

## **Acknowledgments**

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## Abstract

In this paper, we will extend the works of Collins and Trenk, by finding the distinguishing chromatic number of the graph  $C_n \circ I_m$ ,  $n, m \in \mathbb{N}$ . We will first look at a couple of examples, starting with  $m = 2$ , and providing a theorem for  $C_n \circ I_2$ , while also offering an example of a coloring scheme for each  $n$  and explaining why the scheme works. We will then look at the case for  $m = 3$ , followed by the general case.

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# Chapter 1

## Introduction

The notion of the distinguishing chromatic number of a graph  $G$ ,  $\chi_D(G)$ , was first used in the paper by Collins and Trenk, [4]. But before we actually start talking about the distinguishing chromatic number, we first need to understand what the chromatic number and what the distinguishing number are of a graph.

In the simplest terms, the chromatic number of a graph is the smallest number of colors needed to color the vertices of a graph so that if there is an edge between two vertices, the two have to be colored differently. The most common real world problem that deals with chromatic numbers is the scheduling problem. The following is a variation of this problem. Given a list of tasks, say at a high school shop class, we can represent each task by a vertex. If tasks must be done at the same time, then their corresponding vertices would be adjacent. The chromatic number of this graph would then give us the minimum number of stations needed for the class.

Distinguishing number is a term was first introduced in 1996, by Albertson and Collins, [1]. To understand what the distinguishing number really is, we will look at what it was originally introduced for: To solve the classic key ring problem. Let's say we have a key ring with  $n$  different keys. But what if they aren't quite that different. I mean, they open different doors, but sometimes they all just look the same. We would then have to do something to them to differentiate all  $n$  keys. So we assign a color to a key to assist us to tell the difference. But what is the

least number of colors that we will need to actually be able to identify each key? The problem can be solved by finding the distinguishing number of the graph  $C_n$ , a cycle with  $n$  vertices. The results are quite surprising. When  $n$  is equal to 3, 4 or 5,  $D(C_n) = 3$ , while any  $n > 5$  yields a distinguishing number of 2. A proof is provided in [4], by Collins and Trenk.

And finally, we have the distinguishing chromatic number. Like the name suggests, the distinguishing chromatic number is the smallest number of colors needed so that a graph is distinguishable and properly colored. To see how this can be used in the real world, we will again look at the scheduling problem. In the example above, we saw that the chromatic number gave us the minimum number of stations that is needed for the class. The distinguishing number however, would uniquely identify the tasks and the station the task would be performed in. In the Collins and Trenk paper, [4], they proved and gave the distinguishing chromatic number of a few basic graphs, e.g. cycles, complete graphs and paths, while providing bounds for trees and connected graphs.

The rest of this paper will involve all three terms. We will first look at a major theorem in graph theory, the Four Color Theorem, which proves that all planar graphs are 4-colorable. Along with that there are a few selected theorems that I found noteworthy within the realms of chromatic numbers, distinguishing numbers and distinguishing chromatic numbers. Additionally, we will look at the proofs of three variations of Brooks' Theorem, one for each kind of number, which gives us a bound on numbers.

Following that, we will look at a question posed by Thomas Tucker to Karen Collins, which was to find the distinguishing chromatic number of a graph that is a bit more interesting than the ones in [4]: the lexicographic product of an

$n$ -cycle,  $C_n$  and a set of independent vertices of size  $m$ ,  $I_m$ ,  $n, m \in \mathbb{N}$ , denoted by  $C_n \circ I_m$ .

## Chapter 2

### Chromatic Numbers

In this section of the paper we will look at the chromatic number of a few basic graphs. We will then look at a few major theorems concerning the chromatic numbers, which include the Four Color Theorem, Color Mapping Theorem and Brooks' Theorem. Before we continue, we will look at the formal definition of the chromatic number.

**Definition 2.0.1.** The chromatic number of a graph,  $G$ , denoted by  $\chi(G)$ , is the smallest number of colors needed to properly color a graph. A graph is properly colored when adjacent vertices are differently colored.

The following table gives us the chromatic number of a tree, cycle, complete graph and the Petersen graph:

$G$	$\chi(G)$
Path, $P_n$	2
Cycle, $C_{2n}$	2
$C_{2n+1}$	3
$K_n$	$n$
Petersen Graph	3

## 2.1 The Four Color Theorem

One of the most famous theorems in graph theory is the Four Color Theorem. This came about from a question posed by Guthrie in 1852. Guthrie basically asked his teacher, de Morgan, if it is possible to color a planar map with four colors without bordering regions having the same color. For over a century many attempted to provide a proof, but all came up short of actually proving it. But before we talk about the proof of the Four Color Theorem, we will look at a similar theorem, the Five Color Theorem, which was proven by Heawood in 1890.

**Theorem 2.1.1** (The Five Color Theorem - Heawood). *The chromatic number of a planar graph is at most 5.*

A proof of this theorem uses induction on the order of the graph. The base case looks at all graphs with five or fewer vertices, which are all 5-colorable. Then for the inductive step, it uses a variation of Euler's formula, which states that if  $G$  is a simple planar graph with  $|V(G)| \geq 3$ , then  $|E(G)| \leq 3|V(G)| - 6$ , which forces the maximum degree of a vertex,  $v$ , to be at most 5. The proof continues by looking at the worst case scenario, which is when the maximum degree of a vertex equals to 5, and supposing that the five vertices adjacent to  $v$  have different colors. It then looks at the ways in which planar graphs can grow from those six vertices. No matter how the graph is drawn, one of the vertices adjacent to  $v$  can always be colored the same as one of the other four, which frees up that color for  $v$ . Hence all planar graphs are 5-colorable.

Now for the Four Color Theorem.

**Theorem 2.1.2** (The Four Color Theorem). *Every planar graph is 4-colorable.*

One could try to prove this theorem using the same technique used in proving the Five Color Theorem, but the proof would fall apart. This was exhibited in an attempt by Kempe in 1879, which was then proven wrong in 1890 by Heawood. It wasn't until 1976 that a proof of the theorem was created. Using computers, Appel, Haken and Koch came up with a set of 1936 'unavoidable configurations' which all graph triangulations must contain. And from those configurations, an inductive proof showed that all 1936 of them can be 4-colored, and so all planar graphs are 4-colorable. In 1996, Robertson, Sanders, Seymour, and Thomas reduce the number of 'unavoidable configurations' to 633, making calculations a little faster.

## 2.2 Brooks' Theorem for Chromatic Numbers

Stepping out of the world of planar graphs, we would encounter graphs with chromatic numbers greater than four. However, as we will see, all graphs have a bound to their chromatic numbers. This is exhibited in the following theorem by Brooks in 1941. The proof follows the one given in [10].

**Theorem 2.2.1** (Brooks). *Let  $G$  be a connected graph. Let  $\Delta$  be the maximum degree of a vertex in  $G$ . If  $G$  is complete or an odd cycle, then  $\chi(G) = \Delta + 1$ . Otherwise,  $\chi(G) \leq \Delta$ .*

*Proof.* Let  $\Delta = 1$  or 2. If  $\Delta = 1$ , then  $G$  must be  $K_2$ , which has  $\chi(G) = 2$ . If  $\Delta = 2$ , then  $G$  is either a cycle or a path. If  $G$  is an even cycle or a path, then  $\chi(G) = 2$ . Otherwise,  $G$  is an odd cycle, which has  $\chi(G) = 3$ . All these graphs fit our bounds.

Now let  $\Delta \geq 3$ . We will look at two cases:  $G$  is regular and  $G$  is not.

Case 1- If  $G$  is not regular, then there exists some vertex  $v$  such that the degree of  $v < \Delta$ . Using  $v$ , we make a spanning tree, with  $v$  being the root. From  $v$ , every vertex following it can be adjacent to at most  $\Delta - 1$  vertices that are already colored. Therefore we can always color  $v$  with the  $\Delta^{th}$  color.

Case 2- Now let  $G$  be regular. Assume we have a cut vertex,  $v$ . Then we look at one of the components of  $G - v$ , call it  $H$ . Let  $H' = H \cup v$ , so that means  $H'$  consists of the component we picked,  $v$  and all the edges between the two. Since  $G$  is regular,  $deg(v) < \Delta$  in  $H'$ . Therefore we can follow what we did when  $G$  is not regular, since  $H'$  is now not regular. Continue doing this for all the components.

Next, we assume  $G$  is 2-connected. Suppose there exists a  $v$  such that  $v$  is adjacent to both  $x$  and  $y$ , but  $x$  is not adjacent to  $y$ , with  $G - x, y$  connected. We can then follow the case when  $G$  is not regular, since we can free up a color by coloring  $x$  and  $y$  the same color.

So now, all we need is to show is that we can always find a  $v, x$ , and a  $y$  for every 2-connected  $\Delta$ -regular graph. Choose a vertex  $u$  in  $G$ . If  $G - u$  is 2 or more connected, then let  $x$  be  $u$ . Since the graph is not complete and it is regular, there exists a pair of vertices  $v$  and  $y$  such that  $v$  is adjacent to  $x$  and  $y$ , but  $x$  not to  $y$ . Therefore, we have our  $v, x$ , and  $y$ . If  $G - u$  is 1-connected, we let  $v$  be  $u$ . Since  $G$  has no cut vertex and  $G - u$  is 1-connected,  $G - u$  consists of  $\Delta$  blocks such that each block is connected and are connected by a single vertex, so that removing one of these vertices will result in a disconnected graph. That means there are always two blocks that are separated by at least one block. Pick a vertex from each of the two

that isn't a cut vertex in  $G - u$ . Name them  $x$  and  $y$ . Therefore, we have our three vertices, and so proves Brooks' Theorem.

□

### 2.3 Map Color Theorem

So we know that all planar graphs are 4-colorable, but what about graphs that can be embedded into non-planar surfaces? In 1890, Heawood came up with a theorem that determines the chromatic number of graphs that are embeddable in certain genera of surfaces. When looking at the genus of a surface,  $\gamma$ ,  $T_\gamma$  would represent a sphere with  $\gamma$  handles. So if we looked at  $T_0$ , that would be a sphere,  $T_1$  would be a torus,  $T_2$  would be a double torus, and so on.

**Theorem 2.3.1** (Heawood's Formula). *If  $G$  embeds on  $T_\gamma$ , then*

$$\chi(G) \leq \lfloor \frac{7 + \sqrt{1 + 48\gamma}}{2} \rfloor.$$

This is a sketch of the proof provided in [10]. The proof of this theorem starts out pretty much the same as the Five Color Theorem, in that we will use induction on  $|V(G)|$  and use a variation of Euler's formula for surfaces. Let  $c = (7 + \sqrt{1 + 48\gamma})/2$ . Since the theorem looks at the floor function of  $c$ , we know that  $\chi(G) \leq c$  and so we want that the maximum degree of the vertices in  $G$  to be at most  $c - 1$ . Using induction, the base case would then be when  $|V(G)| \leq c$ . To finish the proof, we use Euler's formula for  $T_\gamma$ , which is  $|E(G)| \leq 3(|V(G)| - 2 + 2\gamma)$ . Using simple algebraic calculations and the average degree formula,  $2|E(G)|/|V(G)|$ , we get that the average degree of the vertices is at most  $c - 1$ .

## Chapter 3

### Distinguishing Numbers

We will now look at the distinguishing numbers of the same graphs from the previous section followed by a theorem by Tucker on 3-connected planar graphs and Brooks' Theorem. But first, we will define a distinguishing number.

**Definition 3.0.2.** The distinguishing number of  $G$ , denoted by  $D(G)$ , is the smallest number of colors needed to color  $G$ , so that there does not exist any automorphism of the graph which preserves the vertex coloring.

Here are the distinguishing numbers of some graphs:

$G$	$\chi_D(G)$
$P_n$	2
$C_{n \in \{4,5\}}$	3
$C_{n > 5}$	2
$K_n$	$n$
Petersen Graph	3

#### 3.1 Distinguishing Numbers of 3-Connected Planar Graphs

Recently, Tom Tucker looked at correlations between graph theory and algebraic topology. In the paper, Distinguishability of Maps, [9], Tucker presents some surprising results. One of these results is the following theorem.

**Theorem 3.1.1** (Tucker). *Let  $G$  be a 3-connected planar graph. Then  $D(G) = 2$ , unless  $G$  is  $K_4$ , where  $D(K_4) = 4$ , or if  $G$  is one of a finite number of graphs on 10 or fewer vertices, where then  $D(G) = 3$ .*

Tucker proves his theorem by looking at the distinguishability of vertex, edge and “angle” stabilizers, where “an angle at  $v$  is a triple of vertices  $uvw$  where  $uv$  and  $vw$  are edges,” [9]. Tucker also classifies the graphs when  $D(G) = 3$  in his paper, by studying two cases: vertex transitive, and non-vertex transitive maps.

### 3.2 Brooks Theorem for Distinguishing Numbers

Similar to chromatic numbers, there exists a Brooks’ Theorem for distinguishing numbers as well. The theorem was founded in two separate occasions, one by Collins and Trenk, and the other Klavžar, Wong and Zhu, [8], and both results were published in 2006. We will follow the proof of Collins and Trenk.

**Theorem 3.2.1.** *If  $G$  is a connected graph with maximum degree  $\Delta$ , then  $D(G) \leq \Delta + 1$ . Furthermore, equality is achieved if and only if  $G = K_\Delta, K_{\Delta,\Delta}$ , or  $C_5$ .*

*Proof.* The special cases of when  $G = K_\Delta, K_{\Delta,\Delta}$ , or  $C_5$  are proven by Theorem 2.2 in [4]. Also, [4] provides the distinguishing numbers for when  $\Delta = 1$  and 2, since the graphs that result are  $K_2$ , paths and cycles.

Now, let  $\Delta \geq 3$  for all connected  $G$  other than  $K_{\Delta+1}$  and  $K_{\Delta,\Delta}$ . We will now look at two cases: when  $G$  is not  $\Delta$ -regular and when is.

Case 1- If  $G$  is not regular, then there exists some vertex  $v$  such that  $\deg(v) < \Delta$ .

We then use  $v$  as the root and create a breadth-first spanning tree of  $G$ .

Color  $v$   $\Delta$ . We will use only the remaining colors,  $\{1, 2, 3, \dots, \Delta - 1\}$ , for the

rest of the graph. Now for every vertex that is adjacent to the same vertex (there should be at most  $\Delta - 1$ ), we will color each of them a different color. Since we can do this for all vertices in the spanning tree, we are done. By lemma 4.1 in [4], all connected regular graphs are  $\Delta$ -distinguishable.

Case 2a- Let  $G$  be  $\Delta$ -regular, there be a triple of vertices  $v, x, y$  such that  $vx, vy \in E(G)$ , and  $N(x) - y = N(y) - x$ , where  $N(x)$  denotes the neighborhood of  $x$ , which is the set of all vertices adjacent to  $x$ .

Suppose  $N(z) - y = N(y) - z$  for all  $z$  with  $vz \in E(G)$ . If  $xy \in E(G)$ , then  $G$  is forced to be a complete graph on  $\Delta + 1$  vertices. This violates our assumption that  $G$  is not  $K_{\Delta+1}$ . If  $xy \notin E(G)$ , then our assumption on the neighborhoods of all  $z$ 's would force the graph to become  $K_{\Delta, \Delta}$ , which also violates our earlier assumption that we are no longer looking at the graphs,  $K_{\Delta+1}$  and  $K_{\Delta, \Delta}$ .

So now we look at  $N(z) - y \neq N(y) - z$ . Such  $z, z \neq x, y$  and  $vz \in E(G)$ , exists because  $\Delta \geq 3$ . To make the argument more structured and easier to follow, we will let  $x$  be the first (leftmost) child of  $v$ , while  $y$  and  $z$  are the rightmost or last children of  $v$ . Since  $\deg(v) = \Delta$ , we  $v$  has  $\Delta$  children. We will color  $v$  with  $\Delta$  and the first  $\Delta - 2$  children with the set  $\{1, 2, \dots, \Delta - 2\}$ , with  $x$  being 1, and the siblings after  $x$  in order from 2 to  $\Delta - 2$ . Next, color both  $y$  and  $z$  with  $\Delta - 1$ . Then continue coloring the rest of the graph like we did in Case 1. Note that we can now differentiate all children of  $v$  and its descendants except for  $y$  and  $z$ , meaning we will only need to look at  $y$  and  $z$ , since the rest of the graph is distinguishable. We are done if we can find a way to distinguish  $y$  and  $z$  despite having the same color,  $\Delta - 1$ . We

will now look at an automorphism,  $\sigma$ , between  $y$  and  $z$ . By our assumption, there exists a  $w$  such that  $w \neq z$ , and  $wy \in E(G)$ , but  $wz \notin E(G)$ . This means that  $w = x$  or  $w \in N(x) - y = N(y) - x$ , which implies that  $w$  is one of the first  $\Delta - 2$  children of  $v$  or is a child of  $x$ . Both of these cases give us that  $w$  is distinguishable and that  $\sigma(w) = w$ . But now look at  $y$ . The automorphism  $\sigma(y) = z$ , which then gives us that  $\sigma(w)\sigma(y) = wz \in E(G)$ . Hence we get a contradiction.

Case 2b- Let  $G$  be  $\Delta$ -regular, and  $N(x) - y \neq N(y) - x$  for every triple of vertices  $v, x, y$  such that  $vx, vy \in E(G)$ .

Just like before, choose a vertex  $v$  as the root of our breadth-first spanning tree, color  $v$  with  $\Delta$ , and color the rest of the graph with the set  $\{1, 2, \dots, \Delta - 1\}$ . Again, color the first  $\Delta - 2$  children numerically, and the last two children,  $x$  and  $y$ , with  $\Delta - 1$ . Color the whole graph except for the descendants of  $y$  like in Case 1. This would distinguish all the vertices but not necessarily for  $x, y$ , and their descendants. If there exists no non-trivial automorphism that switches  $x$  to  $y$  or vice versa, then color the descendants of  $y$  distinctly. This would then give us a  $\Delta$ -distinguishable coloring.

Otherwise, there exists some  $\sigma(x)$  that switches  $x$  to  $y$  and vice versa. Let  $S_x$  and  $S_y$  be the set of vertices that are the children of  $x$  and  $y$ , respectively. Since there is an automorphism that interchanges  $x$  and  $y$ , it must then also interchange their children. Now we will show that there is at least one vertex in  $S_y$ , and then show a way to color  $S_y$  in two cases, depending on the size of  $S_y$ .

Since  $N(x) - y \neq N(y) - x$ , there exists some vertex  $z$  such that  $z \neq x$  and

adjacent to  $y$  but not to  $x$ . We know that  $z$  is not distinguishable since there is an automorphism that can switch  $x$  and  $y$ , but since  $z$  is not adjacent to  $x$ , we get that  $z \in S_y$ . So  $|S_y| > 0$ .

**Let**  $1 \leq |S_y| \leq \Delta - 2$ . Since there are less than  $\Delta - 1$  vertices in  $S_y$ , there is always at least 1 color that was used in  $S_x$  that is not in  $S_y$ . This would then distinguish  $x$  and  $y$ . Color the rest of the graph like we have been doing, and we get a  $\Delta$ -distinguishable coloring of  $G$ .

**Let**  $|S_y| = \Delta - 1$ . This means that both  $x$  and  $y$  have  $\Delta - 1$  children and none of which are the same. So we will color the first  $\Delta - 3$  children of  $y$ , 1 thru  $\Delta - 3$  and the last two vertices,  $u_1, u_2$ ,  $\Delta - 2$ . This differentiates all the vertices, including  $x$  and  $y$ , but not  $u_1$  and  $u_2$ , along with their descendants. Now looking at  $u_1$  and  $u_2$ , we will go through all the steps that we have done before: checking if there is a non-trivial automorphism that interchanges  $u_1$  and  $u_2$  and then the appropriate steps afterwards. This process might take a while, but it will stop at some point and give us the case where  $1 \leq |S_y| \leq \Delta - 2$ .

Therefore,  $D(G) \leq \Delta$  and  $D(G) = \Delta + 1$  if and only if  $G = K_\Delta, K_{\Delta, \Delta}$ , or  $C_5$ . □

## Chapter 4

### Distinguishing Chromatic Numbers

Like the chromatic numbers and distinguishing numbers, there is a Brooks' Theorem for distinguishing chromatic numbers, which we will again look at a proof for. We will start this section off by giving a formal definition of distinguishing chromatic number follow by distinguishing chromatic numbers of a few graphs.

**Definition 4.0.2.** The distinguishing chromatic number of a graph,  $G$ , denoted by  $\chi_D(G)$ , is the smallest number such that the graph is properly colored and distinguishable.

The following are the distinguishing chromatic numbers of a few graphs:

$G$	$\chi_D(G)$
$P_{2n}$	2
$P_{2n+1}$	3
$C_{n \in \{4,6\}}$	4
$C_{n > 5}$	3
$K_n$	$n$
Petersen Graph	4

#### 4.1 Brooks Theorem for Distinguishing Chromatic Numbers

Along with the Brooks' Theorem for distinguishing numbers, Collins and Trenk provides the Brooks' Theorem for  $\chi_D(G)$  in their paper, [4]. We will once again follow the proof by the two, which is very similar to their previous proof.

**Theorem 4.1.1.** *For all connected graphs  $G$ , with maximum degree  $\Delta$ ,  $\chi_D(G) \leq 2\Delta$ . Furthermore, equality is achieved if and only if  $G = K_{\Delta,\Delta}$ , or  $C_6$ .*

*Proof.* Again, the special cases,  $G = K_{\Delta,\Delta}$ , or  $C_6$ , are proven by Theorem 2.2 in [4], so we will now show that  $\chi_D(G) \leq 2\Delta - 1$  when  $G \neq K_{\Delta,\Delta}, C_6$ . Like before, [4] provides distinguishing chromatic numbers for graphs with  $\Delta = 1$  or 2, since the graphs that results are  $K_2$ , paths or cycles.

Let  $G$  be a connected graph with  $\Delta \geq 3$ , and not  $K_{\Delta,\Delta}$  nor  $C_6$ . Choose a vertex,  $v \in G$ , and let  $v$  be the root of our breadth-first spanning tree of  $G$ . We will call  $v$  the first generation of the spanning tree, while its children second generation, and so on.

Case 1-  $G$  has a vertex  $v$  of degree less than  $\Delta$ . Color  $v$  with  $2\Delta - 1$ , and use  $\{1, 2, \dots, 2\Delta - 2\}$  for the rest of the graph, making  $v$  distinguishable. We will color each set of vertices in a generation of  $v$  by starting at the left most vertex. For each of these vertices, we will color it with smallest number available that is different from its neighbors and siblings. The maximum number of neighbors and siblings each vertex would have is  $\Delta - 1 + \Delta - 2 = 2\Delta - 3$ , so this coloring will always work. By Lemma 4.1 in [4], this is a proper  $2\Delta - 1$ -distinguishing coloring.

Case 2a-  $G$  is  $\Delta$ -regular and there is a triple of vertices  $v, x, y$  such that  $vx, vy \in E(G)$  and  $N(x) - y = N(y) - x$ .

Suppose  $N(z) - y = N(y) - z$  for all  $z$  with  $vz \in E(G)$ . If  $xy \in E(G)$ , then by our assumption,  $G$  is forced to be a complete graph on  $\Delta + 1$  vertices, which has a  $\chi_D(G) = \Delta + 1 \leq 2\Delta - 1$ . However, if  $N(z) - y \neq N(y) - z$  then we get that  $G = K_{\Delta, \Delta}$ , which violates our earlier assumption.

So now we look at  $N(z) - y \neq N(y) - z$ , with  $z \neq x, y$  and  $vz \in E(G)$ . Let  $x$  be the first child of  $v$  and  $y, z$  the last. We will color  $G$  just like in Case 1, but note that in our second generation set of vertices, vertex  $z$  would get into trouble with the algorithm, since  $z$  has  $\Delta - 1$  neighbors and siblings, making that number  $2\Delta - 2$  total vertices. This means that all of its neighbors and siblings used up all the  $2\Delta - 2$  colors available. There exists a color for  $z$ , unless the siblings of  $z$  are distinct from the neighbors of  $z$ , and they all have different colors. In this case, the  $\Delta - 1$  neighbors of  $z$  is actually in the third generation, which implies that  $y$  and  $z$  aren't adjacent. This gives us that  $N(y) = N(y) - z$  and  $N(z) = N(z) - y$ , which yields  $N(y) \neq N(z)$ .

This would then let us color the first  $\Delta - 2$  vertices  $\Delta - 2$  distinct colors and then  $y$  and  $z$  the same color. But now there is a non-trivial automorphism that interchanges  $y$  and  $z$ . Since  $x$  is the first child of  $v$ , the vertices of  $N(y)$  is either  $x$  or the children of  $x$ . This then forces  $N(y)$  to be colored properly and distinguishable. Putting all the pieces together,  $N(y) \neq N(z)$  and  $N(y)$  is distinguishable, there is no non-trivial automorphism that switches  $y$  and  $z$ . So therefore we have a proper  $(2\Delta-1)$ -distinguishing coloring.

Case 2b-  $G$  is  $\Delta$ -regular, and  $N(x) - y \neq N(y) - x$  for every triple of vertices  $v, c, y$  such that  $vx, vy \in E(G)$ .

Again, choose  $v$  to be the root of our breadth-first spanning tree. We will color the graph like before, starting from the root and starting with the leftmost child, with all vertices distinct from all its neighbors and siblings. If we can do this for the whole spanning tree, then we are done and have a proper  $(2\Delta - 1)$ -distinguishing coloring. However, there will be instances, more often than not, that we will not have enough colors. So color the last two children of  $v$ ,  $x$  and  $y$ , the same color. But again, the set of siblings and neighbors of  $y$  are disjoint. And since  $x$  and  $y$  are not adjacent,  $N(x) = N(x) - y$  and vice versa.

Now, all the vertices are properly colored and distinguishable, except for  $x$ ,  $y$  and their descendants. So any non-trivial automorphism,  $\sigma$  would flip  $x$  and  $y$ . Let  $S_x$  denote the children of  $x$  which doesn't contain any neighbors of  $y$ , and  $S_y$  be the children of  $y$ . Since we are doing a breadth-first spanning tree,  $S_y$  would never contain neighbors of  $x$ . Note that if  $\sigma(x) = y$ , then  $\sigma(S_x) = S_y$ , and vice versa. We will now show that there is at least one vertex in  $S_y$ , and then show a way to properly color them in two cases, depending on the size of  $S_y$ .

Since  $N(x) \neq N(y)$ , there exists some vertex  $z$  such that  $z \neq x$  and adjacent to  $y$ , but not to  $x$ . Then  $z$  is not distinguishable since there is an automorphism that can switch  $x$  and  $y$ , but since  $z$  is not adjacent to  $x$ , we get that  $z \in S_y$ . So  $|S_y| > 0$ .

**Let**  $1 \leq |S_y| \leq \Delta - 2$ . Let  $u$  be the last child of  $y$ . Note that  $u$  is not

adjacent to any of the vertices in the second generation because we are looking at a breadth-first spanning tree. Looking at  $u$ , it can have up to  $\Delta - 2$  siblings and  $\Delta - 1$  neighbors. This means that there is flexibility as to what we color  $u$ . The color used on  $u$ , even with this flexibility, would not affect the coloring of any second generation vertices, except for maybe  $y$ , and it definitely won't affect any generations after the third, by the way we are getting our spanning tree. If we can color  $u$  the same color as  $x$ , we would do so. This would then give us another color for  $y$ , which is different from its neighbors and sibling vertices. Hence we have a proper  $(2\Delta - 1)$ -distinguishable coloring. If we are not able to color  $u$  the same color as  $x$ , then that means  $u$  has a neighbor, not  $y$ , that has the same color as  $x$ . But that would also mean we have an extra color that we can color  $u$  with. Color  $y$  the same color as  $x$ . Now  $S_x$  and  $S_y$  are distinctly colored. A non-trivial automorphism would now preserve the colors of  $S_x$  and  $S_y$  only if the two sets of vertices have the same color set. If they did have the same coloring initially, then the switch in the color of  $u$  would make them different. Therefore  $N(x)$  and  $N(y)$  have different colors and are distinguishable, making this coloring proper and  $(2\Delta - 1)$ -distinguishable.

**Let**  $|S_y| = \Delta - 1$ . This means that both  $x$  and  $y$  have  $\Delta - 1$  children and none of them are the same. If the color set of  $S_x$  and  $S_y$  are different, that means we can color  $x$  and  $y$  the same color, since they are distinguishable. However, they could be the same. If they are the same, then the colors of the children of  $y$  would be disjoint from the colors of the vertices of the second generation. Let vertices  $u_1$  and  $u_2$  be the last children of  $y$ . By the way we have been coloring,  $u_2$  is colored last. If  $u_1$  is adjacent to  $u_2$ , then we can

change the color of  $u_2$  since  $u_1$  is now part of  $u_2$ 's sibling and neighborhood set.

If not, the  $u_1$  and  $u_2$  are not adjacent and we color  $u_2$  the same as  $u_1$ . Doing so would then make the descendants of  $x$  be distinguishable. However, there might now be a  $\sigma$  that can flip  $u_1$  and  $u_2$ . Now looking at  $u_1$  and  $u_2$ , we will go through all the steps that we have done before: checking if there is a non-trivial automorphism that interchanges the two, and then taking the appropriate steps afterwards. Like the previous proof, this process will stop at some point and give us the case where  $1 \leq |S_y| \leq \Delta - 2$ . This would then give us a proper  $(2\Delta - 1)$ -distinguishable coloring.

Therefore  $\chi_D(G) \leq 2\Delta - 1$  and  $\chi_D(G) = 2\Delta$  if and only if  $G = K_{\Delta, \Delta}$  or  $C_6$ . □

Now to some original results.

## Chapter 5

### Distinguishing Chromatic Numbers of $C_n \circ I_m$

This section will showcase a few theorems, corollaries and a lemma by Collins and myself. We will first look at the distinguishing chromatic number of the base case of  $C_n \circ I_m$ , which is when  $m = 2$ . We will then look at a couple more concrete examples, i.e.,  $m = 3, 4$ , and then follow that up with the general case. The results are actually not as easy to come up with as one might think. But before we start providing the theorems and proofs for these graphs, we will define the lexicographical product.

**Definition 5.0.2.** Let  $G$  and  $H$  be graphs. Then the lexicographic product,  $G \circ H$  is defined on  $V(G \circ H) = V(G) \times V(H)$ , two vertices  $(u, x), (v, y)$  of  $G \circ H$  being adjacent whenever  $uv \in E(G)$ , or  $u = v$  and  $xy \in E(H)$ .

**Theorem 5.0.3.**  $\chi_D(C_{n \geq 5} \circ I_2) = 5$ , and  $\chi_D(C_{n \in \{3,4\}} \circ I_2) = 6$ .

Something to note is that there are a couple of general rules to follow.

- Let  $v$  and  $v'$  be in  $I_n$ . Then  $(C_n, v)$  and  $(C_n, v')$  must be colored differently, because the neighborhood of  $(C_n, v)$ ,  $N(C_n, v)$ , is the same as  $N(C_n, v')$ . This implies that there is an automorphism if we switch the two vertices. Hence all vertices in  $I_n$  must be colored uniquely.
- Let  $v$  be a vertex of a graph  $G$ . Then  $v$  is a different color than its neighbors.

*Proof.* We will prove this theorem in 3 cases. First when  $n = 3$ , then  $n = 4$  and finally when  $n \geq 5$ .

**For  $C_3 \circ I_2$ :** Since the graph  $C_3 \circ I_2$  is the complete multipartite graph  $K_{2,2,2}$ , we get that  $\chi_D(C_3 \circ I_2) = |V(C_3 \circ I_2)| = 6$ , by Collins and Trenk, [4].

**For  $C_4 \circ I_2$ :** We will prove that  $\chi_D(C_4 \circ I_2) = 6$  by showing that this graph does not have a distinguishing chromatic number of 5, which would imply that  $\chi_D(C_4 \circ I_2)$  can not be less than 5. This is true since we can simply color a vertex with a fifth color in a four-colored graph without causing adjacent vertices to have the same color. To show that  $\chi_D(C_4 \circ I_2) \neq 5$ , we will use three cases: the top  $C_4$  being colored in two, three and four colors.

Without lost of generality, we will always color a vertex in the top  $C_n$  with 1. Let  $v_1$  be the vertex in the same  $I_2$  set as the vertex on the top  $C_n$  colored 1,  $v_2$  be the vertex to the right of  $v_1$  and so on.

Case- 1 Let the top  $C_4$  be colored with 2 colors. Without lost of generality, let the vertices be colored 1, 2, 1, 2. Since we have 3 colors remaining, we will use all three, 3, 4, 5 respectively on any three consecutive vertices, say  $v_1, v_2$ , and  $v_3$ , because we want to use all five colors. Then  $v_4$  can now be only colored with 2 or 4, since the other three colors are already used for the adjacent vertices. If we color the last vertex with 2 then both vertices in that  $I_2$  have the same color, which violates one of our rules. So we will need to color the last vertex 4, but with this configuration, there exists an automorphism in the graph, in which if we exchange the two  $I_2$ 's with the coloring of 2 and 4, the result would be the original graph. Therefore, we can't color the top  $C_4$  with only two colors.

Case 2- Let the top  $C_4$  be colored with three colors. Without loss of generality, let the vertices be colored 1, 2, 3, 2. Then  $v_1$ , which is on the bottom  $C_4$ , can only be colored with 3, 4 or 5,  $v_2$  with 4 or 5,  $v_3$  with 1, 4 or 5 and  $v_4$  with 4 or 5. We want  $v_2$  and  $v_4$  to have different colors since the corresponding vertices on the top to have the same color. Therefore, WLOG, let  $v_2$  be 4 and  $v_4$  be 5. This then forces  $v_1$  to be 3 and  $v_3$  to be 1. But, similar to Case 1, there exists an automorphism, in that if we exchange the two  $I_2$ 's with the colors 1 and 3, the resulting graph is the same as the original. Therefore, we can't color the top  $C_4$  with only three colors.

Case 3- Let the top  $C_4$  be colored with four colors. Without loss of generality, let the vertices be colored 1, 2, 3, 4. Then  $v_1$  can be colored with 3 or 5,  $v_2$  with 4 or 5,  $v_3$  with 1 or 5, and  $v_4$  with 2 or 5. If we let  $v_1$  be 5, then  $v_2$  must be a 4 and  $v_4$  must be a 2, giving us the same color set in the first and third set of vertices. So let  $v_1$  be colored 3. Then  $v_3$  would be 5 since we don't want the  $I_2$ 's containing  $v_1$  and  $v_3$  to have the same color set. But then this forces the  $I_2$ 's containing  $v_2$  and  $v_4$  to have the same color set. Therefore, we can't color the top  $C_4$  with 4 colors either.

Hence,  $\chi_D(C_4 \circ I_2) > 5$ .

A coloring with 6 can be the following: 1, 2, 3, 4 for the top  $C_4$  and then 5, 6, 5, 6 on the bottom. To check that this coloring works, we look at the vertices that have colors that are used more than once, in our case the vertices colored with 5 and 6, since we can't send 1, 2, 3 or 4 to anywhere else. However, we can't send  $v_1$  to  $v_3$  and get the same graph as the original since  $v_1$  is in the  $I_2$  set with a vertex colored 1 while  $v_3$  with 3. Same goes for  $v_2$  and  $v_4$ . Hence this is a distinguishing

chromatic coloring with 6 colors. Therefore,  $\chi_D(C_4 \circ I_2) = 6$ .

**For  $C_{n \geq 5} \circ I_2$ :** Similar to the argument for  $n = 4$ , we will argue that we can't color  $C_n \circ I_2$  for  $n \geq 5$  with 4 colors. We will argue this with 2 cases: coloring 3 consecutive vertices of the top  $C_n$  with 2 colors and coloring it with 3 colors.

Case- 1 Let three consecutive vertices of the top  $C_n$  be colored with 2 colors, say 1, 2, 1. Using the same naming scheme as above, let  $v_1$  be colored 3. This would force  $v_2$  to be 4. Since we only have 4 colors, we must color  $v_3$  with 3. This would then force the  $I_2$  containing  $v_4$  be colored with colors 2 and 4, the one with  $v_5$  with colors 1 and 3, and so on. If  $n$  is even, then we can interchange any vertices with the same coloring and get an automorphism. If  $n$  is odd, then this force us to color the  $I_2$  containing  $v_n$  with a fifth color. Therefore, we can't color 3 consecutive vertices of the top  $C_n$  with 2 colors.

Case- 2 Let three consecutive vertices of the top  $C_n$  be colored with 3 colors, say 1, 2, 3. This would force  $v_2$  to be colored 4, which implies  $v_1$  is colored 3 and  $v_3$  is colored 1. Like in Case 1, this forces the  $I_2$  containing  $v_4$  to be colored with 2 and 4, the one containing  $v_5$  with 1 and 3, and so on. By the same argument as in Case 1, if the colors of the  $I_2$ 's alternate, we don't get a proper distinguishing coloring.

Therefore,  $\chi_D(C_{n \geq 5} \circ I_2) \geq 5$ .

Now to show that  $\chi_D(C_{n \geq 5} \circ I_2) = 5$ , we will provide coloring schemes for  $n = 5, 6, n$  odd and  $> 5$ , and  $n$  even and  $> 6$ .

**For  $C_5 \circ I_2$ :** Let the top  $C_5$  be colored 1, 2, 3, 1, 2, and the bottom, starting with  $v_1$ , be colored 3, 4, 5, 4, 5. This gives us a proper distinguishing coloring

because each  $I_2$  is a differently colored set, so if we were to interchange two vertices with the same color, we will not get an automorphism of the graph.

**For  $C_6 \circ I_2$ :** Let the top  $C_6$  be colored 1, 2, 1, 2, 1, 2 and the bottom, starting with  $v_1$  be colored 3, 4, 5, 3, 4, 5. Similar to  $n = 5$ , each 1 and each 2 are paired up with a different color in respect to their  $I_2$ 's, so that if we were to interchange two vertices with the same color, we will not get an automorphism of the graph.

Before we provide the coloring schemes for  $n > 6$ , we will prove the following lemma.

**Lemma 5.0.4.** *Given an  $n$ -cycle,  $n \geq 6$ , pick any six consecutive vertices and color them  $A, B, C, D, C, E$ . Let  $\alpha$  be a color not  $A$  or  $E$ , and  $\beta$  not  $A$ . Then the coloring scheme*

$$A, B, C, D, C, E, \alpha, \beta, \alpha, \beta, \text{ etc.}$$

*is a 6-distinguishing coloring if either  $\alpha \in \{B, C, D\}$  or  $\beta \in \{A, B, C, D\}$ , and 7-distinguishing if both  $\alpha \notin \{B, C, D\}$  and  $\beta \notin \{A, B, C, D\}$ .*

*Proof.* Pick any six consecutive vertices, call them  $u_1, u_2, \dots, u_6$ , of a  $C_n$  and color them  $A, B, C, D, C$ , and  $E$ , respectively. Then color the rest of the vertices  $\alpha, \beta, \alpha, \beta$ , and so on. In either cases, 6- or 7-distinguishing, this labeling is distinguishable in that we can identify each vertex by counting how many  $\alpha$ 's or  $\beta$ 's to the right of the vertices that are colored  $A, B, C, D, C, E$ .  $\square$

For the following color schemes, we will denote  $A$  being the color set  $\{1, 5\}$ ,  $B = \{2, 4\}$ ,  $C = \{1, 3\}$ ,  $D = \{4, 5\}$ ,  $E = \{2, 5\}$ ,  $F = \{3, 4\}$ ,  $G = \{1, 4\}$  and  $H = \{2, 3\}$ .

**For odd  $n > 5$ :** Pick 4 consecutive vertices and color them 1, 2, 3, 4 and the rest 3, 2, 3, 2, and so on. Color  $v_1$  with 5,  $v_2$  with 4,  $v_3$  with 1,  $v_4$  with 5,

and  $v_5$  with 1. Then the rest 5, 4, 5, 4, and so on. This scheme give us a proper coloring. Now if we represent each  $I_2$  with its corresponding color set, starting with the one containing  $v_1$ , we get  $A, B, C, D, C, E, F$  with the rest represented by  $E, F, E, F, etc.$  By Lemma 5.0.4, this is distinguishable, since we can identify each vertex by the counting how amount of vertices our target is to the right of the ones with color set  $A, B, C, D$  respectively. Hence this is a proper 5-distinguishing coloring of  $C_n \circ I_2$  for odd  $n > 5$ .

**For even  $n > 6$ :** Pick 6 consecutive vertices and color them 1, 2, 3, 4, 3, 2, and the rest 1, 2, 1, 2, and so on. Color  $v_1$  with 5,  $v_2$  with 4,  $v_3$  with 1,  $v_4$  with 5,  $v_5$  with 1,  $v_6$  with 5, and the rest 4, 3, 4, 3, and so on. This scheme gives us a proper coloring. Similar to the case for odd  $n > 5$ , we will represent the each  $I_2$  with their corresponding color set, starting with the one containing  $v_1$ . We then get,  $A, B, C, D, C, E, G, H$  with the rest represented by  $G, H, G, H, etc.$  Again, by Lemma 5.0.4, this is scheme is distinguishable since we can identify each vertex by counting the vertices to the right of the ones with the color set  $A, B, C, D$ , respectively. Hence this is a proper 5-distinguishing coloring of  $C_n \circ I_2$  for even  $n > 6$ .

And therefore,  $\chi_D(C_n \circ I_2) = 5$  for all  $n \geq 5$ , and  $\chi_D(C_n \circ I_2) = 6$  for  $n = 3$  or 4. □

**Theorem 5.0.5.**  $\chi_D(C_{n \geq 6} \circ I_3) = 7$ ,  $\chi_D(C_{n \in \{4,5\}} \circ I_3) = 8$ , and  $\chi_D(C_3 \circ I_3) = 9$ .

*Proof.* We will prove this theorem in four cases,  $n = 3, 4, 5$ , and  $n \geq 6$ . **For  $C_3 \circ I_3$ :** This graph is the multipartite graph,  $K_{3,3,3}$ , so by Collins and Trenk [4],  $\chi_D(C_3 \circ I_3) = |V(C_3 \circ I_3)| = 9$ .

For the rest of the proof, we will represent the  $I_3$  with the color set  $\{1, 2, 3\}$  as  $u_1$ . Let the  $u_2$  be the  $I_3$  set to the right of  $u_1$ , and so on.

**For  $C_4 \circ I_3$ :** We will prove  $\chi_D(C_4 \circ I_3) = 8$  by showing  $\chi_D(C_4 \circ I_3) > 7$  and providing a scheme for eight. Suppose we can properly color the graph with a 7-distinguishing coloring. Pick an  $I_3$  and color it with the color set  $\{1, 2, 3\}$ , so this set will be our  $u_1$ . Then  $u_2$  must be colored with the color set  $\{4, 5, 6\}$  since all the vertices in this set are adjacent to all the vertices in  $u_1$ . Since we can use seven colors in this graph, without loss of generality, we use 7 in  $u_3$ . Since we can't use 4, 5, or 6, we will color  $u_3$  with the set  $\{1, 2, 7\}$ . This forces us to use  $\{4, 5, 6\}$  for  $u_4$ , which gives us three different color sets, i.e. a 3-coloring of  $C_4$ . Since the  $u_i$ 's,  $i \in \{1, 2, 3, 4\}$ , forms a  $C_4$ , by Collins and Trenk [4], this coloring is not distinguishable, because we need four unique coloring sets. Hence  $\chi_D(C_4 \circ I_3) > 7$ .

A proper distinguishing coloring scheme for  $C_4 \circ I_3$  with eight colors is the following: color  $u_1$  with  $\{1, 2, 3\}$ ,  $u_2$  with  $\{4, 5, 6\}$ ,  $u_3$  with  $\{1, 2, 7\}$ , and  $u_4$  with  $\{4, 5, 8\}$ . This scheme works because the graph is properly colored and each set of colors are different.

Hence  $\chi_D(C_4 \circ I_3) = 8$ .

**For  $C_5 \circ I_3$ :** Similar to  $n = 4$ , we will prove  $\chi_D(C_5 \circ I_3) = 8$  by showing  $\chi_D(C_5 \circ I_3) > 7$  and providing a scheme for eight. Suppose we can properly color the graph with a 7-distinguishing coloring. Pick an  $I_3$  and color it with the color set  $\{1, 2, 3\}$ , so this set will be our  $u_1$ . Then  $u_2$  must be colored with the color set  $\{4, 5, 6\}$ . Again, with seven colors, we can either use it now for  $u_3$  or wait and use it for the other sets.

Case 1- If we use it for  $u_3$ , then, WLOG, we can color it with the set  $\{1, 2, 7\}$ . Then  $u_4$  can be colored with any three of the colors 3, 4, 5, and 6. This would

then force  $u_5$  to be colored with the color not used in  $u_4$ , 7, and one that is in  $u_1$  or  $u_4$ . So there is no proper coloring if we used 7 in  $u_3$ .

Case 2- If we saved 7 for later,  $u_3$  must then be colored with  $\{1, 2, 3\}$ . This implies that  $u_4$  can be colored with any three of the colors 4, 5, 6, 7. This then forces  $u_5$  to be colored with the colored not used in  $u_4$  and 2 colors that are used being used in  $u_1$  or  $u_4$ . So there is no proper coloring if we don't use 7 in  $u_3$ .

Hence  $\chi_D(C_5 \circ I_3) > 7$ .

A proper distinguishing coloring scheme for  $C_5 \circ I_3$  with eight colors is the following: color  $u_1$  with  $\{1, 2, 3\}$ ,  $u_2$  with  $\{4, 5, 6\}$ ,  $u_3$  with  $\{1, 2, 7\}$ ,  $u_4$  with  $\{3, 4, 5\}$ , and  $u_5$  with  $\{6, 7, 8\}$ . This is distinguishing since all five sets are unique and the graph is properly colored.

Hence  $\chi_D(C_5 \circ I_3) = 8$ .

**For  $C_{n \geq 6} \circ I_3$ :** Again, we will show that  $\chi_D(C_{n \geq 6} \circ I_3) = 7$  by showing  $\chi_D(C_{n \geq 6} \circ I_3) > 6$  and providing schemes for seven. It is clear that these graphs cannot be properly colored by six colors and be distinguishable, since the we can only get two unique color sets with six colors. And by Collins and Trenk [4], we need three unique sets. Therefore,  $\chi_D(C_{n \geq 6} \circ I_3) \neq 6$ .

We will again provide schemes for when  $n$  is odd and when  $n$  is even.

**For even  $n \geq 6$ :** Color  $u_1$  with  $\{1, 2, 3\}$ ,  $u_2$  with  $\{4, 5, 6\}$ ,  $u_3$  with  $\{1, 2, 7\}$ ,  $u_4$  with  $\{3, 4, 5\}$ ,  $u_5$  with  $\{1, 2, 6\}$ , and  $u_6$  with  $\{4, 5, 7\}$ . This is a proper distinguishing coloring for  $n = 6$  since all are the sets are unique and properly colored. To get all even  $n$ 's greater than six, we color  $u_i$ , for  $i > 6$  and odd, with  $\{1, 2, 6\}$ , and  $u_j$ , for  $j > 6$  and even, with  $\{4, 5, 7\}$ . This coloring for even  $n > 6$  works

because we can identify each  $I_3$  by counting the number of sets to the right of  $u_1$  through  $u_6$ .

**For odd  $n > 6$ :** We will use the same coloring as in  $n = 6$  for  $u_1$  through  $u_4$ . We will then color  $u_5$  with the set  $\{1, 6, 7\}$ ,  $u_6$  with  $\{2, 3, 4\}$ , and  $u_7$  with  $\{5, 6, 7\}$ . All the sets are unique and proper, so we have a proper distinguishing coloring with seven colors for  $n = 7$ . For  $n > 7$ , we color  $u_i$ , for  $i > 7$  and even, with  $\{2, 3, 4\}$  and  $u_j$ , for  $j > 7$  and odd, with  $\{5, 6, 7\}$ . This coloring for odd  $n > 6$  works, because we can identify each  $I_3$  by counting the number of sets to the right of  $u_1$  through  $u_7$ .

Therefore  $\chi_D(C_{n \geq 6} \circ I_3) = 7$ . □

Before we continue onto the general case,  $C_n \circ I_m$ , we will provide two corollaries. They are for when  $n = 3$  and  $n = 4$ .

**Corollary 5.0.6.**  $\chi_D(C_3 \circ I_m) = |V(C_3 \circ I_m)| = 3m$

*Proof.*  $C_3 \circ I_m$  is a complete multipartite graph. Hence the theorem is true by Collins and Trenk, [4]. □

**Corollary 5.0.7.**  $\chi_D(C_4 \circ I_m) = 2m + 2$

*Proof.* Pick a vertex,  $v_1$ , and color the  $I_m$  which  $v_1$  belongs to, call it  $u_1$ , with the color set  $\{1, 2, \dots, m\}$ . Then  $u_2$  must be colored with  $\{m + 1, m + 2, \dots, 2m\}$ . Since, by Collins and Trenk, [4], a 4-cycle has a distinguishing chromatic number of 4, we need 2 more color sets that are distinct. Therefore, we need 2 extra colors, one for each set, i.e. color  $u_3$  is in with  $\{1, 2, \dots, m - 1, x\}$  and  $u_4$  with  $\{m + 1, m + 2, \dots, 2m - 1, y\}$ . □

**Theorem 5.0.8.**  $\chi_D(C_n \circ I_m) = 2m + 1$  if  $n$  is even or if  $n \geq 2m + 1$ , otherwise  $\chi_D$  is obtained by  $\frac{n+1}{2} > \lceil \frac{m}{\chi_D - 2m} \rceil$ .

*Proof.* We will prove this theorem first for  $n$  is even, then when  $n \geq 2m + 1$ . Let the vertex  $v_i$  be in the set of independent vertices  $u_i$  for  $i \in \mathbb{N}$  and let  $u_i = \{a, b, c, d, \dots : a, b, c, d, \in \mathbb{N}\}$  denote  $u_i$  being colored by the set it is set equal to.

**For  $n$  even:** Let  $n$  be even. We know that  $\chi_D(C_n \circ I_m) \neq 2m$ , because with  $2m$  colors, we can only color all the  $I_m$ 's with two coloring sets. If we were to send an  $I_m$  to another with the same coloring set, we would get an automorphism. Therefore,  $\chi_D(C_n \circ I_m) > 2m$ .

To show that  $\chi_D(C_n \circ I_m) = 2m + 1$ , we will provide a coloring scheme. Pick a vertex on  $C_n$ ,  $v_1$ , and color  $u_1$  with the color set  $\{1, 2, \dots, m\}$ . This forces  $u_2$  to have completely different colors from  $u_1$ , so  $u_2 = \{m + 1, m + 2, \dots, 2m\}$ . We then have,

$$u_3 = \{1, 2, 3, \dots, m - 1, x\},$$

with  $x$  being our  $2m + 1^{\text{th}}$  color,

$$u_4 = \{m, m + 1, m + 2, \dots, 2m - 1\},$$

$$u_5 = \{1, 2, 3, \dots, m - 1, 2m\}, \text{ and}$$

$$u_6 = \{m + 1, m + 2, \dots, 2m - 1, x\}.$$

Then color the rest of the  $u_i$ 's with the same coloring set of  $u_5$  if  $i$  is odd, and  $u_6$  if  $i$  is even. This proper  $2m + 1$ -distinguishing coloring since we can identify each  $I_m$  by counting the number of sets to the right of  $u_1$  through  $u_4$ . Hence,

$\chi_D(C_n \circ I_m) = 2m + 1$  when  $n$  is even.

**For  $n \geq 2m + 1$ :** Since we already have a coloring scheme for all even  $n$ 's, we will just consider all the odd ones greater than or equal to  $2m + 1$ . It should be apparent now that  $\chi_D(C_n \circ I_m) \neq 2m$ , because  $2m$  colors would only provide two different color sets for the  $I_m$ 's which preserves the proper coloring of the graph. Also, since  $n$  is odd, that forces  $u_1$  and  $u_n$  to be colored with the same coloring sets which yields an improper coloring of the graph.

So let us now look at when  $\chi_D(C_n \circ I_m) = 2m + 1$ . Before we give the coloring scheme for the general case, we will look at a concrete example. Let  $m = 4$ . Since  $n \geq 2m + 1 = 9$ , let  $n = 11$ . Then the following coloring scheme is a proper

9-distinguishing coloring:

$$u_1 = \{1, 2, 3, 4\},$$

$$u_2 = \{5, 6, 7, 8\},$$

$$u_3 = \{1, 2, 3, 9\},$$

$$u_4 = \{4, 5, 6, 7\},$$

$$u_5 = \{1, 2, 8, 9\},$$

$$u_6 = \{3, 4, 5, 6\},$$

$$u_7 = \{1, 7, 8, 9\},$$

$$u_8 = \{2, 3, 4, 5\},$$

$$u_9 = \{6, 7, 8, 9\},$$

$$u_{10} = \{2, 3, 4, 5\}, \text{ and}$$

$$u_{11} = \{6, 7, 8, 9\}.$$

Note that  $u_8$  and  $u_9$  can be repeated until the rest of the graph is colored. This coloring scheme works since we can identify each  $I_m$  by counting the number of sets to the right of  $u_1$  through  $u_9$ .

So now for a general  $n \geq 2m + 1$ . We will first give the coloring scheme for

when  $n = 2m + 1$ . Let

$$\begin{aligned} u_1 &= \{1, 2, \dots, m\}, \\ u_2 &= \{m + 1, m + 2, \dots, 2m\}, \\ u_3 &= \{2m + 1, 1, 2, \dots, m - 1\}, \\ u_4 &= \{m, m + 1, m + 2, \dots, 2m - 1\}. \end{aligned}$$

Continuing this algorithm, we get that

$$u_i = \{(j + m(i - 1)) \bmod (2m + 1) : 1 \leq j \leq m\}$$

when  $i$  is odd, and

$$u_i = \{(k + m(i - 2)) \bmod (2m + 1) : m + 1 \leq k \leq 2m\},$$

when  $i$  is even. This gives us a  $2m + 1$ -distinguishing coloring since all the coloring sets differ by at least one color. Unlike the case when  $n$  is even, we need to check if  $u_1$  and  $u_n$  are disjoint. According to our algorithm,  $u_n$  is colored with  $\{(j + 2m^2) \bmod 2m + 1 : 1 \leq j \leq m\}$ , since  $n = 2m + 1$ . This gives us that  $u_n$  is colored with  $\{m + 2, m + 3, \dots, 2m, 2m + 1\}$ . Since  $u_1$ 's coloring set is  $\{1, 2, \dots, m\}$ ,  $u_1$  and  $u_n$  are disjoint, which gives us that this is a proper  $2m + 1$ -distinguishing coloring. Therefore,  $\chi_D(C_n \circ I_m) = 2m + 1$ , when  $n = 2m + 1$ .

Now for when  $n > 2m + 1$ , we will just use the coloring set  $u_{2m}$  for the even  $n$ 's, and  $u_{2m+1}$  for the odd  $n$ 's. This is a proper  $2m + 1$ -distinguishing coloring since we can identify each  $I_m$  by counting the number of sets to the right of  $u_1, \dots, u_{2m-1}$ . Hence,  $\chi_D(C_n \circ I_m) = 2m + 1$  when  $n \geq 2m + 1$ .

**For  $n < 2m + 1$ :** Again, we will just look at the case when  $n < 2m + 1$  and odd, since we covered all even  $n$ 's. To prove that  $\chi_D$  is obtained by  $\frac{n+1}{2} > \lceil \frac{m}{\chi_D - 2m} \rceil$ ,

we will show that there is no proper distinguishing coloring when  $\frac{n+1}{2} \leq \lceil \frac{m}{\chi_D - 2m} \rceil$ .

Before we start the proof, we will look at a lemma.

**Lemma 5.0.9.** *Let  $L(u_i), i \in \mathbb{N}$  denote the set of colors used for  $u_i$ , and  $c$  be the total number of colors used in the  $C_n \circ I_m$ , with  $n < 2m + 1$  and odd. If  $\frac{n+1}{2} \leq \lceil \frac{m}{c-2m} \rceil$ , then  $|L(u_1) \cap L(u_{2j+1})| \geq m - j(c - 2m)$  for  $1 \leq j \leq \frac{n-1}{2}$ .*

*Proof.* We will use induction on  $j$  to prove this lemma. Looking at the base case, we get that  $|L(u_1) \cap L(u_3)| \geq m - (c - 2m)$ . This is true since we need  $m$  colors for  $u_1$ , as well as  $u_3$ , and so there are  $m - (c - 2m)$  colors that are used in both sets, depending on  $c$ .

We will look at one more step, when  $j = 2$ . Suppose  $|L(u_1) \cap L(u_5)| < m - 2(c - 2m) = 5m - 2c$ . Let  $|P|$  denote the overlapping colors in  $u_1$  and  $u_3$ , and  $|Q|$  for  $u_3$  and  $u_5$ . Certainly, we know that  $|L(u_3) \cap L(u_5)| \geq m - (c - 2m)$  by the same argument as the base case. Let  $|R|$  denote the colors in  $u_3$  that are not in either  $u_1$  nor  $u_5$ . Then  $|P \cup Q \cup R| = m$ , since  $P \cup Q \cup R = u_3$ . But,

$$|P \cup Q \cup R| = |P| + |Q| - |P \cap Q| + |R|,$$

because  $|R|$  is disjoint from the other sets. This implies that

$$\begin{aligned} m &= |P| + |Q| - |P \cap Q| + |R| \\ &\geq (m - (c - 2m)) + (m - (c - 2m)) - |P \cap Q| + |R|, \text{ so} \\ |P \cap Q| &\geq 5m - 2c + |R| \\ &= m - 2(c - 2m) + |R|, \end{aligned}$$

which is a contradiction, since we supposed that  $|L(u_1) \cap L(u_5)| < 5m - 2c$ , which means that  $|P \cap Q| < 5m - 2c$  since those are the colors in  $u_5$  that are also in  $u_1$  and  $u_3$ . Hence  $|L(u_1) \cap L(u_5)| \geq m - 2(c - 2m)$ .

Now looking at the general case. Let  $|L(u_1) \cap L(u_{2j+1})| \geq m - j(c - 2m)$  be true up to  $j - 1$ . Much like the case for  $j = 2$ , we get a contradiction if we supposed that  $|L(u_1) \cap L(u_{2j+1})| < m - j(c - 2m)$ . This is true since  $|P| = m - (j - 1)(c - 2m)$  and  $|Q| = m - (c - 2m)$ .

Therefore,  $|L(u_1) \cap L(u_{2j+1})| \geq m - j(c - 2m)$ . □

With this result in mind, we need to show that  $|L(u_1) \cap L(u_n)| \geq m - \frac{n-1}{2}(\chi_D - 2m) > 0$ , to prove that there is no proper  $\chi_D$ -distinguishing coloring, when  $\frac{n+1}{2} \leq \lceil \frac{m}{\chi_D - 2m} \rceil$ . So let  $\frac{n+1}{2} \leq \lceil \frac{m}{\chi_D - 2m} \rceil$ . Note that  $\lceil \frac{m}{\chi_D - 2m} \rceil$  is always positive, since  $\chi_D$  is always bigger than  $2m$ .

$$\begin{aligned}
|L(u_1) \cap L(u_n)| &\geq m - \frac{n-1}{2}(\chi_D - 2m) \\
&= m - \frac{n-1}{2}(\chi_D - 2m) + (\chi_D - 2m) - (\chi_D - 2m) \\
&= m - \frac{n+1}{2}(\chi_D - 2m) + (\chi_D - 2m) \\
&\geq \chi_D - m - \lceil \frac{m}{\chi_D - 2m} \rceil (\chi_D - 2m) \\
&> \chi_D - m - \left( \frac{m}{\chi_D - 2m} + 1 \right) (\chi_D - 2m) \\
&= 0.
\end{aligned}$$

Now to finish off the proof, we will provide a scheme of coloring with  $\chi_D$  colors, such that  $\frac{n+1}{2} > \lceil \frac{m}{\chi_D - 2m} \rceil$ . To get  $\chi_D$  for a specific  $m$  and  $n$ , we will need to

play with the equation for a little bit. Let  $n = 2k + 1$ . Then we get that,

$$k + 1 > \lceil \frac{m}{\chi_D - 2m} \rceil.$$

But notice that both sides of the inequality are integers. With that, and letting  $\epsilon > 0$  we can now rewrite the inequality as,

$$\begin{aligned} k &\geq \frac{m}{\chi_D - 2m} + \epsilon \\ k - \epsilon &\geq \frac{m}{\chi_D - 2m} \\ \chi_D &\geq \frac{m}{k - \epsilon} + 2m. \end{aligned}$$

Remembering that the definition of a chromatic number is the smallest number of colors needed to properly color a graph,  $\chi_D$  will be the smallest number bigger or equal to  $\frac{m}{k - \epsilon} + 2m$ .

Using the same notations as before,  $u_1$  being the set of vertices colored  $\{1, 2, \dots, m\}$ , the first two of these sets will be colored with the  $2m$  colors that we are provided on the right side of the equation. Now we have  $2k - 1$  sets of vertices left to color, so we need to figure out the number of extra colors we would need to introduce so that we rotate the colors in a set  $k$  times, to get that  $L(u_{2k+1}) \cap [L(u_1) \cup L(u_{2k})] = \emptyset$ . This can be done by dividing  $m$  by  $k$ . Since  $\frac{m}{k}$  might not be an integer, we will need to round up, but if  $\frac{m}{k}$  indeed is an integer, then we are set.

So now, let  $c = \frac{m}{k}$ , then  $\chi_D = 2m + c$ . With  $2m + c$  colors, we will color the  $u_1$  and  $u_2$  with the usual colors. The following would then be the coloring for our

graph, for  $i \leq k$ :

$$\begin{aligned}
u_1 &= \{1, 2, 3, \dots, m\} \\
u_2 &= \{m + 1, m + 2, \dots, 2m\} \\
u_3 &= \{1, 2, \dots, m - c, x_1, x_2, \dots, x_c\}, \\
u_4 &= \{m - c + 1, m - c + 2, \dots, 2m - c\}, \\
u_5 &= \{1, 2, \dots, m - 2c, 2m - c + 1, 2m - c + 2, \dots, 2m, x_1, x_2, \dots, x_c\}, \\
&\vdots \\
u_{2i} &= \{m - (i - 1)c + 1, m - (i - 1)c + 2, \dots, 2m - (i - 1)c\} \\
u_{2i+1} &= \{1, 2, \dots, m - ic, 2m - (i - c) + 1, 2m - (i - c) + 2, \dots, 2m, x_1, x_2, \dots, x_c\} \\
&\vdots
\end{aligned}$$

Then at  $u_n$ , we should have that

$$u_n = \{2m + c - kc + 1, 2m + c - kc + 2, \dots, 2m + c\}.$$

However, remembering that  $\frac{m}{k}$  might not be an integer, we would need to round up to the next integer. This might cause the terms that we are subtracting a multiple of  $c$  from  $m$  to be zero or negative, i.e. when  $m - ic < 1$ . So when this does happen, we use the last  $m$  colors for that  $u_{2i+1}$ . Then for  $u_j$  we would use the same color set as  $u_{2i+1}$  when  $j$  is odd, and  $u_{2i}$  when  $j$  is even. This coloring is distinguishing since our  $m$  and  $n$  force at least the first four  $u$ 's to be distinctly colored. This enables us to distinguish each  $u_i$  by counting the sets to the right

of the first four  $u$ 's. Hence  $\frac{n+1}{2} > \lceil \frac{m}{\chi_D - 2m} \rceil$  for  $n > 2m + 1$ .

Therefore,  $\chi_D(C_n \circ I_m) = 2m + 1$  if  $n$  is even or if  $n \geq 2m + 1$ , otherwise  $\chi_D$  is obtained by  $\frac{n+1}{2} > \lceil \frac{m}{\chi_D - 2m} \rceil$ .  $\square$

**Theorem 5.0.10.**  $\chi_D(C_n \circ I_m) = 2m + 1$  if  $n$  is even and  $\chi_D(C_n \circ I_m) = \lceil \frac{m}{\frac{n-1}{2}} \rceil + 2m$ , if  $n$  is odd.

*Proof.* This follows the proof of Theorem 5.0.8.  $\square$

The following table shows  $\chi_D(G)$  for  $2 \leq m \leq 10$  and odd  $n \leq 23$ . Looking at the table, one can see that when  $n \geq 2m$ ,  $\chi_D(C_n \circ I_m) = 2m + 1$ .

$\chi_D(C_n \circ I_m)$	$m = 2$	3	4	5	6	7	8	9	10
$n = 5$	5	8	10	13	15	18	20	23	25
7	5	7	10	12	14	17	19	21	24
9	5	7	9	12	14	16	18	21	23
11	5	7	9	11	14	16	18	20	22
13	5	7	9	11	13	16	18	20	22
15	5	7	9	11	13	15	18	20	22
17	5	7	9	11	13	15	17	20	22
19	5	7	9	11	13	15	17	19	22
21	5	7	9	11	13	15	17	19	21

## Chapter 6

### Open Questions

Like vertices, edges can also be colored. So there should be no surprise that there is also an edge-chromatic number. Similar to the definition of the chromatic number of the graph, the edge-chromatic number, denoted as  $\chi'(G)$ , is the smallest number of colors needed to color the edges of a graph properly. This means that edges that share a common vertex must be colored differently.

Like Brooks' Theorem for chromatic numbers, the edge-chromatic number also has a theorem that gives a bound as to what it can be. However, unlike Brooks' theorem, which just gives a maximum bound, this theorem states that the edge-chromatic number can only be one of two numbers.

**Theorem 6.0.11** (Vizing). *If  $G$  is a simple graph, then  $\chi'(G) = \Delta$  or  $\Delta + 1$ .*

An obvious question now is, since there are Brooks' Theorems for distinguishing and distinguishing chromatic number, are there Vizing's Theorems for those numbers? I think this question would most likely yield some surprising results.

## References

- [1] M. O. Albertson and K. L. Collins, *Symmetry Breaking in Graphs*, Electron. J. of Combin., **3** (1996) #R18.
- [2] B. Bollobás, *Modern Graph Theory*. Springer, New York 1998.
- [3] R. A. Brualdi, *Introductory Combinatorics*, 3rd ed. Prentice-Hall Inc., New Jersey 1999.
- [4] K. L. Collins and A. N. Trenk, *The Distinguishing Chromatic Number*, Electron. J. of Combin., **13** (2006) #R16.
- [5] R. Diestel, *Graph Theory*, 2nd ed. Springer, New York 2000.
- [6] D. S. Dummit and R. M. Foote, *Abstract Algebra*, 3rd ed. John Wiley & Sons, Inc., New York 2004.
- [7] W. Imrich and S. Klavžar, *Product Graphs Structures & Recognition*, John Wiley & Sons, Inc., New York 2000.
- [8] S. Klavžar, T. Wong and X. Zhu, *Distinguishing Labellings of Group Action on Vector Spaces and Graphs*, Journal of Algebra 303 (2006), no. 2, 626-641.
- [9] T. W. Tucker, *Distinguishability of Maps*, in preparation.
- [10] D. B. West, *Introduction to Graph Theory*, 2nd ed. Prentice-Hall Inc., New Jersey 2001.