \cong X \otimes_{\text{Sym}(K)} \text{Sym}(L)$, but we will not make use of this fact. Following [81], we construct a filtration of the map of commutative monoids $X \to P$ as a composition $P_n \to P_{n+1}$ of maps formed by
pushout diagrams in $\mathcal{M}$. Doing so requires the decomposition of $\text{Sym}(K) = \coprod_n \text{Sym}^n(K)$ where $\text{Sym}^n(K) = K^\otimes n / \Sigma_n$.

Thinking of $P$ as formal products of elements from $X$ and from $L$ with relations in $K$ leads to a consideration of $n$-dimensional cubes to build products of length $n$ from the letters $X, K, L$. Because the map $\text{Sym}(K) \to X$ is adjoint to a map $K \to X$, we will in fact only need to consider $n$-dimensional cubes whose vertices are length $n$ words in the letters $K$ and $L$. Formally, for any subset $D$ of $[n] = \{1, 2, \ldots, n\}$ we obtain a vertex $C_1 \otimes \cdots \otimes C_n$ with $C_i = K$ if $i \notin D$ and $C_j = L$ if $j \in D$. The punctured cube is the cube with the vertex $L^\otimes n$ removed. The map $h^\otimes n$ is the induced map from the colimit $Q_n$ of the punctured cube to $L^\otimes n$.

There is an action of $\Sigma_n$ on the cube which permutes the letters in the words (equivalently, which permutes the vertices in the cube in a way coherent with respect to the edges of the cube). Explicitly, the action is defined as follows. Any $\sigma \in \Sigma_n$ sends the vertex defined above to the vertex corresponding to $\sigma(D) \subset [n]$ using the action of $\Sigma_{|D|}$ on the $X$’s and $\Sigma_{n-|D|}$ on the $Y$’s. This action yields a $\Sigma_n$-action on $h^\otimes n : Q_n \to L^\otimes n$, and in a moment we will pass to $\Sigma_n$-coinvariants.

We now show how to obtain $P_n$ (which in this analogy is to be thought of as formal products of length $n$) from the cubes we have just described. The steps in the filtration of $X \to P$ are formed by pushouts of the maps $id_X \otimes h^\otimes n / \Sigma_n$:

$$
\begin{array}{ccc}
X \otimes Q_n / \Sigma_n & \longrightarrow & X \otimes L^\otimes n / \Sigma_n \\
\downarrow & & \downarrow \\
P_{n-1} & \longrightarrow & P_n
\end{array}
$$
The proof that the $P_n$ provide a filtration of $X \to P$ is delayed until Section 6.5.

Assuming the commutative monoid axiom, the maps $h_{\Sigma_n}^n$ are trivial cofibrations. Thus, the map $X \to P$ is a transfinite composite of pushouts of maps in $\mathcal{M} \otimes \{\text{trivial cofibrations}\}$. Hence, by the monoid axiom, $X \to P$ is a weak equivalence. Similarly, for any transfinite composition $f$ of pushouts of maps of the form $\text{Sym}(K) \to \text{Sym}(L)$, we may realize $f$ as a transfinite composition of maps $X \to P$ of the form above. As a transfinite composition of transfinite compositions is still a transfinite composition, the monoid axiom applies again and proves $f$ is a weak equivalence. Lemma 2.5.5 now applies to produce the required model structure on commutative monoids.

To handle the case of commutative $R$-algebras, note that there is an equivalence of categories between $CAlg(R)$ and $(R \downarrow \text{CMon}(\mathcal{M}))$, the category of commutative monoids under $R$. So we may apply the remark after Proposition 1.1.8 of [46] to conclude that this is a model category with cofibrations, fibrations, and weak equivalences inherited from $\text{CMon}(\mathcal{M})$. Note that this is a different approach from the one provided in [81] because we do not pass through $R$-modules en route to commutative $R$-algebras. That $CAlg(R)$ is cofibrantly generated follows from [43], where it is also shown that the generating cofibrations are given by the set $I_R$ of maps in $(R \downarrow \text{CMon}(\mathcal{M}))$ where the map in $\text{CMon}(\mathcal{M})$ is in $I$. Under the equivalence of categories between $CAlg(R)$ and $(R \downarrow \text{CMon}(\mathcal{M}))$, such maps are sent to maps in $R \otimes \text{Sym}(I)$. We can similarly identify the generating trivial cofibrations as $R \otimes \text{Sym}(J)$.

\[ \square \]

Remark 6.1.3. Notice that the proof in fact requires less than the full strength of the hypotheses. Rather than assuming the commutative monoid axiom and the monoid
axiom separately, we could have assumed that transfinite compositions of pushouts of maps in \( \{ \mathcal{M} \otimes h^{\bigtriangleup n}/\Sigma_n \mid h \text{ is a trivial cofibration} \} \) are contained in the weak equivalences. We will refer to this property as the *weak commutative monoid axiom*. Certain model categories discussed in Section 6.3 only satisfy this axiom and not the commutative monoid axiom. However, for reasons which will become clear in Corollary 6.1.8 we have chosen the commutative monoid axiom as the appropriate axiom for our applications.

The full proof in Section 6.5 will in fact prove more than just the theorem. It will also prove the commutative analog to Lemma 6.2 of [81], from which one can deduce the proposition below regarding when cofibrations of commutative monoids forget to cofibrations in \( \mathcal{M} \). It is well-known to experts that obtaining the correct behavior of cofibrations under the forgetful functor is subtle in the commutative setting. Indeed, this was the motivation behind the convenient model structures introduced in [83] and [87]. In order to guarantee the desired behavior we must strengthen the commutative monoid axiom.

**Definition 6.1.4.** A monoidal model category \( \mathcal{M} \) is said to satisfy the *strong commutative monoid axiom* if whenever \( h \) is a (trivial) cofibration in \( \mathcal{M} \) then \( h^{\bigtriangleup n}/\Sigma_n \) is a (trivial) cofibration in \( \mathcal{M} \) for all \( n > 0 \). In particular, we are now assuming that cofibrations are closed under the operation \( (-)^{\bigtriangleup n}/\Sigma_n \).

**Proposition 6.1.5.** Suppose \( \mathcal{M} \) satisfies the strong commutative monoid axiom. Then for any commutative monoid \( R \), a cofibration in \( CAlg(R) \) with source cofibrant in \( \mathcal{M} \) is a cofibration in \( \mathcal{M} \).

See Section 6.5 for a proof of this proposition.
Corollary 6.1.6. Suppose $\mathcal{M}$ satisfies the strong commutative monoid axiom and that $S$ is cofibrant in $\mathcal{M}$. Then any cofibrant commutative monoid is cofibrant in $\mathcal{M}$. If in addition $R$ is cofibrant in $\mathcal{M}$ then any cofibrant commutative $R$-algebra is cofibrant in $\mathcal{M}$.

Corollary 6.1.7. Assume $S$ is cofibrant in $\mathcal{M}$ and that $\mathcal{M}$ satisfies the strong commutative monoid axiom. If $f$ is a cofibration between cofibrant objects then $\text{Sym}(f)$ is a cofibration in $\mathcal{M}$. In particular, if $X$ is cofibrant in $\mathcal{M}$ then $\text{Sym}(X)$ is cofibrant in $\mathcal{M}$.

Proof. Because the model structure on $\text{CMon}(\mathcal{M})$ is transferred from that of $\mathcal{M}$, the functor $\text{Sym}(-)$ is left Quillen, and hence preserves cofibrations. So $\text{Sym}(f)$ is a cofibration of commutative monoids because $f$ is a cofibration in $\mathcal{M}$. If the source $K$ of $f$ is cofibrant then the source of $\text{Sym}(f)$ is a cofibrant commutative monoid, by applying $\text{Sym}(-)$ to the cofibration $\emptyset \xrightarrow{f} K$. By Corollary 6.1.6, the source of $\text{Sym}(f)$ is cofibrant in $\mathcal{M}$. By Proposition 6.1.5, $\text{Sym}(f)$ is a cofibration in $\mathcal{M}$. □

Recall that the point of positive model structures on diagram spectra (e.g. symmetric spectra or orthogonal spectra) is to break the cofibrancy of $S$ and so avoid Lewis’s obstruction [57] to having a model structure on commutative ring spectra. Thus, these corollaries do not apply to positive model categories of spectra. In [83], a variant on the positive model structure is introduced in which cofibrant commutative ring spectra are cofibrant as spectra. This model structure was known in that paper as the convenient model structure, and later as the positive flat model structure. We do not know how
6.1. THE COMMUTATIVE MONOID AXIOM

to obtain this ‘convenient’ property for general model categories. We suspect it has
something to do with forcing the cofibrations to contain the monomorphisms.

The proof of Theorem 6.1.2 makes clear precisely where the monoid axiom is being
used, and hence why the smallness hypotheses are needed. If \( M \) does not satisfy the
monoid axiom, then we can make this step work by assuming \( X \) is a cofibrant commu-
tative monoid. In this case, [45] and [85] make it clear that a semi-model structure can
be obtained. We summarize this as a corollary, so that we may reference it in Chapter

\[ \]

**Corollary 6.1.8.** Let \( M \) be a cofibrantly generated monoidal model category sat-
isfying the commutative monoid axiom, and assume that the domains of the generating
maps \( I \) (resp. \( J \)) are small relative to \((I \otimes M)\)-cell (resp. \((J \otimes M)\)-cell). Then for any
commutative monoid \( R \), the category of commutative \( R \)-algebras is a cofibrantly gener-
ated semi-model category in which a map is a weak equivalence or fibration if and only
if it is so in \( M \).

**Proof.** We begin with the case where \( R = S \), so that we are building a semi-model
structure on \( \text{CMon}(M) \). Consider the proof of Theorem 6.1.2. All of the model category
axioms are purely formal except for factorization of an arbitrary map into a trivial
cofibration followed by a fibration, and except for lifting of trivial cofibrations against
fibrations (which follows from factorization and the retract property). The monoid axiom
is not used until we have already proven that the pushout of commutative monoids

\[
\begin{array}{ccc}
\text{Sym}(K) & \longrightarrow & \text{Sym}(L) \\
\downarrow & & \downarrow \\
X & \longrightarrow & P
\end{array}
\]

can be factored into \(X = P_0 \rightarrow P_1 \rightarrow \cdots \rightarrow P\) where each \(P_{n-1} \rightarrow P_n\) is a pushout of \(X \otimes f^{\square n}/\Sigma_n\). By the commutative monoid axiom, \(f^{\square n}/\Sigma_n\) is a trivial cofibration. Without the monoid axiom it is not clear how to proceed unless \(X\) is cofibrant. If \(X\) is cofibrant, then this map has the form \((\emptyset \rightarrow X) \diamond f^{\square n}/\Sigma_n\) and hence is a trivial cofibration by the pushout product axiom. Thus, the pushout \(P_{n-1} \rightarrow P_n\) must also be a trivial cofibration, and the composite \(X \rightarrow P\) is a composite of trivial cofibrations and hence a trivial cofibration.

From this argument, we may conclude that factorization into a trivial cofibration followed by a fibration works for maps with cofibrant domain. Similarly, lifting of trivial cofibrations \(g\) against fibrations is satisfied if \(g\) has cofibrant domain. In particular, notice that if all objects are cofibrant in \(\mathcal{M}\) then a semi-model structure is the same as a model structure, because adding the assumption that \(X\) is cofibrant changes nothing.

For the case of a general commutative monoid \(R\), observe that the semi-model structure on \(\text{CMon}(\mathcal{M})\) yields a semi-model structure on \(\text{CAlg}(R) = R \downarrow \text{CMon}(\mathcal{M})\) purely formally, since all of the model category axioms are satisfied if and only if they are satisfied in \(\text{CMon}(\mathcal{M})\). To prove the resulting semi-model structure is cofibrantly generated, we again reference [43]. The semi-model structure on \(\text{CAlg}(R)\) is transferred from the structure on \(\text{CMon}(\mathcal{M})\) along the adjunction \(R \otimes - : \text{CMon}(\mathcal{M}) \rightleftarrows \text{CAlg}(R) : U\), and
hence along the adjunction \( \mathcal{M} \cong CAlg(R) \) obtained from composing with the adjunction \( \text{Sym}(-) : \mathcal{M} \cong \text{CMon}(\mathcal{M}) : U \). This yields sets of maps \( FI \) and \( FJ \) which cofibrantly generate a semi-model structure on \( CAlg(R) \).

That the resulting semi-model structure on \( CAlg(R) \) matches the one inherited from \( \text{CMon}(\mathcal{M}) \) follows precisely as in [43] applied to the adjunction \( R \otimes - : \text{CMon}(\mathcal{M}) \cong CAlg(R) : U \). Weak equivalences are clearly the same. In order to prove that fibrations are the same, one must use the fact that in the cofibrantly generated semi-model category \( \text{CMon}(\mathcal{M}) \), a map is a fibration if and only if it has the right lifting property with respect to the generating trivial cofibrations. Thankfully, this is part of the definition of a cofibrantly generated semi-model category given in [85]. Lastly, if two semi-model structures have the same fibrations and weak equivalences then they have the same cofibrations, because a map is a cofibration if and only if it satisfies the left lifting property with respect to trivial fibrations (see [85]). \( \square \)

Observe that if one wishes to obtain on \( CAlg(R) \) a semi-model structure over \( \mathcal{M} \) in the terminology of [85] then one must also assume that \( S \) and \( R \) are cofibrant so that the initial object in \( CAlg(R) \) forgets to a cofibrant object in \( \mathcal{M} \).

As the filtration given in Section 6.3 is related to Harper’s filtration for general operads from Section 7.3 of [36], we pause for a moment to compare these two approaches.
Remark 6.1.9. Harper’s general machinery describes the map $P_{n-1} \to P_n$ as a pushout

$$\begin{array}{ccc}
Com_X(n) \otimes \Sigma_n Q_n & \longrightarrow & Com_X(n) \otimes \Sigma_n L^{\otimes n} \\
\downarrow & & \downarrow \% \\
P_{n-1} & \longrightarrow & P_n
\end{array}$$

where $Com_X$ is the enveloping operad. One may use Proposition 7.6 in [36] to write $Com_X(n) = X$ with the trivial $\Sigma_n$ action. Thus, $P_{n-1} \to P_n$ can be written as the pushout of $X \otimes f^{\otimes n}/\Sigma_n$ and Harper’s filtration makes it clear that the commutative monoid axiom is precisely the right hypothesis.

In a similar way, $Ass_X(n) = X^{\otimes n+1} \cdot \Sigma_n$, i.e. the coproduct of $n!$ copies of $X^{\otimes n+1}$ with the free $\Sigma_n$ action. So in that case the $(- \cdot \Sigma_n) \otimes \Sigma_n (-)$ provides a cancellation and Harper’s filtration reduces to a pushout of $X^{\otimes n+1} \otimes f^{\otimes n}$. We see immediately why the monoid axiom is necessary.

Finally, one could realize commutative $R$-algebras as algebras over the operad $Com_R$ and in this case Harper’s filtration would be a pushout of a map of the form $(Com_R)_A(n) \otimes \Sigma_n f^{\otimes n}$ where $A$ is a commutative $R$-algebra. In this case, the formula in Proposition 7.6 yields $(Com_R)_A(n) = R \otimes A$ and so the maps $P_{n-1} \to P_n$ are pushouts of $(R \otimes A) \otimes f^{\otimes n}/\Sigma_n$. In this way we see that in the presence of the commutative monoid axiom but in the absence of the the monoid axiom we need both $R$ and $A$ to be cofibrant in order to ensure that this map is a trivial cofibration, i.e. to obtain on $CAlg(R)$ a semi-model structure over $\mathcal{M}$. This is the commutative analog of Theorem 3.3 in [45], in which cofibrancy of $R$ was required to achieve a semi-model structure on $R$-algebras. There
the formula \((Ass_R)_A(n) = R \otimes A \cdot \Sigma_n\) means that the relevant pushout takes the form \(R \otimes A \otimes f^{\Sigma n}\) and this makes clear why both \(R\) and \(A\) must be cofibrant in the absence of the monoid axiom.

We conclude this section with a remark comparing our approach and results with the approach outlined by Lurie in [61], in which he proved:

**Theorem 6.1.10.** Let \(\mathcal{M}\) be a left proper, combinatorial, tractable, monoidal model category satisfying the monoid axiom and with a cofibrant unit. Assume further that

\((*)\) If \(h\) is a cofibration then \(h^{\Sigma n}\) is a cofibration in the projective model structure on \(\mathcal{M}^{\Sigma n}\) for all \(n\). Such maps \(h\) are called power cofibrations.

Then \(\text{CMon}(\mathcal{M})\) has a model category structure with weak equivalences and fibrations inherited from \(\mathcal{M}\).

The difference between this result and Theorem 6.1.2 is that in Theorem 6.1.2 we do not require \(\mathcal{M}\) to be left proper, we do not require the unit to be cofibrant, we do not require the model structure to be tractable, we weaken combinatoriality to a much lesser smallness hypothesis, and we weaken \((*)\) to the commutative monoid axiom. We have also discussed how to remove the monoid axiom. Note that Lurie also assumes \(\mathcal{M}\) is simplicial, but never uses this assumption. The assumption that the unit is cofibrant is part of what Lurie requires of a monoidal model category. However, the unit is not cofibrant in the positive and positive flat model structures on categories of spectra. For this reason, Theorem 6.1.10 cannot apply to such examples as stated, but elements of the proof have been made to apply to the positive flat stable model structure in [71].
6.2. ADDITIONAL RESULTS

We refer to condition (*) as Lurie’s hypothesis. It implies the strong commutative monoid axiom as shown in Lemma 4.3.28 of [61]. The key observation is that

\((-)/\Sigma_n : \mathcal{M}^{\Sigma_n} \to \mathcal{M}\)

is the left adjoint of a Quillen pair where the right adjoint is the constant diagram functor (i.e. endows an object with the trivial \(\Sigma_n\) action). Thus, if (*) is satisfied and we apply this map to the projective cofibration \(f^{\square n}\) we obtain the strong commutative monoid axiom. However, (*) assumes strictly more than the strong commutative monoid axiom, as evidenced in Section 6.3 where we show that simplicial sets and topological spaces satisfy the latter but not the former.

Note that Lurie’s Proposition 4.3.21 is slightly more general than what we’ve stated above in that it only requires that there is some combinatorial model structure \(\mathcal{M}_V\) on the relative category \(\mathcal{M}\), and that \(\mathcal{M}_V\) has cofibrations \(V\) generated by cofibrations between cofibrant objects and satisfying (*). In this case \(\mathcal{M}\) is said to be freely powered by \(V\). We could also do our work in that level of generality, but choose not to because it seems unnatural to place a hypothesis on a model category which references the existence of some other model category. The point is that this extra generality does not buy us anything because \(\mathcal{M}\) and \(\mathcal{M}_V\) will be Quillen equivalent by Lurie’s Remark 4.3.20.

Lurie does not prove that it is sufficient to check hypothesis (*) on the generating (trivial) cofibrations, but this has been done in [71].

6.2. ADDITIONAL RESULTS

6.2.1. Functoriality and Homotopy Invariance. We turn now to the question of whether or not the passage from \(R\) to \(CAlg(R)\) is functorial and has good homotopy
theoretic properties. Following [81], we provide a condition so that the homotopy theory of commutative $R$-algebras only depends on the weak equivalence type of $R$.

**Theorem 6.2.1.** Suppose $M$ satisfies the conditions of Theorem 6.1.2.

1. The passage from $R$ to $CAlg(R)$ is functorial: given a ring homomorphism $f : R \to T$, restriction and extension of scalars provides a Quillen adjunction between $CAlg(R)$ and $CAlg(T)$.

2. Suppose that for any cofibrant left $R$-module $N$, the functor $N \otimes_R -$ preserves weak equivalences. Let $f : R \to T$ be a weak equivalence of commutative monoids. Then $f$ induces a Quillen equivalence $CAlg(R) \simeq CAlg(T)$.

**Proof.** Let $f : R \to T$ be a ring homomorphism.

1. The map $f$ makes $T$ into an $R$-module, and provides the extension of scalars functor from $CAlg(R)$ to $CAlg(T)$, i.e. $N \simeq R \otimes_R N \to T \otimes_R N$. Because weak equivalences and fibrations are defined in the underlying category, the right adjoint restriction functor preserves (trivial) fibrations. So they form a Quillen pair and the extension functor preserves (trivial) cofibrations.

2. To check that extension and restriction form a Quillen equivalence in this case, we use Corollary 1.3.16(c) of [46]. First, note that restriction reflects weak equivalences between fibrant objects because the weak equivalences and fibrations in these two categories are the same. Next, suppose $N$ is a cofibrant commutative $R$-algebra. The map $N \simeq R \otimes_R N \to T \otimes_R N$ is a weak equivalence because cofibrant objects are flat. Thus, restriction and extension of scalars form a Quillen equivalence.
An alternative approach for (2) which avoids the need for cofibrant $R$-modules to be flat is suggested by Theorem 2.4 of [45] in the non-commutative case. Note that we do not require the unit to be cofibrant as Hovey did. This is because we do not obtain our model structure on $CAlg(R)$ from $R$-mod. Rather, we obtain it as the undercategory of $CMon(M)$. Via Remark 6.1.9 we may view the generating cofibrations of $CAlg(R)$ as $R \otimes \text{Sym}(I)$ where $I$ is the set of generating cofibrations for $M$.

**Theorem 6.2.2.** Suppose $M$ has a cofibrant unit, satisfies the commutative monoid axiom, and that the domains of the generating cofibrations are cofibrant. Suppose $R$ and $T$ are commutative monoids which are cofibrant in $M$ and suppose $f : R \rightarrow T$ is a weak equivalence. Then extension and restriction of scalars is a Quillen equivalence between $CAlg(R)$ and $CAlg(T)$.

**Proof.** We follow the model of Hovey’s proof in [45]. All that must be shown is that for all cofibrant $R$-modules $M$, $M \rightarrow M \otimes_R T$ is a weak equivalence. Because $M$ is cofibrant we may write $M = \text{colim} M_\alpha$ where $M_0 = 0$ and $M_\alpha \rightarrow M_{\alpha+1}$ is a pushout of a map in $R \otimes \text{Sym}(I)$. For concreteness we will let $K \rightarrow L$ denote the map in $I$ which is used in this pushout.

We show by transfinite induction that $M_\alpha \rightarrow M_\alpha \otimes_R T$ is a weak equivalence for all $\alpha$. The base case is trivial because $M_0 = 0$. For the successor case, apply the left adjoint $- \otimes_R T$ to the pushout square defining $M_\alpha \rightarrow M_{\alpha+1}$ and the result will again be a pushout square. There is also a map from the former pushout square to the latter, induced by the adjunction. We will apply the Cube Lemma (Lemma 5.2.6 in [46]) to the resulting
Here we have canceled $R \otimes_R (\_)$ terms in the right-hand square. Because $\mathcal{M}$ has the commutative monoid axiom and a cofibrant unit, the cofibrancy of $K$ and $L$ implies the cofibrancy of $\text{Sym}(K)$ and $\text{Sym}(L)$ in $\mathcal{M}$. Thus, by Lemma 1.1.12 in [46], smashing with these objects preserves weak equivalences between cofibrant objects, so when we apply this to the weak equivalence $R \to T$ the maps on the upper left and upper right corners in the squares above are weak equivalences. Similarly, the map $\text{Sym}(K) \to \text{Sym}(L)$ is a cofibration and so because $R$ and $T$ are cofibrant the horizontal maps across the top are cofibrations (and hence the bottom horizontals as well, because the are pushouts of cofibrations).

Because all maps $M_\alpha \to M_{\alpha+1}$ are cofibrations and because $M_0$ is cofibrant, all $M_\alpha$ are cofibrant. Because extension of scalars is left Quillen, the objects in the second square are cofibrant. The inductive hypothesis tells us that the map on the lower left corner is a weak equivalence. The Cube Lemma then guarantees us that the map on the lower right corner is a weak equivalence.
For the limit ordinal case, assume that $M_\alpha \to M_\alpha \otimes_R T$ is a weak equivalence for all $\alpha < \lambda$. Then we have a map of sequences

$$
\begin{align*}
&\quad M_0 \quad \longrightarrow \quad M_1 \quad \longrightarrow \quad \cdots \quad \longrightarrow \quad M_\alpha \quad \longrightarrow \quad \cdots \\
&\downarrow \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
&\quad M_0 \otimes_R T \quad \longrightarrow \quad M_1 \otimes_R T \quad \longrightarrow \quad \cdots \quad \longrightarrow \quad M_\alpha \otimes_R T \quad \longrightarrow \quad \cdots
\end{align*}
$$

where all vertical maps are weak equivalences and all all horizontal maps are cofibrations of cofibrant objects. So Proposition 18.4.1 in [42] proves the colimit map $M_\lambda \to M_\lambda \otimes_R T$ is a weak equivalence as well.

\[\square\]

Hovey provides a counterexample which demonstrates that for non-cofibrant $R$ and $T$, and without the hypothesis that cofibrant $R$-modules are flat, it is not true that $R \simeq T$ induces a Quillen equivalence of categories of modules. We do not know whether or not Hovey’s example can be generalized to the case of algebras rather than modules. We do know that the spaces considered in Hovey’s example cannot provide such a counterexample for the question of Quillen equivalence between $CAlg(R)$ and $CAlg(T)$, because commutative monoids in $Top$ are generalized Eilenberg-Mac Lane spaces (as discussed in Example 6.2.4).

The author does not know whether or not it is possible to prove homotopy invariance of $CAlg(R)$ without the hypothesis that cofibrant objects are flat and without having to assume the objects $R$ and $T$ are cofibrant. Note that Corollary 2.4 of [11] does not apply here because the operads $Com$, $Com_R$, and $Com_T$ are not $\Sigma$-cofibrant.
Remark 6.2.3. The results in this section also hold in the absence of the monoid axiom. By Corollary 6.1.8, categories of commutative algebras form semi-model categories and the output of the theorem is a Quillen equivalence of semi-model categories. To see this one need only note that the monoid axiom is not used in the proof, and that the semi-model category analog of 1.3.16 in [46] can be found in Section 12.1.8 of [29].

6.2.2. Rectification. We turn next to the question of rectification. As discussed in [85], categories of algebras over cofibrant operads inherit model structures whenever the monoid axiom is satisfied. Thus, $E_\infty$-algebras in $\mathcal{M}$ will always inherit a model structure in our set-up. There is a weak equivalence $\phi : E_\infty \to Com$, so it is natural to ask whether or not the pair $(\phi^*, \phi_!)$ forms a Quillen equivalence between $E_\infty$-algebras and $Com$-algebras. If there is, then rectification is said to occur.

In all model categories of spectra, if commutative ring spectra form a model category then it is Quillen equivalent to $E_\infty$-algebras. We do not have a statement of this kind for general model categories, and in fact the following counterexample demonstrates that this does not come for free:

Example 6.2.4. Let $\mathcal{M}$ be simplicial sets or topological spaces. We will see in the next section that $\mathcal{M}$ satisfies the strong commutative monoid axiom. The monoid axiom and requisite smallness were verified in [81] for simplicial sets, in [45] for compactly generated spaces, and in [13] for $k$-spaces. Thus, commutative monoids inherit a model structure.

For topological spaces the path connected commutative monoids are weakly equivalent to generalized Eilenberg-Mac Lane spaces, i.e. products of Eilenberg-Mac Lane
spaces. The fact that there are spaces like $QS = \Omega^\infty \Sigma^\infty S^0$ which is $E_\infty$ but is not a GEM demonstrates that rectification between $E_\infty$ and $Com$ fails for spaces.

The rectification results of [11] are phrased so as to apply for very general model categories $\mathcal{M}$, including simplicial sets. However, these results do not apply because $Com$ is not a $\Sigma$-cofibrant operad. If $\mathcal{M}$ satisfies Harper’s hypothesis that all symmetric sequences are projectively cofibrant (e.g. if $\mathcal{M} = Ch(k)$ for $k$ a $\mathbb{Q}$-algebra), then $Com$ is $\Sigma$-cofibrant and so rectification holds.

The key property possessed by good monoidal categories of spectra which allows rectification is

$$(**): \forall$ cofibrant $X$, the map $(E\Sigma_n)_+ \wedge \Sigma_n X^\wedge n \to X^\wedge n / \Sigma_n$ is a weak equivalence.

This property is certainly related to the commutative monoid axiom, but it is not necessary for strictly commutative monoids to inherit a model structure. In particular, it fails for simplicial sets and for topological spaces, so it is impossible to deduce property $(**)$ from the commutative monoid axiom. We now record the correct analogue of this property $(**)$ in general model categories:

**Definition 6.2.5.** Let $\mathcal{M}$ be a monoidal model category which is a $\mathcal{D}$-model category in the sense of [46]. View the unit $S$ of $\mathcal{D}$ as an object in $\mathcal{D}^{\Sigma_n}$ with the trivial $\Sigma_n$ action. Let $q : Q_{\Sigma_n} S \to S$ be cofibrant replacement in $\mathcal{D}^{\Sigma_n}$. Then $\mathcal{M}$ is said to satisfy the **rectification axiom** with respect to operads valued in $\mathcal{D}$ if for all cofibrant $X$ in $\mathcal{M}$, the natural map $Q_{\Sigma_n} S \otimes_{\Sigma_n} X^{\otimes n} \to X^{\otimes n} / \Sigma_n$ is a weak equivalence.

A similar property, requiring certain homotopy orbits to be weakly equivalent to orbits, appears in the axiomatization of good model structures of spectra given by [30].
However, in [30], this condition is equivalent to the condition that all simplicial operads are admissible, and as we have seen that will not be true for general model categories. We pause now to record a proposition about the interplay between the rectification axiom and the commutative monoid axiom which we shall use in Section 6.3.

**Proposition 6.2.6.** Suppose $M$ is a monoidal model category satisfying the rectification axiom. Then $\text{Sym}^n(-)$ takes trivial cofibrations between cofibrant objects to weak equivalences.

**Proof.** Let $f : A \to B$ be a trivial cofibration between cofibrant objects. Note that $f^{\otimes n} : (A)^{\otimes n} \to (B)^{\otimes n}$ is a trivial cofibration in $M$ because it is the composite $A^{\otimes n} \to A^{\otimes n-1} \otimes B \to A^{\otimes n-2} \otimes B^{\otimes 2} \to \cdots \to B^{\otimes n}$. This follows by iteratively applying the fact that $A \otimes -$ and $B \otimes -$ are left Quillen functors.

Furthermore, $Q_{\Sigma_n} S$ is $\Sigma_n$-cofibrant and so when we take the pushout product of $A^{\otimes n} \to B^{\otimes n}$ with $\emptyset \to Q_{\Sigma_n} S$ we obtain a $\Sigma_n$-trivial cofibration, e.g. by Lemma 2.5.2 in [10]. When we pass to $\Sigma_n$-coinvariants we obtain a trivial cofibration $A^{\otimes n} \otimes_{\Sigma_n} Q_{\Sigma_n} S \to B^{\otimes n} \otimes_{\Sigma_n} Q_{\Sigma_n} S$ because $(-)/\Sigma_n$ is left Quillen. Consider the following commutative square, where the bottom horizontal map is $\text{Sym}^n(f)$, the top horizontal map is the map we have just described, and the vertical maps are induced by $Q_{\Sigma_n} S \to S$ and by passage to $\Sigma_n$-coinvariants:

$$
\begin{array}{ccc}
Q_{\Sigma_n} \otimes_{\Sigma_n} A^{\otimes n} & \longrightarrow & Q_{\Sigma_n} \otimes_{\Sigma_n} B^{\otimes n} \\
\downarrow & & \downarrow \\
A^{\otimes m}/\Sigma_m & \longrightarrow & B^{\otimes m}/\Sigma_m
\end{array}
$$
We have shown the top vertical map is a weak equivalence. The vertical maps are weak equivalences by the rectification axiom. Thus, the bottom horizontal map is a weak equivalence by the two out of three property.

In situations arising from topology, where $\mathcal{M}$ is spectra and $\mathcal{D}$ is spaces, the map $Q_{\Sigma_n} S \rightarrow S$ is the cofibrant replacement of the point and so is $E\Sigma_n \rightarrow \ast$ in the unpointed setting and $(E\Sigma_n)_+ \rightarrow S^0$ in the pointed setting. This proposition is used in Section 6.3 to make sure that a particular Bousfield localization respects the commutative monoid axiom.

We have not undertaken a general study of when rectification between $\text{Com}$ and $E_\infty$ holds. The interested reader is encouraged to consult [33] and [77] for more information about rectification for general model categories. We hope to return to the subject of rectification in future work, and in particular to study the effect of applying Bousfield localization to force the Rectification Axiom to hold.

6.3. Examples

In this section we verify the strong commutative monoid axiom for the model categories of chain complexes over a field of characteristic zero, for simplicial sets, for topological spaces, and for positive flat model structures on various categories of spectra. We also discuss precisely what is true for positive (non-flat) model structures of spectra. Throughout this section we make use the following lemma, which is proven in Section 6.4.
Lemma 6.3.1. Suppose $\mathcal{M}$ is a cofibrantly generated monoidal model category and that for all $f \in I$ (resp. $J$) we know that $f^\otimes /\Sigma_n$ is a (trivial) cofibration. Then the strong commutative monoid axiom holds for $\mathcal{M}$.


Consider a field $k$ and $\mathcal{M} = \text{Vect}(k)$. Then $\mathcal{M}$ satisfies the strong commutative monoid axiom if and only if $\text{char}(k) = 0$. Because $\mathcal{M}^\Sigma_n \cong k[\Sigma_n] - \text{mod}$, the projective model structure is nicely behaved (i.e. matches the injective model structure) exactly when $k[\Sigma_n]$ is semisimple, i.e. exactly when $k$ has characteristic zero. Indeed, such $\mathcal{M}$ satisfies the stronger condition required in Theorem 6.1.10. This example generalizes to pertain to $\text{Ch}(R)$ whenever $R$ is a commutative $\mathbb{Q}$-algebra.

The commutative monoid axiom fails over $\mathbb{F}_2$ because $\mathbb{F}_2[\Sigma_2]$ is not projective over $\mathbb{F}_2$ (because now Maschke’s Theorem does not hold) and so the cokernel of $f^\otimes$ does not have a free $\Sigma_n$ action, and this will be an obstruction to $f^\otimes /\Sigma_n$ being a cofibration.

That $\text{CDGA}(k)$ cannot inherit a model structure for $\text{char}(k) = p > 0$ has been known for many years. The fundamental problem is that $\text{Sym}(-)$ does not preserve weak equivalences between cofibrant objects and so cannot be a left Quillen functor. This is because for example $\text{Sym}^p(D(k))$ will not be acyclic even though the disk $D(k)$ is acyclic.

6.3.2. Spaces.

Theorem 6.3.2. The category of simplicial sets satisfies the strong commutative monoid axiom but does not satisfy Lurie’s axiom or the rectification axiom.
Proof. To see that the rectification axiom fails, consider $X = \Delta[0]$. Then the rectification axiom is asking $B\Sigma_n$ to be contractible. To see that Lurie’s axiom fails, consider $f^{G_2}$ where $f : S^0 \to D^1$. This map is not a $\Sigma_2$-cofibration because the action on the cofiber of $f^{G_2}$ is not free. However, to show that we get a cofibration after passing to $\Sigma_2$ coinvariants is easy, because the map is a monomorphism. Furthermore, this line of reasoning generalizes to show that $f^{G_n}/\Sigma_n$ is a cofibration whenever $f$ is a generating (trivial) cofibration. To check that it’s also a weak equivalence if $f$ is a generating trivial cofibration, we use the following theorem of Casacuberta [20]:

**Theorem 6.3.3.** If $f$ is any map in $sSet$, then $\text{Sym}(\cdot)$ preserves $f$-equivalences.

Obviously, this proves much more than we needed, and in fact we use the proof of this theorem in Chapter 7 to see that any monoidal Bousfield localization of $sSet$ also satisfies the strong commutative monoid axiom. The key point in the proof of this theorem is due to an observation of Farjoun [27] which says that for any $X$, $\text{Sym}^n(X)$ can be written as a homotopy colimit of a free diagram formed by the orbits of $\Sigma_n$ where each quotient $\Sigma_n/H$ is sent to the fixed-point subspace $(X^n)^H$. It is then not too much work to see that $\text{Sym}^n(\cdot)$ preserves weak equivalences (and more generally $f$-equivalences). We refer the reader to [20] for the details.

\[\square\]

Observe that the counterexample displaying the failure of Lurie’s axiom and the rectification axiom also applies to $\text{Top}$, $sSet^G$, and $\text{Top}^G$.

**Theorem 6.3.4.** The category of compactly generated topological spaces satisfies the strong commutative monoid axiom.
Proof. In $\text{Top}$, cofibrations are no longer monomorphisms, but the strong commutative monoid axiom still holds. This may be verified by either checking it directly on the generating maps $S^{n-1} \to D^n$ and $D^n \to D^n \times [0,1]$ (a valuable exercise), or by transporting the strong commutative monoid axiom on $\text{sSet}$ to $\text{Top}$ via the geometric realization functor. From [45] we see that $\text{Top}$ satisfies the necessary smallness hypotheses, so Theorem 6.1.2 applies.

In case the reader is interested in checking the commutative monoid axiom on $\text{Top}$ directly, we remark that the interpretation of Farjoun’s work in [20] makes clear that the only property of simplicial sets being used in the argument is that the fixed point subspaces of actions of subgroups of $\Sigma_k$ on $X^k$ are homeomorphic to spaces $X^n$ for some $n \leq k$. So one could apply Farjoun’s work just as well in $\text{Top}$ as in $\text{sSet}$. Indeed, Farjoun’s work provides a way to “free up” any diagram category and view the colimit of a diagram as the homotopy colimit of a different diagram (indexed by the so-called orbit category). In this way good homotopical properties can be achieved in a great deal of generality. The fact that the same argument works in both $\text{Top}$ and $\text{sSet}$ leads us to make the following conjecture.

Conjecture 6.3.5. Suppose that $\mathcal{M}$ is a concretizable Cartesian closed model category in which cofibrations are closed under the operation $(-)^{\Sigma_n}/\Sigma_n$. Then the strong commutative monoid axiom holds in $\mathcal{M}$.

We now turn to equivariant spaces.

Theorem 6.3.6. Let $G$ be a finite group. Then $\text{sSet}^G$ and $\text{Top}^G$ satisfy the strong commutative monoid axiom.
Proof. We begin with $sSet^G$. Note that just as for $sSet$, cofibrations are monomorphisms. Thus, the same proof as for $sSet$ applies. In particular, when applying Farjoun’s trick on $(X^n)^H$ where $H < \Sigma_n$, we simply use the fact that the $G$ action and the $\Sigma_n$ action commute.

To handle the situation of $Top^G$ we may again transfer the strong commutative monoid axiom via geometric realization. Here we really need $G$ to be a finite group. For any simplicial group $G$, a $G$ action on $X \in sSet$ is taken to an action of $|G|$ on $|X|$ by geometric realization. If $G$ is finite then $G = \text{Sing}|G|$ acts on $\text{Sing}|X|$ and we can prove $sSet^G$ is Quillen equivalent to $Top^{[G]}$. However, for non-finite $G$ we do not know whether or not every subgroup $K$ of the topological group $|G|$ is realized as some $|H|$ for $H < G$, so there may be fewer weak equivalences in $Top^{[G]}$ than in $sSet^G$.

□

6.3.3. Symmetric Spectra. The obstruction noticed by Gaunce Lewis and discussed in [57] guarantees that commutative monoids in the usual model structure on symmetric spectra cannot inherit a model structure, because the unit is cofibrant and because the fibrant replacement functor is symmetric monoidal. This second property cannot be changed, but there are model structures on symmetric spectra in which the unit is not cofibrant. The positive model structure was introduced in [50] and [66] and this model structure breaks the cofibrancy of the sphere by insisting that cofibrations be isomorphisms in level 0 (though in other levels they are the same as the usual cofibrations of symmetric spectra). In [83], Shipley found a more convenient model structure which is now called the positive flat model structure. In this model structure the cofibrations are enlarged to contain the monomorphisms, and then the condition in
level 0 is applied. The result is a model structure in which commutative ring spectra inherit a model structure and in which cofibrations of commutative ring spectra forget to cofibrations of spectra.

Note that in [61], Lurie’s axiom is claimed to hold for positive flat symmetric spectra. This is an error, as acknowledged in [63]. Indeed, the example given in Proposition 4.2 of [83] demonstrates this failure conclusively, for both the positive and the positive flat model structures. We will now show that the commutative monoid axiom holds for positive flat (stable) symmetric spectra, and a slight weakening holds for positive (stable) symmetric spectra.

6.3.3.1. Positive Flat Stable Model Structure.

**Theorem 6.3.7.** The strong commutative monoid axiom holds for the positive flat stable model structure on symmetric spectra.

**Proof.** By Lemma 6.3.1, it’s sufficient to check the strong commutative monoid axiom on the generating (trivial) cofibrations. We focus first on the the generating cofibrations, which take the form \( SI^\ell \times S \otimes I^\ell = S \otimes \bigcup_{m>0} G_m(I_{\Sigma_m}) \) where \( G_m \) is the left adjoint to \( Ev_m \) and \( I_{\Sigma_m} \) is the set of generating cofibrations for \( sSet^{\Sigma_m} \). First observe that \((S \otimes f)^{G_m} \) is itself an iterated pushout product of \( f \) with \( \emptyset \to S \) and because the pushout product is symmetric we can pull the \( S \)'s to one side. There we can smash them together because \( S \) is the unit. So \((S \otimes f)^{G_m} \cong S \otimes f^{G_m} \).

Next, \( f \) is some \( G_m(i) \) where \( i : \partial \Delta \to \Delta \). We will prove that \( f^{G_m} \) is \( G_m(i^{G_m}) \). First observe that the domain of \( f^{G_m} \) is a colimit built out of terms of the form \( G_m(X)^i \wedge G_m(Y)^k \), and the codomain is \( G_m(Y)^n \). We next use Proposition 2.2.6 in [50] to rewrite...
$G_m(X) \land G_m(Y)$ as $G_{2m}(X \land Y)$. So the domain can be written as colimit built from terms of the form $G_{nm}(X^j \land Y^k)$ and the range as $G_{nm}(Y^n)$.

Finally, $G_{nm}$ commutes with colimits because it is a left adjoint, so the map $f^{\boxtimes n}$ takes the form $G_{nm}(i^{\boxtimes n}) : G_{nm}(Q_n) \to G_{nm}(Y^{\boxtimes n})$. Thus, because spaces and $\Sigma_n$-spaces satisfy the strong commutative monoid axiom, $i^{\boxtimes n}/\Sigma_n$ is a (trivial) cofibration if $i$ is. Because $G_{nm}$ is left Quillen and commutes with $(-)/\Sigma_n$ we’re done. In particular, Lemma 6.3.1 now implies the class of cofibrations is closed under the operation $(-)^{\boxtimes n}/\Sigma_n$, in either the levelwise or stable model structures. The same argument works for maps in $S J^\ell_+$ (the generating trivial cofibrations for the levelwise model structure), and proves that applying $(-)^{\boxtimes n}/\Sigma_n$ takes such maps to trivial cofibrations.

To complete the proof we must now prove the commutative monoid axiom is satisfied by the other maps in the set of generating trivial cofibrations for the stable model structure. Recall from Theorem 2.4 in [83] and from Definition 3.4.9 in [50] that the stable model structure is obtained via a Bousfield localization of the levelwise model structure with respect to the set of maps $C = \{s_m : G_{m+1}S^1 \to G_mS^0\}$, where $s_m$ is adjoint to the identity map of $S^1$. One could equivalently invert maps in $C' = \{Qf \mid f \in C\}$, where $Qf$ is chosen to be a cofibrant replacement that replaces maps with cofibrations. The generating trivial cofibrations take the form $S^+J = SJ^\ell_+ \cup K$ where $K = \bigcup_{m>0} K_m$, $K_m = c_m \Box I$, and $c_m$ is a stable cofibrant replacement of $s_m$.

Every $f \in K$ is a cofibrations, so $f^{\boxtimes n}/\Sigma_n$ is still a cofibration. That it is also a stable equivalence follows from Theorem 7.1.1. To see that $\text{Sym}^n(s_m)$ is a $C$-local equivalence for all $n, m$, apply Proposition 7.2.2 with the input model category being $L_{C'}(Sp_+)$ (where $Sp_+$ is the positive flat model structure), with $D = sSet$, and using the
observation that in $L_{C'}(Sp_+)$ the maps in $C'$ are trivial cofibrations between cofibrant objects. Observe that this proof is also using the fact that $L_{C'}$ respects the monoidal structure on $Sp_+$, but this can easily be checked using Theorem 4.2.2. This completes the proof.

\[\square\]

We remark that this theorem together with Lewis’s example demonstrate that the commutative monoid axiom need not be preserved by monoidal Quillen equivalences, since the positive flat stable model structure is monoidally Quillen equivalent to the canonical stable model structure. This can be seen via Proposition 2.8 in [83], together with the fact that stable cofibrations are contained in flat cofibrations (Lemma 2.3 in [83]) and the fact that the two model structures have the same weak equivalences. We do not know of a similar example which would demonstrate that the monoid axiom need not be preserved by monoidal Quillen equivalence.

6.3.3.2. Positive Stable Model Structure. Shipley proves in [83] that positive symmetric spectra do not satisfy the property that cofibrations of commutative monoids forget to cofibrations of symmetric spectra. Thus, this model structure cannot satisfy the strong commutative monoid axiom. However, Proposition 4.2 in [83] proves that a cofibration of commutative $R$-algebras forgets to a positive $R$-cofibration (and hence to an $R$-cofibration) even though it is not a positive cofibration in the sense of [66]. This suggests the following result:

**Proposition 6.3.8.** Let $f$ be a (trivial) cofibration in the positive stable model structure. Then $\Sigma f^\otimes n/\Sigma n$ is a (trivial) cofibration in the positive flat stable model structure.
Furthermore, commutative monoids inherit a model structure in the positive stable model structure.

Proof. The proof is identical to the proof that the positive flat stable model structure satisfies the strong commutative monoid axiom. This is because positive cofibrations form a subclass of positive flat cofibrations. For the statement regarding trivial cofibrations, the same logic used above holds, because it is a Bousfield localization with respect to the same class of maps, and the weak equivalences of both the positive stable and positive flat stable model structures are the same. In particular, this observation proves that the positive (stable) model structures satisfy the weak form of the commutative monoid axiom discussed in Remark 6.1.3, so commutative monoids inherit a model structure.

Shipley provides a counterexample which demonstrates that $\text{Sym}(F_1S^1)$ is not positively cofibrant (only positively flat cofibrant) because $[(F_1S^1)^{(2)}/\Sigma_2]_2 = (S^1 \wedge S^1)/\Sigma_2$ and this is not $\Sigma_2$-free. Thus, Proposition 6.1.3 cannot hold as stated. However, for the same reasons as in the proof above (namely, the containment of positive cofibrations in positive flat cofibrations) we can obtain the following weakened form of Proposition 6.1.3.

Proposition 6.3.9. Let $\mathcal{M}$ be the positive stable model structure on symmetric spectra, and let $\text{CAlg}(R)$ be the model structure passed from $\mathcal{M}$ to the category of commutative $R$-algebras (where $R$ is a commutative monoid in $\mathcal{M}$). Suppose $f$ is a cofibration in $\text{CAlg}(R)$ whose source is cofibrant in $\mathcal{M}$. Then $f$ forgets to a cofibration in the positive flat stable model structure.
6.3.4. General Diagram Spectra. In [66], a general theory of diagram spectra is introduced which unifies the theories of $S$-modules, symmetric spectra, orthogonal spectra, $\Gamma$-spaces, and $W$-spaces. For the first, homotopy-coherence is built into the smash product, so commutative monoids immediately inherit a model structure and there is rectification between $Com$-alg and $E_\infty$-alg. For the next two, positive model structures are introduced which allow strictly commutative monoids to inherit model structures. The rectification axiom is then proved and rectification is deduced as a result.

Theorem 6.3.10. The positive flat stable model structure on (equivariant) orthogonal spectra satisfies the strong commutative monoid axiom and the rectification axiom. The positive stable model structure satisfies the weak commutative monoid axiom, Proposition 6.3.8 and Proposition 6.3.9.

Proof. For the positive flat stable model structure on (equivariant) orthogonal spectra, proceed as in the proof of Theorem 6.3.7 but using (equivariant) topological spaces rather than simplicial sets. The rectification axiom is proven in [66] (and in [14] for the equivariant case). For the positive stable model structure proceed as in Proposition 6.3.8 and Proposition 6.3.9.

We turn now to $W$-spaces and $\Gamma$-spaces. Recall that $W$ is the category of based spaces homeomorphic to finite CW-complexes, $\Gamma$ is the category of finite based sets, and $D$-spaces are functors from $D$ to $Top$ (where $D$ is either $W$ or $\Gamma$). The indexing category for $\Gamma$-spaces is a subset of $W$. First, Lewis’s counterexample [57] still applies to rule non-positive model structures out from consideration. This is discussed in the context
of Γ-spaces in Remark 2.6 of [80]. The author has not been able to find a place where this is written down for W-spaces, but it is clear that the same counterexample applies for W-spaces. We must work in positive model structures on W-spaces and Γ-spaces. Such positive model structure are introduced in Section 14 of [66].

Positive flat model structures (also known as convenient model structures) on Γ-spaces and W-spaces may be constructed in the same way as for symmetric spectra and orthogonal spectra. For instance, one can carry out the program of [83] for Γ-spaces (e.g. following the work in [76] and making use of the relationship between Γ-spaces and symmetric spectra as explored in [79]) to obtain the necessary mixed model structure on spaces. From there it is purely formal to construct the appropriate levelwise model structure on diagrams, e.g. using Theorem 6.5 in [66]. The generating cofibrations for W-spaces take the form $F_W I = \{ F_d(i) \mid d \in \text{ske} W, i \in I \}$ where $F_d(-)$ is $W(d,-) : W \to \text{Top}$. The indexing category for Γ-spaces is a subset of W, so an analogous construction works for W-spaces.

The monoidal product is computed levelwise. Passage from the levelwise structure to the positive flat model structure is again formal, and is accomplished by truncating the levelwise cofibrations to force levelwise cofibrations to be isomorphisms in degree 0. Finally, passage to the positive flat stable model structure may be accomplished via Bousfield localization, just as in Section 8 of [66].

**Theorem 6.3.11.** The positive flat model structures on W-spaces and Γ-spaces satisfy the strong commutative monoid axiom. The positive model structure on W-spaces and Γ-spaces satisfies the weak commutative monoid axiom. So commutative monoids inherit model structures in both settings.
The verification of the strong commutative monoid axiom proceeds precisely as for the positive flat model structure on symmetric spectra. In particular, one can reduce the verification to a verification in spaces. We leave the details to the reader. The difficulty comes in the part of the proof when one attempts to pass the commutative monoid axiom to the stable model structure, and that is why the adjective stable is not in the statement of the theorem. In particular, the difficulty is that the rectification axiom is not known to hold for \( \mathcal{D} \)-spaces (where \( \mathcal{D} \) is either \( W \) or \( \Gamma \)). Indeed, we can show that the rectification axiom cannot hold.

First, if the rectification axiom held, then the proof that the strong commutative monoid axiom holds for positive flat stable symmetric spectra (i.e. via Theorem 7.1.1) would prove that \( \mathcal{D} \)-spaces satisfy the commutative monoid axiom. Secondly, because of the rectification axiom the rest of the work in [66] and [83] would prove that commutative \( \mathcal{D} \)-rings were Quillen equivalent to \( E_{\infty} \)-algebras and this would contradict the main theorem of Tyler Lawson’s paper [54].

Lawson produces an \( E_{\infty} \)-algebra in \( \Gamma \)-spaces which cannot be strictified to a commutative \( \Gamma \)-ring. Together with the monoidal functor from \( \Gamma \)-spaces to \( W \)-spaces (developed in [66]), this same counterexample proves that not all \( E_{\infty} \)-algebras in \( W \)-spaces can be strictified to commutative \( W \)-rings.

6.3.5. Other Examples. We have not included a proof that simplicial presheaves satisfy the strong commutative monoid axiom, but this should follow from general facts about diagram categories. Indeed, we hope that this can generalize further to the so-called excellent model categories introduced in [61].
We have not addressed positive model structures on motivic symmetric spectra, but we will say a word about this example in Chapter 9.

There are several other examples which we have not investigated and which we would be curious to learn more about. We list them here:

- Stable module categories over \( \mathbb{Q} \)-algebras.
- Comodules over a Hopf algebroid
- The model for spectra consisting of simplicial functors, in the style of [64].

### 6.4. Sufficiency of Commutative Monoid Axiom on Generators

We prove that if the strong commutative monoid axiom holds for the generating (trivial) cofibrations \( I \) and \( J \) then it holds for all (trivial) cofibrations. This proof is highly technical, and that is why we have placed it in its own section.

**Lemma 6.4.1.** Suppose \( \mathcal{M} \) is a cofibrantly generated monoidal model category and that for all \( f \in I \) (resp. \( J \)) we know that \( f^{\Sigma_n}/\Sigma_n \) is a (trivial) cofibration. Then the strong commutative monoid axiom holds for \( \mathcal{M} \).

We will prove that the class of maps satisfying the condition in the strong commutative monoid axiom is closed under retracts, pushouts, and transfinite compositions. The first two are easy, but the third will require an induction. So we must introduce some new notation, following [36]. Let \( f : X \to Y \) and consider the \( n \)-dimensional cube in which each vertex is a word of length \( n \) on the letters \( X \) and \( Y \).

Recall the action of \( \Sigma_n \) on the diagram which defines \( Q_n \). The vertices of the cube correspond to subsets \( D \) of \( [n] = \{1, 2, \ldots, n\} \) where a vertex \( C_1 \otimes \cdots \otimes C_n \) has \( C_i = X \) if \( i \notin D \) and \( C_j = Y \) if \( j \in D \). Any \( \sigma \in \Sigma_n \) sends the vertex so defined to the vertex
corresponding to $\sigma(D) \subset [n]$ using the action of $\Sigma_{|D|}$ on the $X$’s and $\Sigma_{n-|D|}$ on the $Y$’s. Clearly, this action descends to an action on the colimit $Q_n$.

For inductive purposes, we will need to consider subdiagrams whose vertices consist of words with $\leq q$ copies of the letter $Y$. This subdiagram consists of all vertices of distance $\leq q$ from the initial vertex $X_{\otimes n}$.

We denote the colimit of this subdiagram by $Q^n_q$. The superscript $n$ refers to the fact that this is a subdiagram of the $n$-dimensional cube, so in particular each vertex is a word on $n$ letters. In particular, $Q^n_0 = X_{\otimes n}$ and $Q^n_n = Y_{\otimes n}$. Observe that $Q^n_{n-1}$ is the domain of $f^{\otimes n}$, which we have formerly denoted by $Q_n$. For the purposes of this proof we will now write it as $Q^n_{n-1}(f)$ (or $Q^n_{n-1}$ if the context is clear).

The induction will make use of the maps of colimits $Q^n_{q-1} \to Q^n_q$ which are induced by inclusion of subdiagram. The $\Sigma_n$ action on the cube clearly preserves the size of the subset $D \subset [n]$ and so it restricts to an action of $\Sigma_n$ on each $Q^n_q$. Because this action is a restriction of the $\Sigma_n$-action on the full cube, the map of colimits $Q^n_{q-1} \to Q^n_q$ is automatically $\Sigma_n$-equivariant. Indeed, the map of colimits $Q^n_{q-1} \to Q^n_q$ can be realized by the following pushout:

\[
\begin{array}{ccc}
\Sigma_n \cdot \Sigma_{n-q} \times \Sigma_q X_{\otimes (n-q)} \otimes Q^n_{q-1} & \longrightarrow & Q^n_{q-1} \\
\downarrow & & \downarrow \\
\Sigma_n \cdot \Sigma_{n-q} \times \Sigma_q X_{\otimes (n-q)} \otimes Y_{\otimes q} & \longrightarrow & Q^n_q
\end{array}
\]

(6.1)

where the left vertical map is induced by $f^{\otimes q}$ (see Section 7 of [36] and Remark 4.15 of [35] for a toy case). To explain the notation $\Sigma_n \cdot \Sigma_{n-q} \times \Sigma_q (\cdot)$, first note that for any set $G$ and any object $A$, $G \cdot A = \bigsqcup_{g \in G} A$. When $G = \Sigma_n$ this object inherits a $\Sigma_n$ action.
by permuting the $A \otimes n!$ objects in the coproduct. When we write $\Sigma_n \cdot \Sigma_k \times \Sigma_q (-)$ we are quotienting out by the $\Sigma_k \times \Sigma_q$ action on this object in $\mathcal{M}^{\Sigma_n}$. The result is a coproduct with $n!/k!/q!$ terms because the order of the $k!$ terms to the left of the product (and of the $q!$ terms to the right) do not matter. In particular, applying $\Sigma_n \cdot \Sigma_k \times \Sigma_q (-)$ has the effect of equivariantly building in additional layers of the cube. With this notation in hand we proceed to the proof.

**Proof.** Let $\mathcal{P}$ denote the class of cofibrations $f$ for which $f^\otimes n / \Sigma_n$ is also a cofibration. Let $\mathcal{P}'$ denote the same for trivial cofibrations. We must prove that if $I \subset \mathcal{P}$ then all cofibrations are in $\mathcal{P}$ (and the same for $J \subset \mathcal{P}'$). We will do so by proving the classes $\mathcal{P}$ and $\mathcal{P}'$ are closed under retracts, pushouts, and transfinite compositions.

The simplest to verify is closure under retracts, which follows from the fact that $(-)^\otimes n / \Sigma_n$ is a functor on $\text{Arr}(\mathcal{M})$ so if $f$ is a retract of $g$ (with $g \in \mathcal{P}$ or $\mathcal{P}'$) then $f^\otimes n / \Sigma_n$ is a retract of $g^\otimes n / \Sigma_n$ and hence a (trivial) cofibration.

We next consider closure under pushouts. Suppose $f : X \to Y$ is a pushout of $g : A \to B$ and $g \in \mathcal{P}$ or $\mathcal{P}'$. Then we have a $\Sigma_n$-equivariant pushout diagram

\[
\begin{array}{ccc}
Q_n(g) & \longrightarrow & B^\otimes n \\
\downarrow & & \downarrow \downarrow \searrow \\
Q_n(f) & \longrightarrow & Y^\otimes n
\end{array}
\]

by Proposition 6.13 in [35]. When we pass to $\Sigma_n$-coinvariants we see that $f^\otimes n / \Sigma_n$ is a pushout of $g^\otimes n / \Sigma_n$, e.g. by commuting colimits. Indeed, for any $X \in \mathcal{M}^{\Sigma_n}$, $X \otimes_{\Sigma_n} f^\otimes n$ is a pushout of $X \otimes_{\Sigma_n} g^\otimes n$. So if the latter is assumed to be a (trivial) cofibration because $g \in \mathcal{P}$ or $\mathcal{P}'$ then the former will be as well.
Composition is harder, so we begin with the case of two maps $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{P}$ or $\mathcal{P}'$. We will prove that $Q_{n-1}^n(gf)/\Sigma_n \to Z^{\otimes n}/\Sigma_n$ is a (trivial) cofibration. First note that this map factors through $Q_{n-1}^n(g)/\Sigma_n$ and the hypothesis on $g$ guarantees that $Q_{n-1}^n(g)/\Sigma_n \to Z^{\otimes n}/\Sigma_n$ is a (trivial) cofibration. So we must only prove that $Q_{n-1}^n(gf)/\Sigma_n \to Q_{n-1}^n(g)/\Sigma_n$ is a (trivial) cofibration.

We proceed by realizing both colimit diagrams as subdiagrams of the same diagram, which is a $n$-dimensional cube featuring $3^n$ vertices which are words of length $n$ in the letters $X, Y,$ and $Z$. Formally, this cube is an element of the rectangular diagram category $\text{Fun}((0 \to 1 \to 2)^{\times n}, \mathcal{M})$, and every time we write subdiagram we mean with respect to this cube with $3^n$ vertices. The domain $Q_{n-1}^n(gf)$ of the map we care about is the colimit of the $X - Z$ subdiagram, i.e. the punctured cube formed from vertices which are words in $X$ and $Z$, where all maps are compositions $gf$. The codomain $Q_{n-1}^n(g)$ of the map we care about is the colimit of the $Y - Z$ subdiagram, i.e. the punctured cube formed from vertices which are words in $Y$ and $Z$. So we must again introduce new notation to build this map one step at a time.

The induction will proceed by moving through the rectangle by adding a single $\Sigma_n$-orbit at a time. So we will need to consider $\Sigma_n$-equivariant subdiagrams of the rectangle which contain the $X - Z$ punctured cube and which contain a new vertex $e$ (and hence its entire $\Sigma_n$-orbit).

In order to build this new vertex into the colimit we will also need to consider the subdiagram of the $X - Y - Z$ box which maps to $e$ (but which does not include $e$ itself). This is collection of vertices sitting under $e$ (i.e. of distance strictly less than $e$ from the initial vertex). As with $e$, we wish to consider the $\Sigma_n$-orbit of this subdiagram, which is
equivalently described as all vertices sitting under any vertex in the orbit of \( e \). Now that we have a picture of the subdiagram in mind, we denote the colimit of this subdiagram by \( Q_e \). By construction there is an induced \( \Sigma_n \)-equivariant map \( Q_e \to e \).

We are now ready to consider the diagrams formed when we adjoin the \( Q_e \)-diagram with the \( X - Z \) punctured cube. Let \( Q[0]_{n-1}^n = Q_{n-1}^n(gf) \) denote the colimit of the \( X - Z \) punctured cube. Let \( Q[1]_{n-1}^n \) denote the colimit of the subdiagram containing the \( X - Z \) punctured cube, the orbit of the vertex \( e = Y \otimes Z^{\otimes (n-1)} \), and the vertices in the \( Q_e \) subdiagram. Continue inductively, by adding \( e = Y^{\otimes q} \otimes Z^{\otimes (n-q)} \) and vertices below it to the \( Q[1-1]_{n-1}^n \)-diagram to get the \( Q[q]_{n-1}^n \)-diagram. This process terminates with the whole \( X - Y - Z \) punctured cube whose \( 3^n - 1 \) vertices contain all words in \( X, Y, Z \) except the word \( Z^{\otimes n} \). The colimit of this diagram is denoted \( Q[n]_{n-1}^n \). A cofinality argument shows that this colimit is equal to \( Q_{n-1}^n(g) \), because all factors of \( X \) which appear are mapped to a factor of \( Y \) in the subdiagram and so do not affect the colimit.

The induction will proceed along the maps \( Q[1-1]_{n-1}^n \to Q[q]_{n-1}^n \) induced by containments of subdiagrams. This induction can be thought of as stepping through shells in the cube of increasing distance from the initial vertex \( X^{\otimes n} \) until the information from the entire diagram has been built into the colimit.

Because each step \( Q[q-1]_{n-1}^n \to Q[q]_{n-1}^n \) builds in the information of one new vertex (and its orbit under the \( \Sigma_n \) action on the cube), we may apply Proposition A.4 from
with \( e = Y^q \otimes Z^{(n-q)} \) to write the following pushout diagram:

\[
\begin{array}{ccc}
\Sigma_n \cdot \Sigma_q \times \Sigma_{n-q} & \longrightarrow & Q[q]_n^{n-1} \\
\downarrow & & \downarrow \\
\Sigma_n \cdot \Sigma_q \times \Sigma_{n-q} \cdot Y^q \otimes Z^{(n-q)} & \longrightarrow & Q[q + 1]_n^{n-1}
\end{array}
\]

The left vertical map is induced by \( Q_e \to Y^q \otimes Z^{(n-q)} \) and this is in turn induced by \( f^q \sqcup \Sigma_q \sqcup g^{(n-q)} \) because

\[
Q_e \cong Y^q \otimes Q_{n-q-1}^{n-q-1}(g) \bigsqcup_{Q_{q-1}^{q-1}(f) \otimes Q_{n-q-1}^{n-q-1}(g)} Q_{q-1}^{q-1}(f) \otimes Z^{(n-q)}
\]

To see that the diagram defining \( Q_e \) decomposes into a gluing of the diagrams defining \( Q_{q-1}^{q-1}(f) \otimes Z^{n-q} \) and \( Y^q \otimes Q_{n-q-1}^{n-q-1}(g) \) along the diagram defining \( Q_{q-1}^{q-1}(f) \otimes Q_{n-q-1}^{n-q-1}(g) \), note that every \( X \) in the \( Q_e \) diagram gets mapped to a \( Y \) in the \( Q_e \) diagram and so does not affect the colimit. This is the reason why we insisted upon including the vertices under \( e \) in our construction of the diagram defining \( Q_e \). Furthermore, every \( Z \) in the \( Q_e \) diagram is the image of some \( Y \) and so we may apply a cofinality argument to realize that any map out of the diagram for the left-hand side of (6.3) must factor through the right-hand side, which completes the proof of (6.3).

Now pass to \( \Sigma_n \)-coinvariants in (6.2). Verifying that the left vertical map is a cofibration reduces to verifying that \( f^q \sqcup \Sigma_q \sqcup g^{(n-q)} \sqcup \Sigma_{n-q} \) is a cofibration. This in turn follows from the inductive hypothesis on \( f \) and \( g \). Thus all the maps \( Q[q]_n^{n-1}/\Sigma_n \to Q[q+1]_n^{n-1}/\Sigma_n \) are pushouts of cofibrations and hence are cofibrations themselves. Hence,
their composite \( Q^n_{n-1}(gf)/\Sigma_n \rightarrow Q^n_{n-1}(g)/\Sigma_n \) is a cofibration. This completes the proof that the classes \( \mathcal{P} \) and \( \mathcal{P}' \) are closed under composition.

Finally, we cover the case of transfinite composition. First note that the proof for the composition of two maps proves that the vertical maps and the induced pushout corner map in the following square become cofibrations after passing to \( \Sigma_n\)-coinvariants, by the general machinery of adding a new vertex \( e \) containing only \( Y \)s and \( Z \)s:

\[
\begin{array}{ccc}
Q^n_{i-1}(f) & \rightarrow & Q^n_{i-1}(gf) \\
\downarrow & & \downarrow \\
Q^n_i(f) & \rightarrow & Q^n_i(gf)
\end{array}
\]

Indeed, the same is true of the diagram

\[
\begin{array}{ccc}
Q^n_{n-1}(f) & \rightarrow & Q^n_{n-1}(gf) \\
\downarrow & & \downarrow \\
Y^{\otimes n} & \rightarrow & Z^{\otimes n}
\end{array}
\]

This is the analogous result to Corollary A.7 in [71], which begins with power cofibrations and concludes that the diagram represents a projective cofibration in \( \text{Arr}(\mathcal{M}/\Sigma_n) \).

Recall, e.g. from Definition 2.1 in [22] that a square is a projective cofibration if and only if the vertical maps and the pushout corner map are cofibrations. In our situation we pass to \( \Sigma_n\)-coinvariants on the diagram level and in that way achieve a projective cofibration in \( \text{Arr}(\mathcal{M}) \).

Now let \( X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots \) be a \( \lambda \)-sequence in which each \( f_\alpha \in \mathcal{P} \). Let \( f_\infty : X_0 \rightarrow X_\lambda \) be the composite. To prove that \( f_\infty^{\otimes n}/\Sigma_n \) is a cofibration, we realize this map
as the colimit of a particular diagram. Because colimits commute we can pass to $\Sigma_n$-coinvariants in the diagram and we will see that the colimit of the resulting diagram (which will be $f^{\otimes n}/\Sigma_n$) will be a cofibration. First we realize the domain of $f^{\otimes n}/\Sigma_n$ as a colimit along the sequence $Q^0_{n-1}(f_0) \to Q^1_{n-1}(f_1f_0) \to Q^2_{n-1}(f_2f_1f_0) \to \ldots Q^n_{n-1}(f_{\infty})$.  

Next, we realize $f^{\otimes n}$ as the far right-hand map in:

\begin{eqnarray}
Q^0_{n-1}(f_0) & \longrightarrow & Q^1_{n-1}(f_1f_0) & \longrightarrow & Q^2_{n-1}(f_2f_1f_0) & \longrightarrow & \ldots & \longrightarrow & Q^n_{n-1}(f_{\infty}) \\
X_0^\otimes & \longrightarrow & X_1^\otimes & \longrightarrow & X_2^\otimes & \longrightarrow & \ldots & \longrightarrow & X_\lambda^\otimes
\end{eqnarray}

As in the case for two-fold composition, we pass to $\Sigma_n$-coinvariants in this diagram and realize that the resulting diagram is a projective cofibration in the category of $\lambda$-sequences $\mathcal{M}^\lambda$ because all vertical maps and all pushout corner maps are cofibrations. The colimit of such a diagram must be a cofibration, because colimit is a left Quillen functor from $\mathcal{M}^\lambda \to \mathcal{M}$. This proves that $f^{\otimes n}/\Sigma_n$ is a (trivial) cofibration as desired.

\[\square\]

Before proceeding to the proof of Theorem 6.1.2 we make use of the methods in the proof above to prove the following lemma, which will be of use in Chapter 7.

**Lemma 6.4.2.** Assume that for every $g \in I$, $g^{\otimes n}/\Sigma_n$ is a cofibration. Suppose $f$ is a trivial cofibration between cofibrant objects and $f^{\otimes n}/\Sigma_n$ is a cofibration for all $n$. Then $f^{\otimes n}/\Sigma_n$ is a trivial cofibration for all $n$ if and only if $\text{Sym}^n(f)$ is a trivial cofibration for all $n$. 
Proof. By Lemma 6.3.1, the hypothesis implies that the class of cofibrations is closed under the operation \((-)^{\Omega n}/\Sigma_n\). The pushout diagram \((6.1)\) remains a pushout diagram if we apply \((-)/\Sigma_n\) to all objects and morphisms in the diagram, because \((-)/\Sigma_n\) is a left adjoint and so commutes with colimits. We obtain the diagram

\[
\begin{array}{c}
\text{Sym}^{n-q}(X) \otimes Q_{q-1}^q/\Sigma_q \\
\downarrow \\
\text{Sym}^{n-q}(X) \otimes \text{Sym}^q(Y) \\
\end{array} \quad \rightarrow \quad
\begin{array}{c}
\text{Sym}^n(X) \otimes Q_{q-1}^q/\Sigma_n \\
\downarrow \downarrow \\
Q_{q-1}^q/\Sigma_n \\
\end{array}
\]

(6.5)

We have assumed \(X\) is cofibrant, so \(\text{Sym}^k(X)\) is cofibrant for all \(k\) because the map \(\emptyset \to \text{Sym}^k(X)\) is simply the \(k\text{th}\) iterated pushout product of the map \(\emptyset \to X\). Thus, the left vertical map above is a trivial cofibration as soon as \(f^{\Omega q}\) is a trivial cofibration, by the pushout product axiom.

We are now ready to prove the forwards direction in the lemma. Fix \(n\) and realize \(\text{Sym}^n(f)\) as a composite of maps \(Q_{q-1}^n/\Sigma_n \to Q_q^n/\Sigma_n\) as in \((6.1)\). Assume \(f^{\Omega q}\) is a trivial cofibration for all \(q\) and deduce that each \(Q_{q-1}^n/\Sigma_n \to Q_q^n/\Sigma_n\) is a trivial cofibration, because trivial cofibrations are closed under pushout. Furthermore, because trivial cofibrations are closed under composite, this proves \(\text{Sym}^n(f)\) is a trivial cofibration.

To prove the converse, assume that \(\text{Sym}^k(f)\) is a trivial cofibration for all \(k\). We will prove \(f^{\Omega n}/\Sigma_n\) is a trivial cofibration for all \(n\) by induction. For \(n = 1\) the map is \(f\), which we have assumed to be a trivial cofibration. Now assume \(f^{\Omega i}/\Sigma_i\) is a trivial cofibration for all \(i < n\). As in the proof of Lemma 6.3.1, we will prove \(f^{\Omega n}/\Sigma_n\) is a trivial cofibration via the filtration in \((6.5)\). By our inductive hypothesis, we know that for all
6.5. PROOF OF MAIN THEOREM

\( i < n, \ Q^n_{i-1}/\Sigma_n \to Q^n_i/\Sigma_n \) is a trivial fibration. We therefore have a composite:

\[
\text{Sym}^n(X) = Q^n_0/\Sigma_n \to Q^n_1/\Sigma_n \to \cdots \to Q^n_{n-1}/\Sigma_n \to Q^n_n/\Sigma_n = \text{Sym}^n(Y)
\]

in which each map except the last is a trivial fibration. However, we have assumed \( \text{Sym}^n(X) \to \text{Sym}^n(Y) \) is a trivial fibration, so by the two out of three property the map \( Q^n_{n-1}/\Sigma_n \to Q^n_n/\Sigma_n \) is in fact a weak equivalence. This map is \( f^{\otimes n}/\Sigma_n \), and is a fibration by hypothesis, so it is a trivial fibration. This completes the induction.

\( \square \)

6.5. Proof of Main Theorem

As described in Section 6.1, it is sufficient to prove the statements of Theorem 6.1.2 and Proposition 6.1.5 for the case \( R = S \) of commutative monoids in \( \mathcal{M} \). Before proceeding to the proof, we fix some notation. Given a map \( g : K \to L \) one can form \( g^{\otimes n} : \overline{Q}_n \to L^{\otimes n} \). This map is a (trivial) fibration if \( g \) is such, by the pushout product axiom. The domain and codomain both have an action of \( \Sigma_n \). Modding out by this action gives a map which is denoted by \( f^{\otimes n}/\Sigma_n : Q_n/\Sigma_n \to \text{Sym}^n(L) = L^{\otimes n}/\Sigma_n \).

The proofs of Theorem 6.1.2 and Proposition 6.1.5 follow the proof in [81] that \( \text{Mon}(\mathcal{M}) \) has a model structure inherited from \( \mathcal{M} \). Because that proof is based on the general theory of monads (c.f. Lemma 2.3) it will go through verbatim if Lemma 6.2 in [81] can be generalized to describe pushouts in \( \text{CMon}(\mathcal{M}) \) rather than in \( \text{Mon}(\mathcal{M}) \). We state the analogue to Lemma 6.2:

**Lemma 6.5.1.**

1. If \( \mathcal{M} \) satisfies the commutative monoid axiom then in the category \( \text{CMon}(\mathcal{M}) \), Sym\((J)\)-cell is contained in the collection of maps of the
form \((id_Z \otimes J)\)-cell in \(M\). If in addition \(M\) satisfies the monoid axiom then these maps are weak equivalences in \(M\) and hence in \(C\text{Mon}(M)\).

(2) If \(M\) satisfies the strong commutative monoid axiom then maps in \(\text{Sym}(I)\)-cell with cofibrant domain (in \(M\)) are cofibrations in \(M\).

As in \[81\], the proof of this proposition requires a careful analysis of the filtration on pushouts in the category of commutative monoids. In particular, we must prove the following.

**Proposition 6.5.2.** Given any map \(h : K \to L\) in \(M\), the commutative monoid homomorphism \(X \to P\) formed by the following pushout in \(C\text{Mon}(M)\)

\[
\begin{array}{ccc}
\text{Sym}(K) & \longrightarrow & \text{Sym}(L) \\
\downarrow & & \downarrow \\
X & \longrightarrow & P
\end{array}
\]

factors as \(X = P_0 \to P_1 \to \cdots \to P\) where \(P_{n-1} \to P_n\) is defined by the following pushout in \(M\)

\[
\begin{array}{ccc}
X \otimes Q_n/\Sigma_n & \longrightarrow & X \otimes \text{Sym}^n(L) \\
\downarrow & & \downarrow \\
P_{n-1} & \longrightarrow & P_n
\end{array}
\]

where \(Q_n\) denotes the colimit of the \(n\)-dimensional punctured cube discussed in Section 6.4, which has one vertex for each \(n\)-letter word formed from letters \(K\) and \(L\), except with the \(L^{\otimes n}\) word removed.
This filtration is analogous to the one given in [81], and makes use of the decomposition

\[ \text{Sym}(-) = \bigoplus_{n \geq 0} \text{Sym}^n(-). \]

The map \( g : K \to X \) needed for the construction of \( P_{n-1} \to P_n \) is adjoint to the map \( \text{Sym}(K) \to X \). Note that this description of \( P_n \) is significantly simpler than the one found in [81] because commutativity means one need not consider words with \( X \)'s, \( K \)'s, and \( L \)'s interspersed. Rather, all the \( X \)'s can be shuffled to the left and multiplied at the beginning of the process, rather than at the end as is done in [81]. If we were to keep our notation in line with the notation in [81] then what we call \( Q_n \) would be denoted \( \overline{Q_n} \), but we will avoid this unnecessary shift in notation, because we will have no need for colimits of cubes formed from words in the letters \( X, K, L \).

Once we prove this proposition, we will restrict attention to the case when \( h = j \) is a trivial cofibration to prove the first statement in Lemma 6.5.1 and we will restrict to when \( h = i \) is a cofibration and \( X \) is cofibrant for the second statement. This is done at the end of the section.

**Proof of Proposition 6.5.2.** We begin by describing the left vertical map in the diagram which defines the maps \( P_{n-1} \to P_n \). This will be done inductively. Because \( X \otimes - \) commutes with colimits (because it’s a left adjoint), the map \( X \otimes Q_n / \Sigma_n \to P_{n-1} \) may be defined componentwise on the vertices of the cube defining \( X \otimes Q_n \).

For the \( n = 1 \) case the map \( X \otimes K \to X \otimes X \to X = P_0 \) is \( g \) followed by \( \mu_X : X \otimes X \to X \).

Let \( D \) be a proper subset of \( [n] = \{1, \ldots, n\} \) and define \( W(D) = C_1 \otimes \cdots \otimes C_n \) where \( C_i = K \) if \( i \notin D \) and \( C_i = L \) if \( i \in D \). These are the vertices of the cube defining \( Q_n \).
Given a vertex $X \otimes W(D)$ define a map by first applying $g$ to all factors of $K$ (call this map $g^*$), then shuffling all the factors of $X$ so obtained to the left by a permutation $\sigma_D$, then multiplying these factors together. This map takes $X \otimes W(D)$ to $X \otimes L^\otimes[D]$ and hence to $X \otimes \text{Sym}^{|D|}(L)$ by passing to $\Sigma_n$-coinvariants. Induction then gives a map to $P_{|D|}$ and hence to $P_{n-1}$ because $D$ was a proper subset of $[n]$.

The map above is well-defined (i.e. respects the $\Sigma_n$ action on the cube defining $X \otimes Q_n$) because a permutation $\sigma$ which takes $W(D)$ to a different vertex $W(T)$ for some $T$ of the same size as $D$ yields the following commutative diagram:

\[
\begin{array}{ccc}
X \otimes W(D) & \longrightarrow & X \otimes L^\otimes[D] \\
\downarrow 1 \otimes \sigma & & \downarrow \quad X \otimes L^\otimes[D]/\Sigma_n \\
X \otimes W(T) & \longrightarrow & X \otimes L^\otimes[D]
\end{array}
\]

The left square commutes because the top left horizontal map is $\sigma_D \circ g^*$ and the bottom left horizontal map is $\sigma_T \circ g^*$, so the dotted arrow can be defined as $\sigma|_D$ on the $|D|$ factors of $L$ and as $\sigma|_{[n]-D}$ on the $n-|D|$ factors of $X$ (using the fact that $X$ is commutative). Thus, both ways of going around are simply doing $g^*, \sigma$, and the shuffling of $X$'s to the left. The right pentagon commutes $X$ is commutative (so the order of factors doesn’t matter) and because passage to $\Sigma$-coinvariants means the order of factors of $L$ does not matter either.

These maps from vertices assemble to a map $X \otimes Q_n \to P_{n-1}$ because taking $i \notin D$ and defining the map $X \otimes W(D \cup \{i\}) \to P_{n-1}$ as above gives a diagram, which we will
show commutes:

\[
\begin{array}{ccc}
X \otimes W(D) & \rightarrow & X \otimes L^{\otimes |D|} \\
\downarrow & & \downarrow \\
X \otimes W(D \cup \{i\}) & \rightarrow & X \otimes L^{\otimes (|D|+1)}
\end{array}
\]

\[
\begin{array}{ccc}
& & P_{|D|} \\
& & \\
& & P_{|D|+1}
\end{array}
\]

\[
\begin{array}{ccc}
& & P_{|D|+1}
\end{array}
\]

The upper left horizontal map is \( \mu_X \circ \sigma_D \circ g^* \) so we may factor it as \( X \otimes W(D) \rightarrow X^{\otimes (n-|D|-1)} \otimes K \otimes L^{\otimes |D|} \rightarrow X \otimes L^{\otimes |D|} \) where \( K \) is the \( i \)th factor of the original \( W(D) \).

Since this factor becomes an \( L \) in the bottom row we have the following diagram:

\[
\begin{array}{ccc}
X \otimes W(D) & \rightarrow & X \otimes K \otimes L^{\otimes |D|} \\
\downarrow & & \downarrow \\
X \otimes W(D \cup \{i\}) & \rightarrow & X \otimes L^{\otimes (|D|+1)}
\end{array}
\]

\[
\begin{array}{ccc}
& & P_{|D|} \\
& & \\
& & P_{|D|+1}
\end{array}
\]

The difference between the two ways of going around the left-hand square is the order of factors in the \( L \) component (the order in the \( X \) component doesn’t matter).

Thus, this square will commute upon passage to \( P_{|D|+1} \) because of passage to \( \Sigma_{|D|+1} \)-coinvariants. Recall that \( P_{|D|+1} \) is a pushout of \( X \otimes Q_{|D|+1} \), which is itself a pushout of vertices \( X \otimes W(R) \). Because a pushout of a pushout is again a pushout, the right-hand square commutes by the basic property of pushouts.

This completes the inductive definition of \( P_n \). Setting \( P \) to be the colimit of the \( P_n \) (taken in \( \mathcal{M} \)) completes the analysis. Note that in \([81]\) the pushout is the free product of \( T(L) \) and \( X \) over \( T(K) \) for the free monoid functor \( T \), whereas in the commutative setting \( P \) is the (conceptually simpler) tensor product \( \text{Sym}(L) \otimes_{\text{Sym}(K)} X \). In an analogous way to the versions of these statements in \([81]\) we now prove
(1) \( P \) is naturally a commutative monoid.

(2) \( X \to P \) is a map of commutative monoids.

(3) \( P \) has the universal property of the pushout in the category of commutative monoids.

As in [81] the unit for \( P \) is the map \( S \to X \to P \) and the multiplication on \( P \) is defined from compatible maps \( P_n \otimes P_m \to P_{n+m} \) by passage to the colimit. These maps are defined inductively using the following pushout diagram (which is simply the product of the two pushout diagrams defining \( P_n \) and \( P_m \), where for spacing reasons we let \( \overline{Q_n} \) denote \( Q_n/\Sigma_n \) and let \( \overline{L^n} \) denote \( L^n/\Sigma_n \):

\[
\begin{array}{ccc}
(X \otimes \overline{Q_n}) \otimes (X \otimes \overline{L^m}) & \longrightarrow & (X \otimes \overline{L^n}) \otimes (X \otimes \overline{L^m}) \\
\downarrow & & \downarrow \\
(P_{n-1} \otimes P_m) \bigcup (P_n \otimes P_{m-1}) & \longrightarrow & P_n \otimes P_m
\end{array}
\]

This is a pushout square by Lemma 4.1 in [70].

The lower left corner has a map to \( P_{n+m} \) by induction. The upper right corner is mapped there by shuffling the middle \( X \) to the left-hand side, multiplying the two factors of \( X \), passing to \( \Sigma_{n+m} \)-coinvariants, and using the definition of \( P_{n+m} \). To show \( P \) is a commutative monoid one must verify the following diagrams:

\[
\begin{array}{ccccccc}
S \otimes P & \longrightarrow & P \otimes P & \longrightarrow & P \otimes P & \mu \otimes 1 & \longrightarrow & P \otimes P \\
\downarrow & & \downarrow \otimes 1 & & \downarrow 1 \otimes \mu & & \downarrow \tau & & \downarrow \\
P & \longrightarrow & P \otimes P & \longrightarrow & P & P \otimes P & \longrightarrow & P \otimes P
\end{array}
\]

The leftmost diagram commutes because the left-hand factor of \( P \) is \( P_0 \), coming from a map \( S \to X \), and so if we replace the other factors of \( P \) by \( P_m \) we see that this
The diagram commutes before passage to colimits. In particular, the diagram defining the map \( P_0 \otimes P_m \to P_m \) collapses in the following way. The upper left corner is \( X \otimes X \otimes L^{\otimes m}/\Sigma_m \) because \( X \otimes Q_0 = X \). The upper right corner is also \( X \otimes X \otimes L^{\otimes m}/\Sigma_m \) because \( X \otimes L^{\otimes 0} = X \). Thus, the upper horizontal map is the identity. Similarly the bottom horizontal map is the identity on \( P_0 \otimes P_m \). Recalling that the \( P_0 \) comes from a map \( S \to X \) where \( S \) is the monoidal unit we may write

\[
P_0 \otimes P_m = (P_0 \otimes P_{m-1}) \bigotimes_{S \otimes X \otimes Q_m/\Sigma_m} (S \otimes X \otimes L^{\otimes m}/\Sigma_m) = P_{m-1} \bigotimes_{X \otimes Q_m/\Sigma_m} (X \otimes L^{\otimes m}/\Sigma_m) = P_m
\]

Where \( P_0 \otimes P_{m-1} = P_{m-1} \) by induction, and the other factors of \( S \) disappear because \( S \) is the unit for \( X \). This proves the commutativity of the leftmost diagram.

The middle diagram also commutes on the level of individual \( P_i \). In particular, the two ways of getting from \( P_n \otimes P_m \otimes P_k \) to \( P_{n+m+k} \) (i.e. via \( P_{n+m} \otimes P_k \) and via \( P_n \otimes P_{m+k} \)) are the same. The key observation to show this is that all maps in the diagram are of the form \( \text{Pushout} \otimes \text{Identity} \), and the pushout of a pushout is a pushout. Thus, both ways of going around are pushouts, and the universal property of pushouts shows that they must be isomorphic.

The rightmost diagram also commutes on the level of individual \( P_i \), i.e. \( P_n \otimes P_m \to P_{n+m} \) is the same as \( P_n \otimes P_m \to P_m \otimes P_n \to P_{m+n} \). To see this, look at the diagram defining \( \mu_P \) and consider what happens if the \( n \) factors and \( m \) factors are swapped. This causes no harm to the upper right corner because the map \( (X \otimes L^{\otimes m}/\Sigma_m) \otimes (X \otimes L^{\otimes n}/\Sigma_n) \) requires passage to \( \Sigma_{m+n} \)-coinvariants, so changing the order of the \( L \) factors has no effect on \( \mu_P \). Similarly there is no harm to the lower left corner because of induction.
The upper left corner is hardest, but either way of going around to $P_{m+n}$ will render the swapping of factors meaningless. One way around requires passage to $\Sigma_{m+n}$-coinvariants and the other way goes to $P_i \otimes P_j$ factors for $i, j < n, m$ and so will hold by induction. This completes the proof of statement (1).

To verify that the map $X \to P$ is a map of commutative monoids one must only verify that it’s a map of monoids and that the two monoids in question are commutative. This means verifying the commutativity of the following diagrams:

\[
\begin{array}{ccc}
X \otimes X & \longrightarrow & X \\
\downarrow & & \downarrow \\
P \otimes P & \longrightarrow & P
\end{array}
\quad \begin{array}{ccc}
S & \longrightarrow & X \\
\downarrow & & \downarrow \\
P & \longrightarrow & P
\end{array}
\]

The map $P \otimes P \to P$ is induced by passage to colimits of the multiplication $P_n \otimes P_m \to P_{n+m}$ and so by definition the obvious diagram with $P_n \otimes P_m$, $P_{n+m}$, $P \otimes P$, and $P$ commutes for all $n, m$. The point is that defining $P \otimes P \to P$ requires one to go to $P_n \otimes P_m$, so the commutativity is tautological. In particular, it commutes for $n = m = 0$ and this proves the left-hand diagram above commutes, since $X = P_0$. The right-hand diagram commutes by definition of the map $S \to P$ as coming from $X$. This completes the proof of statement (2).

To prove that $P$ satisfies the universal property of pushouts in the category of commutative monoids requires one to define a map $P \to M$ which completes the following diagram, where $M$ is a commutative monoid, $X \to M$ is monoidal, and $L \to M$ is a map in $\mathcal{M}$. The reason one works with $K$ and $L$ rather than $\text{Sym}(K)$ and $\text{Sym}(L)$ is that the data of a map of commutative monoids $\text{Sym}(K) \to M$ is the same as that of a map
from $K$ to $M$, by the free-forgetful adjunction.

\[
\begin{array}{ccc}
K & \rightarrow & L \\
\downarrow & & \downarrow \\
X & \rightarrow & P \\
\downarrow & & \downarrow \\
& & M
\end{array}
\]

The existence of maps $K \rightarrow X \rightarrow M$ and $L \rightarrow M$ defines maps from $X \otimes W(D) \rightarrow M$ for all $D$ and all $n$. Commutativity of the outer diagram forces the maps $X \otimes W(D) \rightarrow M$ to be compatible, i.e. commutativity of the square diagram featuring $X \otimes W(D), X \otimes W(D \cup \{i\}), M$, and $M$. This is because the left-vertical map in that diagram is $K \rightarrow L$ and the right vertical map is $K \rightarrow X \rightarrow M$ (which is easy to see when thinking of commutativity of the outer diagram above as defining a word in $M$). Furthermore, these maps respect the $\Sigma_n$ action on the cube defining $Q_n$ because $M$ is commutative. Thus, by induction on $n$ we may define a map $P_n \rightarrow M$ because the diagram featuring $X \otimes Q_n/\Sigma_n, X \otimes L^{\otimes n}/\Sigma_n, P_{n-1}$, and $M$ commutes. In this diagram we use induction to define the map $P_{n-1} \rightarrow M$ and we using the fact that $M$ is commutative to define the map $X \otimes L^{\otimes n}/\Sigma_n \rightarrow M$.

Commutativity of this diagram is due to the fact that $X \otimes W(D) \rightarrow M$ factors through $X \otimes L^{\otimes |D|}/\Sigma_{|D|}$ and hence through $P_{n-1}$ via $P_{|D|}$. The unique maps $P_n \rightarrow M$ assemble to a unique map $P \rightarrow M$.

Commutativity of the triangle featuring $X, P$, and $M$ follows by definition of $P$ as a colimit and of $X$ as $P_0$. Commutativity of the other triangle follows because it holds
with \( P_n \) substituted for \( P \), for all \( n \). This is because commutativity holds in the triangle which defines the map \( P_n \to M \) for all \( n \), so it holds in the (first) \( L \) factor of \( X \otimes L^\otimes n/\Sigma_n \), i.e. \( L \to M \) is the same as \( L \to P_n \to M \) for all \( n \). This completes the proof of statement (3) and hence of the proposition. \( \square \)

We move now to homotopy theoretic considerations, and use the proposition to prove Lemma 6.5.1.

**Proof.** To prove statement (1), recall that the commutative monoid axiom tells us that if \( h \) is a trivial cofibration then \( h^{\otimes n}/\Sigma_n \) is a trivial cofibration for all \( n > 0 \).

So suppose \( h = j : K \xhookrightarrow{\sim} L \). Because \( j \) is a trivial cofibration, the map \( j^{\otimes n}/\Sigma_n : Q_n/\Sigma_n \to \text{Sym}^n(L) \) is a trivial cofibration. Thus, the map \( X \otimes j^{\otimes n}/\Sigma_n \) is of the form required by the monoid axiom. This means transfinite compositions of pushouts of such maps are weak equivalences, so in particular \( X \to P \) is a weak equivalence in \( \mathcal{M} \) and hence in \( \text{CMon}(\mathcal{M}) \). Any map in \( \text{Sym}(J) \)-cell is a transfinite composite of pushouts of maps in \( \text{Sym}(J) \). We have seen that all such pushouts are of the form required by the monoid axiom, and a transfinite composite of a transfinite composite is still a transfinite composite, so the monoid axiom applied again proves that \( \text{Sym}(J) \)-cell is contained in the weak equivalences. This completes the proof of (1).

For (2), suppose \( h = i : K \xhookrightarrow{\sim} L \) and suppose \( X \) is cofibrant in \( \mathcal{M} \). By the strong commutative monoid axiom, the maps \( i^{\otimes n}/\Sigma_n \) are cofibrations for all \( n \), so \( X \otimes i^{\otimes n}/\Sigma_n \) are cofibrations for all \( n \). Since pushouts of cofibrations are again cofibrations, the maps \( P_{n-1} \to P_n \) are cofibrations for all \( i \). Because \( P_0 = X \) is cofibrant, this means all the \( P_k \) are cofibrant and also \( X \to P \) is a cofibration (so \( P \) is cofibrant) because transfinite
compositions of cofibrations are again cofibrations (see Proposition 10.3.4 in [42]). Every map in Sym(I)-cell which has cofibrant domain is a transfinite composite of pushouts of maps of the form above, and so is in particular again a cofibration in $\mathcal{M}$.

$\square$

In the proof above, we make use of a particular filtration on the map $X \to P$. We could also have followed [61] and filtered the map $\text{Sym}(f)$ as

$$\text{Sym}(K) = B_0 \to B_1 \to \cdots \to \text{Sym}(L)$$

where each $B_n$ is a $\text{Sym}(K)$-module. This makes it clear that the map $X \to P$ is a map of $X$-modules, and thus makes it easier to check that $P$ is in fact a monoid. However, this filtration requires special knowledge of $\text{Com}$, namely that it is generated by $\text{Com}(2)$-swaps (i.e. functions of arity two) so that $\text{Com}$-algebras can be multiplied with themselves. The author chose the approach presented here because it admits an operadic generalization.

### 6.6. Operadic Generalization

The commutative monoid axiom has a natural generalization to an arbitrary operad $P$. Recall that cofibrancy may be defined for $P$-algebras via a lifting property, even if the category of $P$-algebras is not a model category.

**Definition 6.6.1.** Let $P$ be an operad. A monoidal model category $\mathcal{M}$ is said to satisfy the $P$-algebra axiom if for all cofibrant $P$-algebras $A$ and for all $n \geq 0$, $P_A(n) \otimes_{\Sigma_n} (-)^{\Box n}$ preserves trivial cofibrations (where $P_A$ is the enveloping operad).
Theorem 6.6.2. Let $P$ be an operad and suppose $\mathcal{M}$ is a combinatorial model category satisfying the $P$-algebra axiom. Then the category $P\text{-}\text{alg}(\mathcal{M})$ inherits a semi-model structure from $\mathcal{M}$.

Proof. As usual, this semi-model structure will be transferred along the free-forgetful adjunction $(P, U)$ via Lemma 2.5.5. Because $\mathcal{M}$ is combinatorial, the smallness hypotheses of Lemma 2.5.5 are automatically satisfied. Let $j : K \rightarrow L$ be a trivial cofibration in $\mathcal{M}$ and consider the pushout of $P(j)$ along a $P$-algebra homomorphism $P(K) \rightarrow A$. Denote the resulting map $\gamma : A \rightarrow B$. Factor this map as in the Section 7.3 of [36], recalled in Remark 6.1.9. So $\gamma$ is a transfinite composite of pushouts of maps of the form $P_{A}(n) \otimes_{\Sigma_{n}} j^{\text{on}}$. By hypothesis all such maps are trivial cofibrations, so $\gamma$ is a trivial cofibration. As in Corollary 6.1.8 this completes the proof. □

While the $P$-algebra axiom gives a minimal condition on $\mathcal{M}$ so that $P$-algebras inherit a semi-model structure, it is not clear that this condition can be checked in practice because of the presence of $P_{A}$ in the hypotheses. However, we can generalize the commutative monoid axiom to find a new family of axioms on $\mathcal{M}$ which do not make reference to $P_{A}$. This line of reasoning has been used in [85] (where conditions are given so that $\Sigma$-cofibrant operads are admissible) and in [36] (where conditions are given so that all operads are admissible).

These two examples demonstrate that in order for the category of $P$-algebras to inherit a semi-model structure, a cofibrancy hypothesis on either $\mathcal{M}$ or $P$ will be needed. The following result will unify all previous results on admissibility into a single framework.
and provide new results for levelwise cofibrant operads, where the cofibrancy hypotheses are evenly distributed between the operad and the model category.

**Theorem 6.6.3.** Let $\mathcal{M}$ be a combinatorial monoidal model category. Let $f$ run through the class of trivial cofibrations. Consider the following hypothesis, where $X$ is an object with a $\Sigma_n$-action that runs through some class of objects $K$:

**Hypothesis**: $X \otimes_{\Sigma_n} f^{\Sigma_n}$ is a trivial cofibration for all $X \in K$.

In each row of the following table, placing this hypothesis on $\mathcal{M}$ for the class of objects $K$ listed in the left column gives a semi-model structure on $P$-algebras for all $P$ satisfying the hypotheses in the right column.

<table>
<thead>
<tr>
<th>Hypothesis on $\mathcal{M}$</th>
<th>Class of operad</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = {\Sigma_n$-projectively cofibrant objects $}$</td>
<td>$(\Sigma-)Cofibrant$</td>
</tr>
<tr>
<td>$K = {\text{objects cofibrant in } \mathcal{M}}$</td>
<td>Levelwise cofibrant</td>
</tr>
<tr>
<td>$K = {\text{objects in } \mathcal{M}^{\Sigma_n}}$</td>
<td>Arbitrary</td>
</tr>
</tbody>
</table>

The hypotheses going down the left column are increasing in strength, while the hypotheses in the right column are decreasing. The last row says that if $\mathcal{M}$ is combinatorial, monoidal, satisfies the monoid axiom, and has the property that $\forall X \in \mathcal{M}^{\Sigma_n}$, $X \otimes_{\Sigma_n} f^{\Sigma_n}$ is a (trivial) cofibration, then all operads are admissible. This generalizes the main theorem from [36], which states that if all symmetric sequences in $\mathcal{M}$ are projectively cofibrant then all operads are admissible. Similarly, the first row recovers Spitzweck's Theorem (Theorem 5.0.6), since it follows from the pushout product axiom that for any $\Sigma_n$-projectively cofibrant $X$, the map $X \otimes_{\Sigma_n} f^{\Sigma_n}$ is a trivial cofibration. The row regarding levelwise cofibrant operads is new.
Proof. The proof proceeds as in Remark \textcolor{red}{6.1.9} but using $P_A(n)$ rather than $Com_A(n)$.

The hypothesis in the theorem guarantees this procedure will work as soon as $P_A(n)$ is known to be in the class of objects considered in the left-hand column. For the bottom row this condition is automatic. For the top row one may use Proposition 5.17 in \textcolor{red}{37}.

For the middle row, we must show $P_A(n)$ is cofibrant in $\mathcal{M}$ if $P$ is levelwise cofibrant and $A$ is a cofibrant $P$-algebra. The proof in Proposition 5.17 (and Proposition 5.44a, on which it relies) in \textcolor{red}{37} goes through mutatis mutandis.

We are still working to reduce the hypotheses on $\mathcal{M}$ so that combinatoriality is not required. This will come down to better understanding what the free $P$-algebra functor does to the domains of the generating trivial cofibrations.

These hypotheses on $\mathcal{M}$ are not too difficult to check. For example, the one for levelwise cofibrant operads holds for $sSet$, even though the hypothesis in the bottom row does not hold for $sSet$. The bottom row holds for $Ch(k)$ for $k$ a field of characteristic 0 and for the positive flat model structure on symmetric spectra (by arguments analogous to those found in Section \textcolor{red}{6.3} above).

To get from a semi-model structure to a full model structure we would need to add a new hypothesis on $\mathcal{M}$. By way of analogy, note that to do this for cofibrant operads or for $Com$, the monoid axiom was needed. This is because the filtration by $P_A$ is simpler in these cases. In general, we need a hypothesis similar to the monoid axiom but which takes the $\Sigma_n$ action into account.

**Definition 6.6.4.** Let $\mathcal{D}_\Sigma$ be the class of maps in $\mathcal{M}^{\Sigma_n}$ which are trivial cofibrations in $\mathcal{M}$. We say $\mathcal{M}$ satisfies the $\Sigma_n$-equivariant monoid axiom if transfinite
compositions of pushouts of maps of the form \((\mathcal{D}_\Sigma_n) \otimes \Sigma_n X\) are contained in the weak equivalences for all \(X \in \mathcal{M}^\Sigma_n\).

It is clear from the filtration argument given in Section 7.3 in \[36\] that this hypothesis will imply the semi-model structures are actually model structures. However, this hypothesis is in fact so strong that it alone proves all operads are admissible, regardless of the hypotheses in Theorem 6.6.3. We summarize:

**Corollary 6.6.5.** Suppose \(\mathcal{M}\) is a combinatorial monoidal model category satisfying the \(\Sigma_n\)-equivariant monoid axiom. Then for any operad \(P\), algebras over \(P\) inherit a model structure from \(\mathcal{M}\).

A simpler hypothesis to check, which also works to improve a semi-model structure to a model structure, is the hypothesis that all objects are cofibrant. Combined with our earlier observation about \(sSet\) this implies all levelwise cofibrant operads (hence all operads) are admissible when \(\mathcal{M} = sSet\).
Bousfield Localization and Commutative Monoids

In this chapter we turn to the interplay between monoidal Bousfield localizations and commutative monoids, using our results from the previous chapter. In order to apply Corollary 6.1.8 to verify the hypotheses of Corollary 3.3.2 we must give conditions on the maps $\mathcal{C}$ so that if $\mathcal{M}$ satisfies the commutative monoid axiom then so does $L_C(\mathcal{M})$.

7.1. Bousfield Localization and the Commutative Monoid Axiom

As with the main results of Chapter 4, our method will be to apply Lemma 4.3.2, which is just the universal property of Bousfield localization. However, $(-)^{\otimes n}/\Sigma_n$ is not a functor on $\mathcal{M}$, but rather on $\text{Arr}(\mathcal{M})$. Lemma 6.4.2 lets us instead work with the functor $\text{Sym}^n : \mathcal{M} \to \mathcal{M}$ defined by $\text{Sym}^n(X) = X^{\otimes n}/\Sigma_n$.

**Theorem 7.1.1.** Assume $\mathcal{M}$ is a tractable monoidal model category satisfying the strong commutative monoid axiom. Suppose that $L_C(\mathcal{M})$ is a monoidal Bousfield localization with generating trivial cofibrations $J_C$. If $\text{Sym}^n(f)$ is a $\mathcal{C}$-local equivalence for all $n \in \mathbb{N}$ and for all $f \in J_C$, then $L_C(\mathcal{M})$ satisfies the strong commutative monoid axiom.

**Proof.** It is proven in Lemma 6.3.1 that if $(-)^{\otimes n}/\Sigma_n$ takes generating (trivial) cofibrations to (trivial) cofibrations, then it takes all (trivial) cofibrations to (trivial) cofibrations. The generating cofibrations of $L_C(\mathcal{M})$ are the same as those in $\mathcal{M}$ and $\mathcal{M}$
satisfies the strong commutative monoid axiom, so the class of cofibrations of $L_C(\mathcal{M})$ is closed under the operation $(-)^\square_n/\Sigma_n$.

Suppose now that $f : X \to Y$ is a generating trivial cofibration of $L_C(\mathcal{M})$. Because $\mathcal{M}$ is tractable and tractability is preserved by Bousfield localization (see Proposition 4.3 in [48]), we may assume $f$ has cofibrant domain and codomain. In particular, the proof of Lemma 6.4.2 implies $\text{Sym}^n(f)$ is a cofibration, because $f^\square_k/\Sigma_k$ is a cofibration for all $k$ and the domain $X$ of $f$ is cofibrant.

By hypothesis, $\text{Sym}^n(f)$ is a trivial cofibration of $L_C(\mathcal{M})$ for all $n$. We are therefore in the situation of Lemma 6.4.2 and may conclude that $f^\square_n/\Sigma_n$ is a trivial cofibration for all $n$. We now apply Lemma 6.3.1 to conclude that all trivial cofibrations of $L_C(\mathcal{M})$ are closed under the operation $(-)^\square_n/\Sigma_n$. □

Remark 7.1.2. It is tempting to try to prove the theorem using Lemma 4.3.2 i.e. using the universal property of Bousfield localization. After all, just as in Theorem 4.2.2 we are assuming the property we need on the maps in $\mathcal{C}$ and trying to deduce this property for all $\mathcal{C}$-local equivalences between cofibrant objects. However, $\text{Sym}^n$ is not a left adjoint. One could attempt to get around this by applying Lemma 4.3.2 with the functor $\text{Sym} : \mathcal{M} \to \text{CMon}(\mathcal{M})$, but this would require the existence of a model structure on $\text{CMon}(\mathcal{M})$ in which the weak equivalences are $\mathcal{C}$-local equivalences. As this is what we’re trying to prove by obtaining the commutative monoid axiom on $L_C(\mathcal{M})$, this approach is doomed to fail.

If we know more about $\mathcal{M}$ in the statement of the theorem above then we can in fact get a sharper condition regarding the generating trivial cofibrations $J_\mathcal{C}$. One way
to better understand the trivial cofibrations in $L_C(M)$ is via the theory of framings. Definition 4.2.1 of [42] defines the full class of horns on $C$ to be the class

$$\Lambda(C) = \{ \tilde{f} \square i_n \mid f \in C, n \geq 0 \}$$

where $i_n : \partial \Delta[n] \to \Delta[n]$ and $\tilde{f} : \tilde{A} \to \tilde{B}$ is a Reedy cofibration between cosimplicial resolutions. In the case where $C$ is a set and $M$ is cofibrantly generated, Definition 4.2.2 of [42] defines an augmented set of $C$-horns to be $\overline{\Lambda(C)} = \Lambda(C) \cup J$. Finally, 4.2.5 defines a set $\overline{\Lambda(C)}$ to be a set of relative $I$-cell complexes with cofibrant domains obtained from $\overline{\Lambda(C)}$ via cofibrant replacement.

Theorem 3.11 in [2] states that if $M$ is proper and stable, if the $C$-local objects are closed under $\Sigma$ (such $L_C$ are called stable), and if $C$ consists of cofibrations between cofibrant objects then $J_C$ is $J \cup \Lambda(C)$. The last hypothesis is a standing assumption for this thesis. The key input to Theorem 3.11 is the observation that for such $M$, a map is a $C$-fibration if and only if its fiber is $C$-fibrant.

**Corollary 7.1.3.** Suppose $M$ is a stable, proper, simplicial model category satisfying the strong commutative monoid axiom. Suppose that $L_C$ is a stable and monoidal Bousfield localization such that for all $n \in \mathbb{N}$ and $f \in C$, $\text{Sym}^n(f)$ is a $C$-local equivalence. Then $L_C(M)$ satisfies the strong commutative monoid axiom.

**Proof.** By Theorem [7.1.1] we must only check that $\text{Sym}^n$ takes maps in $J_C = J \cup \Lambda(C)$ to $C$-local equivalences. By the commutative monoid axiom on $M$, maps in $J$ are taken to weak equivalences, so we must only consider maps in $\Lambda(C)$.
The reason for the hypothesis that $\mathcal{M}$ is simplicial is Remark 5.2.10 in [46], which states that the functor $\tilde{A}^m = A \otimes \Delta[m]$ is a cosimplicial resolution of $A$ (at least, when $A$ is cofibrant). We further observe that the model structure on $L_C(\mathcal{M})$ is independent of the choice of cosimplicial resolution. Thus, we may take our map in $\Lambda(C)$ to be of the form $(f \otimes \Delta[m]) \Box i_n$ where $f : A \to B$ is in $C$.

The map $(f \otimes \Delta[m]) \Box i_n$ can be realized as the corner map in the diagram

$$
\begin{align*}
A \otimes \Delta[m] \otimes \partial \Delta[n]_+ & \rightarrow B \otimes \Delta[m] \otimes \partial \Delta[n]_+ \\
A \otimes \Delta[m] \otimes \Delta[n]_+ & \rightarrow \text{dom}((f \otimes \Delta[m]) \Box i_n) \\
& \rightarrow B \otimes \Delta[m] \otimes \Delta[n]_+
\end{align*}
$$

If we can prove that $(g \otimes K)_{\Box n}/\Sigma_n$ is a $C$-local trivial cofibration for any $C$-local trivial cofibration $g$ between cofibrant objects then we can apply the same reasoning from the proof of Proposition [4.3.4] to deduce that the corner map becomes a $C$-local trivial cofibration after applying $(-)_{\Box n}/\Sigma_n$. This reasoning goes by proving that after applying $(-)_{\Box n}/\Sigma_n$ the lower curved map and the top horizontal map are $C$-local trivial cofibrations, so the bottom horizontal map is as well (because it is a pushout), and hence the corner map is a weak equivalence by the two out of three property. This reasoning works because whenever $f$ is a pushout of $g$ then $f_{\Box n}/\Sigma_n$ is a pushout of $g_{\Box n}/\Sigma_n$ as shown in Lemma [6.3.1].

Because $g \otimes K$ is a $C$-local trivial cofibration between cofibrant objects, we may apply Lemma [6.4.2] to reduce the final step to checking that if $\text{Sym}^n(g)$ is a $C$-local
trivial cofibration for all $n$ then so is $\text{Sym}^n(g \otimes K)$. This is proven as Lemma 27 in [31].

When the hypotheses of stability and properness are dropped one can no longer easily write down the set $J_C$. However, Theorem 4.1.1 (and its proof, notably 4.3.1) in [42] demonstrate that the class of maps $X \to L_C(X)$ are contained in $\overline{\Lambda(C)}$-cell. Given a $C$-local trivial cofibration $g : X_1 \to X_2$ between cofibrant objects, applying fibrant replacement $L_C$ results in a map $L_C(g)$ which is a weak equivalence between cofibrant objects. An appeal to Ken Brown’s lemma on the functor $\text{Sym}^n$ and to the two out of three property reduces the verification that $(-)^{Cn}/\Sigma_n$ takes $g$ to a $C$-local equivalence to verifying that $(-)^{Cn}/\Sigma_n$ takes $X_i \to L_C(X_i)$ to $C$-local equivalences.

Since such maps are in $\overline{\Lambda(C)}$-cell, by Lemma 6.3.1 one must only show that maps in $\overline{\Lambda(C)}$ are taken to $C$-local equivalences by $(-)^{Cn}/\Sigma_n$ (that they are taken to cofibrations is immediate by the strong commutative monoid axiom on $\mathcal{M}$). This observation leads to the following result, which we have recently learned was independently discovered as Theorem 28 in version 3 of the preprint [31].

**Theorem 7.1.4.** Suppose $\mathcal{M}$ is a cofibrantly generated, tractable, simplicial model category satisfying the strong commutative monoid axiom. Suppose that for all $n \in \mathbb{N}$ and $f \in C$, $\text{Sym}^n(f)$ is a $C$-local equivalence. Then $L_C(\mathcal{M})$ satisfies the strong commutative monoid axiom.

As the proof of this Theorem appears in [31], we will content ourselves with the sketch of the proof given above and we refer the interested reader to [31] for details. With a careful analysis of $\overline{\Lambda(C)}$ the author believes one could remove the need for $\mathcal{M}$ to
be simplicial. However, lacking equations of the sort found in Remark 5.2.10 of [46], he
does not know how to proceed.

Remark 7.1.5. Recall from Definition [6.6.1] that the commutative monoid axiom
has a natural generalization to an arbitrary operad $P$, namely that for all cofibrant $P$-
algebras $A$ and for all $n$, $P_A(n) \otimes_{\Sigma_n} (-)^{\Sigma_n}$ preserves trivial cofibrations (where $P_A$ is the
enveloping operad).

We hope in the future to study the types of localizations which preserve the $P$-algebra
axiom for general $P$, so that Corollary [3.3.2] may be applied to deduce preservation results
for arbitrary operads $P$. We conjecture in Conjecture [9.1.2] that the correct condition
on a localization is that for all $f \in \mathcal{C}$, for all $A \in P$-alg, and for all $n$, then $P_A(n) \otimes_{\Sigma_n} f^{\Sigma_n}$
is contained in the $\mathcal{C}$-local equivalences.

7.2. Preservation of Commutative Monoids

We turn now to the question of preservation under Bousfield localization of commu-
tative monoids. We will be applying Theorem [7.1.1] and Corollary [3.3.2] for this purpose
in a moment, but we first remark on a simpler case where the hypotheses of Theorem
[7.1.1] are not necessary.

7.2.1. Spectra. Preservation of commutative monoids by monoidal Bousfield lo-
calizations is easy in certain categories of spectra, because of the property that for all
cofibrant $X$ in $\mathcal{M}$, the map $(E\Sigma_n)_+ \land_{\Sigma_n} X^{\land_n} \to X^{\land_n}/\Sigma_n$ is a weak equivalence. Recall
that we generalized this property in Definition [6.2.5]. It is clear that the rectification
axiom automatically holds on $L_\mathcal{C}(\mathcal{M})$ if it holds on $\mathcal{M}$, because the cofibrant objects
are the same and the weak equivalences are contained in the $\mathcal{C}$-local equivalences. We
now prove that in the presence of the rectification axiom, preservation results for commutative monoids are particularly nice.

**Theorem 7.2.1.** Let $QCom$ denote a $\Sigma$-cofibrant replacement of $\text{Com}$ in $\mathcal{M}$. Let $\mathcal{M}$ be a monoidal model category in which the rectification axiom implies that $QCom$ and $\text{Com}$ rectify. Let $L_C$ be a monoidal Bousfield localization. Then $L_C$ preserves commutative monoids. In particular,

- For positive symmetric spectra, positive orthogonal spectra, or $\mathbb{S}$-modules, $QCom$ is $E_\infty$ and any monoidal Bousfield localization preserves strict commutative ring spectra.
- For positive $G$-equivariant orthogonal spectra, $QCom$ is $E^G_\infty$ and any monoidal Bousfield localization preserves strict commutative equivariant ring spectra.

**Proof.** Let $E$ be a commutative monoid, so in particular $E$ is a $QCom$ algebra via the map $QCom \to \text{Com}$. Because $QCom$ is $\Sigma$-cofibrant, $QCom$-algebras in both $\mathcal{M}$ and $L_C(\mathcal{M})$ inherit semi-model structures, so Corollary 3.3.2 implies $L_C(E)$ is weakly equivalent to some $QCom$-algebra $E_Q$. The rectification axiom in $L_C(\mathcal{M})$ now implies $E_Q$ is weakly equivalent to a commutative monoid $\widehat{E}$. □

Currently, this result is only known to apply to the categories of spectra listed in the statement of the theorem. We conjecture that the rectification axiom implies rectification between $QCom$ and $\text{Com}$ for general $\mathcal{M}$. If this conjecture is proven then the theorem will apply to all $\mathcal{M}$ which satisfy the rectification axiom. Even if the conjecture is false, the following proposition demonstrates that when $\mathcal{M}$ satisfies the rectification axiom
then the conditions of Theorem 7.1.1 are satisfied and so any monoidal localization preserves commutative monoids.

**Proposition 7.2.2.** If $L_C(M)$ is a monoidal Bousfield localization and $M$ satisfies the rectification axiom, then $L_C$ preserves commutative monoids.

**Proof.** We will apply Proposition 7.2.2 to the model category $L_C(M)$, using our observation that the rectification axiom holds on $L_C(M)$ whenever it holds on $M$. Thus, $\text{Sym}^n : L_C(M) \to L_C(M)$ takes $C$-local trivial cofibrations between cofibrant objects to $C$-local equivalences. In particular, the hypotheses of Theorem 7.1.1 are satisfied and we may deduce from Corollary 3.3.2 that $L_C$ preserves commutative monoids. □

**7.2.2. Spaces.** We turn our attention now to simplicial sets and topological spaces. Example 6.2.4 demonstrates that rectification fails in this setting. So even though Proposition 5.1.1 proves all localizations are monoidal, this does not immediately tell us that all localizations preserve commutative monoids. Recall that the path connected commutative monoids are weakly equivalent to generalized Eilenberg-Mac Lane spaces (GEMs). Preservation of commutative monoids has been proven for pointed CW complexes as Theorem 1.4 in [20].

**Theorem 7.2.3.** Let $M$ be the category of pointed CW complexes. Let $C$ be any set of maps. Then $\text{Sym}(-)$ preserves $C$-local equivalences and $L_C$ sends GEMs to GEMs.

The proof of this theorem is based on work of Farjoun which appears in Chapter 4 of [27], so will also hold for $M = sSet$. In Subsection 6.3.2 we generalized this work to hold for $k$-spaces. So we may extend the theorem above to $k$-spaces as well. Observe that the
Theorem above implies both $sSet$ and $k$-spaces satisfy the conditions of Theorem 7.1.1 because $\text{Sym}^n$ is a retract of $\text{Sym}$. We summarize

**Theorem 7.2.4.** Let $\mathcal{M}$ be either simplicial sets or $k$-spaces. Then every Bousfield localization preserves GEMs.

Thus, we have extended the result above and Theorem 4.B.4 in [27] to a wider class of topological spaces than spaces having the homotopy type of a CW complex.

### 7.2.3. Chain Complexes.

**Proposition 7.2.5.** Let $k$ be a field of characteristic zero. The only Bousfield localizations of $\text{Ch}(k)_{\geq 0}$ are truncations.

**Proof.** Over any PID, the homotopy type is determined by $H_*$, so this means adding weak equivalences is equivalent to killing some object. Thus, all localizations are nullifications. All objects are wedges of spheres, and killing $k^2$ in degree $n$ is the same as killing $k$ in degree $n$. Thus, the localization is completely determined by the lowest dimension in which the first nullification occurs. The localization is therefore equivalent to $0 \to V$ where $V$ is the sphere on $k$ in that dimension. \hfill $\square$

**Corollary 7.2.6.** All Bousfield localizations of $\text{Ch}(k)_{\geq 0}$ are monoidal and hence preserve algebras over cofibrant operads.

**Remark 7.2.7.** For unbounded chain complexes, truncations need not preserve algebraic structure. For example, if $f : S^{-2} \to D^{-3}$ gets inverted then just as with the Postnikov section, an algebra will be taken to an object with no unit.
Quillen proved in Proposition 2.1 of Appendix B of [72] that bounded chain complexes over a field of characteristic zero satisfies the commutative monoid axiom. The proof that all quasi-isomorphisms are closed under $\text{Sym}^n$ goes via cofiber and the 5-lemma on homology groups. The key observation is that $\text{Sym}^n(\cdot)$ preserves group isomorphisms. The same proof demonstrates that $\text{Sym}^n$ preserves $\mathcal{C}$-local equivalences for all $L_\mathcal{C}$ as above. Hence, all Bousfield localizations of $\text{Ch}(k)_{\geq 0}$ preserve commutative differential graded algebras. Of course, this can also be seen directly from the description of $L_\mathcal{C}$ as a truncation.

### 7.2.4. Equivariant Spectra

We conclude this section by returning to the result of Hill and Hopkins which we stated as Theorem 5.2.2. In [40], several equivalent conditions are given in order for a localization to preserve commutative structure. We may therefore restate Theorem 5.2.2 using the condition most related to our approach in Section 7.2.

**Theorem 7.2.8.** Suppose $L$ is a localization. If $\text{Sym}^n(\cdot)$ preserves $L$-acyclicity for all $n$ then $L$ preserves commutativity.

Preservation of $L$-acyclics is the same as preservation of $L$-local equivalences as can be seen for example via the rectification axiom and the property that cofibrant objects are flat. So we see that when we specialize Theorem 7.2.1 to the model category of equivariant spectra and to localizations of the form $L$ we precisely recover the theorem of Hill and Hopkins.

Recall that in Theorem 5.2.3 we gave minimal conditions for a Bousfield localization to preserve $E_\infty^G$-structure. Theorem 7.2.1 implies the same conditions will guarantee
preservation of strict commutative structure because equivariant spectra satisfy rectification (see the Appendix to \[14\]). Thus, we have improved on Theorem 5.2.2 and obtained sharper, easier to check conditions.
CHAPTER 8

Bousfield Localization and the Monoid Axiom

As shown by Corollary 3.3.2, our preservation results do not require $L_C(M)$ to satisfy the monoid axiom. However, having found conditions so that the pushout product axiom, commutative monoid axiom, and property that cofibrant objects are flat transfer to $L_C(M)$, we feel we should include a word on how to obtain the monoid axiom for $L_C(M)$ in case the reader is interested in studying the monoidal model category $L_C(M)$ for a purpose other than the preservation of operad-algebra structure.

We remark that Proposition 3.8 of [1] proves that $L_C(M)$ inherits the monoid axiom from $M$ if $L_C$ takes a special form similar to localization at a homology theory. In contrast, our result will place no hypothesis on the maps in $C$ at all, beyond our standing hypothesis that these maps are cofibrations.

8.1. h-Monoidal Model Categories

The Monoid Axiom is fundamentally a statement about how much tensoring with an arbitrary object $X$ changes the trivial cofibrations, since it is well known that transfinite compositions of pushouts of trivial cofibrations are again trivial cofibrations. In $L_C(M)$ we have lost control of the trivial cofibrations, but we can fix this by adding a hypothesis stating that the maps we’re interested in behave enough like trivial cofibrations to get closure under pushout and transfinite composition. Most proofs in the literature that some model category satisfies the monoid axiom come down to knowing that the
cofibrations are monomorphisms and that monomorphisms behave nicely with respect to transfinite composition. In this spirit, we will assume that our $\mathcal{C}$-local equivalences behave like the closed neighborhood deformation retracts from classical topology. We now define what we mean by this last sentence.

**Definition 8.1.1.** A map $f: A \to B$ is called a *homotopical-cofibration* if every pushout along $f$ is a homotopy pushout. This means that for any map $g: A \to W$, if $Z$ is the colimit of the pushout square formed by $f$ and $g$ and $Z'$ is the colimit of the cofibrant replacement of the diagram, then the map $Z' \to Z$ induced by the pushout property is a weak equivalence:

\[ \begin{array}{ccc}
QA & \xleftarrow{f} & QB \\
| & \downarrow & | \\
A & \xrightarrow{g} & B \\
| & \downarrow & | \\
QW & \xrightarrow{f} & Z' \\
| & \downarrow & | \\
W & \xrightarrow{g} & Z
\end{array} \]

It is clear that any trivial cofibration is a homotopical cofibration, by the two out of three property. If $\mathcal{M}$ is left proper then any cofibration is a homotopical-cofibration. If $\mathcal{M}$ is a stable model category then a map $f$ is a homotopical cofibration if and only if the map from the cofiber of $Qf$ to the cofiber of $f$ is a weak equivalence. In $Ch(Z)$, such maps are simply monomorphisms. Although $Top$ is not stable, maps which are closed neighborhood deformation retracts have this property. Furthermore, a theorem
of Steenrod states that for \( A \subset B \) and \( j : A \to B \) inclusion, \( j \) is a cofibration if and only if \( A \) is a neighborhood deformation retract of \( B \).

Recall that the maps considered in the monoid axiom take the form (Trivial-Cofibrations \( \otimes \mathcal{M} \))-cell. Assume for the moment that \( \mathcal{M} \) is a model category satisfying the property that all maps of the form \( h \otimes X \), where \( h \) is a trivial cofibration and \( X \) is any object, are homotopical-cofibrations. Because the pushout of a pushout is a pushout, it is not too much work to see that homotopical-cofibrations are closed under pushout by arbitrary maps. We are therefore most of the way towards proving that such a category \( \mathcal{M} \) satisfies the monoid axiom (we are only missing the part about transfinite compositions). So our plan will be to introduce a new property on \( \mathcal{M} \) to guarantee the required behavior for transfinite compositions, then to prove that any monoidal Bousfield localization \( L_{\mathcal{C}} \) preserves these two properties we have assumed on \( \mathcal{M} \), and then to deduce that \( L_{\mathcal{C}}(\mathcal{M}) \) satisfies the monoid axiom.

**Remark 8.1.2.** A similar notion to homotopical-cofibrations has recently appeared in [8] under the name \( h \)-cofibration. A map \( f : X \to Y \) is an \( h \)-cofibration if the functor \( f_* : X/\mathcal{M} \to Y/\mathcal{M} \) given by base change along \( f \) preserves weak equivalences, i.e. in any diagram as below in which both squares are pushout squares and \( w \) is weak equivalence, then \( w' \) is also a weak equivalence:

\[
\begin{array}{ccc}
X & \longrightarrow & A \longrightarrow B \\
\downarrow f & & \downarrow \quad \quad \downarrow \\
Y & \longrightarrow & A' \longrightarrow B'
\end{array}
\]
Note that this is a different usage of the term $h$-cofibration than the usage in [26] where it means ‘Hurewicz cofibration.’ When $\mathcal{M}$ is left proper, Proposition 1.5 in [8] proves that $f$ is an $h$-cofibration if and only if $f$ is a homotopical-cofibration in the sense of Definition [8.1.1]. In light of this Proposition and the fact that our $\mathcal{M}$ will always be left proper in this chapter, we will henceforth adhere to the existing terminology in the literature and refer to homotopical cofibrations as $h$-cofibrations.

Proposition 1.5 of [8] also characterizes $h$-cofibrations in left proper model categories as maps $f$ such that there is a factorization of $f$ into a cofibration followed by a cofiber equivalence $w: W \to Y$, i.e. for any map $g: W \to K$ the right-hand vertical map in the following pushout diagram is a weak equivalence:

\[
\begin{array}{ccc}
W & \longrightarrow & K \\
\downarrow^w & & \downarrow \\
X & \longrightarrow & T
\end{array}
\]

Lemma 1.3 in [8] proves that $h$-cofibrations are closed under composition, pushout, and finite coproduct. We will make use of these various properties of $h$-cofibrations in this section. The purpose for introducing $h$-cofibrations is to make the following definition, which should be thought of as saying that the cofibrations in $\mathcal{M}$ behave like inclusions of closed neighborhood deformation retracts.

**Definition 8.1.3.** $\mathcal{M}$ is said to be *$h$-monoidal* if for each (trivial) cofibration $f$ and each object $Z$, $f \otimes Z$ is a (trivial) $h$-cofibration.

We will find conditions so that Bousfield localization preserves $h$-monoidality, and we will then use this to deduce when Bousfield localization preserves the monoid axiom.
In [8], \( h \)-monoidality is verified for the model categories of topological spaces, simplicial sets, chain complexes over a field (with the projective model structure), symmetric spectra (with the stable projective model structure), and several other model categories not considered herein. We now verify \( h \)-monoidality for the remaining model structures we have discussed in this thesis. We remind the reader that an \textit{injective model structure} has weak equivalences and cofibrations defined levelwise, and fibrations defined by the right lifting property.

\textbf{Proposition 8.1.4.} The following model structures on symmetric spectra are \( h \)-monoidal:

\begin{enumerate}
  \item The levelwise projective model structure (of Theorem 5.1.2 in [50]).
  \item The positive model structure (of Theorem 14.1 in [66]).
  \item The flat (a.k.a. \( S \)-)model structure (of Proposition 2.2 in [83]).
  \item The positive flat model structure (obtained by redefining the cofibrations from the model structure above to be isomorphisms in level 0).
  \item The stable projective model structure (this is proven to be \( h \)-monoidal in Proposition 1.14 of [8]).
  \item The positive stable model structure (of Theorem 14.2 in [66]).
  \item The flat stable model structure (of Theorem 2.4 in [83]).
  \item The positive flat stable model structure (of Proposition 3.1 in [83]).
\end{enumerate}

\textbf{Proof.} We appeal to Proposition 1.9 in [8], and make use of the injective (or injective stable for (5)-(8)) model structure on symmetric spectra, introduced in Definition 5.1.1 (resp. after Definition 5.3.6) of [50]. The references above prove that all eight of
the model structures above are monoidal and that both injective model structures are left proper (e.g. because all objects are cofibrant). The final condition in Proposition 1.9 is that for any (trivial) cofibration \( f \) and any object \( X \), the map \( f \otimes X \) is a (trivial) cofibration in the corresponding injective model structure. The cofibration part of this is Proposition 4.15(i) in version 3 of Stefan Schwede’s book project [79], since for all eight of the model structures above the cofibrations are contained in the flat cofibrations and for any \( X \) the map \( \emptyset \to X \) is an injective (a.k.a. levelwise) cofibration. The trivial cofibration part is Proposition 4.15(iv) in [79], which includes statements for both levelwise and stable weak equivalences.

We turn now to orthogonal and equivariant orthogonal spectra. We first need a lemma regarding the existence of injective model structures. Let \( \mathcal{Sp}_\Delta \) denote orthogonal spectra built on \( \Delta \)-generated spaces (an overview of this category may be found in [25]). Let \( G \) be a compact Lie group and let \( G\mathcal{Sp}_\Delta \) denote \( G \)-equivariant orthogonal spectra built on \( \Delta \)-generated spaces.

**Lemma 8.1.5.** The following model structures exist and are left proper and combinatorial: the levelwise injective model structure on \( \mathcal{Sp}_\Delta \), the stable injective model structure on \( \mathcal{Sp}_\Delta \), the levelwise injective model structure on \( G\mathcal{Sp}_\Delta \), and the stable injective model structure on \( G\mathcal{Sp}_\Delta \).

**Proof.** Left properness will be inherited from \( \Delta \)-generated spaces. For existence, we proceed as in Theorem 5.1.2 and Lemma 5.1.4 of [50]. Verification of lifting and factorization make use of a set \( C \) (resp. \( tC \)) containing a map from each isomorphism class of (trivial) cofibrations \( i : X \to Y \) where \( Y \) is a countable spectrum. The use of
Zorn’s Lemma in Lemma 5.1.4 and the requisite countability from Lemmas 5.1.6 and 5.1.7 hold in this setting because of our decision to work with $\Delta$-generated spaces. The rest of Lemma 5.1.4 goes through mutatis mutandis, using properties of topological fibrations and using Lemma 12.2 in [66] when checking that injective cofibrations are closed under smashing with an arbitrary object.

The sets $C$ and $tC$ serve as generating (trivial) cofibrations. Together with the fact that a category of spectra built on a locally presentable category is again locally presentable, this proves the model structures are combinatorial. The stable injective structures are obtained by Bousfield localization in the usual way, which exists because the levelwise structures are left proper and combinatorial. $\square$

**Proposition 8.1.6.** Work over $\Delta$-generated spaces. Fix a compact Lie group $G$ and fix a universe $U$ which we take to mean a $G$-universe when working equivariantly. The following model structures are $h$-monoidal. The citations are where existence of each model structure is proven:

1. Levelwise projective model structure on orthogonal spectra $Sp^O$ (Theorem 6.5 in [66]).
2. Positive model structure on $Sp^O$ (Theorem 14.1 in [66]).
3. Flat model structure on $Sp^O$ (Proposition 1.3.5 in [87]).
4. Positive flat model structure on $Sp^O$ (Proposition 1.3.10 in [87]).
5. Projective model structure on $G$-equivariant orthogonal spectra $Sp^G$ (Theorem III.2.4 in [65]).
6. Positive model structure on $Sp^G$ (Theorem III.2.10 in [65]).
7. Flat model structure on $Sp^G$ (Theorem 2.3.13 of in [87]).
(8) Positive flat model structure on $Sp^G$ (cofibrations are flat cofibrations which are isomorphisms in level 0).

(9) Stable projective model structure on $Sp^O$ (Theorem 9.2 in [66]).

(10) Positive stable model structure on $Sp^O$ (Theorem 14.2 in [66]).

(11) Flat stable model structure on $Sp^O$ (Theorem 2.3.27 in [87]).

(12) Positive flat stable model structure on $Sp^O$ (Theorem 2.3.27 in [87]).

(13) Stable model structure on $Sp^G$ (Theorem III.4.2 in [65]).

(14) Positive stable model structure on $Sp^G$ (Theorem III.5.3 in [65]).

(15) Flat stable model structure on $Sp^G$ (Theorem 2.3.13 of in [87]).

(16) Positive flat stable model structure on $Sp^G$ (Theorem 2.3.27 in [87]).

Proof. The proof proceeds just as it does for Proposition 8.1.4, i.e. by comparison to the injective (stable) model structures in each of these settings. For the statement that for any cofibration $f$ and any object $X$, the map $f \otimes X$ is a cofibration in the corresponding injective model structure, we appeal to Lemma 12.2 of [66] (which works equally well in the equivariant context).

Finally, we turn to the statement that for any trivial cofibration $f$ and any object $X$, the map $f \otimes X$ is a weak equivalence in the corresponding injective model structure. For the levelwise model structures above this property is inherited from spaces, e.g. by Lemma 12.2 in [66]. For the stable model structures we appeal to the monoid axiom on all of the model structures in the theorem and to the fact that projective (stable) equivalences are the same as injective (stable) equivalences. The monoid axiom has been verified in [87] for all these model structures by Theorems 1.2.54 and 1.2.57 (both...
originally proven in [66], 1.3.10, 2.2.46 and 2.2.50 (both originally from [65]), and 2.3.27.

8.2. Preservation of the Monoid Axiom

We return now to the question of the monoid axiom. It is proven in Proposition 2.5 of [8] that if $\mathcal{M}$ is left proper, $h$-monoidal, and the weak equivalences in $(\mathcal{M} \otimes I)$-cell are closed under transfinite composition, then $\mathcal{M}$ satisfies the monoid axiom. We will use this to find conditions on $\mathcal{M}$ so that $L_C(\mathcal{M})$ satisfies the monoid axiom. First, we improve Proposition 2.5 from [8] by replacing the third condition with the hypothesis that the (co)domains of $I$ are finite relative to the class of $h$-cofibrations (in the sense of Section 7.4 of [46]).

**Proposition 8.2.1.** Suppose $\mathcal{M}$ is cofibrantly generated, left proper, $h$-monoidal, and the (co)domains of $I$ are finite relative to the class of $h$-cofibrations. Then $\mathcal{M}$ satisfies the monoid axiom.

**Proof.** We follow the proof of Proposition 2.5 in [8]. Consider the class $\{f \otimes Z \mid Z \in \mathcal{M}, f \in J\}$. As $\mathcal{M}$ is $h$-monoidal, this is a class of trivial $h$-cofibrations. By Lemma 1.3 in [8], $h$-cofibrations are closed under pushout. By Lemma 1.6 in [8], because $\mathcal{M}$ is left proper, trivial $h$-cofibrations are closed under pushouts (e.g. because weak equivalences are closed under homotopy pushout). In order to prove $\{f \otimes Z \mid Z \in \mathcal{M}, f \in J\}$-cell is contained in the weak equivalences of $\mathcal{M}$ we must only prove that transfinite compositions of trivial $h$-cofibrations are weak equivalences. Consider a $\lambda$-sequence $A_0 \to A_1 \to \cdots \to A_\lambda$ of trivial $h$-cofibrations. Let $j_\beta$ denote the map $A_\beta \to A_{\beta+1}$ in this $\lambda$-sequence.
As in Proposition 17.9.4 of [42] we may construct a diagram

\[
\begin{array}{cccccc}
A'_0 & \longrightarrow & A'_1 & \longrightarrow & \cdots & \longrightarrow & A'_\beta & \longrightarrow & \cdots \\
\downarrow q_0 & & \downarrow q_1 & & \cdots & & \downarrow q_\beta & & \\
A_0 & \longrightarrow & A_1 & \longrightarrow & \cdots & \longrightarrow & A_\beta & \longrightarrow & \cdots 
\end{array}
\]

in which each $A'_\beta$ is cofibrant, all the maps $A'_\beta \to A_\beta$ are trivial fibrations, and all the maps $A'_\beta \to A'_{\beta+1}$ are trivial cofibrations. Construction of this diagram proceeds by applying the cofibration-trivial fibration factorization iteratively to every composition $j_\beta \circ q_\beta : A'_\beta \to A_\beta \to A_{\beta+1}$ in order to construct $A'_{\beta+1}$. As $j_\beta$ and $q_\beta$ are both weak equivalences, so is their composite and so the cofibration $A'_\beta \to A'_{\beta+1}$ produced by the cofibration-trivial fibration factorization is a weak equivalence by the two out of three property.

We now show that the map $q_\lambda : A'_\lambda \to A_\lambda$ is a weak equivalence, following the approach of Lemma 7.4.1 in [46]. Consider the lifting problem

\[
\begin{array}{ccc}
X & \longrightarrow & A'_\lambda \\
f \downarrow & & \downarrow q_\lambda \\
Y & \longrightarrow & A_\lambda 
\end{array}
\]

Where $f$ is in the set $I$ of generating cofibrations. Because the domains and codomains of maps in $I$ are finitely presented we know that the map $X \to A'_\lambda$ factors through some finite stage $A'_{\lambda'}$. Similarly, $Y \to A_\lambda$ factors through some finite stage $A_m$. Let $k = \max(n, m)$. The map $A'_k \to A_k$ is a trivial fibration so there is a lift
8.2. PRESERVATION OF THE MONOID AXIOM

Define $h : Y \to A'_\lambda$ as the composite with $A'_k \to A'_\lambda$.

Both triangles in the left-hand square commute by definition of lift. The triangle featuring $g$ and $h$ commutes because it is a composition. So the triangle featuring $f$ and $h$ commutes. The right-hand square commutes by construction of $A'_\lambda$ and $A_\lambda$, so the trapezoid containing $g$ and $q_\lambda$ commutes. Thus, the triangle featuring $h$ and $q_\lambda$ commutes.

The existence of this lift $h$ for all $f \in I$ proves that $A'_\lambda \to A_\lambda$ is a trivial fibration. Now consider that transfinite compositions of trivial cofibrations are always trivial cofibrations, so $A'_0 \to A'_\lambda$ is a weak equivalence. Furthermore, the vertical maps $q_0 : A'_0 \to A_0$ and $q_\lambda : A'_\lambda \to A_\lambda$ are trivial fibrations. So by the two out of three property, the map $A_0 \to A_\lambda$ is a weak equivalence as required. \hfill $\square$

It is shown in [8] that the compactness hypothesis of the proposition is satisfied for topological spaces, simplicial sets, equivariant and motivic spaces, and chain complexes. Similarly, it holds for all our categories of structured spectra because the sphere spectrum is $\aleph_0$-compact as a spectrum. Lastly, it holds for all the stable analogues of these structures because the compactness hypothesis is automatically preserved by any Bousfield localization (the set of generating cofibrations of $L_C(M)$ is simply $I$ again).

Remark 8.2.2. The proof above only uses the fact that the maps $j_\beta$ were $h$-cofibrations in order to factor $Y \to A_\lambda$ through some finite stage. So if the (co)domains of
8.2. PRESERVATION OF THE MONOID AXIOM

I are finite relative to the class of weak equivalences then the proof above demonstrates that weak equivalences are preserved under transfinite composition.

Proposition 8.2.3. Suppose $\mathcal{M}$ is tractable, left proper, $h$-monoidal, such that the (co)domains of $I$ are finite relative to the class of $h$-cofibrations and cofibrant objects are flat. Let $L_C$ be a monoidal Bousfield localization. Then $L_C(\mathcal{M})$ is $h$-monoidal.

Proof. Suppose $f : A \to B$ is a cofibration in $L_C(\mathcal{M})$ and $Z$ is any object of $L_C(\mathcal{M})$. We must show $f \otimes Z$ is an $h$-cofibration in $L_C(\mathcal{M})$. Because $L_C(\mathcal{M})$ is left proper, Proposition 1.5 in [8] reduces us to proving that there is a factorization of $f \otimes Z$ into a cofibration followed by a cofiber equivalence $w : X \to B \otimes Z$, i.e. for any map $g : X \to K$ the right-hand vertical map in the following pushout diagram is a $C$-local equivalence:

$$
\begin{array}{ccc}
X & \rightarrow & K \\
\downarrow{w} & & \downarrow{} \\
B \otimes Z & \rightarrow & T \\
\end{array}
$$

Because $f$ is a cofibration in $\mathcal{M}$, the $h$-monoidalty of $\mathcal{M}$ guarantees us that $f \otimes Z$ is an $h$-cofibration in $\mathcal{M}$. Apply the cofibration-trivial fibration factorization in $\mathcal{M}$. Note that this is also a cofibration-trivial fibration factorization of $f \otimes Z$ in $L_C(\mathcal{M})$ because cofibrations and trivial fibrations in the two model categories agree. The resulting $w : X \to B \otimes Z$ is a trivial fibration in either model structure. Because $\mathcal{M}$ is left proper we know that the map $w$ is a cofiber equivalence in $\mathcal{M}$ by Proposition 1.5 in [8] applied to the $h$-cofibration $f \otimes Z$. So in any pushout diagram as above the map $K \to T$ is a weak
8.2. PRESERVATION OF THE MONOID AXIOM

equivalence in $\mathcal{M}$, hence in $L_C(\mathcal{M})$. Thus, $w$ is a fiber equivalence in $L_C(\mathcal{M})$ and its existence proves $f \otimes Z$ is an $h$-cofibration in $L_C(\mathcal{M})$.

Now suppose $f$ were a trivial cofibration in $L_C(\mathcal{M})$ to start. We must show that $f \otimes Z$ is a $\mathcal{C}$-local equivalence. We do this first in the case where $f$ is a generating trivial cofibration. Because $L_C(\mathcal{M})$ is tractable, $A$ and $B$ are cofibrant. Apply cofibrant replacement to $Z$:

\[
\begin{array}{ccc}
A \otimes QZ & \longrightarrow & B \otimes QZ \\
\downarrow & & \downarrow \\
A \otimes Z & \longrightarrow & B \otimes Z
\end{array}
\]

The fact that cofibrant objects are flat in $L_C(\mathcal{M})$ implies the vertical maps are $\mathcal{C}$-local equivalences (because $A$ and $B$ are cofibrant) and that the top horizontal map is a $\mathcal{C}$-local equivalence (because $QZ$ is cofibrant). By the two out of three property the bottom horizontal map is a $\mathcal{C}$-local equivalence.

By Lemma 1.3 in [8], the class of $h$-cofibrations is closed under cobase change and retracts. By Lemma 1.6, the class of trivial $h$-cofibrations is closed under cobase change (because $L_C(\mathcal{M})$ is left proper). Weak equivalences are always closed under retract. Finally, by Proposition 8.2.1 the class of trivial $h$-cofibrations is closed under transfinite composition by our compactness hypothesis on $\mathcal{M}$ (equivalently, on $L_C(\mathcal{M})$). So for a general $f$ in the trivial cofibrations of $L_C(\mathcal{M})$, realize $f$ as a retract of $g \in J_C$-cell, so that $g \otimes Z$ is a transfinite composite of pushouts of maps of the form $j \otimes Z$ for $j \in J_C$. We have just proven that all $j \otimes Z$ are trivial $h$-cofibrations and closure properties imply $g \otimes Z$ and hence $f \otimes Z$ are trivial $h$-cofibrations as well.
8.2. PRESERVATION OF THE MONOID AXIOM

Theorem 8.2.4. Suppose $\mathcal{M}$ is a tractable, left proper, $h$-monoidal model category such that the (co)domains of $I$ are finite relative to the $h$-cofibrations and cofibrant objects are flat. Then for any monoidal Bousfield localization $L_C$, the model category $L_C(\mathcal{M})$ satisfies the monoid axiom.

Proof. Apply Proposition [8.2.1] to the category $L_C(\mathcal{M})$. By the proposition just proven, $L_C(\mathcal{M})$ is $h$-monoidal. It is left proper because $\mathcal{M}$ is left proper.

The argument of Proposition [8.2.1] is to be applied to $\lambda$-sequences of maps which are pushouts of maps in $\{ f \otimes Z \mid f \text{ is a trivial cofibration in } L_C(\mathcal{M}) \}$. Such maps are $h$-cofibrations in $\mathcal{M}$ because $\mathcal{M}$ is $h$-monoidal, $f$ is a cofibration in $\mathcal{M}$, and $h$-cofibrations are closed under pushout. Thus, the hypothesis that the (co)domains of $I$ are finite relative to the $h$-cofibrations in $\mathcal{M}$ is sufficient to construct the lift in Proposition [8.2.1] and to prove the transfinite composition part of the proof of the monoid axiom. \qed
CHAPTER 9

Conclusion

We have introduced the notion of a monoidal Bousfield localization as one which preserves the pushout product axiom and the axiom that cofibrant objects are flat. We have characterized such localizations, en route introducing a lemma which helps prove model structures are tractable. We have provided a general result regarding when monoidal Bousfield localizations preserve algebras over operads, we have applied this to recover and generalize numerous known results regarding preservation of algebras over cofibrant operads in spaces and spectra, and we have obtained new preservation results in the context of equivariant spectra. We have introduced a collection of equivariant operads which interpolates between naive $E_\infty$ operads and genuine $E_\infty$ operads, and we have proven preservation results in this context as well.

We have provided general conditions under which the category of commutative monoids in a monoidal model category $\mathcal{M}$ inherits a model structure and further conditions so that a cofibration of commutative monoids forgets to a cofibration in $\mathcal{M}$. This resolves a problem which has been open for fifteen years. We have provided numerous examples of model categories which satisfy the commutative monoid axiom, unifying many situations in which commutative monoids were known to inherit a model structure and also providing several new examples, resolving an open problem from [66]. We have introduced a new axiom on a model category which we conjecture will imply rectification between the operads $Com$ and $E_\infty$, and we have discussed this conjecture in the known...
examples. Additionally, we have proven results regarding functoriality of the passage from \( R \) to commutative \( R \)-algebras, the interplay between the rectification axiom and the \( \text{Sym} \) functor, and when Bousfield localization preserves the commutative monoid axiom. We have given numerous examples of localizations which preserve commutative monoids, again recovering known examples and providing new examples. We have conjectured how to extend this work to general non-cofibrant operads.

Lastly, we have provided conditions under which Bousfield localization preserves the monoid axiom. En route we introduce the notions of \( h \)-cofibrations and \( h \)-monoidal model categories (which as appeared independently in \([8]\)) and provided conditions so that \( h \)-monoidality is preserved by Bousfield localization. We have collected numerous new examples of \( h \)-monoidal model categories, en route introducing injective model structures on (equivariant) orthogonal spectra. We end now by stating some questions which we hope to work on in the near future.

### 9.1. Bousfield Localization and Operad Algebras

In Section 6.1 we produced an inherited model structure on commutative monoids. In Chapter 7 we provided conditions so that Bousfield localization preserves the commutative monoid axiom. One of the greatest frustrations of this thesis was the inability to remove the assumption that \( \mathcal{M} \) is simplicial in Theorem 7.1.4

**Conjecture 9.1.1.** Suppose \( \mathcal{M} \) is a tractable model category satisfying the strong commutative monoid axiom. Suppose that for all \( n \in \mathbb{N} \) and \( f \in \mathcal{C} \), \( \text{Sym}^n(f) \) is a \( \mathcal{C} \)-local equivalence. Then \( L_{\mathcal{C}}(\mathcal{M}) \) satisfies the strong commutative monoid axiom.
This may be possible, but it may require some hard work involving the theory of framings. In Section 6.6 we also provided axioms which allow for inherited model structures on algebras over general operads, but we said nothing about preservation of these axioms under localization. This leads to our next three problems.

**Conjecture 9.1.2.** Suppose $\mathcal{M}$ is a nice model category satisfying the $P$-algebra axiom of Definition 6.6.1. Suppose the set of maps $\mathcal{C}$ satisfies the property that for all $f \in \mathcal{C}$, for all $A \in P\text{-alg}$, and for all $n$, then $P_A(n) \otimes \Sigma_n f^{\otimes n}$ is contained in the $\mathcal{C}$-local equivalences. Then $P$-algebras valued in $L_{\mathcal{C}}(\mathcal{M})$ inherit a model structure from $L_{\mathcal{C}}(\mathcal{M})$.

Surely this will require some hypotheses on $\mathcal{M}$, e.g. that $\mathcal{M}$ is simplicial, $h$-monoidal, tractable, and has compact (co)domains for the generating cofibrations $I$. Our next two problems are related:

**Problem 9.1.3.** Find conditions on $\mathcal{M}$ and $\mathcal{C}$ so that the general operad axioms of Theorem 6.6.3 are passed from $\mathcal{M}$ to $L_{\mathcal{C}}(\mathcal{M})$.

**Problem 9.1.4.** Generalize Theorem 6.6.3 to colored operads, and perhaps even to PROPs. After this is done, address the conjecture above in these contexts.

Another problem related to monoidal Bousfield localizations was mentioned in Chapter 4. Recall that [19] provides another condition on a Bousfield localization which implies that it preserves certain algebraic structure.

**Problem 9.1.5.** Determine the connection between monoidal Bousfield localizations and the closed localizations of [19].
The work in this thesis allows for the consideration of the category $P$-$\text{alg}(L_C(M))$ of $P$-algebras in the model category $L_C(M)$, for $P$ a cofibrant operad (Chapter 5), for $P = \text{Com}$ (Chapter 7), and potentially for general non-cofibrant operads $P$. In [5], Michael Batanin and Clemens Berger have provided conditions so that algebras over various operads admit Bousfield localization, i.e. so that the model categories $P$-$\text{alg}(M)$ are left proper. To be precise, the following theorem is proven:

**Theorem 9.1.6.** For any tame polynomial monad $T$ in sets and any compactly generated monoidal model category $M$ fulfilling the monoid axiom, the category of $T$-algebras in $M$ admits a relatively left proper transferred model structure. This model structure is left proper if $M$ is strongly $h$-monoidal.

This allows for the consideration of model categories of the form $L_C(P$-$\text{alg}(M))$, where $C$ is a set of $P$-algebra homomorphisms. It is therefore natural to ask to what extent passage to $P$-algebras and localization commute.

**Conjecture 9.1.7.** Suppose $M$ is a monoidal model category satisfying the hypotheses of Theorems 6.1.2 and 9.1.6. Suppose $P$ is an operad such that the free operad functor is a tame polynomial monad (e.g. $P = \text{Ass}$). Suppose $C$ is a monoidal Bousfield localization which preserves the $P$-algebra axiom. Then there is a Quillen equivalence between commutative monoids in $L_C(M)$ and the model category $L_{P(C)}(P$-$\text{alg}(M))$.

It is possible that if $C$ is a set of $P$-algebra homomorphisms already then it will be unnecessary to apply the free $P$-algebra functor to $C$. This question merits further study.

In fact, as Theorem 9.1.6 applies to more general monads than those coming from free operad-algebra functors, one can attempt to generalize in a different direction.
9.1. Bousfield Localization and Operad Algebras

**Question 9.1.8.** Can axioms similar to those needed to ensure model categories on algebras over an operad be given so that algebras over a (tame) polynomial monad inherit a model structure? What conditions are needed on \( \mathcal{C} \) so that \( L_\mathcal{C} \) preserves such axioms?

Once this question has been answered we can attempt to resolve Conjecture 9.1.7 in the full generality of Theorem 9.1.6. One can then apply this work to the ongoing work of Batanin and Berger towards understanding \( n \)-operads, Day-Street convolutions, the Stabilisation Hypothesis of Baez-Dolan, and the general intersection of categorical algebra, localization, and homotopy theory.

There are also still problems related to Bousfield localization in non-monoidal settings which we hope to study. For instance, it has recently been proven in [34] that under mild conditions coreflective subcategories of a model category \( \mathcal{M} \) may be endowed with a model structure in which weak equivalences, fibrations, and cofibrations are inherited from \( \mathcal{M} \). One way to view the image of a Bousfield localization is as the subcategory of fibrant objects, which in this case is a reflexive subcategory.

**Question 9.1.9.** Are there conditions so that a reflexive subcategory of a model category \( \mathcal{M} \) inherits a model structure from \( \mathcal{M} \)? How is this model structure related to \( L_\mathcal{C}(\mathcal{M}) \)?

Normally, the category of fibrant objects in a model category (e.g. \( L_\mathcal{C}(\mathcal{M}) \)) is a fibration category, so the question above is related to the rigidity chain mentioned in Problem 9.2.9 below.
In this thesis we have not addressed right Bousfield localizations (a.k.a. colocalizations) at all. The theory expounded in [38] allows for the consideration of coalgebras over comonads.

**Problem 9.1.10.** Dualize the results in this thesis to provide conditions so that right Bousfield localization preserves the structure of coalgebras over comonads.

### 9.2. Rectification and Infinity Categories

Recall that for any cofibrantly generated model category $M$ we can define the cofibrant replacement $QCom$ for $Com$ and $QCom$-algebras will inherit a model structure if $M$ is a monoidal model category satisfying the monoid axiom (by Theorem 5.0.7).

Recall the rectification axiom of Definition 6.2.5 which generalized the following property enjoyed by good monoidal categories of spectra: for all cofibrant $X$, $(E\Sigma_n)^+ \wedge \Sigma_n X^{\wedge n} \to X^{\wedge n}/\Sigma_n$.

**Conjecture 9.2.1.** Suppose $M$ is a nice model category satisfying the commutative monoid axiom and the rectification axiom. Then there is a Quillen equivalence between the model categories of commutative monoids in $M$ and $QCom$-algebras in $M$.

As mentioned in Subsection 6.2.2, it appears to be possible to force the rectification axiom to hold by applying Bousfield localization.

**Problem 9.2.2.** Work out the theory of the nearest model structure to $M$ satisfying the rectification axiom via Bousfield localization. Determine what this procedure does to known examples where the rectification axiom fails, e.g. $Top$, $sSet$, $W$-spaces.
There is also an ∞-category theoretic version of the rectification problem, and it is open even for the case of spectra. Here by ∞-category we mean quasi-category in the sense developed by Joyal ([53], [52]) and Lurie ([62]). A rectification result was part of Lurie’s goal in [61], but due to the error noticed in [63], Lurie’s machinery cannot be used to resolve this rectification problem for symmetric spectra.

Let us fix some notation. If \( M \) is a symmetric monoidal model category then let \( M[\mathcal{W}^{-1}] \) denote the corresponding ∞-category. Lurie proves in [60] that \( M[\mathcal{W}^{-1}] \) is a symmetric monoidal ∞-category, so we denote it \( M[W^{-1}]^\otimes \). Let \( \mathcal{O} \) be a simplicial operad and denote by \( \mathcal{O}^\otimes \) the corresponding ∞-operad (this is a small abuse of notation as we are hiding the use of the homotopy coherent nerve).

If \( \text{Alg}_\mathcal{O}(M) \) is a model category then one gets an ∞-category \( \text{Alg}_\mathcal{O}(M)[W^{-1}_\mathcal{O}] \), because work of Lurie in [60] shows how to get from a strict \( \mathcal{O} \)-algebra to an \( \mathcal{O}^\otimes \)-algebra in the ∞-category \( M[W^{-1}]^\otimes \). One should always get an ∞-category \( \text{Alg}_{\mathcal{O}^\otimes}(M[W^{-1}]^\otimes) \) because algebras over any ∞-operad in any symmetric monoidal ∞-category again form an ∞-category.

**Problem 9.2.3.** When is the natural map \( \text{Alg}_\mathcal{O}(M)[\mathcal{W}^{-1}_\mathcal{O}] \to \text{Alg}_{\mathcal{O}^\otimes}(M[\mathcal{W}^{-1}]^\otimes) \) an equivalence of ∞-categories?

When \( \mathcal{O} = \text{Com} \), Lurie proves that the answer to this question would be affirmative if symmetric spectra satisfied his version of the commutative monoid axiom.

**Conjecture 9.2.4.** The problem above has an affirmative answer for the operad \( \mathcal{O} = \text{Com} \) when \( M \) satisfies the commutative monoid axiom and the rectification axiom.
9.2. Rectification and Infinity Categories

The first step towards resolving this conjecture is to work it out in the case where $\mathcal{M}$ is the positive flat model structure on symmetric spectra. The author believes he can prove it in this case. The rectification axiom is needed because the statement should not be true for the infinity category of spaces, by Example 6.2.4. This example raises another question, regarding how best to deal with non-cofibrant operads in the $\infty$-categorical setting.

**Question 9.2.5.** Let $\mathcal{O}$ be an arbitrary operad and let $\mathcal{O}^\otimes$ be the corresponding $\infty$-operad. Is there an $\mathcal{O}'$, weakly equivalent to $\mathcal{O}$, such that $\text{Alg}_{\mathcal{O}}(\mathcal{M})[\mathcal{W}^{-1}]$ equivalent to $\text{Alg}_{\mathcal{O}^\otimes}(\mathcal{M}[\mathcal{W}^{-1}]^\otimes)$ as an $\infty$-category?

A related problem is to determine just how nice the situation is for cofibrant operads.

**Problem 9.2.6.** Given a cofibrant operad $P$ prove that $\text{Alg}_P(\mathcal{M})$ is equivalent as an $\infty$-category to $\text{Alg}_P(\mathcal{M}[\mathcal{W}^{-1}])$, i.e. it doesn’t matter if we view $P$ as an $\infty$-operad or as an operad.

Another question which is natural, given the considerations in this thesis, is:

**Question 9.2.7.** What can be said when $\text{Alg}_{\mathcal{O}}$ is only a semi-model category?

Because the $\infty$-category cannot tell the difference between a model category and a semi-model category (e.g. because the difference only manifests in the trivial cofibrations), there should be an $\infty$-category associated to any semi-model category. It is not clear whether or not one should expect rectification results to hold in this context.

A different form of rectification is recovering a model category from an $\infty$-category. A consequence of work of Dugger [23] is that every presentable $\infty$-category $\mathcal{D}$ gives rise...
to a combinatorial model category \( \mathcal{M} \) whose associated \( \infty \)-category is equivalent to \( \mathcal{D} \). It would be nice to know if there is a monoidal version of this result:

**Problem 9.2.8.** Prove that any symmetric monoidal \( \infty \)-category rectifies into a symmetric monoidal combinatorial model category.

In fact, this sort of consideration opens up a number of related problems.

Semi-model structures are not the only level of structure less than that of a model structure but more than an \( \infty \)-category. Other times only a Brown category structure can be obtained (e.g. for \( C^\ast \)-algebras in \([90]\)). Dually, there is the notion of a cofibration category (e.g. for the category of topological spaces with proper maps in \([88]\)), and the weaker notion of a category of cofibrant objects (due to Waldhausen \([92]\)). Recently, different models for \((\infty,1)\)-theory have been proposed by Barwick and Kan, including partial model categories \([3]\) and relative categories \([4, 5]\). From the point of view of infinity categories these differences are invisible.

**Problem 9.2.9.** Develop the theory of the chain of 2-categories connected by forgetful 2-functors:

\[
\begin{array}{ccc}
\text{ModelCat} & \xrightarrow{\text{CofibrationCat}} & \text{CatOfCofibObjects} \\
\downarrow & & \downarrow \\
\text{SemiModelCat} & \xrightarrow{\text{CofibrationCat}} & \text{CatOfCofibObjects} \\
\downarrow & & \downarrow \\
\text{BrownCat} & \xrightarrow{\text{PartialModelCat}} & \text{RelCat}
\end{array}
\]
Question 9.2.10. Can these 2-categories be made into \((\infty, 2)\)-categories? Are there \((\infty, 2)\)-functors between them?

This chain gives a number of rigidity problems which one may study. For instance, if two model categories are Quillen equivalent as semi-model categories then that Quillen equivalence lifts to the model category level. There are also rigidity problems for categories rather than functors:

Question 9.2.11. To what extent can a given semi-model structure be embedded canonically into a model structure?

Recent work of Ching and Riehl \([21]\) proves that every combinatorial model category is Quillen equivalent to one in which all objects are cofibrant. Recalling that a semi-model category in which all objects are cofibrant is a model category, it is natural to try to use this to resolve the following problem:

Problem 9.2.12. Prove that every combinatorial semi-model category is Quillen equivalent to a model category.

Question 9.2.13. Is there a monoidal version of this Quillen equivalence to a model category with all objects cofibrant?

It would be good to determine how much structure can be lifted in other spots of the chain.

All of the 2-categories above can be viewed as models for the homotopy theory of homotopy theories. In \([46]\), Hovey proves that ModelCat is a 2-category and one of his
vistas is to develop a notion of 2-model category. The author does not know how to do this, but it does seem likely that there is a 2-relative category of model categories. If one accepts this philosophy then it gives a way to approach a different vista in [46], namely whether or not the homotopy theory of model categories is equivalent to the homotopy theory of simplicial model categories.

**Problem 9.2.14.** Prove that there are 2-relative categories of partial model categories and of simplicial partial model categories. Prove that they are equivalent as 2-relative categories.

It seems likely that the category $sPartialModelCat$ of simplicial partial model categories is a good model for $(\infty, 1)$-categories, i.e. satisfies the axioms of the Unicity Theorem of Barwick and Schommer-Pries [7]. If so, then this would provide a Quillen equivalence between $sPartialModelCat$ and $PartialModelCat$, which would give a place to start towards the solution of this problem.

There are also still open questions on the model category level. Twice large classes of model categories have been proven to admit Quillen equivalent simplicial model categories: [24] and [75].

**Question 9.2.15.** Is every model category Quillen equivalent to a simplicial model category?

Are there classes of monoidal model categories which admit monoidally Quillen equivalent simplicial monoidal model categories?
9.3. Equivariant Stable Homotopy Theory

**Problem 9.3.1.** Prove that there are positive flat stable model structures $Sp^f$. Prove that these model structures satisfy the commutative monoid axiom.

We note that Theorem 2.3.30 in [87] has produced positive flat stable model structures relative to a family under certain hypotheses on the family and on the weak equivalences. So this may be a natural place to begin this problem. Once this has been done, one can attempt to generalize Theorem A.5 from [14] to the family setting:

**Problem 9.3.2.** Prove that rectification holds between the operad $E^\mathcal{F}_\infty$ and $Com$ in $Sp^f$.

In the vein of better understanding the connection between this thesis and [14], we pause for a moment to discuss the $N_\infty$-operads which are studied in [14].

**Definition 9.3.3.** An $N_\infty$-operad is a $G$-operad $P$ such that $P_0$ is $G$-contractible, the action of $\Sigma_n$ on $P_n$ is free, and $P_n$ is the universal space for a family $\mathcal{N}_n(P)$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times 1$.

Three related problems:

**Problem 9.3.4.** Use the techniques in this thesis to find model structures on $G$-Oper and $Sp^G$ so that $N_\infty$-operads become cofibrant and hence admissible.

**Problem 9.3.5.** Characterize $E^\mathcal{F}_\infty$-algebras and understand the relationship between $E^\mathcal{F}_\infty$-operads and the $N_\infty$-operads of [14].
Problem 9.3.6. Use the methods in this thesis to approach the main open problem left by [14], i.e. to determine whether or not all homotopy types of $N_{\infty}$-operads are realized by the equivariant little disks or equivariant linear isometries operads.

A first step towards solving these problems might be to introduce generalized $N_{\infty}$-operads which allow for the families $\mathcal{F}_n(P)$ to intersect $G$ in a family of subgroups $\mathcal{F}$ of $G$ other than the family $\mathcal{F} = \{All\}$.

Definition 9.3.7. Let $\mathcal{F}$ be a family of closed subgroups of $G$. A generalized $N_{\infty}$-operad is a $G$-operad $P$ such that $P_0$ is $G$-contractible, the action of $\Sigma_n$ on $P_n$ is free, and $P_n$ is the universal space for a family $\mathcal{N}_n(P)$ of subgroups of $G \times \Sigma_n$ which contains all subgroups of the form $H \times 1$ where $H \in \mathcal{F}$.

$N_{\infty}$-operads capture universe-related information while $E^{\mathcal{F}}_{\infty}$-operads capture family-related information. Therefore generalized $N_{\infty}$-operads may see the entirety of the stable equivariant homotopy category and may be helpful in answering the following series of questions.

Problem 9.3.8. Better understand the interplay between family model structures and universe model structures.

This is a huge problem. The first step is to note that for any universe $\mathcal{U}$ one can consider the family of subgroups with isotropy in $\mathcal{U}$. Similarly, one may obtain a universe from a given family. A comparison of the resulting model structures would be the natural place to begin this problem.
Another place to approach this problem is the following construction. Let $\mathcal{U}$ be a complete universe and look at the $\mathcal{U}$-level model structure (with respect to the family $\mathcal{F} = \{ All \}$) from [51]. Use Remark 3.13 from [78] to define weak equivalences relative to any fixed subuniverse $\mathcal{V}$ of $\mathcal{U}$ and to any family $\mathcal{K}$. These homotopy groups are bigraded, with one grading for the family and one for the universe. Choose a universe $V$ and a family $\mathcal{F}$. Consider the class of maps $\mathcal{M}_{\mathcal{F}, \mathcal{V}}$ for which those $(\mathcal{F}, \mathcal{V})$ homotopy groups are isomorphisms.

**Question 9.3.9.** Is it possible to apply Bousfield localization to this class?

If so, then this construction yields a bigraded sequence of model structures on the category of $G$-spectra. The initial model structure is the $\mathcal{U}$-level model structure (it has the fewest weak equivalences) and the terminal one is in some sense a naive model structure on $G$-spectra. This construction should also work stably.

**Question 9.3.10.** Are there Quillen equivalences between any of the model structures in this bigraded sequence and any existing model structures on $G$-spectra? Are there Quillen equivalences between any two different choices of $\mathcal{F}, \mathcal{V}$? If so, what does this tell us about the relationship between $\mathcal{F}$ and $\mathcal{V}$? Are these model structures are related to $\mathcal{E}_{\infty}^{\mathcal{F}}, N_{\infty}$, and generalized $N_{\infty}$-operads?

### 9.4. Further Examples

In this thesis we did not consider model categories of simplicial presheaves. This example is becoming increasingly important. In [81] the monoid axiom passed from $sSet$ to simplicial presheaves purely formally.
Problem 9.4.1. Determine whether or not the commutative monoid axiom holds for simplicial presheaves. Determine whether or not the operadic generalizations of the commutative monoid axiom introduced in Section 6.6 hold for simplicial presheaves.

Furthermore, we hope that this theory could generalize all the way to the so-called excellent model categories of [60], again in a mostly formal way.

We have seen that all operads valued in simplicial sets are admissible.

Problem 9.4.2. Is there a canonical way to replace an operad by a simplicial operad whose algebras are the same? Are there conditions on \( \mathcal{M} \) to guarantee such replacement holds?

Once we understand the situation for simplicial presheaves we can attempt to generalize the work in this thesis to hold for motivic situations.

Problem 9.4.3. Verify the commutative monoid axiom for motivic spaces over any base scheme \( B \).

Problem 9.4.4. Find model structures of motivic symmetric spectra in which the commutative monoid axiom holds.

The author has already made some progress on the latter problem. In joint work with Markus Spitzweck, the author has developed positive flat model structures on symmetric spectra built on a combinatorial model category \( \mathcal{M} \). However, a similar program has been carried out in [44], requiring different hypotheses on the base category \( \mathcal{M} \). So the first step towards solving this problem would be to compare the two approaches.
We conclude with one more problem related to motivic spectra. In Subsection 2.6.3 we developed model structures on the category of $G$-operads and obtained operads $E^\otimes$ via cofibrant replacement. The same program could be carried out in the motivic setting. The category of motivic spaces is combinatorial, so Theorem 12.2.A from [29] may be used to place a semi-model structure on the category of operads valued in motivic spaces over a base scheme $B$. We denote this semi-model structure by $\text{Oper}_B$.

**Definition 9.4.5.** Define the motivic $E_\infty$-operad $E^B_\infty$ to be the cofibrant replacement of $\text{Com}$ in the semi-model category $\text{Oper}_B$.

If this operad is going to behave like $E^\otimes$ then Chapter 4 of [69] hints at what the $n^{th}$ space should be and mentions an open problem which we state below.

**Conjecture 9.4.6.** The $n^{th}$ space of $E^B_\infty$ is homotopy equivalent to the étale classifying space of $\Sigma_n$ in the category of motivic spaces over $B$.

**Problem 9.4.7.** Use the operads $E^B_\infty$ to resolve the $\mathbb{P}^1$-infinite loop space recognition problem, i.e. characterize $\mathbb{P}^1$-infinite loop spaces up to homotopy as the group-like $E^B_\infty$ algebras in the category of $B$-spaces.

We hope to pursue these questions in the future.
Bibliography


