Non-Hermitian Wave Mechanics:
Application to
Integrated Photonics and Electronics

by

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Dedication

To my dear ♥Neda♥, for her endless love and support. My journey would not be successful without her.
Abstract

Non-Hermitian wave mechanics is a natural mathematical framework that allows us to describe propagation of classical waves in the presence of gain and loss mechanisms. Recognizing this fact, we aim in this thesis to utilize complex potentials (impedance profiles) with a special spatio-temporal symmetries such that allow us to manage wave propagation. A special class of this type of synthetic structures are the ones which are invariant under joint space-time reflection or $\mathcal{PT}$ symmetries. Our work focus on the novel transport properties of these structures. Whenever appropriate, we point the potential technological implications in the frameworks of optics and electronics circuitry. Many of our predictions have already found experimental confirmations. We will also discuss these experimental verifications.
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# Contents

1 Introduction 1

2 \(\cal{P}\cal{T}\) Symmetry 4

2.1 In Action: The Simple Case of a Two Level System 5

2.1.1 \(\cal{P}\cal{T}\) Symmetric Two Level System 6

2.1.2 Stationary Solutions: Eigenvectors and Eigenvalues 7

2.1.3 Dynamics 8

2.2 Experimental Observations of \(\cal{P}\cal{T}\) Symmetry 10

2.2.1 \(\cal{P}\cal{T}\) Symmetric Photonic via Synthetic Materials 11

2.2.2 Pseudo \(\cal{P}\cal{T}\) Symmetric Systems 13

2.2.3 \(\cal{P}\cal{T}\) Symmetric Electronics 15

2.2.4 Asymmetric Spatio-Temporal Dynamics and State Manipulation 17

2.3 \(\cal{P}\cal{T}\) Symmetry and Non-reciprocity 20

2.3.1 Integrated \(\cal{P}\cal{T}\) Symmetric Non-linear Isolator 22

2.4 Summary 24

3 Extended \(\cal{P}\cal{T}\) Symmetric Structures 26

3.1 Beam Dynamics in \(\cal{P}\cal{T}\) Lattices 28

3.1.1 One Dimensional \(\cal{P}\cal{T}\) Dimeric Chain 29

3.1.2 Revivals in Extended \(\cal{P}\cal{T}\) Systems 32

3.1.3 Conical Diffraction in \(\cal{P}\cal{T}\) Symmetric Honeycomb Lattices 35
3.2 $\mathcal{PT}$ symmetric Mesh Lattices ........................................ 36
3.3 Summary ........................................................................... 41

4 Generalized $\mathcal{PT}$ Symmetry in the Presence of Magnetic Materials 42

4.1 Electrodynamics of Stratified Media ................................. 44
  4.1.1 Propagation Matrix Method ..................................... 44
  4.1.2 Transmittance and Reflectance of a Stratified Medium 48
  4.1.3 Dispersion Relation of Periodic Stratified Media ....... 50
  4.1.4 Transfer Matrix ...................................................... 55

4.2 Transport Properties of $\mathcal{PT}$ Systems .......................... 57

4.3 Experimental Demonstration of Scattering in Periodic $\mathcal{PT}$ systems ... 60
  4.3.1 Experimental demonstration of a unidirectional reflectionless $\mathcal{PT}$ metamaterial at optical frequencies . 64
  4.3.2 Temporal Resolved Beam Dynamics in $\mathcal{PT}$ lattices 65

4.4 Summary ........................................................................... 67

5 Conclusion .......................................................................... 68

A Selected Publications .......................................................... 70

B Complete List of Publications ............................................. 135
# List of Figures

2.1 Schematic of the experimental set-up ................................. 11
2.2 Experimental results of beam dynamics in $\mathcal{PT}$ dimer .......................... 12
2.3 Cross section of the double well passive $\mathcal{PT}$ symmetric waveguides ..... 14
2.4 $\mathcal{PT}$ symmetric electronic $LRC$ circuit .............................. 15
2.5 Eigenfrequencies of a $\mathcal{PT}$-$LRC$ dimer vs. the gain and loss parameter . 17
2.6 Brachistochrone dynamics in $\mathcal{PT}$ systems ............................. 19
2.7 Faraday isolator ........................................................................ 21
2.8 Isolation action in non-linear $\mathcal{PT}$ dimer .............................. 23

3.1 $\mathcal{PT}$ dimeric lattice and its dispersion relation ......................... 30
3.2 Talbot carpet in the presence of imperfections in a $\mathcal{PT}$ lattice .......... 34
3.3 $\mathcal{PT}$-Honeycomb photonic lattice .................................. 35
3.4 Conical diffraction at the exceptional point ................................. 36
3.5 Experimental set-up of emulated $\mathcal{PT}$ network ......................... 38
3.6 Emulated one dimensional $\mathcal{PT}$ lattice ................................. 39
3.7 Band structure and dynamics of the experimental $\mathcal{PT}$ lattice ........ 40

4.1 Symmetric dispersion relation ........................................... 52
4.2 Asymmetric dispersion relation ......................................... 53
4.3 Mirrorless Unidirectional Laser ........................................... 54
4.4 Generalized $\tilde{\mathcal{PT}}$ symmetric micro-cavity .......................... 58
4.5 Scattering characteristic of $\mathcal{PT}$ cavity ................. 59
4.6 Enhanced polarization independent isolation ..................... 61
4.7 Scattering properties of a $\mathcal{PT}$ symmetric Bragg grating .... 63
4.8 Experimental implementation of the passive loss Bragg grating .... 65
4.9 Experimental measured reflections of the passive loss Bragg grating . 66
4.10 Scattering from the passive Bragg grating and passive loss Bragg grating 66
Chapter 1

Introduction

High speed optical communication and data processing, utilizing integrated photonic device poses a monumental challenge to modern science and engineering. In this endeavor, great progress has been made over the course of the last twenty years. Two examples of these achievements, are photonic crystals and more recently meta-materials which allow for the flow of light to be tailored and thus the transport of information can be controlled. Despite of plethora of burgeoning of new ideas, and their ingenious implementations in on-chip scale optical devices, the existence of loss spoils the efficiency of such devices; thus rendering the endeavor of photon management and signal processing a challenging task. At the same time, the standard approach of incorporating gain into these structures results in problems like uncontrollable amplification, which occasionally destroys these devices.

Currently a different viewpoint is emerging which aims to manipulate absorption via a judicious design. This involves the combination of delicately balanced amplification and absorption mechanisms, and the manipulation of the refraction index. The goal of which is to achieve new classes of synthetic photonic structures with altogether intriguing physical behavior and novel functionality. This idea can potentially have a vast range of applicability to other physical frameworks, ranging from acoustics, to transmission
lines and (nano-)antenna arrays.

First in optics [1–3] and more recently in electronics [4–6] structures with delicately balanced gain and loss, have been proposed demonstrating unconventional phenomena such as power oscillation, invisibility, and non-reciprocity. These systems are described by an effective non-Hermitian Hamiltonian which commutes with the joint parity-time $\mathcal{PT}$ symmetry operator. Parity-time symmetric Hamiltonians were originally proposed and mathematically investigated by Bender, and colleagues, in the framework of quantum field theories [7]. Although in quantum mechanics these $\mathcal{PT}$-theories continue to be a controversial theme for researchers, in the fields of optics and electronics they have been established during the last few years as a mathematical tool which allow us to describe realistic structures with peculiar properties. The goal of this thesis is to advocate for these theories and develop/establish $\mathcal{PT}$-concepts that have technological applications in the optics and electronics framework.

This thesis consists of three parts where the main theoretical $\mathcal{PT}$-concepts are presented together with a highlight of our related original contributions, which can be found in the appendix A. At the same time, whenever appropriate, we provide and explain the available experimental realizations associated with the theoretical advances discussed in each specific chapter. Below we provide a rough description of the context of each chapter.

In chapter 2 we review stationary and dynamical properties of the simplest $\mathcal{PT}$ symmetric system, consisting of two coupled $\mathcal{PT}$ symmetric levels. The experimental realization of this toy structure, both in the framework of optics and electronics is presented in parallel. $\mathcal{PT}$ symmetric phenomena such as power oscillations, brachistochrone dynamics and spontaneous $\mathcal{PT}$ symmetric phase transitions are theoretically discussed and illustrated via the experimental realizations. The possibility to implement a $\mathcal{PT}$ symmetric nonlinear structure for optical isolation and the potential technological advances of such a proposal are discussed at the end of the chapter.
In the next chapter we stretch the study of $\mathcal{PT}$ symmetric structures to extended systems. Coupled $\mathcal{PT}$ binary lattices are the first example of this category. Specifically, we show how the gain and loss parameter can be used to tailor the dispersion relation of these systems. We demonstrate the consequences of this capability in the framework of optical revivals (Talbot effect) and beam conical diffraction. Finally, we review a recent experimental configuration associated with the beam dynamics in $\mathcal{PT}$ optical fiber loops, emulating the behavior of extended lattices.

In chapter we lay down the theoretical groundwork describing the electrodynamics of stratified media, in the presence of birefringent magneto-optical materials. We develop a scattering formalism and introduce propagation and transfer matrices. Via this formalism we calculate the scattering characteristics of these media. In the case of periodic stratified media we show how one can find the dispersion relation. After that we discuss the notion of an asymmetric dispersion relation and the emergence of an inflection point. We show that co-existence between the inflection point and gain mechanisms leads to unidirectional lasing action. Next we exploit transport properties of the generalized $\tilde{\mathcal{PT}}$ systems, where we have polarization independent asymmetric wave propagation. Eventually, we introduce the notion of unidirectional invisibility and its experimental ramifications.

Our conclusions will be given in the last chapter of the thesis. Our aim is to convince the reader that the field of non-Hermitian wave mechanics is largely unexploited. Via specific examples we have shown how the appropriate amount of attenuation (the “evil” in mainstream studies) together with amplification can lead to structures that possess new behavior and functionality. The study of $\mathcal{PT}$ symmetric optics and electronic circuitry provide an experimentally simple and fertile ground to test many $\mathcal{PT}$ ideas. This is a first start to our investigation of the unexplored "lands" of non-Hermitian wave mechanics.
Chapter 2

\( \mathcal{PT} \) Symmetry

The traditional Dirac-von Neumann formulation of quantum mechanics requires that all physical observables (and thus also the Hamiltonian) should be represented by Hermitian operators [8]. Such a constraint guarantees the reality of physical observables; a natural requirement for any theory to be valid. As a consequence of this constraint we also get that the spectrum of any physical operator is real, the eigenvectors are orthogonal to each other and the evolution operator is unitary i.e. it preserves the norm. Self-adjoint operators, however are not the only ones whose spectrum are real. It has been shown that non-Hermitian Hamiltonians that respect anti-linear symmetries, i.e. commutes with an anti-linear operator like the joint parity-time (\( \mathcal{PT} \)) symmetry operator, may have real spectra while the generated dynamics is (pseudo-) unitary [7]. This observation led Carl M. Bender and colleagues to propose an extension of quantum mechanics based on non-Hermitian but \( \mathcal{PT} \) symmetric operators[9].

While these ideas – in the framework of quantum mechanics – are still debatable, it was recently suggested that optics and electronics can provide a particularly fertile ground where \( \mathcal{PT} \)-related concepts can be realized and experimentally investigated [2, 3, 10]. What makes this possible in optics is the combination of the formal equivalence between the Schrödinger equation in quantum mechanics and the optical wave equation
(in the paraxial approximation), and of the possibility to simultaneously manipulate loss, gain, and index of refraction. By another route, one can also show that there is an isomorphism between the Kirchoff’s laws and Schrödinger equation. This latter framework has proved a very fruitful groundwork the last couple of years as it can provide experimental opportunities to test novel $\mathcal{PT}$ symmetric ideas and phenomena. This can be achieved via combining $LRC$ circuits that balance attenuation and amplification in a $\mathcal{PT}$ symmetric manner. In this case the complex antilinear potentials can be synthesized with the help of resistors and amplifiers.

The structure of this chapter is as follows. In section 2.1 we present the simplest possible $\mathcal{PT}$ symmetric quantum system consisting of a two level system. Via this presentation we will introduce the notions of parity ($\mathcal{P}$) and time ($\mathcal{T}$) symmetries. Then, in section 2.2 we will extend the notion of $\mathcal{PT}$ symmetry to photonics and electronics and we will review the experimental realization of $\mathcal{PT}$ systems in these frameworks. In section 2.3 we will exploit the notion of non-reciprocity and show how $\mathcal{PT}$ symmetry together with non-linearity leads to isolation action. A summary will be presented at the last section 2.4.

2.1 In Action: The Simple Case of a Two Level System

Parity ($\mathcal{P}$) or space-reflection is a linear operator which changes the sign of the coordinate operator $\hat{x}$, $\mathcal{P}\hat{x}\mathcal{P} = -\hat{x}$ and momentum operator $\hat{p}$, $\mathcal{P}\hat{p}\mathcal{P} = -\hat{p}$ while it leaves the fundamental commutation relation between them invariant, $[\hat{x}, \hat{p}] = i\hbar$. Time reversal operator ($\mathcal{T}$) is an anti-linear operator which leaves the $\hat{x}$ invariant $\mathcal{T}\hat{x}\mathcal{T} = \hat{x}$ but changes the sign of the $\hat{p}$, $\mathcal{T}\hat{p}\mathcal{T} = -\hat{p}$. However it does not affect the commutation relation between them. This requires a conjugation on the complex number $i$. By definition of the anti-linear operators the anti-linear operator $\mathcal{T}$ satisfies the following property:

$$\mathcal{T}\{c_1\phi_1 + c_2\phi_2\} = c_1^*\mathcal{T}\phi_1 + c_2^*\mathcal{T}\phi_2 . \tag{2.1}$$
where $c_1$ and $c_2$ are complex numbers and $\phi_1, \phi_2$ any two test wavefunctions.

Recently it has been shown that a certain class of complex non-Hermitian Hamiltonians could exhibit entirely real spectrum provided they respect parity-time symmetry, i.e. their Hamiltonian commutes with the joint $\mathcal{PT}$ operator $[\mathcal{PT}, H] = 0$ \cite{7, 9}. This means that in a $\mathcal{PT}$ symmetric Hamiltonian, the complex potential profile $V(x) = V_R(x) + i\gamma V_I(x)$ satisfies the following relation

$$V(x) = V^*(-x)$$

which imposes the constrains that the real part of the potential $V_R(x)$ should be an even function while the imaginary part $V_I(x)$ should be an odd function. The parameter $\gamma$ controls the degree of non-Hermiticity of $H$. In the classical wave limit, the complex part of the potential in a $\mathcal{PT}$ symmetric system represents the existence of a mechanism that balances gain and loss process. For $\gamma$-values below a critical value $\gamma_{\mathcal{PT}}$ known as spontaneous $\mathcal{PT}$ symmetry breaking point, the spectrum of the Hamiltonian is real and the system is in the so called exact phase. In the exact phase $\gamma < \gamma_{\mathcal{PT}}$ the $\mathcal{PT}$ operator and $H$ share the same eigenfunctions. However the so called broken $\mathcal{PT}$ symmetric phase where $\gamma > \gamma_{\mathcal{PT}}$, the eigenfunctions of $H$ cease to be eigenfunctions of the $\mathcal{PT}$ operator, despite the fact that $H$ and the $\mathcal{PT}$ operator still commute \cite{7, 9}. As a result, in the broken $\mathcal{PT}$ symmetric phase, the spectrum becomes partially or completely complex.

### 2.1.1 $\mathcal{PT}$ Symmetric Two Level System

A two level system with balanced dissipation and amplification is the simplest example of a $\mathcal{PT}$ symmetric configuration. Using the Pauli matrices $\sigma_j$ ($j = 1, 2, 3$), the Hamiltonian of such a system can be written in the following form:

$$H = \begin{pmatrix} v_0 + i\gamma & \kappa \\ \kappa & v_0 - i\gamma \end{pmatrix} = v_0 \mathbf{1} + \kappa \sigma_1 + i\gamma \sigma_3$$

(2.3)
where $1$ is a $2 \times 2$ identity matrix. The parameters $v_0$, $\gamma$ are real and correspond to the real, and imaginary part of the potential, respectively. The last real parameter $\kappa$ is the coupling strength between the two energy levels. It is obvious that this Hamiltonian is not Hermitian, but one can easily verify that it is $\mathcal{PT}$ symmetric, where the parity operator, $\mathcal{P}$, is simply the Pauli matrix, $\sigma_1$, and $\mathcal{T}$ performs complex conjugation.

### 2.1.2 Stationary Solutions: Eigenvectors and Eigenvalues

Hamiltonian (2.3) is non-Hermitian and as a result it has left $\langle L_n |$ and right $| R_n \rangle$ eigenvectors ($n = 1, 2$) which are defined as the solutions of the following set of equations:

$$
\langle L_n | H = \langle L_n | E_n \quad \text{and} \quad H | R_n \rangle = E_n | R_n \rangle.
$$

These eigenstates are bi-orthogonal, i.e. the left and right eigenvectors – unlike the Hermitian case – are distinct and $\langle L_n | \neq | R_n \rangle$. Therefore, they do not respect the Euclidian ortho-normalization condition. Specifically the normalized right eigenvectors associated with the $H$ given in Eq.(2.3) are

$$
| R_1 \rangle = \frac{1}{\sqrt{2 \cos \alpha}} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix} \quad \text{and} \quad | R_2 \rangle = \frac{1}{\sqrt{2 \cos \alpha}} \begin{pmatrix} ie^{-i\alpha/2} \\ -ie^{i\alpha/2} \end{pmatrix}
$$

While the left eigenvectors are

$$
\langle L_1 | = \frac{1}{\sqrt{2 \cos \alpha}} \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix}^T \quad \text{and} \quad \langle L_2 | = \frac{1}{\sqrt{2 \cos \alpha}} \begin{pmatrix} -ie^{-i\alpha/2} \\ ie^{i\alpha/2} \end{pmatrix}^T.
$$

The normalization constant $\frac{1}{\sqrt{2 \cos \alpha}}$ with $\sin(\alpha) = \frac{\gamma}{\kappa}$ can be found via the following normalization condition

$$
\langle L_n | R_m \rangle = \delta_{nm}.
$$

¹The system described by Eq. (2.3) is still symmetric and thus there is a further relation between left and right eigenvectors, i.e. $| R \rangle = | L \rangle^*$. 

The completeness relation is
\[ \sum_n |R_n\rangle \langle L_n| = 1. \quad (2.8) \]

The eigenvalues of the Hamiltonian \((2.3)\) are given by a simple diagonalization of this Hamiltonian and have the form
\[ \mathcal{E}_\pm = v_0 \pm \sqrt{\kappa^2 - \gamma^2}. \quad (2.9) \]

Equation \((2.9)\) shows that the ratio \(\gamma/\kappa\) controls the reality of the eigenvalues of the Hamiltonian \(H\). For \(|\gamma/\kappa| < 1\) eigenvalues are real and the system is in the exact phase. Moreover from Eqs. \((2.5)\) and \((2.6)\) in this parameter domain one can confirm that \(H\) and \(\mathcal{P}\mathcal{T}\) operator share the same eigenfunctions. \(\mathcal{P}\mathcal{T}\) spontaneous phase transition occurs at \(|\gamma/\kappa| = 1\) where eigenvectors and eigenvalues of the Hamiltonian \(H\) becomes degenerate. Whereas for larger values \(|\gamma/\kappa| > 1\) the system is in the broken phase where eigenvalues become complex and eigenvectors of the Hamiltonian \(H\) cease to be the eigenvectors of the \(\mathcal{P}\mathcal{T}\) operator.

### 2.1.3 Dynamics

The evolution of any generic wave-function \(|\Psi(t)\rangle\) associated with the \(\mathcal{P}\mathcal{T}\) symmetric Hamiltonian \(H\) is given by the evolution operator \(\hat{U}(t) = e^{-iHt}\). To this end one can expand the wave-function at time \(t = 0\) in the basis of the \(H\) and then apply the evolution operator on the expansion:
\[ |\Psi(t)\rangle = e^{-iHt} \sum_n c_n |R_n\rangle = \sum_n c_n e^{-i\mathcal{E}_n t} |R_n\rangle \quad (2.10) \]

where \(c_n = \langle L_n|\Psi(0)\rangle\) is the expansion coefficient.

In the case of a two level system with \(v_0 = 0\), the dynamics can be found using a different approach. Specifically the Hamiltonian \((2.3)\) can be written as
\[ H = \mathcal{E} \sigma \cdot \hat{n} \quad (2.11) \]
with $E = |E_\pm| = \sqrt{\kappa^2 - \gamma^2}$ and the unit vector $\hat{n}$ which is defined as $\hat{n} = \frac{1}{2}(\kappa, 0, \gamma)$. Using matrix expansion of the $\hat{U}(t)$ and the Taylor series for sine and cosine we will have

$$\hat{U}(t) = \exp(-iHt) = \cos(Et)I - i \sin(Et)H/E$$  \hspace{1cm} (2.12)$$

where $I$ is the $2 \times 2$ identity matrix. We get that a generic initial state evolving under the non-Hermitian Hamiltonian (2.11) takes the following form

$$|\psi(t)\rangle = \hat{U}\{c_1|\chi_+\rangle + c_2|\chi_-\rangle\} = \frac{1}{\cos \alpha} \begin{pmatrix} c_1 \cos \left(\frac{Et}{2} - \alpha\right) - c_2 i \sin \left(\frac{Et}{2}\right) \\ c_2 \cos \left(\frac{Et}{2} + \alpha\right) - c_1 i \sin \left(\frac{Et}{2}\right) \end{pmatrix}.$$  \hspace{1cm} (2.13)$$

Above $|\chi_+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\chi_-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $c_1, c_2$ are generic coefficients that respect the normalization and define the initial condition.

From Eq.(2.13) we easily get that the total norm $I(t) = \langle \psi(t)|\psi\rangle$ is given by

$$I(t) = \frac{1}{2 \cos^2 \alpha} \left( \cos^2 \left(\frac{Et}{2} - \alpha\right) + 2 \sin^2 \left(\frac{Et}{2}\right) + \cos^2 \left(\frac{Et}{2} + \alpha\right) \right).$$  \hspace{1cm} (2.14)$$

From the above equation and also the numerical simulations presented in our article [11], we observe that, unlike the passive case, in a $\mathcal{PT}$ symmetric system with $\gamma \neq 0$ the total norm is no longer a constant of motion. Instead, one finds that there is another conserved quantity which is coined the pseudo-norm [12] and is defined as

$$\langle \chi_+|\psi(t)\rangle\langle \psi(t)|\chi_-\rangle + \langle \chi_-|\psi(t)\rangle\langle \psi(t)|\chi_+\rangle = 2c_1c_2 \cos^2(\alpha).$$  \hspace{1cm} (2.15)$$

At the same time, we have to realize that the physical observable in any experimental realization of a $\mathcal{PT}$ symmetric system is still the standard norm. For example, in optics it describes the total light intensity remaining in the system, while in electronics can describe the total energy in the circuit. Obviously, the standard norm is given by Eq.(2.14) where in the exact phase experience an oscillatory behavior.

Interestingly, the pattern of the dynamics differs depending on whether the initial excitation is on the gain or at loss site. In other words, the beam dynamics for an excitation
in channel 1 does not mirror the beam dynamics of an input at channel 2. This property does not exist at \( \gamma = 0 \), where the superposition of two (symmetric and anti-symmetric) eigenstates of the Hermitian Hamiltonian leads to a symmetric wave propagation\cite{11, 13}. This “asymmetric” dynamics is a novel characteristic of \( \mathcal{PT} \) systems and can be of extreme importance for technological applications (for example integrated optical diodes, switches and optical gates) when it is combined with non-linearity (see section 2.3 and our articles \cite{6, 11}).

At the exceptional point, the Hamiltonian \( (2.3) \) is defective and does not have a complete basis of eigenvectors. In this case the evolved generic state is given by

\[
\psi(t) = (c_1 + c_2 t) \begin{pmatrix} 1 \\ -i \end{pmatrix} + c_2 \begin{pmatrix} -i/\gamma \\ 0 \end{pmatrix}
\]

which clearly indicates that the norm of the system grows in a power law manner \cite{14}. In the broken phase with \( \gamma > k \), one can deduce from Eq.(2.14) that the norm grows exponentially\cite{11, 13, 15}.

### 2.2 Experimental Observations of \( \mathcal{PT} \) Symmetry

Instead of being a mathematical curiosity, experimental realizations – such as coupled optical waveguides \cite{2, 10}, coupled LRC electrical circuits \cite{4, 5, 16}, periodically switching gain and loss in coupled fiber loops \cite{17}, \( \mathcal{PT} \) symmetric absorptive silicon based waveguide \cite{18} and parity-time symmetric whispering gallery microcavities \cite{19} – elevated the \( \mathcal{PT} \) symmetric systems to accessible physical scenarios. In this section we review the experimental realizations of two level systems in the framework of optics and electronics.
Section 2.2. Experimental Observations of $\mathcal{PT}$ Symmetry

Figure 2.1: Schematic of the experimental set-up. An Ar$^+$ laser beam is coupled into the arms of the structure fabricated on a photo-refractive LiNbO$_3$ substrate. One waveguide experiences gain while the other with loss is masked to avoid any amplification. The CCD camera at the end monitors the intensity and phases at the output. Figure taken from [2] and referenced herein.

### 2.2.1 $\mathcal{PT}$ Symmetric Photonic via Synthetic Materials

Using couple mode theory and under paraxial approximation, light propagation in coupled waveguides can be described via Schrödinger-like differential equations where the index of refraction plays the role of the complex potential and the paraxial propagation distance plays the role of the time [10]. More specifically, the optical-field dynamics in a system of two coupled waveguides are described by the following set of equations:

\begin{align}
  i \frac{d\psi_1}{dz} + \psi_2 - i\gamma \psi_1 &= 0; \quad (a) \\
  i \frac{d\psi_2}{dz} + \psi_1 + i\gamma \psi_2 &= 0; \quad (b)
\end{align}

(2.17)

where $\psi_{1,2}$ are modal electric field amplitudes in the amplifying (Eq.2.17(a)) and lossy (Eq.2.17(b)) waveguide channels, $z$ represents a dimensionless propagation distance – normalized in units of coupling lengths – and $\gamma$ is a scaled gain and loss coefficient, also normalized to the coupling strength. This isomorphism provides a productive test bed where the notions of parity-time symmetry can be experimentally explored and exotic properties can be observed. A first demonstration of a $\mathcal{PT}$ symmetric photonic structure has been developed by C. E. Rüter et.al. [2] where they observed intensity/power oscillations in the exact phase and asymmetric light propagation as predicted in section
Section 2.2. Experimental Observations of $\mathcal{PT}$ Symmetry

2.1.1. Figure 2.2: Experimental results of light beam propagation the active "$\mathcal{PT}$ symmetric system".

In the above figures, the left/right panels correspond to an initial excitation at the left/right channel. The left channel corresponds to the gain channel while the right channel corresponds to the loss channel. (a) A conventional system corresponding to $\gamma = 0$. This propagation is reciprocal. (b) $\gamma < \gamma_{\mathcal{PT}}$ corresponding to the exact $\mathcal{PT}$-phase. In this case, we observed asymmetric dynamics. (c) $\gamma > \gamma_{\mathcal{PT}}$ corresponding to the broken $\mathcal{PT}$-phase. Figure taken from \cite{2}.

Specifically, the authors of Ref. \cite{2} fabricated two coupled $\mathcal{PT}$ symmetric waveguides from iron-doped LiNbO$_3$, each supporting one propagating mode, as shown in Fig.(2.1). The symmetric index profile $n_R(x)$ is made of Titanium in-diffusion. The Hamiltonian of this system is similar to the one in Eq.(2.3) where now, $v_0$ corresponds to the real part of the refractive index and $\kappa$ is the evanescent coupling between the two waveguides. One of these waveguides is being optically pumped to provide gain, $\gamma_G$, for the guided light, while the neighboring waveguide experiences an equal amount of loss, $\gamma_L$. Optical gain $\gamma_G$ has a typical magnitude of a few cm$^{-1}$ in Fe-doped LiNbO$_3$ and is provided through two-wave mixing using the material’s photorefractive non-linearity. Amplification is provided in only one channel by a mask on top of the sample which partially block the pump light in the other channel. A CCD (charge coupled device) camera monitored both the output intensity and the phase relation between the two channels (using interference with a plane reference wave). Optical excitation of electrons from Fe$^{2+}$ centers to the conduction band produce the losses $\gamma_L$.

The diffraction dynamics of the optical mode electric field amplitude as depicted in Fig.(2.2) is described mathematically by the results/model of the previous section.
2.2.2 Pseudo PT Symmetric Systems

While PT symmetry requires the existence of a balanced gain and loss mechanism, using a transformation we can show that the Hamiltonian of a passive-loss structure can be transformed to a PT symmetric one. To be more precise, propagation of the field in two coupled waveguides, with one being passive and the other with loss, is described via the following equation [10]:

\[
\frac{d}{dz} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 & \kappa \\ \kappa & -i\gamma \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}
\]  

(2.18)

Via a transformation of the form \( \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = e^{-\gamma t/2} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \) we can rewrite the above equation in the form of a PT symmetric one given in Eq. (2.17). Starting from this observation, Guo et al. [10] designed a non-Hermitian passive/loss optical double well structure as shown in Fig. (2.3a,b). The two waveguides were fabricated through a multilayer of Al\(_x\)Ga\(_{1-x}\)As with a complex refractive index distribution, \( n(x) = n_0 + n_R(x) + in_I(x) \), where \( n_0 \) is the constant background index, \( n_R(x) \) is the real index profile of the structure, and \( n_I(x) \) stands for the loss component and depends on the controlled loss parameter, \( \gamma \). In the experiment, loss is introduced to the right waveguide by a thin layer of Chromium. At the wavelength of operation (\( \lambda_0 = 1.55 \mu m \)), Chromium is heavily lossy while keeping the detuning between the two waveguides at a minimum level. This helps to overcome restrictions from the Kramer-Kronig relation. As a monochromatic light at 1550 nm is sent vertically to the non-lossy waveguide, the total intensity or transmission, \( T = |\phi_1|^2 + |\phi_2|^2 \), from both waveguides at some propagation distance \( z \) were measured and plotted as a function of the loss parameter, \( \gamma \).

Intuitively, one would predict the total intensity to decrease as loss increased in the system. However, the result shown in Fig. (2.3c) is contrary to our intuition! According
Section 2.2. Experimental Observations of $\mathcal{PT}$ Symmetry

Figure 2.3: (a) Schematic of a cross section of the double well structure of passive $\mathcal{PT}$ symmetric waveguides. Light propagates in the left guide and remains in the top layer. The yellow slab on the right, represents the Chromium which introduces loss, $\gamma$, into the system. (b) Scanning electron microscopy picture of the finalized passive $\mathcal{PT}$ device with the Cr stripe shown on the right. (c) The total transmission of a passive $\mathcal{PT}$ symmetric dimer as the loss in the lossy waveguide is increased. The dots correspond to experimental results and the solid line corresponds to the theoretical predictions. Notice that above a certain loss value ($\sim 6\text{cm}^{-1}$), the total transmission increases. Figure taken from [10].

to Fig. (2.3c), there is an initial decrease in the total transmission; however, above a certain critical loss value, the total transmission increases! This loss enhanced transmission is a direct manifestation of a $\mathcal{PT}$ non-Hermitian system. Below the critical value, the system is in the exact phase and according to Eq. (2.14) the “normalized norm” $I(z) * e^{-\gamma z}$ has an oscillatory behavior and light is continuously exchanged between the waveguides. Therefore increasing the loss will lead to a decreased output. At the spontaneous $\mathcal{PT}$ symmetry breaking point, the coupling between the waveguide channels begins to reduce. As the loss is increased further, the system enters the broken phase, the supermodes become increasingly asymmetric, and more localized on each of the waveguides. The power exchange that allows the incident light beam to experience loss when propagating along the lossy waveguide, is reduced and the total transmission is increased.
2.2.3 \( \mathcal{PT} \) Symmetric Electronics

The optical proposals in the previous section \[2.2\] are designed to study the spatial dynamics. However a desirable scenario is to study \( \mathcal{PT} \) systems in the spatio-temporal domain. In this respect active \( LRC \) circuits can be a fertile framework, allowing us to study spatio-temporal evolution both theoretically and experimentally in great detail.

The parity-time symmetric \( LRC \) system consists of a pair of coupled electronic oscillators, one with gain and the other one with equal amount of loss. The loss is simply generated with a resistance \( R \) and gain is introduced via negative resistance \(-R\) (amplifier). Figure \[2.4\] demonstrates such a \( \mathcal{PT} \) symmetric configuration. Notice that the electronics \( \mathcal{PT} \) symmetric dimer is not invariant under parity or time reversal individually. The parity operator switches the places of the amplifier and the resistance. On the other hand, the time reversal operator changes the sign of the resistances, thus turning the positive resistance to a negative one (i.e. an amplifier) and vice versa. Hence the dimer is invariant under both \( \mathcal{P} \) and \( \mathcal{T} \) operation. The details of experimental implementation of \( \mathcal{PT} \) electronics is given in \[4\] and our articles \[5, 16\].

![Schematic of the \( \mathcal{PT} \) symmetric electronic \( LRC \) circuit.](image)

**Figure 2.4**: Schematic of the \( \mathcal{PT} \) symmetric electronic \( LRC \) circuit. The gain and loss sides are coupled through capacitive and mutual inductive coupling. Figure is taken from \[5\].

Using Kirchhoff’s laws we can derive a set of coupled second order differential equations
which describes the relation between the charges and currents at the capacitors:

\[
V_1 = i\omega (LI_1 + MI_2) I_1 - \frac{V_1}{R} + i\omega CV_1 + i\omega C_c(V_1 - V_2) = 0 \quad (2.19)
\]

\[
V_2 = i\omega (LI_2 + MI_1) I_2 + \frac{V_2}{R} + i\omega CV_2 + i\omega C_c(V_2 - V_1) = 0. \quad (2.20)
\]

Above \(L\) describes the inductances and \(C\) gives the capacitances in the left (gain) and right (loss) side of the dimer, \(M\) is the mutual coupling between the inductances. The gain and loss side of the \(LRC\) circuit are also coupled via a capacitor \(C_c\). Voltages and currents in the left and right capacitors are given by \(V_{1,2}\) and \(I_{1,2}\) respectively. Eliminating the currents from the relations, scaling frequency and time by \(\omega_0 = 1/\sqrt{LC}\), and taking \(\mu = M/L\) and \(c = C_c/C\) gives the matrix equation:

\[
\begin{pmatrix}
\frac{1}{\omega(1-\mu^2)} - \omega(1+c) - i\gamma & \omega c - \frac{\mu}{\omega(1-\mu^2)} \\
\omega c - \frac{\mu}{\omega(1-\mu^2)} & \frac{1}{\omega(1-\mu^2)} - \omega(1+c) + i\gamma
\end{pmatrix}
\begin{pmatrix}
V_1 \\
V_2
\end{pmatrix} = 0. \quad (2.21)
\]

Above \(\gamma = R^{-1}\sqrt{\frac{L}{C}}\) is the gain and loss parameter. From the above equation we can derive the stationary and dynamical properties of \(\mathcal{PT}\) symmetric \(LRC\) dimer \[4, 5\]. Specifically the linear, homogeneous system \(2.21\) has four normal mode frequencies given by

\[
\omega_{1,2} = \pm \sqrt{\gamma_c^2 - \gamma^2 + \sqrt{\gamma_{\mathcal{PT}}^2 - \gamma^2}}; \quad \omega_{3,4} = \pm \sqrt{\gamma_c^2 - \gamma^2 - \sqrt{\gamma_{\mathcal{PT}}^2 - \gamma^2}}; \quad (2.22)
\]

with the \(\mathcal{PT}\) symmetry breaking point identified as

\[
\gamma_{\mathcal{PT}} = \left| \frac{1}{\sqrt{1-\mu}} - \sqrt{\frac{1+2c}{1+\mu}} \right| \quad (2.23)
\]

and the upper critical point by

\[
\gamma_c = \frac{1}{\sqrt{1-\mu}} + \sqrt{\frac{1+2c}{1+\mu}}. \quad (2.24)
\]

These theoretical results together with our experimental results \[4, 5\], as depicted in Fig.\(2.5\), shows that this “active” dimer allows for a direct observation of all the characteristics of \(\mathcal{PT}\) systems such as power oscillation in the exact phase, power law behavior at the exceptional point and exponential growth of the total energy in the capacitors in the broken phase. Like other \(\mathcal{PT}\) systems transition from exact phase to the broken phase can be controlled by a gain and loss parameter \(\gamma\).
2.2. Experimental Observations of $\mathcal{PT}$ Symmetry

Figure 2.5: (Left) Parametric evolution of the experimentally measured eigenfrequencies, vs. the normalized gain and loss parameter $\gamma/\gamma_{PT}$. A comparison with the theoretical results of Eq. (2.22), indicates an excellent agreement. In all cases, we show only the $Re(\omega_l)>0$ eigenfrequencies. The open circles in the lower panel are reflections of the experimental data (lower curve) with respect to the $Im(\omega) = 0$ axis. (Right) Experimentally measured temporal dynamics of the capacitance energy $E_{C}^{tot}(\tau)$ of the total system for various $\gamma$-values. As $\gamma \to \gamma_{PT}$ the $\tau^2$ behavior, a signature of the spontaneous $\mathcal{PT}$-symmetry breaking, is observed. Figure is taken from [5].

2.2.4 Asymmetric Spatio-Temporal Dynamics and State Manipulation

It is instructive to perform an alternate analysis of the dimer which can be accomplished by recasting Kirchoff’s laws, into a “rate equation” form by making use of a Liouvillian formalism. This approach leads to the introduction of a state vector $\Psi \equiv (Q_1, Q_2, \dot{Q}_1, \dot{Q}_2)^T$, which is constructed from the charges $Q_{1,2} = CV_{1,2}$ and displacement currents $\dot{Q}_{1,2}$ at the capacitors in each dimer. The time evolution of this state vector is generated by an effective non-Hermitian Hamiltonian $H_{eff}$ defined as

$$H_{eff} = i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\alpha\beta & \alpha\zeta & (1+c)\gamma & c\gamma \\ \alpha\zeta & -\alpha\beta & -c\gamma & -(1+c)\gamma \end{pmatrix}$$ (2.25)
where $\alpha = 1/(1 - \mu^2)$, $\beta = 1 + c + c\mu$, $\zeta = c + \mu + c\mu$. The effective Hamiltonian (2.25) is symmetric with respect to generalized $\mathcal{P}_0\mathcal{T}_0$ transformations, i.e. $[\mathcal{P}_0\mathcal{T}_0, H_{\text{eff}}] = 0$, where $\mathcal{P}_0 = \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$ and $\mathcal{T}_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathcal{K}$ where $\mathcal{K}$ denotes the operation of complex conjugation. Furthermore, it can be shown that the effective Hamiltonian $H_{\text{eff}}$ is related to a transposition symmetric, $\mathcal{PT}$ symmetric Hamiltonian $H$ (see our articles [5, 16]).

The above formulation allows us to discuss and experimentally test one of the earliest challenges in calculus of variation, the quantum brachistochrone dynamics. In its classical version the brachistochrone is associated with the problem of least time for a point like particle to travel from point A to a point B, under the influence of gravity [20]. The elevation of this problem to the quantum realm has recently attracted attention in the framework of the emerging field of quantum computation where one studies the possibility of using quantum dynamical processes to solve computational problems [21]. Here, one has to quantify the various costs (for example energy, and time) that the desired computation may require. The search for optimal time evolutions, with limited resources, is a natural problem that arises in such a context. More specifically, in quantum mechanics the brachistochrone is associated with the search for a (time-independent) Hamiltonian $H$ such that it minimizes the time $\tau_{fpt}$ (first passage time) needed for evolution $|\Psi_i\rangle \rightarrow |\Psi_f\rangle = e^{-iHt}|\Psi_i\rangle$ between two orthogonal states $|\Psi_i\rangle$ and $|\Psi_f\rangle$ while keeping the difference between the eigenvalues $\Delta E = |E_1 - E_2|$ of the corresponding Hamiltonian fixed [22]. Such a constraint is appropriate since a rescaling of the Hamiltonian as $H \rightarrow \alpha H$, with $\alpha > 1$, would make $\Delta E$, and hence the transition rates, large. This corresponds to the fact that, physically, only a finite amount of resources (e.g. a finite magnetic field) are typically available.

The quantum brachistochrone was addressed by a number of researchers (see for example [21, 24]) where it was shown that, the minimal time $\tau_{fpt}$ to perform the required
transformation is
\[ \tau_{fpt} = \frac{2\hbar}{\Delta E} \arccos(|\langle \Psi_i | \Psi_f \rangle|). \] (2.26)

Note that \( \tau_{fpt} \) is always strictly positive (except of course the trivial case \(|\Psi_i\rangle = |\Psi_f\rangle\)) and inversely proportional to \( \Delta E \).

\textbf{Figure 2.6:} (a) Typical temporal dynamics of the displacement current \( I_2(t) \) when the initial condition corresponds to an excitation of the gain side. (b) The first passage time \( \tau_{fpt} \) versus the gain and loss parameter \( \gamma \). Circles are experimental data while lines (of similar color) indicate numerical data. The black dashed line denotes \( \tau_{fpt} \) for \( \gamma = 0 \). The measurements are done under the constraint \( \Delta \omega / \omega_0 \approx 0.375 = \text{constant} \). Figure is taken from [16].

It was shown in [25] that the lower bound for \( \tau_{fpt} \) can be made arbitrarily low if, with the same \( \Delta E \) limitations, we allow the use of pseudo Hermitian Hamiltonians [25–27].

Our experimental \( \mathcal{PT} \) symmetric electronics [16] was the first and simplest possible parity-time symmetric system which allowed us to study brachistochrone dynamics in non-Hermitian \( \mathcal{PT} \) systems. We found that this system bypasses the lower bound of Eq.(2.26), reducing the first passage time \( \tau_{fpt} \) while retaining a real valued spectrum (\( \gamma < \gamma_{\mathcal{PT}} \)) and fixed eigenfrequency difference \( \Delta \omega \). The initial conditions used in both the theoretical and experimental study are \( I_2(0) = 1 \) and all the other dynamical variables zero. The first passage time \( \tau_{fpt} \) is then defined as the time for which the envelope of the current \( I_2 \) oscillations is zero (see Fig.(2.6a)). An operative definition used in our preliminary studies pertain to the total energies \( E_1(t) \) and \( E_2(t) \) of each circuit: the first passage time is defined as the time for which \( E_1(\tau_{fpt}) = E_2(\tau_{fpt}) \). We have checked numerically that this definition gives the same qualitative results as the criterion for the current envelope. In Fig.(2.6b) we report our measurements together.
with our theoretical predictions for the case of a simple $\mathcal{PT}$ symmetric dimer where an initial excitation has been introduced in the gain (red symbols and dashed line) or loss (green symbols and dashed line) side. In the former case, the passage time is increased with respect to the Hermitian case (indicated with dashed black line) while in the latter (brachistochrone case) the time $\tau_{fpt}$ decreases monotonically as the gain and loss parameter $\gamma$ increases.

The theoretical analysis and experimental details of the $\mathcal{PT}$ symmetric brachistochrone dynamics realized by our $LRC$ electronic circuitry can be found in our contribution Ref. [16].

2.3 $\mathcal{PT}$ Symmetry and Non-reciprocity

We will close this chapter with an important application of $\mathcal{PT}$ symmetry in integrated photonics, namely the possibility of creating non-reciprocal light transport. Reciprocity states that the effect produced at point $P$ by a source $P_0$ is the same as the effect produced at $P_0$ if we place a source of equal intensity at point $P$ [28]. In optics, realization of non-reciprocal elements and devices are of great importance since they can prevent back-reflections. Non-reciprocity is feasible by means of magneto-optical effects [29] where rotation of the plane of polarization induced by the permittivity tensor of the magnetic material leads to the isolation.

In the presence of an external magnetic field or spontaneous magnetization, the permittivity tensor of a magneto-optical material is antisymmetric and its off-diagonal elements are responsible for the rotation of the plane of the polarization [30]. The rotation of the plane of the polarization in magneto-optical materials is known as the Faraday effect. According to the Faraday effect, the angle of rotation $\theta$ is given by Becquerel’s formula

$$\theta = \nu B d \quad (2.27)$$
where $\nu$ is a wavelength-dependent characteristic of the material which is known as the Verdet constant, $B$ is the magnetic field, and $d$ is the length of the Faraday rotator. In most materials, the Verdet constant is very small. In this respect, in order to have a large degree of rotation, the size of the Faraday rotator or the magnetic field should be large.

At the same time, the rotation of the plane of polarization is in the same direction for forward and backward propagating polarized waves. Under the time reversal, a right circularly polarized wave propagating to the forward direction transforms to a right circularly polarized wave propagating in the backward direction [29]. This mechanism is the primary idea behind the realization of a Faraday isolator.
A Faraday isolator, as presented in Fig. (2.7), is composed of a magneto-optical material “sandwiched” between two polarizer: one vertical (input polarizer) and the other (analyzer) aligned at an angle $\theta$. Light traveling in the forward direction is polarized vertically by the input polarizer, rotates by $\theta = 45^\circ$ via the Faraday rotation and then passes through the analyzer. On the way back, the $45^\circ$ polarized light rotates additional $45^\circ$ and becomes horizontally polarized. Thus, the final outgoing signal does not pass through the input polarizer. Faraday isolators are very common and widely used; however, they tend to be bulky. This restriction prevents them to be good candidates for integrated photonics.

### 2.3.1 Integrated $\mathcal{PT}$ Symmetric Non-linear Isolator

The asymmetric dynamics of the linear $\mathcal{PT}$ symmetric optical coupler discussed in section 2.1.1 does not break Lorentz reciprocity in paired input-output channels. However, a combination of asymmetric dynamics with the non-linearity leads to non-reciprocal transport. The simplest possible realistic non-linear $\mathcal{PT}$ set-up consists of two coupled $\mathcal{PT}$-waveguides with Kerr non-linearity $\chi$. Each of the waveguides support one propagating mode $\psi_{1,2}$; one provides gain for the guided light while the other an equal amount of loss. The following equations describe the dynamics in this system:

$$\frac{dS_0}{dz} = \vec{E} \cdot \vec{S}; \quad \frac{d\vec{S}}{dz} = S_0 \vec{E} + \vec{S} \times \vec{B}$$  \hspace{1cm} (2.28)

where $S_0 = |\psi_1|^2 + |\psi_2|^2$ is the total norm and $S_3 = |\psi_1|^2 - |\psi_2|^2$ is the imbalance intensity between the two waveguides. The remaining two parameters $S_1$ and $S_2$ describe the phase difference of the field in the two waveguides and are given by $S_j = \vec{\psi}^\dagger \sigma_j \vec{\psi}$ ($\vec{\psi} = (\psi_1, \psi_2)^T$). The two real vectors $\vec{E} = (0, 0, 2\gamma)$ and $\vec{B} = (2, 0, \chi S_3)$ in Eqs. (2.28) can be interpreted as a pseudo-electric and pseudo-magnetic fields, respectively.

In Figure (2.8) we report two dynamics scenarios (Rabi-type oscillation, Figs. (2.8a,b), and diode behavior, Figs. (2.8c,d)) that occur for two representative values of the non-linearity strength $\chi$. In contrast to the $\gamma = 0$ case, the dynamics in Fig. (2.8a,b)
Figure 2.8: Beam propagation in two coupled nonlinear waveguides with non-linearity strength $\chi$ and a complex $\mathcal{PT}$ symmetric refractive index profile. Waveguides are colored, indicating balanced gain (red) and loss (green) regions. Left columns correspond to an initial excitation at the loss waveguide port while right columns correspond to an initial excitation at the gain waveguide. (a,b) the non-linearity is below critical value while for (c,d), the non-linearity strength is above. Figure taken from Ref.[11].

are asymmetric with respect to the axis of symmetry of the system. While this is true for any value of non-linearity strength $\chi$, it is much more pronounced for the case of Figs.(2.8c,d). In the latter case, in the forward direction (Fig.(2.8c)), starting from the lossy waveguide (green) the output beam leaves the structure from the gain (red) waveguide. In the corresponding reversed excitation where the beam starts from the gain waveguide (Fig.(2.8d)), the light does not end up at the reciprocal (lossy) waveguide but it again leaves the structure at the gain waveguide. This novel unidirectional propagation is the key mechanism for creation of optical valves. We have found in Ref.[11] that the critical non-linearity above which such behavior is observed is given by $\chi_d = 4 - 2\pi\gamma$.

The isolation action in the nonlinear $\mathcal{PT}$ dimer is a result of dynamical decoupling. Here we explain the underlying physical mechanism. Above the critical value of non-linearity, the input beam from the lossy channel experiences a low index of refraction, therefore...
it tunnels to the gain waveguide. On the other hand, once the light is at the gain waveguide, it experiences a high index of refraction which confines it to this waveguide. As a result it cannot tunnel back to the lossy waveguide. In the backward process, the beam initiated from the gain waveguide is self-trapped from the very beginning and there is no tunneling to the lossy waveguide. A detailed theoretical analysis together with numerical modeling is presented in our article [11]. The nonlinear $\mathcal{PT}$ isolator as presented here is not polarization sensitive, does not work based on higher harmonic generation and can also be fabricated as an on-chip device. Furthermore, it can provide broadband nonreciprocal action which is highly desirable in optical isolators.

Dynamical decoupling is not the only mechanism to produce asymmetric transport. Excitation of $\mathcal{PT}$ nonlinear resonances is another way that can lead to optical isolation. This resonance mechanism has been recently observed experimentally via the electronics $\mathcal{PT}$ symmetric Van der Pol oscillators (see our article [6]) and micro-disks resonators [19]. The resonances have functional dependence of the local nonlinear impedance profiles which leads to an asymmetry in the output signal. However, in contrast to the passive nonlinear proposals where there is competition between the strength of output signal and degree of asymmetry [32, 33], in the nonlinear $\mathcal{PT}$ resonators the asymmetry can be enhanced via the gain and loss parameter without compromising the strength of the output signal.

2.4 Summary

In this chapter we introduced the notion of parity-time $\mathcal{PT}$ symmetry. Using a prototype two level system, we showed that $\mathcal{PT}$ structures can have different phases characterized by the degree of their “non-Hermiticity”: exact phase for which the spectrum is real and broken phase for which the spectrum is complex. Moreover, we found the dynamical features of each phase, namely power oscillations in the exact phase, exponential power growth in the broken phase, and quadratic power growth at the spontaneous symme-
try breaking point. The latter being the transition point between exact and broken phase. We have reviewed some of the first experimental realizations of $\mathcal{PT}$ systems in optics and electronics and showed the agreement between theoretical and experimental results. Via the electronic set-up, we exploited the brachistochrone dynamics in $\mathcal{PT}$ systems where the passage time decreases monotonically as the gain and loss parameter $\gamma$ increases. Finally, we showed that by combining the parity-time symmetry and non-linearity, we can achieve non-reciprocity. The proposed nonlinear $\mathcal{PT}$-symmetric isolator is broadband, polarization independent, and does not depend on the excitation of higher harmonics.
Chapter 3

Extended $\mathcal{P}\mathcal{T}$ Symmetric Structures

In chapter 2 we discussed experimental realizations of $\mathcal{P}\mathcal{T}$ symmetric geometries composed of only two active elements such as optical $\mathcal{P}\mathcal{T}$ symmetric coupled waveguides and $\mathcal{P}\mathcal{T}$ symmetric electronics coupled oscillators. Naturally, the next step is to investigate extended systems which possess parity-time symmetries. Examples of such extended systems are 1D chains of coupled $\mathcal{P}\mathcal{T}$ symmetric dimers and their 2D extensions.

An intriguing feature of periodic extended systems is the existence of a dispersion relation which allows for the creation of propagating bands and gaps. In dispersive media, plane waves with different wavevectors have different propagation velocities. This means that the propagation constant is a function of the wavevector. Depending upon the material and/or geometric arrangements of these extended systems, we can create different dispersion relations, which result in unusual light propagation. Such effect include light confinement, Bloch oscillations, anomalous diffraction, etc. It is needless to argue about the technological implications of our capabilities to manipulate light propagation. After all, during the last twenty years, our capability to engineer photonic structures with
predefined dispersion relations has led to the creation of photonic lattices, negative refraction metamaterials, etc which can be used to manage light propagation.

One question that we are addressing here is whether or not we can adopt the existing managing techniques to the case of $\mathcal{PT}$ symmetric systems. Do the gain and loss elements, characterized by $\gamma$ as a controlled parameter, help us to further manipulate the dispersion relation? And even more importantly, what new types of beam dynamics will be observed in structures that satisfy the $\mathcal{PT}$ symmetries? After all, as discussed in the previous chapter, in the simplest cases of non-Hermitian two $\mathcal{PT}$-elements, the eigenvectors are bi-orthogonal (skewed) while the corresponding eigenvalues (propagation constants) can be real or complex pairs.

In this chapter we will discuss extended optical $\mathcal{PT}$ symmetric systems with special emphasis on coupled waveguide arrays. Our specific contribution in this direction can be found in Refs. [34, 35].

Below, we will see that manipulation of gain and loss in $\mathcal{PT}$ symmetric lattices, allow us to control the band-structure. More specifically, the gain/loss parameter can be used to control the width of the gap between two consecutive bands. For a specific amount of $\gamma = \gamma_{\mathcal{PT}}$, the gap size becomes zero. This leads to a degeneracy in the dispersion relation. At the same time, the associated eigenvectors will coalesce – a characteristic of the so called exceptional point. Below $\gamma_{\mathcal{PT}}$, the system is in the exact phase and its spectrum is real. In this phase, the corresponding effective Hamiltonian and the $\mathcal{PT}$ operator share the same set of eigenfunctions. For larger values of gain and loss parameter $\gamma > \gamma_{\mathcal{PT}}$, the system will undergoes a phase transition to the broken phase where the spectrum becomes partially or completely complex depending upon the value of the gain and loss parameter. The complex part of the eigenvalues is responsible for an exponential amplification in the total power of light propagating inside the system, which in many cases is not a desirable outcome.

In extended $\mathcal{PT}$ systems, the dispersion relation $\mathcal{E}$ is a function of the geometrical
characteristics of the structure, wavevector \( k \) and the gain and loss parameter \( \gamma \). As a result the group velocity, \( \frac{\partial c}{\partial k} \), will be a function of \( \gamma \) and the wave packet dynamics will be affected as well by \( \gamma \).^1

Although a detailed study has been done on the extended \( \mathcal{PT} \) systems (for example see \[1, 14, 36–39\] and our articles Refs. [34, 35]), experimental realizations of such systems is still a challenging task. In optics, realization of \( \mathcal{PT} \) structures that simultaneously exhibit a symmetric refractive index distribution and an antisymmetric gain and loss profile has been hampered by technical difficulties. Therefore, it would be interesting to propose new classes of \( \mathcal{PT} \) symmetric platforms, which allow us to experimentally observe spatially extended \( \mathcal{PT} \) symmetric dynamics. In this regard, and for the completeness of the presentation, we discuss in section 3.2 the recent experimental work of Ref.[17] which allows to emulate the dynamics in one dimensional \( \mathcal{PT} \) symmetric lattices using two coupled fiber-loops.

The structure of the chapter is as follows: in section 3.1 we discuss the beam dynamics in \( \mathcal{PT} \) lattices. Later we will discuss revivals in 1D parity-time symmetric lattices and we will show how the gain and loss can affect the revival pattern in such lattices. Finally in this section we exploit the effect of the exceptional point on the wave packet dynamics. In section 3.2 we present an experimental realization of parity-time symmetric extended systems where discretization in time helped the experimentalists to fabricate a large scale \( \mathcal{PT} \) symmetric network [17].

### 3.1 Beam Dynamics in \( \mathcal{PT} \) Lattices

Motivated by the experimental realizations of Refs. [2, 4, 5, 10], we first consider a periodic 1D dimeric lattice consisting of coupled \( \mathcal{PT} \) symmetric waveguides where each

^1Notice that in our study of group velocity we only consider the exact phase where the dispersion relation is real. The extension of the notion of the group velocity to the broken phase is an unexplored problem.
waveguide is assumed to support only one mode. Each dimer consists of two types of waveguides: type (A) involving a gain material and type (B) exhibiting an equal amount of loss. As depicted in Fig. (3.1a), within each dimer, the waveguides are coupled through optical tunneling with a coupling constant $K$. Dimers are coupled to each other by a coupling $C$. Next, we extend our study to the 2D optical honeycomb lattices with $\mathcal{PT}$ symmetry and investigate the possibility of abnormal diffraction.

### 3.1.1 One Dimensional $\mathcal{PT}$ Dimeric Chain

In the paraxial limit, the evolution of the electric field $\Psi_n = (a_n, b_n)^T$ in the $n$-th dimer along the propagation $z$-axis is described via

$$
\begin{align*}
    i \frac{da_n(z)}{dz} & = \epsilon a_n(z) + Kb_n(z) + Cb_{n-1}(z) \\
    i \frac{db_n(z)}{dz} & = \epsilon^* b_n(z) + Ka_n(z) + Ca_{n+1}(z)
\end{align*}
$$

(3.1)

where $a_n(b_n)$ is the electric field envelope in the gain (loss) waveguide and $\epsilon = \epsilon_0 + i\gamma$ is related to the complex refractive index. Without any loss of generality, we will assume below that $\epsilon_0 = 0$, $\gamma > 0$ and $C < K$. The first study that addressed the spectral properties and dynamics in such extended lattices was done in Ref. [14].

Following the work of Ref. [14] we can describe the wave packet dynamics either in space or in momentum representation. Specifically space and momentum dynamics are related via a unitary Fourier transform

$$
\begin{align*}
    a_n(z) & = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \tilde{a}_k(z) \exp(ink) \\
    b_n(z) & = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \tilde{b}_k(z) \exp(ink)
\end{align*}
$$

(3.2)

that performs the transformation between the two spaces. In Eq. (3.2) $k$ is the wavevector and the integral is taken over the Brillouin zone $-\pi \leq k \leq \pi$. This allows us to perform the evolution in Fourier space and then evaluate the spatial representation by a backward transformation i.e.

$$
\Psi_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_k(z)e^{ink}dk.
$$

(3.3)
Section 3.1. Beam Dynamics in PT Lattices

Figure 3.1: (a) 1D dimeric lattice consisting of coupled PT symmetric waveguides with in-dimer coupling $K$ and inter-dimer coupling $C$. Sublattice (gain waveguide) $A$ is shown by the red rectangular cuboids while sublattice (lossy waveguide) $B$ is shown by green rectangular cuboids. Each dimer is distinguished by the index $n$. (b) Dispersion relations for various $\gamma$-values. At $\gamma = \gamma_{PT}$, the gap between the two bands disappears and an exceptional point singularity is created. Figure taken from our article Ref.[34]

where $\psi_k(z) \equiv (\tilde{a}_k(z), \tilde{b}_k(z))^T$ is the Fourier component associate to the field amplitudes of the n-th dimer i.e. $\Psi_n = (a_n, b_n)^T$.

Because of the translational invariance of the system Eq.(3.1), the equations of motion in the Fourier representation break up into $2 \times 2$ blocks. The two-component wavefunctions for different $k$ values are decoupled, thus allowing for a simple theoretical description of the system. The dynamics of the system in the Fourier space is expressed in terms of a set of coupled differential equations

$$\frac{i}{d} \frac{d}{dz} \begin{pmatrix} \tilde{a}_k(z) \\ \tilde{b}_k(z) \end{pmatrix} = H_k \begin{pmatrix} \tilde{a}_k(z) \\ \tilde{b}_k(z) \end{pmatrix}; \quad H_k = \begin{pmatrix} \epsilon & v_k \\ v_k^* & \epsilon^* \end{pmatrix}$$

(3.4)

where in Eq.(3.4) $v_k = K + C \cdot e^{-ik}$.

Hamiltonian (3.4) is a non-Hermitian Hamiltonian and as a result it has left and right eigenvectors. The eigenvalues $\mathcal{E}(k)$ of the system are given by assuming a solution of
the form \((a_n, b_n)^T = \exp(-iEz)(A, B)^T\) for the differential equations (3.4):

\[
E_l = l\sqrt{(K - C)^2 + 4KC\cos^2(k/2) - \gamma^2}, \quad (l = \pm 1).
\] (3.5)

From Equation (3.5) it is obvious that the transition point where the eigenvalues become complex (exceptional point) is given by \(\gamma_{PT} = K - C\) and \(k = \pm \pi\). We have plotted this dispersion relation in Fig.(3.1) for various \(\gamma\).

Using the same mathematical methods as the one used in subsection 2.1.3 we can write the evolution operator in the following form:

\[
\hat{U}_k = \cos(|E_l|z)1 - i\left\{\frac{\sin(|E_l|z)}{|E_l|}\right\}H_k.
\] (3.6)

Equation (3.6) has the same form as Eq.(2.12), except that now it is wavevector dependent. In this regard we can predict some general characteristics of wave-packet dynamics in this lattice based on the discussion that we had in chapter 2 for a two level system.

For example for the wavevector \(k = \pm \pi\) the dispersion relation \(E_l\) in Eq.(3.5) is given by

\[
E_l = l\sqrt{(K - C)^2 - \gamma^2}.
\] (3.7)

which has exactly the same form as the propagation constant of a two level system \(E = l\sqrt{\kappa^2 - \gamma^2}\) (see Eq.(2.9) in previous chapter). We therefore conclude that at the exceptional point the total field intensity will grow in a power law manner. Moreover we can deduce that in the broken phase the norm of the system grows exponentially (see Refs. [14, 17]). In Ref.[14] it has been shown that the total intensity \(P\) as a function of the axial propagation distance, \(z\), behaves as

\[
P(z) \sim \begin{cases} 
\bar{K} & \gamma < \gamma_{PT} \\
\frac{1}{2M} \sum_n \langle R_n | R_n \rangle \langle L_n | L_n \rangle & \gamma = \gamma_{PT} \\
e^{2\sqrt{\gamma^2 - \gamma_{PT}^2}z} & \gamma > \gamma_{PT}
\end{cases}
\] (3.8)

where \(\bar{K} = \frac{1}{2M} \sum_n \langle R_n | R_n \rangle \langle L_n | L_n \rangle\) is the average diagonal Petermann factor defined via the left \(|L_n\rangle\) and right \(|R_n\rangle\) eigenvectors of \(H\) and \(M\) is the number of the waveguides.
3.1.2 Revivals in Extended PT Systems

The phenomenon of periodic replication of a wave-packet during the time evolution is known as the quantum revivals (for a review on quantum revivals see [40]). One form of Quantum revivals is the so called Talbot effect [41, 42]. Talbot effect is a self-imaging process of a periodic pattern illuminated by a monochromatic coherent light. These images are located at even integer multiples of the so-called Talbot distance $z_T = 2a^2/\lambda$, where $a$ represents the spatial period of the initial pattern and $\lambda$ is the light wavelength. Talbot effect has applications in fields ranging from image processing and synthesis, photolithography [43], and optical testing and metrology [44] to spectrometry and optical computing [45]. However all the above achievements are limited in studying the properties of the input beams and using real gratings for imaging. Bypassing these limitations will not only enrich the conventional self-imaging research, but also offers new methods for imaging technologies. It is therefore extremely desirable to investigate and propose self-imaging architectures which incorporate gain or/and loss mechanisms and more specifically investigate Talbot effect in the PT symmetric dimeric lattice of the previous sub-section. This task has been undertaken in our study Ref.[34]. Below we summarize some of the basic results of Ref. Ref.[34].

We recall that in order the Talbot effect to occur, the input field distribution should be periodic. Specifically the input field has to satisfy the relation $\Psi_n(0) = \Psi_{n+N}(0)$ where $N$ represents the spatial period of the input field. Because of this periodic initial condition the allowed wave-numbers, $k$ can take values only from the discrete set

$$k_m = \frac{2m\pi}{N}, \quad m = 0, 1, 2, ..., N - 1. \quad (3.9)$$

At the same time the evolution of any initial wave packet $\psi_k(0)$ is given by (see Eq[2.10])

$$\psi_k(z) = e^{-iHz}\psi_k(0) = \sum_{l=\pm 1} c_l^k e^{-iE_l(k)z} |R_l(k)\rangle \quad (3.10)$$

This condition is general and in any (passive or active) discrete lattice is applicable.
where $c^k_l = \langle L_l(k)|\psi_q(0)\rangle$ is the expansion coefficient and $\langle L_l(k)|$ is the left eigenvector associated with eigenvalue $\mathcal{E}_l(k)$. Substituting the constraint 3.9 in Eq. (3.10) we get the following expression for the evolved field at the $n$-th dimer

$$
\Psi^{(N)}_n(z) = \sum_{l=\pm 1; m=1}^{N-1} c^{k_m}_l e^{-i\mathcal{E}_l(k_m)z} |R_l(k_m)\rangle
$$

(3.11)

It is therefore clear that field revivals are possible at intervals $z$ such that $\mathcal{E}(k_m)z_T = 2\pi\nu$ where $\nu$ is an integer. Therefore the ratio of any two eigenvalues $\mathcal{E}_m \equiv \mathcal{E}(k_m)$ has to be a rational number, i.e.

$$
\frac{\sqrt{(K-C)^2 + 4KC\cos^2\left(\frac{m\pi}{N}\right)} - \gamma^2}{\sqrt{(K-C)^2 + 4KC\cos^2\left(\frac{m'\pi}{N}\right)} - \gamma^2} = \frac{\alpha}{\beta}
$$

(3.12)

where $\alpha$ and $\beta$ are relatively prime integers. At the same time, revivals in the field intensity are ensured provided that $(\mathcal{E}_m - \mathcal{E}_m')/(\mathcal{E}_m' - \mathcal{E}_m) = \alpha/\beta$ where the indices belong to the set $\{0, 1, ..., N-1\}$ and are taken at least three at a time. It is straightforward to show that this condition is trivially satisfied for the same set of $N$-input pattern periodicities as for the fields [34].

Using Eq. (3.12) and some trigonometric relations (see our article [34]) we found that in the exact phase specific periodicities associated to, $N = 1, 2, 3$ result to revivals while at the exceptional point we get self imaging for $N = 1, 3$.

Specifically in the exact phase and for $N = 1$ the Talbot length $z_T$ depends on the gain and loss parameter as $z_T = 2\pi/\mathcal{E}_0 = 2\pi/\sqrt{\gamma^2_{PT} + 4KC - \gamma^2}$ and therefore it varies by changing $\gamma$. Such reconfigurable behavior of the Talbot length is characteristic of the exact phase $\gamma < \gamma_{PT}$ and can be found also for the $N = 2, 3$-period input patterns. For $N = 2$ (corresponding to eigenvalue indices $m = 0, 1$ in Eq. (3.9)) one can show that for fixed $K, C$ and $\gamma_{PT} = K-C$ such that $\gamma_{PT}/(K+C) > \alpha/\beta$, Eq. (3.12) is satisfied provided that $\gamma = \sqrt{\gamma^2_{PT} - 4KC\alpha^2/(\beta^2 - \alpha^2)}$ (we assume that $\alpha < \beta$). Similarly for $N = 3$, the Talbot revivals are possible provided that $\gamma = \sqrt{\gamma^2_{PT} + KC[1 - 4(\alpha/\beta)^2]/[1 - (\alpha/\beta)^2]}$ where $0.5 < \alpha/\beta < \sqrt{1 - 3KC/(K+C)^2}$. In
Figure 3.2: Talbot intensity "carpets" for period-$N$ input field patterns at the exact phase $\gamma < \gamma_{PT}$ in the presence of structural imperfections. Everything is measured in units of the "average" inter-dimer coupling $C_0 = 1$ while the "average" intra-dimer coupling is $K_0 = 4$. (a) $\gamma = 0.1$ with 0% structural imperfections; (b) $\gamma = 0.1$ with ±2% disorder in the couplings around the average values $C_0, K_0$; (c) $\gamma = 0.1$ with ±5% disorder in the couplings. In all cases the input pattern has periodicity $N = 1$ and it is chosen to be $\{1,0,1,0,1,0,\cdots\}$; (d) $\gamma = \sqrt{11/3}$ with 0% coupling disorder; (e) $\gamma = \sqrt{11/3}$ with coupling disorder as in (b); (f) $\gamma = \sqrt{11/3}$ with coupling disorder as in (c). Now the input pattern has periodicity $N = 2$ and it is chosen to be $\{1,1,0,0,1,1,\cdots\}$.

Both cases the corresponding Talbot length is $\gamma$-dependent and it is given by the largest period $z_T = 2\pi/|E_j - E_j'| \sim 2\pi/E_0$ that results from the eigenvalues involved in the initial pattern. At the spontaneous $\mathcal{PT}$ symmetry breaking point the image revivals occur at Talbot lengths governed by the characteristics of the passive lattice.

Importantly, Talbot revivals are robust against structural imperfections. In Fig. (3.2) we confirm numerically that for realistic values of positional imperfections (up to 5% of the inter-dimer coupling) Talbot revivals are only slightly distorted. More specifically, revivals with short Talbot length $z_T$ are unaffected while the one with larger Talbot length due to the distortion of the delicate balance between the mode amplitudes and phases that eventually dominate the evolution are fragile.
Section 3.1. Beam Dynamics in $\mathcal{PT}$ Lattices

3.1.3 Conical Diffraction in $\mathcal{PT}$ Symmetric Honeycomb Lattices

One dimensional lattices are not the only possible geometries that one can consider for the study of $\mathcal{PT}$-dynamics. Another geometry that we have investigated in Ref. [35] is the two dimensional $\mathcal{PT}$ symmetric honeycomb lattice. Such a lattice (see Fig. (??) consists of two types of waveguides: type (A) made from lossy material (green) whereas type (B) exhibits the equal amount of gain (red). Their arrangement in space is such that they form coupled (A-B) dimers with inter and intra-dimer couplings $t_a$ and $t$ respectively. Passive honeycomb lattices show anomalous beam propagation originating from their special dispersion relation. One of the characteristics of their dispersion relation is the existence of the Diabolic points. Diabolic points (DP) are singular points, for which the direction of the group velocity is not uniquely defined. They are characteristic singularities of Hamiltonian systems, for which the spectrum shows degeneracy. Around the DP, dispersion relation has a relativistic form and is linear, $E \propto k$. Due to this linear behavior an initial Gaussian beam with a momentum at the vicinity of a diabolic points diffracts in the lattice such that attains the shape of a ring. The radius of this ring grows linearly while its thickness does not broaden [46]. This phenomenon is known as conical

Figure 3.3: Honeycomb photonic lattice structure with intra-dimer coupling $t$ and inter-dimer coupling $t_a = 1$. Sub-lattice (lossy waveguide) $a_{n,m}$ is shown by green circles while sub-lattice (gain waveguide) $b_{n,m}$ is shown by the red circles. Each dimer is distinguished by index $n$ and $m$. The field is coupled evanescently between the waveguides. Figure taken from Ref. [35].
diffraction (CD) and first has been predicted by Sir William Hamilton [47].

Figure 3.4: Propagation of a Gaussian superposition of Bloch modes at the vicinity of an exceptional point in a $\mathcal{P}\mathcal{T}$ symmetric honeycomb lattice. Shown is the beam intensity at normalized propagation distances of (a) $z = 0$; (b) $z = 10$; (c) $z = 15$, for $\gamma_{\mathcal{P}\mathcal{T}} = 1$ and $t = 3$. The input bell-shaped beam transforms into a ring-like structure of light with a non-varying thickness. Figure taken from Ref.[35].

Exceptional point on the other hand is another class of degeneracies, which arises in the dispersion relation of non-Hermitian Hamiltonians. In our article [35] we have shown that one can observe a new type of conical diffraction at the exceptional point in $\mathcal{P}\mathcal{T}$ honeycomb lattices. The diffraction happens such that an initial Gaussian beam, with

the wavevectors associated to the exceptional point, released into the $\mathcal{P}\mathcal{T}$ symmetric honeycomb lattice transforms to a ring-like shape (see Fig.(3.4)). By increasing the $\gamma$, the resulting ring-like shape becomes brighter. Moreover the diffracted ring travels along the lattice with a transverse speed proportional to $\sqrt{\gamma}$. For the detailed discussion on CD in $\mathcal{P}\mathcal{T}$ symmetric honeycomb lattices we refer the reader to our contribution in Ref.[35].

3.2 $\mathcal{P}\mathcal{T}$ symmetric Mesh Lattices

We have seen that gain and loss have a great impact on the dispersion relation and, it will affect the light propagation in extended lattices. In this regard, it is interesting to demonstrate these novel propagation properties of the parity-time symmetric lattices
experimentally. However, experimental realizations of extended parity-time symmetric systems are not easily accessible. In optics real and imaginary parts of the index of refraction are not unrelated. This is a direct consequence of the Kramers-Kronig relations coming from Cauchy’s residue theorem for complex integration. Kramers-Kronig relations are bidirectional and connect the real and imaginary parts of a complex function. However as it has been mentioned before, in $\mathcal{PT}$ systems the real part of the complex potential should be an even function, while its imaginary part should be an odd function. On the other hand, by introducing active elements (imaginary part of the index of refraction), we also affect the symmetry constraints of the real part of the index of refraction via Kramers-Kronig relations. In electronics, although Kramers-Kronig relations do not cause a problem, other issues, such as non-linear effects, hysteresis and parasitic inductance, arise. In the following we discuss an optical scenario, where discretization in time and space helped experimentalists to bypass the challenge in optics and fabricate a 1D extended $\mathcal{PT}$ symmetric mesh.

In Ref. A. Regensburger, et.al proposed an experimental set-up depicted in Fig. (3.5) to facilitate the realization of extended $\mathcal{PT}$ systems. Such realization was achieved by the discretization in the transverse $n$ and longitudinal $m$ directions. In their experiment they have considered a structure composed of two coupled fiber loops. The upper and lower loops have a length difference $\Delta L = L_{\text{min}}$, where $L_{\text{min}}$ is the length of the shorter loop. The two loops are coupled together with a 50% splitter. A signal pulse which is injected into the short loop will perform one round trip during the time $\Delta T = L_{\text{min}}/c$. At the end of this period the signal will be coupled to the longer loop via the splitter. At this point 50% of the beam is traveling in the long loop while the remaining 50% continues to stay in the shorter loop. Notice that, during this process, the pulse at the shorter loop will “go-around” twice while the pulse at the longer loop will make only one round trip. The time evolution of this multiplex process is measured in terms of discrete $\Delta T$ steps, i.e. $t = m\Delta T$. After the time step $m = 3$, the pulse associated with the longer loop will reach the splitter and it will interfere with the pulse that have
already made two rounds in the short loop.

In the above scheme, the spatial discrete evolution $n$ is defined via the splitting process during which a coherent light pulse traveling in the shorter (longer) loop makes a step to the left (right) in an isomorphic mesh network. The latter is indicated in Fig. 3.6c, together with the equivalent process occurring at the two coupled fiber loops.

Next we discuss the introduction of complex optical potential in the double fiber loop. The even, real part of the optical $\mathcal{PT}$ potential ($\Re\{n(x)\}$) can be discretely introduced using phase modulation $\pm \phi_0$. This can be achieved by an electro-optic phase modulator,
Figure 3.6: (a) Coupled fiber loops with periodically alternating gain and loss. PM is the phase modulator. (b) Passages of the evolution of a pulse in the $\mathcal{PT}$ symmetric time domain modulation. (c) The equivalent $\mathcal{PT}$ lattice. The real part gain a phase shift $\phi_\pm$. The red color indicate the gain waveguides while the lossy ones are indicated with the blue color. Figure taken from [17].

which is driven by an arbitrary waveform generator that creates a phase shift $\phi(n, m)$ to the pulse propagating in the lower loops. This phase modulation is an effective position dependent index of refraction.

Finally a semiconductor optical amplifier (SOA) produces gain in the signal. The introduced gain is large enough that in the gain loop it can compensate the intrinsic losses and even have net amplification. As a result the antisymmetric, imaginary component of the effective potential ($\Im\{n(x)\}$) is induced by some gain/loss factor $G$. In each round trip the gain and loss is switched between the upper and lower loops via the Acousto-optic modulator (AMO). In this respect the loops are repeatedly switched between gain and loss by equal amounts.

The periodic nature of the aforementioned structure in $n$ and $m$ leads to a band structure depicted in Fig. (3.7). Notice that the passive band structure of this system does not have any gap as depicted in Fig. (3.7a) [49] and the system, hence, has no $\mathcal{PT}$ threshold. Therefore, even a small amount of gain and loss $G$ abruptly forces the bands to merge at the exceptional points, where the transition to imaginary eigenvalues occurs (Fig. (3.7b)).
Section 3.2. $\mathcal{PT}$ symmetric Mesh Lattices

Figure 3.7: The first row shows the band structure of the (a) passive, (b) active lattice with no phase modulation (broken phase) where the bands are merged, (c) $\mathcal{PT}$ lattice at the exceptional point and (d) $\mathcal{PT}$ lattice in the exact phase. Second row shows the beam dynamics for the corresponding cases. The third row shows the total norm of the system which is according to what has been predicted in chapter 2. Figure taken from [17].

To establish a finite threshold, a symmetric phase potential $\phi(n)$ has to be introduced into the time lattice. The presence of such a potential forces the bands to move away from one another, thus creating a band gap. In this arrangement, the spectrum is again entirely real, in spite of the fact that the system is not Hermitian. By probing the entire band via a single signal excitation the authors of Ref.[17] observed the theoretical predictions of the total power in $\mathcal{PT}$ systems described in section 3.1, namely, in the broken phase (Fig. (3.7b)) the total norm grows exponentially in time while at the exceptional point it increase quadratically (Fig.)(3.7c)). In the exact phase (Fig. (3.7d)), unlike the passive case where the total norm is conserve, due to the bi-orthogonality of the Bloch modes, power oscillation has been observed.

While this achievement is undoubtedly a very important step towards the realization of $\mathcal{PT}$ symmetric extended systems, it is our hope that in the near future actual ex-
tended $\mathcal{PT}$-lattices will be fabricated that will allow us to investigate $\mathcal{PT}$ dynamics in depth.

\section*{3.3 Summary}

In this chapter we have introduced and reviewed the notion of $\mathcal{PT}$ symmetric extended systems. We began by a 1D lattice composed of $\mathcal{PT}$ symmetric dimeric waveguides coupled to each other. We demonstrated that one can manipulate the dispersion relation and control the gap size in the band structure by controlling the gain and loss parameter. Furthermore we showed that one can have reconfigurable Talbot effects in this lattice by managing only the gain and loss. An elevation of dynamics to two-dimensional lattices, like the honeycomb lattice, allowed us to investigate a new type of conical diffraction arising from the exceptional point dynamics. The resulting diffraction pattern is brighter with respect to the passive case. In addition, the transverse velocity of the diffracted wave packet was found to be a function of gain and loss parameter. Finally we reviewed an experimental realization of extended parity-time symmetric optical system.
Advanced communication science and technology and computer science have increased the demand for transforming and processing large volumes of data in a short time. The existence of miniaturized integrated photonics with high bandwidth and low power consumption makes integrated optics a promising candidate for this request. In this endeavor the creation of high efficiency on-chip optical isolators, one of the most important building blocks of integrated optics, is still a challenging task for researchers. In chapter 2 we exploited the isolation action in the presence of $\mathcal{PT}$ symmetry and non-linearity. However it is more desirable to have non-reciprocity in the linear domain, where there is no need to have high power input signal. In the linear domain there are pathways that allow for optical non-reciprocity. As discussed in chapter 2 non-reciprocity is possible by making use of Faraday rotation due to the interaction of light with a magneto-optical medium. Another approach, relies on time dependent modulation of the refractive index. Realization of the latter scenario is very complicated \[50\, 51\] and with the current technology is not commercially viable. While the former does not have this drawback it suffers from another problem, namely it is incompatible
with on-chip integration at a micrometer dimension. The origin of this difficulty is that at optical frequencies, the Faraday effect is very weak. As a result in order to perform a sufficient isolation action, Faraday isolators demand either large real-estate, high Verdet constants, or very strong magnetic fields (recall Eq. (2.27)). A natural way to enhance the Faraday rotation is to incorporate the magneto-optic materials into a resonator. In fact, a high-Q resonator increases the residence time of the photons inside the cavity. As a result, photons interact with the magnetic element for longer times/paths. Unfortunately, these photonic structures have undesirable side effects such as enhanced absorption, linear birefringence, and nonlinear effects. The most deleterious one, enhanced absorption, can dramatically affect the functionality of the optical devices. In addition, power loss and degraded quality factor of the optical resonator are inevitable. One way to avoid this problem is to incorporate gain elements to balance the losses. This observation motivated us to extend the notion of standard $\mathcal{PT}$ symmetric structures to include cases where time-reversibility is broken due to the presence of a magnetic field. This led us to the definition of a new family of $\mathcal{PT}$ symmetric systems which we call generalized $\tilde{\mathcal{PT}}$ symmetry.

In this chapter we will introduce the notion of $\tilde{\mathcal{PT}}$ symmetry. We will explore scattering properties of media with $\tilde{\mathcal{PT}}$ symmetry. Later, being faithful to our original motivation, we will propose a photonic structure, which belongs to this symmetry class, and can be used for optical isolation.

\footnote{Verdet constant depends on the dispersion of the refractive index, $dn/d\lambda$ where $n$ is the index of refraction $\lambda$ is the wavelength.

\[
v = -\frac{1}{2} \frac{e \lambda}{m c} \frac{dn}{d\lambda}
\] (4.1)

Here $e/m$ is the charge to mass ratio of the electron and $c$ is the speed of light. With the exception of some paramagnetic materials, the quantitative observations are in excellent agreement with Becquerel’s equation (2.27). Typically, in the range of wavelengths $\lambda = 6 \times 10^{-5}$ to $7 \times 10^{-5}$ cm, the refractive index changes by about $10^{-2}$. Hence (in MKS units) $dn/d\lambda \approx 10^5 m^{-1}$, $\lambda/c \approx 2 \times 10^{-15}$ seconds, and $\frac{\lambda}{m} = 1.76 \times 10^{11}$, gives $v \sim 17.6$ radians per tesla-meter, or $v \sim 0.06$ minutes of arc per gauss-cm.}
The structure of the chapter is as follows: First, in section 4.1 we review propagation and transfer matrix approach, which can be used to characterize scattering properties of linear magneto-optical materials with birefringence. This method is general and can be applied even in the presence of active elements. By incorporating the gain and loss in section 4.2 we extend the notion of parity-time symmetry to the generalized $\hat{\mathcal{PT}}$ symmetry. Finally in section 4.3 we present experimental observation of unidirectional reflectionless and invisible $\mathcal{PT}$ symmetric systems.

4.1 Electrodynamics of Stratified Media

4.1.1 Propagation Matrix Method

In general, a stratified medium can be a composed of two or more different type of layers. Here, we consider composite cases where one of the layers is an anisotropic dielectrics denoted bellow by $\mathcal{A}$ and, the other is a ferromagnetic material indicated with $\mathcal{F}$. The former, with an anisotropy in the $xy$ plane, is described via a permeability tensor, $\hat{\mu} = 1$ where $1$ is the unit matrix. Moreover the permittivity tensor $\hat{\epsilon}_A$ of such a $\mathcal{A}$ layer is given by

$$\hat{\epsilon}_A = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{xy}^* & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon + \delta \cos 2\phi & \delta \sin 2\phi & 0 \\ \delta \sin 2\phi & \epsilon - \delta \cos 2\phi & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix}. \quad (4.2)$$

In Eq. (4.2) the parameter $\delta$ describes the magnitude of in-plane anisotropy, while the angle $\phi$ defines the orientation of the principle axes of tensor $\hat{\epsilon}_A$ in the $xy$ plane. The orientation $\phi$ may vary from layer to layer. All the components of the $\hat{\epsilon}_A$ except for $\epsilon$ are assumed to be real. The imaginary part of $\epsilon$ represents the amount of gain ($\Im(\epsilon) < 0$) or loss ($\Im(\epsilon) > 0$) associated with the layer.

The Ferromagnetic $\mathcal{F}$ layers have a magnetization $M_0$ parallel to the $z$ direction (which is assumed to be fixed in the direction of propagation) without any in-plane anisotropy.
The permittivity, $\hat{\epsilon}_F$, and permeability, $\hat{\mu}_F$, are given by the following tensors:

$$
\hat{\epsilon}_F = \begin{bmatrix}
\epsilon & i\alpha & 0 \\
-i\alpha & \epsilon & 0 \\
0 & 0 & \epsilon_{zz}
\end{bmatrix}, \quad \hat{\mu}_F = \begin{bmatrix}
\mu & i\beta & 0 \\
-i\beta & \mu & 0 \\
0 & 0 & \mu_{zz}
\end{bmatrix} \tag{4.3}
$$

The $xy$ component and its corresponding transpose in the permittivity and permeability tensors are responsible for the Faraday rotation. The diagonal components can be in general complex numbers.

The electric ($\vec{E}$) and magnetic fields ($\vec{H}$) inside and outside such a stratified medium are given by the Maxwell equations:

$$
\nabla \times \vec{E} = i\omega c \vec{B}, \quad \nabla \times \vec{H} = -i\omega c \vec{D} \tag{4.4}
$$

where $\vec{E}(z) = \begin{pmatrix} E_x(z) \\ E_y(z) \end{pmatrix}$, $\vec{H}(z) = \begin{pmatrix} H_x(z) \\ H_y(z) \end{pmatrix}$. From the Maxwell equations (4.4) we derive the “reduced time harmonic” Maxwell equations \[53\]

$$
\sigma_y \frac{\partial}{\partial z} \vec{E} = \frac{\omega}{c} (\hat{\mu} \vec{H}), \quad \sigma_y \frac{\partial}{\partial z} \vec{H} = -\frac{\omega}{c} (\hat{\epsilon} \vec{E}) \tag{4.5}
$$

where $\sigma_y$ is the Pauli matrix. Equations (4.5) govern the propagation of a plane monochromatic wave along the $z$ direction in a 1D stack of $A$ and/or $F$ layers.

Notice that both electric and magnetic fields are perpendicular to the $z$ direction. At each interface the permeability and permittivity tensors are discontinuous, however the electric and magnetic field $\vec{E}$ and $\vec{H}$ are continues as there is no free charge and current. Upon assuming plane wave solutions $\vec{E}(z) = \vec{E}_0 e^{ikz}$ and $\vec{H}(z) = \vec{H}_0 e^{ikz}$ within a layer, where $k$ is the wavevector, Eq. (4.5) leads to four linear equations

$$
in \sigma_y \vec{E} + \hat{\mu} \vec{H} = 0, \quad \hat{\epsilon} \vec{E} - in \sigma_y \vec{H} = 0 \tag{4.6}
$$

where $n = \frac{c}{\omega}$. Solutions of Eq. (4.6) describe four electromagnetic eigenmodes, $\vec{E}_{1,2}$ and $\vec{H}_{1,2}$, within the layer \[53\]. Within each layer the solution of the Maxwell equations can
be written as

\[
\begin{pmatrix}
\vec{E}_{1,2}(z) \\
\vec{H}_{1,2}(z)
\end{pmatrix}
= e^{ik_{1,2}z}
\begin{pmatrix}
\vec{E}_{1,2} \\
\vec{H}_{1,2}
\end{pmatrix},
\begin{pmatrix}
\vec{E}_{1,2}(z) \\
\vec{H}_{1,2}(z)
\end{pmatrix}
= e^{-ik_{1,2}z}
\begin{pmatrix}
\vec{E}_{1,2} \\
\vec{H}_{1,2}
\end{pmatrix}
\quad (4.7)
\]

with the wavevector \( k_j = \frac{\omega}{c} n_j (j = 1, 2) \), where \( n_j \) are the eigenvalues of the tensor

\[ \hat{\eta} = \sigma_y \hat{\mu} \sigma_y \hat{\epsilon}. \]  

(4.8)

We can write the solution of the Maxwell equations Eq.(4.7) in terms of the propagation matrix of the layer, \( \hat{T}(z) \), and in terms of the electromagnetic eigenmodes \( \vec{E}_j \) and \( \vec{H}_j \),

\[
\begin{pmatrix}
\vec{E}_{1,2}(z) \\
\vec{H}_{1,2}(z)
\end{pmatrix}
= \hat{T}(z)
\begin{pmatrix}
\vec{E}_{1,2} \\
\vec{H}_{1,2}
\end{pmatrix}.
\quad (4.9)
\]

Using equations (4.9) and (4.7) we can construct the elements of the propagation matrix \( \hat{T}(z) \) associated with any layer with the eigenvectors \( \vec{E}_j \) and \( \vec{H}_j \). Specifically

\[
\hat{T}(z) = \hat{T}(z_2 - z_1) = \hat{W}(z_2)\hat{W}^{-1}(z_1)
\]

(4.10)

where \( \hat{W}(z) \) is a 4 \times 4 matrix defined \[53\]

\[
\hat{W}(z) =
\begin{pmatrix}
\mathcal{E}_{1,x} e^{ik_1z} & \mathcal{E}_{1,x} e^{-ik_1z} & \mathcal{E}_{2,x} e^{ik_2z} & \mathcal{E}_{2,x} e^{-ik_2z} \\
\mathcal{E}_{2,y} e^{ik_1z} & \mathcal{E}_{1,y} e^{-ik_1z} & \mathcal{E}_{2,y} e^{ik_2z} & \mathcal{E}_{2,y} e^{-ik_2z} \\
\mathcal{H}_{1,x} e^{ik_1z} & -\mathcal{H}_{1,x} e^{-ik_1z} & \mathcal{H}_{2,x} e^{ik_2z} & -\mathcal{H}_{2,x} e^{-ik_2z} \\
\mathcal{H}_{1,y} e^{ik_1z} & -\mathcal{H}_{1,y} e^{-ik_1z} & \mathcal{H}_{2,y} e^{ik_2z} & -\mathcal{H}_{2,y} e^{-ik_2z}
\end{pmatrix}.
\quad (4.11)
\]

Equation (4.6) together with the Eq.(4.11) and appropriate choice of \( \hat{\epsilon} \) and \( \hat{\mu} \) can result in the propagation matrix associated with the \( \mathcal{A} \) or \( \mathcal{F} \) layer. For example, for the \( \mathcal{A} \) layer with the permittivity \( (4.2) \) and \( \hat{\mu} = 1 \) inserted in the Eq.(4.6) we get the following solutions

\[
\vec{E}_1 = \begin{pmatrix}
\cos \phi \\
\sin \phi
\end{pmatrix} = \frac{\mathcal{H}_2}{n_2}, \quad \vec{E}_2 = \begin{pmatrix}
-\sin \phi \\
\cos \phi
\end{pmatrix} = \frac{\mathcal{H}_1}{n_1}
\quad (4.12)
\]

with

\[
n_{1,2} = \sqrt{\epsilon \pm \delta}
\quad (4.13)
\]
Section 4.1. Electrodynamics of Stratified Media

where sub-indices 1 and 2 are associated to the + and − sign respectively. The final step in order to calculate the propagation matrix of the \( A \) slab with thickness \( A \) is to construct the \( \hat{W}(z) \) and its inverse from the eigenvectors (4.12). After some algebraic steps we get [53]

\[
\begin{pmatrix}
(u^2 v_1 + v^2 u_1) 1 + uv(u_1 - u_2)\sigma_x & iuv(n_2 v_2 - n_1 v_1)\sigma_z + X \\
uw(n_2 v_2 - n_1 v_1)\sigma_z + (n_1 u^2 v_1 + n_2 v^2 u_2)\sigma_y & (u^2 u_1 + v^2 v_1) 1 - uv(u_1 - u_2)\sigma_x
\end{pmatrix}
\]

(4.14)

where \( 1 \) is the \( 2 \times 2 \) unit matrix, \( \sigma_x,\sigma_y,\sigma_z \) are the Pauli matrices and

\[
X = \begin{pmatrix}
0 & i(u^2 v_1 + v^2 u_2) \\
-i(\frac{v^2 u_1}{n_1} + \frac{u^2 v_2}{n_2}) & 0
\end{pmatrix}.
\]

(4.15)

Other parameters are defined as

\[
u_j = \cos(k_j A), \quad v_j = \sin(k_j A) \\
u = \cos \phi, \quad v = \sin \phi.
\]

(4.16)

A similar approach can be used in order to derive the propagation matrix of a ferromagnetic layer \( F \) with the thickness \( F \). Specifically in this case Eq.(4.6), with the permittivity and permeability (4.3), leads to the following solutions

\[
\begin{pmatrix}
\vec{E}_1 \\
\vec{H}_1
\end{pmatrix} = \begin{pmatrix}
1 & \frac{\zeta^-}{-i} \\
-i& 1
\end{pmatrix} = \hat{\mathcal{H}}_1,
\begin{pmatrix}
\vec{E}_2 \\
\vec{H}_2
\end{pmatrix} = \begin{pmatrix}
-i & \frac{\zeta^+}{-i} \\
1& 1
\end{pmatrix} = \hat{\mathcal{H}}_2
\]

(4.17)

where \( \zeta_{\pm} = \sqrt{(\epsilon \pm \alpha)(\mu \pm \beta)^{-1}} \). The propagation matrix \( \hat{T}_F \) reads as [53]

\[
\hat{T}_F = \begin{pmatrix}
(\frac{U_1 + U_2}{2}) 1 - (\frac{U_1 - U_2}{2})\sigma_y & (\zeta_1^{-1} V_1 - \zeta_2^{-1} V_2) 1 - (\zeta_1^{-1} V_1 + \zeta_2^{-1} V_2)\sigma_y \\
(\zeta_1 V_2 - \zeta_2 V_1) 1 + (\zeta_1 V_2 + \zeta_2 V_1)\sigma_y & (\frac{U_1 + U_2}{2}) 1 - (\frac{U_1 - U_2}{2})\sigma_y
\end{pmatrix}
\]

(4.18)

with

\[
U_j = \cos(q_j F), \quad V_j = \sin(q_j F)
\]

(4.19)

and the wavevectors \( q_j = \frac{\omega}{c} n_j \quad (j = 1, 2) \) \footnote{We used \( q \) as a notation for the wave-vector in \( F \) layers in order to distinguish them from the wavevectors in \( A \) layers.} In this, wavevectors \( n_{1,2} \) are the eigenvalues.
of the tensor $\hat{\eta}$ (see Eq. (4.8) for its definition) given by

$$n_{1,2} = \sqrt{(\epsilon \pm \alpha)(\mu \pm \beta)}$$  (4.20)

where sub-index 1 and 2 are associated with + and − respectively.

The total propagation matrix $\hat{T}$ of any complex multilayer heterostructure can be obtained via the appropriate multiplication of the propagation matrix of the constituting layers. In fact, for computational purposes, calculation of the $\hat{T}$ matrix is enough and from that we can calculate the transmission and reflection.

### 4.1.2 Transmittance and Reflectance of a Stratified Medium

In general, an incoming electromagnetic field can be written in the following vector form

$$\vec{\psi}_{in}(0) = \begin{pmatrix} \vec{E}_{in}(0) \\ \vec{H}_{in}(0) \end{pmatrix} = \begin{pmatrix} A_x \\ A_y \\ -A_y \\ A_x \end{pmatrix}.$$  (4.21)

The reflected field can also be written as

$$\vec{\psi}_{ref}(0) = \begin{pmatrix} \vec{E}_{ref}(0) \\ \vec{H}_{ref}(0) \end{pmatrix} = \begin{pmatrix} B_x \\ B_y \\ B_y \\ -B_x \end{pmatrix}.$$  (4.22)

The total electromagnetic field in the side of incidence (let’s say left) of the structure is

$$\vec{\psi}_t = \vec{\psi}_{in}(0) + \vec{\psi}_{ref}(0).$$  (4.23)
The transmitted electromagnetic field at the other side (let’s say right) of the heterostructure with a total length of $L$ at frequency $\omega$ is given by

$$
\vec{\psi}_r = \vec{\psi}_{tr}(L) = \begin{pmatrix} \vec{E}_{tr}(L) \\ \vec{H}_{tr}(L) \end{pmatrix} = \begin{pmatrix} P_x e^{i n_0 \omega L} \\ P_y e^{i n_0 \omega L} \\ -P_y e^{-i n_0 \omega L} \\ P_x e^{-i n_0 \omega L} \end{pmatrix}
$$

(4.24)

where $c$ is the velocity of light and $n_0$ is the background index of refraction out of the heterostructure. The total propagation matrix $\hat{T}$ connects the electromagnetic field $\vec{\psi}_l$ at the left side to the transmitted electromagnetic vector $\vec{\psi}_r$ at the right side of the multilayer medium,

$$
\vec{\psi}_r = \hat{T} \vec{\psi}_l
$$

(4.25)

Solving Eq.(4.25) for the unknown coefficients $B_{x,y}$ and $P_{x,y}$ we get

$$
\begin{pmatrix} B_x \\ B_y \\ P_x \\ P_y \end{pmatrix} = - \begin{pmatrix} T_{11} - T_{14} & T_{12} + T_{13} & -e^{i n_0 \omega L} & 0 \\ T_{21} - T_{24} & T_{22} + T_{23} & 0 & -e^{i n_0 \omega L} \\ T_{31} - T_{34} & T_{32} + T_{33} & 0 & e^{i n_0 \omega L} \\ T_{41} - T_{44} & T_{42} + T_{43} & -e^{i n_0 \omega L} & 0 \end{pmatrix}^{-1} \times \begin{pmatrix} A_x(T_{11} + T_{14}) + A_y(T_{12} - T_{13}) \\ A_y(T_{21} + T_{24}) + A_y(T_{22} - T_{23}) \\ A_x(T_{31} + T_{34}) + A_y(T_{32} - T_{33}) \\ A_y(T_{41} + T_{44}) + A_y(T_{42} - T_{44}) \end{pmatrix}
$$

(4.26)

The real valued energy flux or Poynting vector associated with a time-harmonic electromagnetic field is

$$
\vec{S} = \frac{1}{2} \mathcal{R}(\vec{E}^* \times \vec{H}).
$$

(4.27)

The transmission $T$ (reflection $R$) coefficient is defined as the ratio of the $z$ components of the transmitted (reflected) Poynting vector and incoming energy flux

$$
T = \frac{[\vec{S}_{tr}]_z}{[\vec{S}_{in}]_z} = \frac{|P_x|^2 + |P_y|^2}{|A_x|^2 + |A_y|^2}, \quad R = \frac{[\vec{S}_{ref}]_z}{[\vec{S}_{in}]_z} = \frac{|B_x|^2 + |B_y|^2}{|A_x|^2 + |A_y|^2}.
$$

(4.28)
4.1.3 Dispersion Relation of Periodic Stratified Media

One of the characteristics of periodic media as it has been discussed in chapter 3 is the dispersion relation $\omega(k)$ which determines their transport properties. For example a symmetric dispersion relation ensures us that the light propagation is reciprocal. Furthermore, the gaps inform us about the frequencies at which we have zero transmission and complete reflection. In this section we describe how we can calculate the dispersion relation of a stratified medium using the transfer matrix formalism discussed previously. Moreover, we will show that a composite periodic layered medium with a magneto-optic unit cell results to a dispersion relation which is asymmetric \[53\]. It turns out that such spectral asymmetry is a necessary but not a sufficient condition for transport non-reciprocity.

Consider an infinite periodic 1D stratified medium with a primitive cell of length $L$. The transport properties of this structure can be described via the propagation matrix $\hat{T}_p$. We used sub-index $p$ to indicate that our structure is periodic. Using the Bloch theorem, and due to the periodicity of the structure we conclude that the Bloch electromagnetic eigenmodes satisfy the relation

$$
\begin{pmatrix}
\vec{E}(L) \\
\vec{H}(L)
\end{pmatrix}
= e^{ikL}
\begin{pmatrix}
\vec{E}(0) \\
\vec{H}(0)
\end{pmatrix}.
$$

(4.29)

Using the propagation matrix $\hat{T}_p$ and Eq. (4.29) we have

$$
[\hat{T}_p - e^{ikL}1]
\begin{pmatrix}
\vec{E}(0) \\
\vec{H}(0)
\end{pmatrix} = 0.
$$

(4.30)

From Eq.(4.30) we see that the Bloch eigenmodes coincide with the eigenmodes of the propagation matrix $\hat{T}_p$. The dimensionless wavevectors $K$ which are functions of $\omega$ are given by:

$$
K_j(\omega) = i \ln\left(\frac{1}{\lambda_j(\omega)}\right), \quad j = 1, 2, 3, 4
$$

(4.31)
with $\lambda_j(\omega)$ being the eigenvalues of the $\hat{T}_p$. Dispersion relation $\omega(\vec{K})$ (component of the vector $\vec{K}$ are $K_j, j = 1, 2, 3, 4$) of the system can be obtained from Eq. (4.31) by finding the inverse function.

The dispersion relation in the absence of the scatterer is given by $\omega = |\vec{K}, (c = 1)|$. The presence of the periodic medium breaks this dispersion relation to bands which are separated by non-transmitting gaps. Bands correspond to a frequency window where transmittance is unity, while a gap correspond to a frequency window with zero transmittance. Usually the gaps are referred in the literature as reflected mirrors domains. In the gaps the so called evanescent modes associated to the frequency $\omega$ are complex which means they are not propagation modes.

In Fig. (4.1a) we have shown the basic unit of a periodic structure. In this representative example we used an $A$ and an $F$ layer next to each other to construct the basic unit. The corresponding dispersion relation of this heterostructure is depicted in Fig. (4.1b) where we highlighted the gaps.

The dispersion relation in Fig. (4.1) is an example of the symmetric dispersion relations where $\omega(\vec{K}) = \omega(-\vec{K})$. The lack of such a symmetry in the dispersion relation namely $\omega(\vec{K}) \neq \omega(-\vec{K})$ is known as spectral non-reciprocity. It has been shown that magnetically periodic media which do not support space inversion and/or time reversal symmetry might have spectral non-reciprocity \[54, 56\]. The magnetic layer induces magnetic non-reciprocity which is associated with the breaking of time reversal symmetry due to a static magnetic field or spontaneous magnetization. However as it has been mentioned, breaking time-reversal symmetry is not sufficient condition in order to obtain spectral asymmetry and the absence of space inversion is also required \[53, 57\]. A possible way to achieve this second condition is via two birefringent layers.

The simplest structure with spectral non-reciprocity as depicted in Fig. (4.2a) is a periodic arrangement with a primitive unit cell composed of three components: a central magnetic layer “sandwiched” between two misaligned anisotropic dielectrics \[53\]. Fig-
Figure 4.1: (a) Basic unit of a representative periodic medium. The heterostructure is an infinite layered medium composed of many basic units. (b) Dispersion relation of the periodic structure in part a. We used the following parameters is the permittivity of the A layer: $\epsilon = 3, \delta = 1, \phi = \sqrt{\pi}$. In the F layer for the permittivity we used: $\epsilon = 1.525, \alpha = 0.925, \mu = 1, \beta = 0$.

Figure 4.2b demonstrates a representative spectrum of such configuration. Interestingly enough we observe that one of the spectral branches develops a spectral singularity known as stationary inflection point (SIP) at $\omega = \Omega$. At SIP the first and second derivative of the dispersion relation with respect to the wavevector are zero. The associated mode is known as frozen mode as it has zero group velocity in one direction. A distinctive characteristic of these modes is that, in contrast to traditional Fabry-Perot (FP) resonances, they are virtually independent of the size and geometry of the confined photonic structure [58]. Utilizing these modes in our article [59] we proposed to create a Mirrorless Unidirectional Laser (MUL) which emits the outgoing optical field into a single direction. This new mechanism for unidirectional lasing action relies on the co-existence of gain and highly non-reciprocal SIP-related frozen modes.

Specifically we used the periodic structure of Fig.4.3a, where its constitute unit composed of two birefringent layers and a magnetic layer. In Fig.4.3b we have shown
the development of the SIP occurring at $\omega_0 \approx 5463.5$, marked by a circle. At this frequency there are two propagating Bloch waves: one with $k_0 \approx 0.613$ and the other with $k_1 \approx -2.452$. Obviously, only one of the two waves can transfer electromagnetic energy – the one with $k = k_1$ and corresponding group velocity pointing in the positive $\hat{z}$ direction i.e. $\vec{v}(k_1) > 0$. The Bloch eigenmode with $k = k_0$ has zero group velocity $\vec{v}(k_0) = 0$ and therefore does not transfer energy. The latter propagating mode is the frozen mode associated with the SIP.

We have shown in our contribution Ref. [59] that by incorporating a specific amount of gain lasing action happens. We confirm this by evaluating in Fig. (4.3c) an overall response function, defined as the total intensity of outgoing (reflected or transmitted)
waves for either a left or right injected wave; that is \( \Theta_{L,R}(\omega) = T_{L,R} + R_{L,R} \) where \( T_{L/R} \) and \( R_{L/R} \) are the respective left and right transmittances and reflectances averaged over polarization. For a lossless passive medium, one always has \( \Theta(\omega) = 1 \) due to power conservation, whereas \( \Theta(\omega) > 1 \) indicates that an overall amplification has been realized. Near the lasing frequency \( \Theta(\omega) \) takes large values, diverging as the lasing threshold is attained. Furthermore in the insets of Fig. 4.3c we report the left and
right transmittances in the regime of regular FP resonances (left inset) and at a SIP-related frozen mode (right inset). While for FP resonances $T_L$ and $T_R$ exhibit a moderate asymmetry, for a SIP-related mode the asymmetry between them increases dramatically ($T_R \gg T_L$ by more than two orders of magnitude) indicating strongly asymmetric transport. Even more this asymmetry is further enhanced as the size of the system becomes larger. This is in contrast to the FP resonances where the left/right asymmetry subsides.

A comparison between Fig.(4.3b) and Figs.(4.3c) indicates that the lasing threshold frequency is very close to $\omega_0$ associated with the SIP. While, at this frequency, a right-moving propagating wave (associated to $k_1$ and having large group velocity $v(k_1) > 0$) releases most of its energy outside the photonic structure, the mode associated to $k \approx k_0$ has extremely small group velocity and allows for a long residence time of the photons inside the structure. The interaction of these photons with the gain medium results in strong amplification which in turn leads to a lasing action. Once the lasing threshold is reached, the outgoing lasing beam is emitted predominantly from the left side of our structure (opposite side from $v(k_1) > 0$), therefore producing unidirectional lasing.

4.1.4 Transfer Matrix

As it has been shown in subsection 4.1.1, the propagation matrix $\hat{T}$ is a powerful tool to calculate the transmission and reflection coefficients of a stratified medium. However in some theoretical considerations it is beneficial to work with the transfer matrix $\hat{M}$. The transfer matrix connects the forward and backward electric field coefficients on the left and right side of the stratified medium. Let us assume that the stratified medium is placed in such a way that it extends in the interval $z = [-L, L]$. The propagation matrix $\hat{T}$ connects the fields at the left side, $z = -L$, to the fields at the right side,
$z = L$, of the heterostructure $^3$

\[ \vec{\psi}_r(L) = \hat{T} \vec{\psi}_l(-L) \]  

(4.32)

where the field vectors are

\[
\vec{\psi}_r(L) = \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}_r = \begin{pmatrix} A_x^r e^{ikL} + B_x^r e^{-ikL} \\ A_y^r e^{ikL} + B_y^r e^{-ikL} \\ -A_y^r e^{ikL} + B_y^r e^{-ikL} \\ A_x^r e^{ikL} - B_x^r e^{-ikL} \end{pmatrix}
\]

(4.33)

and

\[
\vec{\psi}_l(-L) = \begin{pmatrix} E_x \\ E_y \\ H_x \\ H_y \end{pmatrix}_l = \begin{pmatrix} A_x^l e^{-ikL} + B_x^l e^{ikL} \\ A_y^l e^{-ikL} + B_y^l e^{ikL} \\ -A_y^l e^{-ikL} + B_y^l e^{ikL} \\ A_x^l e^{-ikL} - B_x^l e^{ikL} \end{pmatrix}
\]

(4.34)

where we used the index $l, r$ to identify left and right side of the structure. Then the transfer matrix $\hat{M}$ can be obtained from Equation (4.32) by re-writing it in the basis of backward and forward coefficients $A_{x,y}$ and $B_{x,y}$

\[
\begin{pmatrix} \vec{A}^r \\ \vec{B}^r \end{pmatrix} = \hat{Q}(L)^{-1} \hat{T} \hat{Q}(-L) \begin{pmatrix} \vec{A}^l \\ \vec{B}^l \end{pmatrix}
\]

(4.35)

where the $4 \times 4$ matrix $\hat{Q}(z)$ is given by

\[
\hat{Q}(z) = \begin{pmatrix} e^{ikz} & 1 \\ -e^{-ikz} & 1 \\ ie^{ikz} \times \sigma_y & ie^{-ikz} \times \sigma_y \end{pmatrix}.
\]

(4.36)

Above $\vec{A}^{r,l} = (A_x^{r,l}, A_y^{r,l})^T, \vec{B}^{r,l} = (B_x^{r,l}, B_y^{r,l})^T$ while $1$ is a $2 \times 2$ identity matrix and $\sigma_y$ is the Pauli matrix. From Eq. (4.35) it is obvious that the transfer matrix $\hat{M}$ is nothing else than

\[
\hat{M} = \hat{Q}(L)^{-1} \hat{T} \hat{Q}(-L).
\]

(4.37)

$^3$There is no preference in the choice of coordinates such that $z = [-L,L]$, however we will see that this choice makes the final form of the transfer matrix $\hat{M}$ to be symmetric, see Eq. (4.37).
Section 4.2. Transport Properties of \( \mathcal{PT} \) Systems

Elements of the \( \hat{M} \) are related to the transmission and reflections. In order to see this let us re-write \( \hat{M} \) in terms of \( 2 \times 2 \) blocks in the following form:

\[
\hat{M} = \begin{pmatrix}
  M_{11} & M_{12} \\
  M_{21} & M_{22}
\end{pmatrix}.
\] (4.38)

From Eq.(4.35) it is obvious that the elements of the transfer matrix \( \hat{M} \), are related to the transmission and reflection coefficients at the left and right side of the structure via the following relations:

\[
\begin{align*}
  r^l &= -M_{22}^{-1}M_{21}, & r^r &= M_{12}M_{22}^{-1} \\
  t^l &= M_{11} - M_{12}M_{22}^{-1}M_{21}, & t^r &= M_{22}^{-1}.
\end{align*}
\] (4.39)

In order to have asymmetric wave propagation we need to have \( t^l \neq t^r \). From Eq.(4.39) we can show that this cannot be the case if we have

\[
[M_{21}, M_{22}] = 0 \quad \text{and} \quad M_{11}M_{22} - M_{12}M_{21} = 1
\] (4.40)

where \([...]\) is the commutator of its corresponding elements.

4.2 Transport Properties of \( \mathcal{PT} \) Systems

In Ref.[60] we have shown that a generalized parity time \( \mathcal{PT} \) symmetric magneto-optical layered medium can break relation \( (4.40) \) and leads to an asymmetric propagation. The simplest example of such a medium is composed of three layers: a dielectric layer with misalignment \( \phi_1 \), anisotropy \( \delta \) and gain \(-i\gamma\), a ferrite with Faraday characteristics \( \alpha, \beta \) placed in the middle, and another anisotropic misaligned dielectric with loss and parameters \( \phi_2, \delta \) and \(+i\gamma\). In Fig.(4.4a) we demonstrate our design schematically.

Before considering further the transport properties of the structure presented in Fig.(4.4a), lets find new conservation relations associated with such \( \mathcal{PT} \) symmetric systems. The time reversal operator \( \mathcal{T} \) is an anti-linear operator which performs transpose complex
conjugation while the linear operator $\hat{P} = \mathcal{P}\Theta$ consist of the parity operator $\mathcal{P}$ which represents a spatial inversion $\vec{r} \rightarrow -\vec{r}$ (note though that in our case spatial inversion is the same as reflection), and the exchange operator $\Theta$ which changes the angle of misalignment at $z$ with the one at $-z$ and similarly $\phi_1 \leftrightarrow \phi_2$. Permittivity and permeability of this structure is invariant under the generalized $\hat{P}T$ transformation. This simply means that if $\tilde{E}_{l,r}(\alpha, \beta, \phi_1, \phi_2)$ is a solution of the Maxwell equations then $(\tilde{E}_{l,r}(-\alpha, -\beta, \phi_2, \phi_1))^*$ is also a solution. Transfer matrix $\hat{M}$ connects the left to the right independent of which solution is chosen, i.e.

$$
\begin{pmatrix}
\tilde{A}(\alpha, \beta, \phi_1, \phi_2) \\
\tilde{B}(\alpha, \beta, \phi_1, \phi_2)
\end{pmatrix}
= \hat{M}
\begin{pmatrix}
\tilde{A}(\alpha, \beta, \phi_1, \phi_2) \\
\tilde{B}(\alpha, \beta, \phi_1, \phi_2)
\end{pmatrix}^*
\begin{pmatrix}
\tilde{A}(\alpha, -\beta, \phi_2, \phi_1) \\
\tilde{B}(\alpha, -\beta, \phi_2, \phi_1)
\end{pmatrix}
= \hat{M}
\begin{pmatrix}
\tilde{A}(\alpha, -\beta, \phi_2, \phi_1) \\
\tilde{B}(\alpha, -\beta, \phi_2, \phi_1)
\end{pmatrix}^*
$$

From Eq. (4.41) it is obvious that in such a structure inverse of the transfer matrix is equal to the transfer matrix with the negative Faraday rotation and exchanging the misalignments. In other words our structure satisfies the following conservation
Section 4.2. Transport Properties of \( \tilde{PT} \) Systems

relation:

\[
\hat{M}(\alpha, \beta, \phi_1, \phi_2) \hat{M}^*(-\alpha, -\beta, \phi_2, \phi_1) = 1.
\] (4.42)

Figure 4.5: (a) The left/right reflectances vs. \( \omega \). (b) The difference \( |\langle T_l \rangle - \langle T_r \rangle| \) between the left and right transmittances vs. \( \omega \). (c) The quality factor \( Q_T \) vs. \( \omega \). All layers have the same thickness \( d \), the phase misalignment is \( \Delta \phi_{\text{active dielectric}} = \sqrt{\pi} \), the anisotropy is \( \delta = 1 \), while the gain/loss parameter is \( \gamma = 0.0005 \). Furthermore the real part of permittivity for the birefringent layers is \( \epsilon = 9 \), for the magneto-optical layer is \( \epsilon = 1.525 \) while we assume that \( \epsilon_0 = 1 \). The Faraday gyrotropic parameters are \( \alpha = 0.925, \beta = 0, \) and \( \mu = 1 \). Figure taken from our article at [60].

Next we consider the asymmetric transport properties of the \( \tilde{PT} \) symmetric structure presented in Fig.(4.6a). We present in Fig.(4.5a) the result of our numerical simulation for the average (left/right) reflectance \( \langle R^{l,r} \rangle_p \) while in Fig.(4.5b) we report the left-right transmittance difference \( |\langle T^l \rangle_p - \langle T^r \rangle_p| \) (where \( R \equiv |r|^2 \) and \( T = |t|^2 \)) associate with the mentioned \( \tilde{PT} \) three layered structure. In all these simulations the average \( \langle \cdot \rangle_p \) is taken over all possible polarizations of the incoming wave. We find that the reflectances and transmittances for left/right incident waves are different from one another. Although an asymmetric left-right reflection is a characteristic property of systems with anti-linear symmetry [3], the asymmetry in the transmittances is a new element which essentially requires the presence of both a magneto-optical material and loss and/or gain materials. To further quantify the asymmetric behavior of our structure
we report in Fig. (4.5c) the asymmetry quality factor $Q_T$ which is defined as

$$Q_T = \left| \frac{<T_l - T_r>_p}{<T_l + T_r>_p} \right|. \quad (4.43)$$

The asymmetry of the $\mathcal{PT}$ symmetric magneto-optical micro-cavity can be further amplified by embedding it between two identical anisotropic Bragg mirrors, see Fig. (4.4b) [60]. The anisotropy at the Bragg gratings creates pseudo-gaps at the transmission spectrum as shown in Fig. (4.6a). We have found that at these frequency windows the non-reciprocity is enhanced. In Fig. (4.6b) we report the $Q_T$-factor for a structure shown in Fig. (4.4b) with a grating consisting of only 45 layers. The frequency domains of polarization independent asymmetric transport are marked with (green) shadowed areas and coincide with the pseudo-gaps of the grating.

The enhancement of asymmetry can be understood intuitively once we realize that our micro-cavity behaves as a high-Q optical resonator filled with magneto-optical material once it is embedded in a Bragg grating. In this case, each individual photon resides in the magnetic material much longer compared to the same piece of magnetic material placed outside the resonator. Since the sign of Faraday rotation is independent of the direction of light propagation, one can assume that the total amount of Faraday rotation is proportional to the photon residence time in the magnetic material.

### 4.3 Experimental Demonstration of Scattering in Periodic $\mathcal{PT}$ systems

In recent years the subject of cloaking physics has attracted considerable interest, specifically in connection to transformation optics [61]. In Ref. [3] we have introduce an alternative notion associated with unidirectional transparency and invisibility. As opposed to surrounding a scatterer with a cloak medium, in our case the invisibility arises because of $\mathcal{PT}$ symmetry. This is accomplished via a judicious design that involves a combination of optical gain and loss regions and the process of index modulation.
Section 4.3. Experimental Demonstration of Scattering in Periodic $\mathcal{PT}$ systems

Figure 4.6: (Upper subfigure) Dispersion relation $\omega(k)$ of the infinite periodic anisotropic Bragg reflector with permittivity contrasts $\epsilon = 3$ and $\epsilon = 12$ and $\delta = 1, \Delta \phi = 0$. (Lower subfigure) Quality factor $Q_T$ (red dashed line) of a $\mathcal{PT}$ symmetric non-reciprocal magneto-optical cavity when it is embedded in the anisotropic Bragg reflector associated with the set-up of the upper sub-figure. The micro-cavity has the same constitute parameters as in Fig.(4.5). The transmittance spectrum of the periodic Bragg is also shown in order to identify the pseudo-gap frequency windows (green shadowed areas) where the $Q_T$-factor take large values. Figure taken from our article at [60].

Specifically, we considered scattering from $\mathcal{PT}$ synthetic Bragg structures and investigated the consequences of $\mathcal{PT}$ symmetry in the scattering process. It is well known that passive gratings (involving no gain or loss) can act as high efficiency reflectors around the Bragg wavelength. Instead, we find that at the $\mathcal{PT}$ symmetric breaking point, the system is reflectionless over all frequencies around the Bragg resonance when light is incident from one side of the structure while from the other side its reflectivity is enhanced. Furthermore, we show that in this same regime the transmission phase vanishes-a necessary condition for evading detectability. Even more surprisingly, is the fact that these effects persist even in the presence of Kerr non-linearities.

The invisible $\mathcal{PT}$ symmetric heterostructure that we considered is a multi-layered optical
slab with each layer of width $d$. The layers being arranged such that for $|z| \leq L/2$ we have

$$n(z) = n_0 + n_1 \text{sign}[\cos(2\beta z)] + in_2 \text{sign}[\sin(2\beta z)]$$  \hspace{1cm} (4.44)

where $\beta = \frac{\pi}{4d}$, $n_0$ is the background index of refraction and $n_{1,2}$ are the real and imaginary part of the modulation in the scattering domain. Outside of this range $|z| \geq L/2$ the index of refraction is $n_0$. This grating exhibits a spontaneous $\mathcal{PT}$ symmetry breaking for $n_1 = n_2$. For $n_2 = 0$ the periodic modulation of refractive index leads to Bragg reflection at frequencies close to the Bragg frequency, $\omega = c\beta/n_0$ where $c$ is the speed of light in vacuum. The steady state scattering solution for the electric field $E(z)$ with frequency $\omega$ obeys the Helmholtz equation

$$\frac{d^2 E}{dz^2} + \frac{\omega^2}{c^2} n^2(z) E = 0$$  \hspace{1cm} (4.45)

For $|z| < L/2$ outside the grating region, Eq. (4.45) admits a solution $E_0^-(z) = A_- e^{ikz} + B_- e^{-ikz}$ for $z < -L/2$ and $E_0^+(z) = A_+ e^{ikz} + B_+ e^{-ikz}$ for $z > L/2$ where $k = n_0 \omega/c$. The transmission and reflection coefficients, for left (right) incident waves correspond to $B_+ = 0 (A_- = 0)$ and are given by $T_l = |A_+ / A_-|^2 (T_l = |B_- / B_+|^2)$, $R_l = |B_- / A_-|^2 (R_r = |A_+ / B_+|^2)$.

A theoretical analysis can be perform by approximating the index of refraction by

$$n(z) = n_0 + n_1 \cos(2\beta z) + in_2 \sin(2\beta z).$$  \hspace{1cm} (4.46)

By decomposing the electric field inside the scattering domain $|z| < L/2$, in terms of forward $E_F(z)$ and backward $E_B(z)$ traveling envelopes as $E_F(z) e^{ikz} + E_B(z) e^{-ikz}$ and by employing coupled mode theory, we find exact expressions for the transmission and reflection coefficients near the Bragg frequency [3]

$$T = \frac{|\lambda|^2}{|\lambda|^2 \cos^2(\lambda L) + \delta^2 \sin^2(\lambda L)}$$  \hspace{1cm} (4.47)

$$R_l = \frac{(n_1 - n_2)^2 k^2 / 4n_0^2}{\delta^2 + |\lambda|^2 \cot^2(\lambda L)}; \quad R_r = \frac{(n_1 + n_2)^2 k^2 / 4n_0^2}{\delta^2 + |\lambda|^2 \cot^2(\lambda L)}$$
Section 4.3. Experimental Demonstration of Scattering in Periodic $\mathcal{PT}$ systems

where $\lambda = \sqrt{\delta^2 - k^2(n_1^2 - n_2^2)/4n_0^2}$ and $\delta = \beta - k$ is the detuning from the Bragg wave-vector. For $n_2 = 0$ one recovers the standard scattering features of periodic Bragg structures (see also our numerical results in Fig. (4.7a)). Namely, $R_l = R_r$ while at the same time $T_r = T_l$. At the exceptional point $n_1 = n_2$ we find that $T_l = T_r = 1$ and $R_r = 0$ (i.e. perfect transmission) while $R_l \propto L^2$ (see Fig. (4.7b)). The latter algebraic amplification of $R_l$ is a characteristic of $\mathcal{PT}$ media at the exceptional point as discussed in chapter 2. As depicted in Fig. (4.7c), for $n_2 > n_1$ we will have deviation from unidirectional invisibility.

![Figure 4.7](image)

**Figure 4.7:** (a) Standard left/right transmission of the passive Bragg grating, $n_2 = 0$. (b) By introducing the gain and loss $0 \neq n_2 < n_1$ the reflections start deviating from each other. (c) Unidirectional invisibility, at the exceptional point where $n_1 = n_2$ transmissions are equal to one, reflection right is zero and reflection left is proportional to the length square of the system. The inset shows the power law behavior of the reflection versus the system size (d) Above the exceptional point the scattering features deviate from the invisibility.

Detailed simulations together with Eq. (4.45) (see Ref. [4]), allow us to conclude that the unidirectional invisibility survives even for a broad range of frequencies in the vicinity of the Bragg frequency and is robust against small deviations from the exceptional point. In the following we present experimental ramification of our proposal in different configurations.
Section 4.3. Experimental Demonstration of Scattering in Periodic PT systems

4.3.1 Experimental demonstration of a unidirectional reflectionless PT metamaterial at optical frequencies

In chapter 2 we discussed that a passive loss structure can be mapped to a PT symmetric system. While for such a passive loss system the transmission decays from unity, there exist an exceptional point such that the reflection diminished as predicted in Ref. [3]. Indeed we should have a unidirectional reflectionless potential. This idea has been experimentally proved in Ref. [18]. Specifically, Feng, et al used a spatial periodic passive-absorptive part Si – Ge/Cr modulation on a metal oxide semiconductor silicon base. In their experiment, as depicted in Fig.(4.8), they fabricated a passive loss PT-silicon metamaterial width 800(nm) and thickness 220(nm) embedded inside SiO$_2$ waveguide. The waveguide supports a wavevector $k_1 = 2.96k_0$ which is the fundamental mode of the waveguide. $k_0$ is the wavevector associate with the propagation constant of air. The modulation has the form

$$\Delta n(z) = \cos(2k_1z) - i\zeta \sin(2k_1z), \quad \frac{2n\pi}{k_1} - \frac{2\pi}{k_1} \leq z \leq \frac{2n\pi}{k_1} + \frac{2\pi}{k_1} \quad (4.48)$$

with the periodicity $\frac{2\pi}{k_1}$. Si modulation creates the real part contrast while the Ge/Cr is responsible for the imaginary part contrast. Using the slowly varying envelop approximation where second derivative of electric field with respect to propagation direction is zero $\frac{dE}{dz} = 0$, the forward $A_+$ and backward $A_-$ fundamental amplitudes satisfy the following coupled differential equations:

$$\frac{dA_\pm}{dz} = \mp \frac{\xi}{2\pi} A_\pm \mp i \frac{1}{8} \zeta A_\mp \kappa A_\mp. \quad (4.49)$$

In equation (4.49) $\xi$ and $\kappa$ denote attenuation and mode coupling between $A_\pm$. Elements of the transfer matrix $M$ which connects $A_\pm(0)$ to $A_\pm(L)$ is given by

$$M_{11} = \cosh(\rho L) - \frac{\xi}{2\pi \rho} \sinh(\rho L), \quad M_{12} = i(1 - \zeta^2) \frac{\kappa}{8 \rho} \sinh(\rho L)$$
$$M_{21} = -i(1 + \zeta) \kappa \frac{\xi}{8 \rho} \sinh(\rho L), \quad M_{22} = \cosh(\rho L) - \frac{\xi}{2\pi \rho} \sinh(\rho L) \quad (4.50)$$

with $\rho = \sqrt{\left(\zeta \xi / 2\pi \right)^2 + (1 - \zeta^2) \kappa^2 / 64}$. The forward and backward reflections are given by $R_f = |\frac{M_{21}}{M_{22}}|^2$ and $R_b = |\frac{M_{12}}{M_{22}}|^2$ respectively. From Eq.(4.50) it is obvious that at $\zeta = 1$
4.3. Experimental Demonstration of Scattering in Periodic $\mathcal{PT}$ systems

Figure 4.8: (a) Schematic of the experimental implementation of the passive loss waveguides on the silicon oxide insulator platform. The single mode directional coupler with the coupling ratio $3\text{dB}$ maximize the detected signal. (b) and (c) SEM picture of the whole and zoomed modulation profile. Figure taken from Ref.[18].

which is the exceptional point of the system the backward reflection is diminished. Experimental demonstration of this phenomenon is shown in Fig.(4.9).

4.3.2 Temporal Resolved Beam Dynamics in $\mathcal{PT}$ lattices

In the previous subsection we discussed the possibility of observing the unidirectional reflectionless scattering at the exceptional point in the passive loss $\mathcal{PT}$ systems. However unidirectional invisibility is possible only if we incorporate gain into the system. As it has been discussed in chapter 3, A. Regensburger et. al [17] achieved to incorporate gain and loss in a balanced manner in a discrete temporal $\mathcal{PT}$ symmetric structure fabricated via fiber optic loops (for a discussion of this experimental realization, see chapter 3). Using this setup they have been demonstrated the unidirectional invisibility phenomenon that we have predicted in Ref. [3], in the temporal domain. In order to
Section 4.3. Experimental Demonstration of Scattering in Periodic $\mathcal{PT}$ systems

Figure 4.9: (a) Experimental measured reflections of the parity time passive loss Bragg grating. The red and blue curves are the Gaussian fits to the row data of the forward (black) and backward (green) reflections, respectively (b) contrast ratio of the reflectivities obtained from fitting data in (a). Figure taken from Ref.[18].

perform such an experiment they have created a Bragg scatterer using a periodic phase modulation in a finite time domain. Outside this domain traveling pulses does not experience the $\mathcal{PT}$ potential. In the passive case the potential act as a Bragg reflector, Fig.(4.10a), while in the $\mathcal{PT}$ case the transmission is complete with zero reflection in one side, Fig.(4.10b), and an enhancement of reflection from other side, Fig.(4.10c) as it has been predicted in the theory.

Figure 4.10: (a) Scattering from the passive system with a periodic phase modulation $2\phi_0 = 0.36\pi$). The structure shows a complete reflection. (b) The scattering from the left side of the system at the exceptional point. The complete transmission with zero reflection is observed. (c) the same experiment as (b) but from the right side of the system. An enhancement of reflection with complete transmission is observed.
4.4 Summary

In this chapter we introduced the notion of the generalized parity-time $\tilde{\mathcal{PT}}$ symmetry. Using the machinery developed in this chapter we calculated the transport properties of the $\tilde{\mathcal{PT}}$ systems, namely the asymmetric dispersion relation and non-reciprocity in transport. Moreover we have shown that passive-gain magneto-optical structures can exhibit asymmetric dispersion relation and a stationary inflection point simultaneously. Such structures can result to unidirectional lasing modes. Finally we discussed the unidirectional invisibility and its experimental ramifications in optical structures.
Chapter 5

Conclusion

In this thesis, we have explored transport properties of complex media with parity-time ($\mathcal{PT}$) symmetry. This new type of materials combines, in a judicious manner, spatial domains with balanced amplification and attenuation mechanisms. The associated impedance profile (index of refraction in the optics language) is mathematically represented via a function that satisfies the symmetry relation: $V(x) = V^*(-x)$. We have shown, via specific examples, that these symmetries result in signal propagation with intriguing features, and many promising technological applications. The most prominent of these features is the possibility of non-reciprocal transport when $\mathcal{PT}$-symmetry and non-linearities coexist.

We investigated beam dynamics in a new family of $\mathcal{PT}$-dimeric lattices. Furthermore we identified new management methods, involving the manipulation of the gain and loss parameter, allowing us to control the dispersion relation, and thus the beam diffraction. In this respect, we developed the notions of $\mathcal{PT}$ symmetric conical diffraction taking place in photonic honeycomb lattices, and the $\mathcal{PT}$ symmetric Talbot effects associated with reconfigurable revivals of an initial periodic pattern.

In the scattering domain we extended the scattering formalism towards systems with
(generalized) $\mathcal{PT}$-symmetry and derived new conservation relations. These were used to develop notions like unidirectional invisibility, and polarization independent non-reciprocity. The later is produced in the case where gyrotropic elements are also present in the $\mathcal{PT}$-structure. Furthermore, using the conditions for non-reciprocal transport we have developed the notion of mirror-less unidirectional lasing structures.

Many of our theoretical predictions, which were derived in the framework of photonics have been tested and confirmed using the fertile framework of $\mathcal{PT}$ symmetric electronics. Arrays of $LRC$ elements where loss, via the resistances, is balanced with gain, provided by an amplifier, can easily be fabricated and monitored. They can be used as prototype systems for the creation of (nano) antennas and meta-material arrays (like split-ring resonator arrays) with novel transport characteristics.

The field of $\mathcal{PT}$ symmetric wave propagation, and more generally non-Hermitian wave mechanics, is at its infancy. As a consequence there is still a long road ahead. As the saying goes, “joy lies in the fight, in the attempt, and in the suffering involved, not in the victory itself”, so is the scientific endeavor in this direction bound to bring us exciting insights and discoveries.
Appendix A

Selected Publications

A.1 Unidirectional Nonlinear $\mathcal{PT}$-symmetric Optical Structures

A.2 Unidirectional invisibility induced by $\mathcal{PT}$-symmetric periodic structures

A.3 Bypassing the bandwidth theorem with $\mathcal{PT}$ symmetry

A.4 Exceptional-point dynamics in photonic honeycomb lattices with $\mathcal{PT}$ symmetry

A.5 $\mathcal{PT}$-symmetric Talbot effects
A.6 $\mathcal{PT}$-symmetric electronics


A.7 Taming the flow of light via active magneto-optical impurities


A.8 Observation of asymmetric transport in structures with active nonlinearities


A.9 Unidirectional Lasing Emerging from Frozen Light in Non-Reciprocal Cavities

Hamidreza Ramezani, S. Kalish, I. Vitebskiy, T. Kottos, (available online at arxiv.org) Submitted 2013

A.10 Optical Asymmetry Induced by $\mathcal{PT}$-symmetric Nonlinear Fano Resonances

F. Nazari, N. Bender, Hamidreza Ramezani, M. K. Moravvej-Farshi, D. N. Christodoulides, T. Kottos, (available online at arxiv.org) Submitted 2013
I. INTRODUCTION

Transport phenomena and in particular directed transport are at the heart of many fundamental problems in physics, chemistry, and biology [1]. At the same time they are also of great relevance to technological applications based on a variety of transport-based devices such as rectifiers, pumps, particle separators, molecular switches, and electronic diodes and transistors. Of special interest is the realization of novel classes of integrated photonic devices that allow diodes and transistors. Of special interest is the realization of pumps, particle separators, molecular switches, and electronic on a variety of transport-based devices such as rectifiers, also of great relevance to technological applications based on asymmetric nonlinear absorption [3], second harmonic generation in asymmetric waveguides [4], nonlinear photonic crystals [5], and photonic quasicrystals and molecules [6].

In this article, we propose a mechanism for unidirectional optical transport based on configurations involving nonlinear optical materials with parity ($\mathcal{P}$) and time ($\mathcal{T}$) reflection. This is possible by judiciously interleaving gain and loss regions, in such a way that the (complex) refractive index $n(x) = n_g(x) + i\gamma n_l(x)$ profile satisfies the condition $n^*(-x) = n(x)$. A first experimental realization of such (linear) arrangements has been recently reported in Refs. [7,8] where a $\mathcal{PT}$ dual coupled structure was fabricated and the beam dynamics was investigated. Here we show that the interplay of nonreciprocal dynamics arising from $\mathcal{PT}$ symmetry [8], and self-trapping phenomena associated with Kerr nonlinearities [9], can mold the flow of light in a surprising way. Such directed dynamics can be exploited in the realization of a new generation of optical isolators or diodes.

Even though the validity of our arguments can be demonstrated for a variety of nonlinear $\mathcal{PT}$ configurations below, we will highlight its basic principles, using the simplest possible arrangement, consisting of two $\mathcal{PT}$-coupled waveguide elements with Kerr nonlinearity of strength $\chi$. Each of the waveguides is single moded—one providing gain and the other an equal amount of loss. We have obtained the phase diagram in the $\chi$-$\gamma$ plane for which our system acts as an optical diode, and we have identified the minimum propagation length needed, in order to achieve this unidirectional functionality. Detail numerical simulations support our theoretical predictions.

This article is structured as follows. In Sec. II an overview of the linear $\mathcal{PT}$-symmetric dimer is presented. The nonlinear $\mathcal{PT}$-symmetric dimer will be introduced in Sec. III, where the equations of motion are given in terms of Stokes parameters. In subsection III A, we present both our theoretical and numerical results on the dynamics of the nonlinear $\mathcal{PT}$-symmetric dimer. In Sec. III B we calculate the critical value of the nonlinearity for which diode action is possible. Finally we will draw our conclusions in Sec. IV.

II. LINEAR $\mathcal{PT}$-SYMMETRIC DIMER: AN OVERVIEW

In this section we will briefly review the basic properties of the linear $\mathcal{PT}$-symmetric dimer [7,8,10]. In integrated optics this simple $\mathcal{PT}$ element can be realized in the form of a coupled system, with only one of the two parallel channels being optically pumped to provide gain $\gamma$ for the guided light, whereas the neighbor arm experiences equal amount of loss (see Fig. 1). Under these conditions, and by using the coupled-mode approach, the optical-field dynamics in the two coupled waveguides are described by the following set of equations:

$$\frac{d\psi_1}{dz} + \psi_2 - i\gamma \psi_1 = 0; \quad (1a)$$
$$\frac{d\psi_2}{dz} + \psi_1 + i\gamma \psi_2 = 0; \quad (1b)$$

where $\psi_{1,2}$ are modal electric field amplitudes in the amplifying [Eq. (1a)] and lossy [Eq. (1b)] waveguide channels, $z$ represents a dimensionless propagation distance, normalized in units of coupling lengths, and $\gamma$ is a scaled gain(loss) coefficient, also normalized to the coupling strength.

The Hamiltonian corresponding to the linear problem of Eq. (1), is written as:

$$\mathcal{H} = \begin{pmatrix} iy & 1 \\ -1 & -i\gamma \end{pmatrix}$$

(2)
As the gain (loss) parameter \( \gamma \) of the coupling units, the spontaneous waveguides. For the parameters of the simulation (we use normalized scale is logarithmic) in both waveguides, while the beam propagation distribution of the modes is asymmetric, one of them living predominantly in the amplifying waveguide and the other in the lossy one. At the phase-transition point \( \gamma = \gamma_T \) the two eigenfunctions and their corresponding eigenvalues coalesce leading to an “exceptional” point singularity [20].

The beam dynamics associated with Eq. (1) were investigated theoretically in Refs. [7,10,11] while direct measurements were performed in Refs. [7,8]. These authors recognized that as the gain (loss) parameter \( \gamma \) increases above \( \gamma_T \), the total beam power starts growing exponentially, while for \( \gamma < \gamma_T \) power oscillations are observed [see Figs. 1(b) and 1(c)]. The most dramatic effect in the beam evolution is the appearance of nonreciprocal wave propagation [see Figs. 1(c)–1(f)]. Specifically, the beam propagation pattern differs depending on whether the initial excitation is on the left or right waveguide. This is contrasted with the \( \gamma = 0 \) case [Figs. 1(a) and 1(b)], where the beam propagation is insensitive to the initial condition.

III. NONLINEAR \( \mathcal{PT} \)-SYMMETRIC DIMER

We begin our analysis by providing the mathematical model that describes optical wave propagation in a Kerr nonlinear \( \mathcal{PT} \)-symmetric coupled dual waveguide arrangement (see Fig. 2). The two modal field amplitudes are governed by the

Nonlinear waveguides with nonlinearity strength \( \chi \) and a complex \( \mathcal{PT} \)-symmetric refractive index profile. Waveguides are color coded, indicating balanced gain (red, left) and loss (green, right) regions. Gray waveguides indicate a passive (\( \gamma = 0 \)) system.

and commutes with the combined \( \mathcal{PT} \) operator. A surprising result associated with this class of problems is the possibility that such a \( \mathcal{PT} \)-symmetric Hamiltonian \( \mathcal{H} \) can have an entirely real energy spectrum, despite the fact that, in general, they are non-Hermitian [7,8,10–19]. For the specific example of the non-Hermitian Hamiltonian of Eq. (5), a direct diagonalization gives the following set of eigenvalues:

\[
\lambda_{\pm} = \pm \sqrt{1 - \gamma^2},
\]

which are real as long as the gain (loss) parameter \( \gamma \) is smaller than some critical value, \( \gamma_T = 1 \) (exact \( \mathcal{PT} \)-symmetric phase). As the gain (loss) parameter \( \gamma \) increases above \( \gamma_T \), the eigenvalues becomes complex (broken \( \mathcal{PT} \)-symmetric phase).

The corresponding eigenvectors of the Hamiltonian Eq. (2) are

\[
|+\rangle = \begin{pmatrix} e^{i\sigma} \\ e^{-i\sigma} \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} ie^{-i\sigma} \\ -ie^{i\sigma} \end{pmatrix}, \quad \sin \sigma = \gamma.
\]

In the exact \( \mathcal{PT} \)-symmetric phase, both the \( \mathcal{H} \) and \( \mathcal{PT} \) operators share the same set of eigenvectors. In this regime, the mode intensity is symmetric with respect to the mirror axis of the two waveguides. As \( \gamma \) increases above \( \gamma_T \) the eigenfunctions of \( \mathcal{H} \) cease to be eigenfunctions of the \( \mathcal{PT} \) operator, despite the fact that \( \mathcal{H} \) and the \( \mathcal{PT} \) operator still commute. This happens because the \( \mathcal{PT} \) operator is antilinear, and thus the eigenstates of \( \mathcal{H} \) may or may not be eigenstates of \( \mathcal{PT} \). In the broken \( \mathcal{PT} \)-symmetric phase, the spacial distribution of the modes is asymmetric, one of them living predominantly in the amplifying waveguide and the other in the lossy one. At the phase-transition point \( \gamma = \gamma_T \) the two eigenfunctions and their corresponding eigenvalues coalesce leading to an “exceptional” point singularity [20].

FIG. 1. (Color online) Beam propagation in two coupled linear waveguides. For the parameters of the simulation (we use normalized coupling units), the spontaneous \( \mathcal{PT} \)-breaking take place at \( \gamma_T = 1 \).

In all cases, left (right) panels correspond to an initial excitation at the left (right) channel. (a),(b) A passive system corresponding to \( \gamma = 0 \). The propagation is reciprocal; (c),(d) \( \gamma = 0.4 \gamma_T \) corresponding to the exact \( \mathcal{PT} \) phase. A nonreciprocal beam propagation is evident. Although the dynamics is non-Hermitian, the evolution is “pseudo-unitary” and the total beam power remains bounded. (e),(f) For \( \gamma = 1.5 \gamma_T \), the beam power grows exponentially (vertical scale is logarithmic) in both waveguides, while the beam propagation is again nonreciprocals with respect to the mirror axis of the two waveguides. Waveguides are color coded, indicating balanced gain (red, left) and loss (green, right) regions. Gray waveguides indicate a passive (\( \gamma = 0 \)) system.

FIG. 2. (Color online) Beam propagation in two coupled nonlinear waveguides with nonlinearity strength \( \chi \) and a complex \( \mathcal{PT} \)-symmetric refractive index profile. Waveguides are color-coded, indicating balanced gain (red, left) and loss (green, right) regions. Gray waveguides indicate a passive (\( \gamma = 0 \)) system.
evolution equations:

\[ i \frac{d\psi_1}{dz} + \psi_2 - i\gamma \psi_1 + \chi |\psi_1|^2 \psi_1 = 0; \quad (5a) \]
\[ i \frac{d\psi_2}{dz} + \psi_1 + i\gamma \psi_2 + \chi |\psi_2|^2 \psi_2 = 0; \quad (5b) \]

where \( \chi \) is the strength of the Kerr nonlinearity.

Equations (5) can be rewritten in terms of the (real) Stokes parameters \( S_i = \psi_i^* \psi_j \), where \( \psi_i \) is the Pauli spin matrices [21]. In this representation, the total field intensity is given by \( S_0 = |\psi_1|^2 + |\psi_2|^2 \), and \( S_i = |\psi_i|^2 - |\psi_j|^2 \) is the intensity imbalance between the two waveguides, while \( S_1 = \psi_1^* \psi_2 + \psi_1 \psi_2^* \) and \( S_2 = i(\psi_1^* \psi_2 - \psi_1 \psi_2^*) \). In this representation Eqs. (5) take the form:

\[ \frac{dS_0}{dz} = \frac{\sqrt{E} \cdot \tilde{S}}{S}; \quad \frac{d\tilde{S}}{dz} = S_0 \frac{\sqrt{E}}{S} + \frac{\sqrt{E}}{S} \times \tilde{B}, \quad (6) \]

where we have introduced the two real vectors \( \vec{E} = (0, 0, 2\gamma) \) and \( \vec{B} = (2, 0, \chi S_1) \), and the three-dimensional Stokes vector \( \tilde{S} \equiv (S_1, S_2, S_3) \). We note that the condition \( S_1^2 - \vec{S} \cdot \vec{S} = 0 \) is always satisfied. It is worth mentioning that Eqs. (6) are identical to the equation of motion of a relativistic negatively charged particle with zero mass, in a pseudoelectromagnetic field \( (\vec{E}, \vec{B}) \), where \( (S_0, \tilde{S}) \) represents the energy and three-dimensional momentum of the particle, while the propagation distance \( z \) has the role of the time.

Nonlinear \( PT \)-symmetric optical coupled systems can be realistically synthesized on semiconductor waveguides known for their high Kerr-like nonlinearities [22]. As in Ref. [8], coupling lengths as low as \( L_c = 1 \) mm can be obtained, in which case a gain (loss) level below \( 30 \) cm\(^{-1} \) (readily available in such materials) will suffice to keep the arrangement in the \( PT \) phase. In addition, critical switching \( (\chi \sim 1) \) can also occur at milliwatt power levels in multiquantum well configurations.

### A. Dynamics

For \( \gamma = 0 \), Eqs. (6) admit two constants of motion: the total input power \( S_0 \) and the total energy \( H = (\chi / 2) S_1^2 + 2 S_0 \). These two constants allow for an analytic solution of the Stokes vector \( \tilde{S} \) in terms of elliptic functions [9]. Depending on the initial preparation and strength of nonlinearity \( \chi \), we observe two distinct dynamical behaviors. For example, if the initial beam of total input power \( S_0(0) = 1 \), is prepared in one of the two waveguides [i.e., \( S_0(0) = \pm 1 \)], we observe either Rabi oscillations or self-trapping [9]. The former case corresponds to \( \chi < 4 \) and results in beam oscillations between the two waveguides, while the latter case occurs for \( \chi > 4 \) and leads to localization of the field (for all times) at the waveguide that was initially placed. In both cases, symmetric initial preparation will result in a dynamics which is reciprocal with respect to the axis of symmetry of the two coupled waveguides.

For \( \gamma \neq 0 \), the energy \( H \) and the beam power \( S_0 \) are no longer conserved quantities. Nevertheless, \( PT \) symmetry enforces two other constants of motion \( C, J \):

\[ C^2 = (\chi S_1 - 2)^2 + (\chi S_0)^2, \quad (7a) \]
\[ J = S_0 \pm \frac{2\gamma}{\chi} \sin^{-1} \left( \frac{\chi S_1 - 2}{C} \right), \quad (7b) \]

thus indicating that the system of Eqs. (5) is fully integrable. Below we will consider the case where initially \( S_0(0) = 1 \), \( S_1(0) = \pm 1 \), while \( S_1(0) = S_0(0) = 0 \). In this case, the constants of motion, as defined in Eqs. (7), take the values \( C_\pm = \pm 2 \) and \( J_\pm = 1 \mp \gamma \pi / \chi \).

Using \( C \) and \( J \), in this particular case we can express the components of the Stokes vector in terms of \( S_0(z) \) in the following way:

\[ \chi S_1 = 2 \left[ 1 - \cos \left( \frac{1 - S_0(z)}{2\gamma} \right) \right], \quad (8a) \]
\[ \chi S_2 = 2 \sin \left( \frac{1 - S_0(z)}{2\gamma} \right), \quad (8b) \]
\[ \chi S_3 = \pm \sqrt{\frac{(\chi S_0(z))^2 - \left( 8 \sin \left( \frac{\chi S_1(z)}{4\gamma} \right) \right)^2}{2}}. \quad (8c) \]

Substituting the expression for \( S_1 \) from Eq. (8c), to the first of the Eqs. (6), we get that

\[ \pm \int_{S_0(0)}^{S_0(z)} dS_0 \sqrt{(\chi S_0^2 - \left( 4 \sin \left( \frac{\chi S_1(z)}{4\gamma} \right) \right)^2)} = \frac{2\gamma}{\chi} z. \quad (9) \]

Even though the problem is soluble by quadratures, the integral in Eq. (9) cannot be evaluated further and thus a closed expression for \( S_0(z) \) is not possible. It is therefore instructive at this point to gain insight on the properties of the dynamics of this \( PT \) nonlinear coupler by numerically solving Eqs. (5) and (6). The accuracy of the numerical integration was checked via the conservation laws Eqs. (7), which were satisfied up to \( 10^{-10} \).

Examples of the resulting beam dynamics for \( \gamma = 0.1 \) and two representative nonlinearity strengths \( \chi = 1.9 \) and \( \chi = 8 \) are reported in Figs. 2(a) and 2(b) and then 2(c) and 2(d) respectively. In contrast to the \( \gamma = 0 \) case [9], now the dynamics is nonreciprocal with respect to the axis of symmetry of the system. While this is true for both values of nonlinearity strength \( \chi \), it is much more pronounced for the case of Figs. 2(c) and 2(d). In this latter case, the output field always leaves the sample from the waveguide with gain (red-colored) irrespective of the preparation of the input beam. At the same time the output beam intensity at the lossy waveguide approaches zero for longer waveguides. It is important to stress that in the case of the linear \( PT \) dimer [see Figs. 1(e) and 1(f)] the beam intensity at the lossy waveguide never goes to zero. Instead, it increases exponentially, albeit with a smaller prefactor with respect to the one of the gain waveguide. This novel unidirectional propagation of the \( PT \)-symmetric nonlinear dimer is the key mechanism for establishing optical isolators (diodes). It has to be contrasted with the corresponding cases shown in Figs. 2(a) and 2(b) where the output beam depends on the input state, i.e., an initial excitation at the gain waveguide results in an output field at the lossy guide and vice versa.
To quantify the ability of our setup to act as an optical nonreciprocal device, we have defined the efficiency factor $Q$ of unidirectional propagation as

$$Q(z) = 1 - |T_{z},+ S_{z},-|,$$  (10)

where $T_{z},+ S_{z},-$ is the normalized transmission coefficient associated with the gain ($+$) waveguide of length $z$. In our definition we have always assumed that the initial input beam has total power $S_{0}(z = 0) = 1$, while the beam is launched either in the gain ($+$) or in the loss ($-$) waveguide. The efficiency factor takes values from $0 \leq Q \leq 1$: a perfect diode corresponds to $Q = 1$ (since the term inside the absolute value in Eq. (10) will be zero), while the opposite limit of $Q = 0$ indicates total revival of the field. In the inset of Fig. 3 we report our numerical findings for the efficiency factor $Q$ as a function of the nonlinearity strength $\chi$ for three different waveguide lengths $z = 10, 20$, and $30$, and for a fixed value of the gain (loss) parameter $\gamma = 0.1$. It is clear that an optimal diode is achieved once the nonlinearity strength $\chi$ is larger than a critical value $\chi_c$.

B. Critical nonlinearity

Next we present a heuristic argument that aims to estimate the critical nonlinearity strength $\chi_d$ (as a function of $\gamma$), above which the $PT$ symmetric nonlinear dimer acts as an optical diode of high efficiency factor $Q = 1$. To this end we focus our analysis on the temporal behavior of the total power $S(z)$.

In the case of (Rabi-like) oscillations $S(z)$ is bounded between a minimum and a maximum value. Instead, in the regime where the coupled system acts as an optical diode, $S(z)$ is bounded only from below, while asymptotically it grows in an exponential fashion [23]. Using the first of Eq. (6) together with Eq. (8c), and requesting the extrema condition $dS/dz = 0$ (which is equivalent to $S_{0}(z) = 0$) together with the condition $d^2S/dz^2 < 0$ for the existence of a global maxima, we find that $S(z)$ shows oscillatory behavior (i.e., Rabi-like oscillations) if the nonlinearity $\chi$ is smaller than $\chi_d$, given by

$$\chi_d = 4 - 2\pi \gamma.$$  (11)

In the main panel of Fig. 3 we compare Eq. (11) with the numerical values found for $\chi_d$. The latter has been evaluated via a direct integration of Eq. (5) for systems sizes up to $z = 10^6$. The critical nonlinearity $\chi_d$ was evaluated up to fourth-digit accuracy as the nonlinearity strength for which the total power $S(z)$ is bounded. In all cases the accuracy of the integration scheme has been guaranteed by requesting that the constants of motion Eq. (7) are conserved with accuracy up to $10^{-5}$. A nice agreement between the theoretical and numerical value of $\chi_d$ is evident for small values of the gain (loss) parameter $\gamma$, while deviations from the theoretical prediction start to be visible as $\gamma$ approaches the $PT$ transition point (i.e., $\gamma = 1$) of the linear system.

Finally, we investigate the minimal waveguide length $z_d$ which is required in order to have a high-$Q$ diode. From Figs. 2(c) and 2(d) we see that the beam evolution follows two distinct scenarios depending on the initial conditions: if the beam is launched initially at the gain waveguide, the propagation is mainly along this channel. If, on the other hand, the beam excites the lossy waveguide, there is a minimum propagation distance $z_d$ which is required before the light intensity is concentrated in the gain waveguide. We have found that $z_d$ is proportional to the “first passage distance” $z_{fpd}$ associated with the point that $S_1$ becomes zero for the first time. In Fig. 4 we report the results of our simulations for $z_d \sim z_{fpd}$ for various $\chi$ values or input power levels.

An intriguing feature of $z_{fpd}$ is the existence of singularities (peaks in the $z_{fpd}$) for some characteristic values of the gain (loss) parameter $\gamma$. To understand the origin of these singularities, we have plotted the evolution of the Stokes vector $S$ by making use of the rescaled variables $\mathcal{F} = S / S_0$. In this representation, the magnitude $|\mathcal{F}|$ remains constant, and thus we can visualize the evolution on the Bloch sphere (see Fig. 5). It should be emphasized that the Bloch trajectories can in general show self-intersections, as they are a projection from a higher-dimensional phase space. One must also distinguish between closed orbits and those approaching an asymptotic state, as this is in general connected to broken and unbroken $PT$ symmetry [12]. We note that closely related Bloch dynamics appear in different physical model systems like the ones reported in Ref. [24]. Our analysis indicated that the

![FIG. 3. (Color online) (Main figure) A semilogarithmic plot of $\chi_d$ vs $\gamma$. For the numerical evaluation of $\chi_d$ we have integrated Eq. (5). (Inset) The efficiency factor $Q$ vs nonlinearity strength $\chi$, for a fixed gain (loss) parameter $\gamma = 0.1$ and three different waveguide lengths $z = 10, 20$, and $30$. For nonlinearity strength $\chi = \chi_d \approx 3.4$ the isolator reaches its optimal efficiency.](image1)

![FIG. 4. (Color online) The numerically extracted first passage distance $z_{fpd}$ versus the gain (loss) parameter $\gamma$. The initial conditions are chosen to be $S_0(0) = 1$ and $S_0(0) = -1$. An inverse power law is observed. (Inset) The proportionality coefficient $f(\gamma)$ is plotted versus the nonlinearity strength $\chi$ for $\chi > \chi_d$. The red line correspond to the best linear fit.](image2)
FIG. 5. (Color online) Dynamics of the rescaled Stokes variables $F$ for $\chi = 9$ and various gain (loss) parameters $\gamma$: dashed blue lines correspond to $\gamma = 0.157$; pink lines ($\bullet$) correspond to $\gamma = 0.15$; solid yellow lines correspond to $\gamma = 0.12$; and red lines ($\blacksquare$) to $\gamma = 0.174$. The green line ($\bigcirc$) corresponds to the passive system $\gamma = 0$ with critical nonlinearity $\chi = 4$, where the motion of the trajectory is on the separatrix. The trajectory associated with $\gamma = 0.157$ (see the red arrow in Fig. 4) is typical to the cases where $z_{fpd}$ diverge and correspond to the closest one to the separatrix of the passive system.

singularities in $z_{fpd}$ are associated with trajectories that, during their evolution, stay close to the separatrix associated to the critical value $\chi = 4$ (transition between Rabi oscillations and self-trapping) of the passive system.

Leaving aside the issue of the singularities, we have found that for all $\chi$ values larger than $\chi_d$, the first passage distance $z_{fpd}$ follows an inverse power law, i.e.,

$$z_{fpd} = f(\chi)/\gamma,$$

(12)

where the proportionality factor $f(\chi)$ is $\chi$ dependent. A best least-squares fit allows us to extract the various $f(\chi)$ which is in this case $f(\chi) = -0.6 + 0.5\ln(\chi)$ (see inset of Fig. 4).

IV. CONCLUSIONS

In conclusion, we have proposed a mechanism for directed transport in nonlinear optical coupled systems that relies at the interplay between nonlinearity and $PT$ reflection symmetries. More specifically, we have observed that above a critical nonlinearity strength, the beam evolution is unidirectional, i.e., the output beam remains in the gain channel, irrespective of initial conditions. Such behavior implies that these systems can be used to realize new classes of optical diodes and other unidirectional photonic elements. Of great interest will be to extend these notions to more involved arrangements like nonlinear $PT$ lattices where nonlinear excitations are expected to lead to even more intriguing phenomena.

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[23] In this case, the system of Eq. (5) act as a set of two uncoupled waveguides, i.e., in the long time limit one can assume that the coupling constant is essentially zero. As a result the intensity at the gain waveguide (which is approximately the same as the total intensity) increases exponentially, while the one at the lossy waveguide decays exponentially. We have checked the accuracy of this statement with direct numerical simulations.

Unidirectional Invisibility Induced by $\mathcal{PT}$-Symmetric Periodic Structures

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Parity-time ($\mathcal{PT}$) symmetric periodic structures, near the spontaneous $\mathcal{PT}$-symmetry breaking point, can act as unidirectional invisible media. In this regime, the reflection from one end is diminished while it is enhanced from the other. Furthermore, the transmission coefficient and phase are indistinguishable from those expected in the absence of a grating. The phenomenon is robust even in the presence of Kerr nonlinearities, and it can also effectively suppress optical bistabilities.

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In the last few years considerable research effort has been invested in developing artificial materials appropriately engineered to display properties not found in nature. In the electromagnetic domain, such metamaterials make use of their structural composition, which in turn allows them to have complete access of all four quadrants of the real $\epsilon$-$\mu$ plane. Several exotic effects ranging from negative refraction to superlensing and from negative Doppler shift to reverse Cherenkov radiation can be envisioned in such systems [1]. Quite recently, the possibility of synthesizing a new family of artificial optical materials that instead rely on balanced gain and loss regions has been suggested [2–6]. This class of optical structures deliberately exploits notions of parity ($\mathcal{P}$) and time ($\mathcal{T}$) symmetry [7–9] as a means to attain altogether new functionalities and optical characteristics [2]. Under $\mathcal{PT}$ symmetry, the creation and absorption of photons occurs in a controlled manner, so that the net loss or gain is zero. In optics, $\mathcal{PT}$ symmetry demands that the complex refractive index obeys the condition $n(\mathbf{r}) = n^*(\mathbf{r})$, in other words the real part of the refractive index should be an even function of position, whereas the imaginary part must be odd. $\mathcal{PT}$-symmetric materials can exhibit several intriguing features. These include among others, power oscillations [2,4,10], absorption enhanced transmission [5], double refraction, and nonreciprocity of light propagation [2]. In the nonlinear domain, such pseudo-Hermitian nonreciprocal effects can be used to realize a new generation of on-chip isolators and circulators [6]. Other exciting results within the framework of $\mathcal{PT}$ optics include the study of Bloch oscillations [11], and the realization of coherent perfect laser absorbers [12] and nonlinear switching structures [13].

To date, most of the studies on optical realizations of $\mathcal{PT}$ synthetic media have relied on the paraxial approximation which maps the scalar wave equation to the Schrödinger equation, with the axial wave vector playing the role of energy. This formal analogy allows one to investigate experimentally fundamental $\mathcal{PT}$ concepts that may impact several other areas, ranging from quantum field theory and mathematical physics [7–9], to solid state [14] and atomic physics [15]. Among the various themes that have fascinated researchers, is the existence of spontaneous $\mathcal{PT}$ symmetry breaking points (exceptional points) where the eigenvalues of the effective non-Hermitian Hamiltonian describing the dynamics of these systems abruptly turn from real to complex [9]. Recently, interest in $\mathcal{PT}$-scattering configurations [16–19] has been revived in connection with using such devices under a dual role, that of a lasing and a perfect coherent absorbing cavity [12,20].

In this Letter we explore the possibility of synthesizing $\mathcal{PT}$-symmetric objects which can become unidirectionally invisible at the exceptional points. In recent years the subject of cloaking physics has attracted considerable interest, specifically in connection to transformation optics [1,21]. Here, our notion of invisibility stems from a fundamentally different process. As opposed to surrounding a scatterer with a cloak medium, in our case the invisibility arises because of spontaneous $\mathcal{PT}$-symmetry breaking. This is accomplished via a judicious design that involves a combination of optical gain and loss regions and the process of index modulation. Specifically, we consider scattering from $\mathcal{PT}$-synthetic Bragg structures (see Fig. 1) and investigate the consequences of $\mathcal{PT}$ symmetry in the scattering process. It is well known that passive gratings (involving no gain or loss) can act as high efficiency reflectors around the Bragg wavelength. Instead, we find that at the $\mathcal{PT}$ symmetric breaking point, the system is reflectionless over all frequencies around the Bragg resonance when light is incident from one side of the structure while from the other side its reflectivity is enhanced. Furthermore, we show that in this same regime the transmission phase vanishes—a necessary condition for evading detectability. Even more surprising, is the fact that these effects persist even in the presence of Kerr nonlinearities.

To demonstrate these effects we consider an optical periodic structure or grating having a $\mathcal{PT}$-symmetric
refractive index distribution \( n(z) = n_0 + n_1 \cos(2\beta z) + in_2 \sin(2\beta z) \) for \( |z| < L/2 \). This grating is embedded in a homogeneous medium having a uniform refractive index \( n_0 \) for \( |z| > L/2 \) (see Fig. 1). Here \( n_1 \) represents the peak real index contrast and \( n_2 \) the gain and loss periodic distribution. In practice, these amplitudes are small, e.g., \( n_1, n_2 < n_0 \). The grating wave number \( \beta \) is related to its spatial periodicity \( \Lambda \) via \( \beta = \pi/\Lambda \) and in the absence of any gain modulation \( (n_2 = 0) \) the periodic index modulation leads to a Bragg reflection close to the Bragg angular frequency \( \omega_B = c/\beta n_0/c \) (where \( c \) is the speed of light in vacuum). In this arrangement, a time-harmonic electric field of frequency \( \omega \) obeys the Helmholtz equation:

\[
\frac{\partial^2 E(z)}{\partial z^2} + \omega^2/c^2 n^2(z) E(z) = 0. \tag{1}
\]

For \( |z| > L/2 \), Eq. (1) admits the solution \( E_0(z) = E'_L \exp(ikz) + E'_R \exp(-ikz) \) for \( z < -L/2 \) and \( E_0(z) = E'_L \exp(ikz) + E'_R \exp(-ikz) \) for \( z > L/2 \) where the wave vector \( k = n_0 \omega/c \). The amplitudes of the forward and backward propagating waves outside of the grating domain are related through the transfer matrix \( M \):

\[
\begin{pmatrix}
E'_L \\
E'_R
\end{pmatrix}
= \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
E'_L \\
E'_R
\end{pmatrix}. \tag{2}
\]

The transmission and reflection amplitudes for left (L) and right (R) incidence waves, can be obtained from the boundary conditions \( E'_L = 0 \) (\( E'_R = 0 \)) respectively, and are defined as \( t_L = \frac{E'_L}{E'_0} \); \( r_L = \frac{E'_L}{E'_0} \); \( t_R = \frac{E'_R}{E'_0} \); \( r_R = \frac{E'_R}{E'_0} \). These can be expressed in terms of the transfer matrix elements as follows [16,17]

\[
t_L = t_R = t = \frac{1}{M_{22}}; \quad r_L = -\frac{M_{21}}{M_{22}}; \quad r_R = \frac{M_{12}}{M_{22}}. \tag{3}
\]

While the transmission for left or right incidence is the same, this is not necessarily the case for the reflection. From the above relations one can deduce the form of the scattering matrix \( S \) [17] in terms of the \( M \)-matrix elements. For \( PT \)-symmetric systems, the eigenvalues of the \( S \) matrix either form pairs with reciprocal moduli or they are all unimodular. In the latter case the system is in the exact \( PT \) phase while in the former one it is in the broken-symmetry phase [12,16]. For the complex periodic structure considered here, the transition from one phase to another (spontaneous \( PT \)-symmetry breaking point) takes place when \( n_1 = n_2 \) [22].

To analyze this structure we decompose the electric field inside the scattering domain \( E(z) \), in terms of forward \( E_f(z) \) and backward \( E_b(z) \) traveling envelopes as

\[
E(z) = E_f(z) \exp(ikz) + E_b(z) \exp(-ikz). \tag{4}
\]

Next we employ slowly varying envelopes for the field, i.e., \( E_f(z) = E'_f(z) \exp(i\delta z) \) and \( E_b(z) = E'_b(z) \exp(-i\delta z) \), where \( \delta = \beta - k \) is the detuning. Substituting these expressions in Eq. (1), and keeping only synchronous terms while eliminating second order corrections in \( n_1 \) and \( n_2 \), we then express the field at a point \( z \) inside the sample in terms of the field at \( z = -L/2 \). For \( k = \beta \) close to the Bragg point, we get

\[
\begin{pmatrix}
E'_f(z) \\
E'_b(z)
\end{pmatrix}
= \begin{pmatrix}
\bar{E}'_f \exp(\omega z) \\
\bar{E}'_b \exp(-\omega z)
\end{pmatrix}
= \begin{pmatrix}
E'_f(-\frac{L}{2}) \\
E'_b(-\frac{L}{2})
\end{pmatrix}
\tag{5}
\]

where \( \bar{E} = \cos(\lambda(z + L/2)) - i \sin(\lambda(z + L/2)) \delta \cdot \hat{e}, \hat{e} \) are the Pauli matrices, and the unit vector \( \hat{e} \) is defined as \( \hat{e} = (1/\lambda)(-kn_2/2n_0; -ikn_1/2n_0; \delta) \), while \( \lambda = \sqrt{\delta^2 - k^2(n_1^2 - n_2^2)/4n_0^2} \). By imposing continuity of the field at \( z = \pm L/2 \), Eq. (5) becomes equivalent to Eq. (2). The transmission \( T = |t|^2 \) and reflection coefficients \( R_L = |r_L|^2 \) and \( R_K = |r_K|^2 \) are in this case

\[
T = \frac{|\lambda|^2}{|A^2\cos^2(\lambda L) + \delta^2| \sin(\lambda L)|^2}, \tag{6}
\]

\[
R_L = \frac{(n_1 - n_2)^2k^2/4n_0^2}{\delta^2 + |A^2\cot(\lambda L)|^2}; \quad R_K = \frac{(n_1 + n_2)^2k^2/4n_0^2}{\delta^2 + |A^2\cot(\lambda L)|^2}.
\]

For \( n_1 = 0 \) one recovers the standard scattering features of periodic Bragg structures. Namely, \( R_L = R_K \), while close to the Bragg point \( \delta = 0 \) the reflection (transmission) becomes unity (zero) (in the large \( L \) limit), see Fig. 2. Instead, if \( n_1 \neq 0 \), an ”asymmetry” in the left (right) reflection coefficient starts to develop [22]. We would like to note that the \( PT \) arrangement considered here is fundamentally different from that encountered in distributed feedback lasers (DFBs) [23]. In DFB systems both the index and gain and loss profile vary in phase and thus no \( PT \)-symmetry breaking is possible.

At \( n_1 = n_2 \), this asymmetry becomes most pronounced. Even more surprising is the fact that at the Bragg point \( \delta = 0 \), the transmission is identically unity, i.e., \( T = 1 \), while the reflection for left incident waves is \( R_L = 0 \) (see Fig. 2). This is a direct consequence of the \( PT \) nature of this periodic structure. At the same time, the reflection for right incident waves grows with the size \( L \) of the sample as.
Eq. (5), we deduce that the phase will fail to detect this periodic structure. Although the above theoretical analysis is performed close to the exceptional point when \( n_2 = n_1 \), in this case, \( R_k \) is diminished (up to \( n_2^2 \approx 10^{-6} \) — see inset) for a broad frequency band, while \( R_k \) is enhanced, in excellent agreement with our theoretical predictions.

\[
R_k = L^2 \left( \frac{n_1^2}{n_0^2} \right)^2 \left( \frac{\sin(L\delta)}{L\delta} \right)^2 \approx -L^2 \left( \frac{n_1}{n_0} \right)^2. \tag{7}
\]

Such quadratic increase of the field intensity is a hallmark of exceptional point dynamics [10]. This behavior is directly confirmed by our numerical simulations. We will refer to this phenomenon as unidirectional reflectivity. Furthermore, Eqs. (6) indicate that a transformation \( n_2 \to -n_2 \), reverts the reflectivity of the system, allowing for reflectionless behavior for right incident waves, i.e., \( R_k = 0 \), while the reflection from the left \( R_k \) is now following the prediction of Eq. (7). In other words, the phase lag between the real and imaginary refractive index dictates the unidirectional reflectivity of the system.

Reflectionless potentials in one-dimensional scattering configurations are not in general invisible. This is due to the fact that the phase of the transmitted wave might depend on energy, thus leading to wave packet distortion after the potential barrier. In this respect, a transparent potential can be detected from simple time-of-flight measurements. It is therefore crucial to examine the phase \( \phi_t \) of the transmission amplitude \( t = |t| \exp(i\phi_t) \) and compare it with the phase acquired by a wave propagating in a grating-free environment (\( \phi_t = 0 \)) [24]. Using Eq. (5), we deduce that the phase \( \phi_t \) close to the Bragg point is

\[
\phi_t = \arctan \left( -\frac{\delta}{\lambda} \right) + L\delta. \tag{8}
\]

At \( n_2 = n_2 \), we find that \( \delta = \lambda \), which results in a transmission phase \( \phi_t = 0 \). Thus interference measurements will fail to detect this periodic structure. Although the above theoretical analysis is performed close to the Bragg point \( \delta = 0 \), our numerical results reported in Fig. 3(a), indicate that these effects are valid over a very broad range of frequencies. For comparison, we also report in Fig. 3(a), \( \phi_t \), for the case of a passive \( n_2 = 0 \) Bragg grating.

Next, we analyze the dependence of the transmission delay time \( \tau_t = \frac{d \phi_t}{dk} [25,26] \), on the detuning \( \delta \). This quantity provides valuable information about the time delay (or advancement) experienced by a transmitted wave packet when its average position is compared to the corresponding one in the absence of the scattering medium. Using Eq. (8) we find that at the spontaneous \( \mathcal{PT} \)-symmetry breaking point the transmission delay time is \( \tau_t = 0 \). In Fig. 3(b), we show results for a \( \mathcal{PT} \) structure at \( n_1 = n_2 \) together with those expected from the passive case.

It is also interesting to investigate the robustness of the above phenomena in the presence of Kerr nonlinearities. To this end, we assume the presence of a Kerr term in the refractive index profile, i.e., \( n(z) = n_0 + n_1 \cos(2\beta z) + in_2 \sin(2\beta z) + \chi |E(z)|^2 \). By decomposing the optical field into two counterpropagating waves and by considering only synchronous terms [22,27], we can then obtain a set of equations describing the field envelopes \( \tilde{E}_j(z) \) and \( \tilde{E}_j(z) \), in terms of Stokes variables [28]:

\[
\tilde{S}_j(z) = 2\kappa \tilde{S}_j; \quad \tilde{S}_1(z) = 2 g \tilde{S}_1; \quad \tilde{S}_2(z) = 2 \delta \tilde{S}_1 - 3 \rho \tilde{S}_j; \quad \tilde{S}_3(z) = -2 \delta \tilde{S}_2 + 3 \rho \tilde{S}_j - 2 \delta \tilde{S}_1 - 2 \delta \tilde{S}_2 \quad \text{where} \quad \rho = k\chi/n_0, \quad \kappa = kn_1/2n_0 \text{ and } \gamma = kn_2/2n_0. \]

It can be shown [22] that this nonlinear system has the following conserved quantities:

\[
g \tilde{S}_0 = \kappa \tilde{S}_1 - \tilde{C}_1, \quad \rho \tilde{S}_0 \tilde{S}_j - 4 \kappa \delta \tilde{S}_j + 4 \kappa \delta \tilde{S}_j - \tilde{C}_2.
\]

Using these constants of the motion, one can solve exactly the Stokes equations. Because of lack of space we will not discuss the derivations in detail here [22] but rather cite the final results for the transmission and reflection coefficients.
We have shown that the interplay of Bragg scattering and PT symmetry allows for unidirectional invisibility which can be observed over a broad range of frequencies around the Bragg point. This process was found to be robust against perturbations. In the presence of nonlinearities this unidirectional invisibility still persists and non-reciprocal transmission is possible. Of interest will be to investigate if these phenomena can also occur in higher dimensions and under vectorial conditions.

In contrast to the linear case, now \( T_L \neq T_R \) for \( n_1 \neq n_2 \) indicating a diode action [6,22] (see Fig. 4). However, of interest here is the behavior of the system at the exceptional point \( n_1 = n_2 \). We find that \( T_L = T_R = 1 \), while \( R_L = 0 \), as in the linear case. These results are valid for any input intensity as shown in Fig. 4. At the same time we have found that the transmission phase is again independent of the detuning \( \delta \) and equal to \( \phi_0 = 0 \). We thus conclude that the phenomenon of unidirectional invisibility of the PT-periodic system at the exceptional point is entirely unaffected by the presence of Kerr nonlinearities.

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[22] H. Ramezani et al., (to be published).
[24] We do not consider the trivial phase \( k_L \) associated with free propagation.
[28] The Stokes variables are defined as \( S_0 = |\mathbf{E}_0|^2 + |\mathbf{E}_1|^2 \), \( S_1 = |\mathbf{E}_0|^2 - |\mathbf{E}_1|^2 \), \( S_2 = \mathbf{E}_1 \mathbf{E}_0^* + \mathbf{E}_1 \mathbf{E}_2^* \), \( S_3 = i(\mathbf{E}_1 \mathbf{E}_0^* - \mathbf{E}_1 \mathbf{E}_2^*) \), and they satisfy the identity \( S_1^2 + S_2^2 + S_3^2 = S_0^2 \).
[29] We checked [via exact numerical evaluations of \( T, R_L, R_R \), and \( \phi_0 \), using Eq. (1)], the generality of our results for various Bragg scattering potentials. In the current simulations we present results for PT-slab grating.
Bypassing the bandwidth theorem with $\mathcal{PT}$ symmetry

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The beat time $\tau_{\text{fpt}}$ associated with the energy transfer between two coupled oscillators is dictated by the bandwidth theorem which sets a lower bound $\tau_{\text{fpt}} \sim 1/\delta \omega$. We show, both experimentally and theoretically, that two coupled active LRC electrical oscillators with parity-time ($\mathcal{PT}$) symmetry bypass the lower bound imposed by the bandwidth theorem, reducing the beat time to zero while retaining a real valued spectrum and fixed eigenfrequency difference $\delta \omega$. Our results foster design strategies which lead to (stable) pseudounitary wave evolution, and may allow for ultrafast computation, telecommunication, and signal processing.

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One of the fundamental principles of wave physics is the bandwidth theorem [1] which in quantum mechanics takes the form of the celebrated energy-time Heisenberg uncertainty relation [2]. A direct consequence of this principle is the fact that the form of the celebrated energy-time Heisenberg uncertainty relation $\tau_{\text{fpt}} \sim 1/\delta \omega$. We show, both experimentally and theoretically, that two coupled active LRC electrical oscillators with parity-time ($\mathcal{PT}$) symmetry bypass the lower bound imposed by the bandwidth theorem, reducing the beat time to zero while retaining a real valued spectrum and fixed eigenfrequency difference $\delta \omega$. Our results foster design strategies which lead to (stable) pseudounitary wave evolution, and may allow for ultrafast computation, telecommunication, and signal processing.

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to be a simple realization of a $\mathcal{PT}$-symmetric dimer. Each inductor is wound with 75 turns of no. 28 copper wire on 15 cm diameter PVC forms in a 6 × 6 mm loose bundle for an inductance of $L = 2.32$ mH. The coils are mounted coaxially with a bundle separation adjusted for the desired mutual inductance $M$. The isolated natural frequency of each coil is $\omega_0 = 1/\sqrt{LC} = 2 \times 10^3$ s$^{-1}$.

The actual experimental circuit deviates from Fig. 1 in the following ways: (1) a resistive component associated with coil wire dissipation is compensated by an equivalent gain component applied to each coil; (2) a small capacitance trim is included to aid in circuit balancing; and (3) additional LM356 voltage followers are used to buffer the voltages $V_1$ and $V_2$, captured with a Tektronix DPO2014 oscilloscope.

The linear nature of the system requires a balance of $\mathcal{PT}$ symmetry only to the extent that component stability over time allows for a measurement. All circuit modes either exponentially grow to the nonlinearity limit of the buffers, or shrink to zero. Transient data is obtained respecting these time scales.

Kirchhoff’s laws lead to the following set of equations for the charge $Q_1$ ($Q_2$) on the capacitor corresponding to the amplified (lossy) side:

$$
\begin{align*}
\frac{d^2Q_1}{dt^2} &= -\alpha Q_1 + \mu \alpha Q_2 + \gamma \frac{dQ_1}{dt}, \\
\frac{d^2Q_2}{dt^2} &= \mu \alpha Q_1 - \alpha Q_2 - \gamma \frac{dQ_2}{dt},
\end{align*}
$$

where $t \equiv 0t_0$, $\alpha = 1/(1 - \mu^2) \geq 1$, $\gamma = R^{-1} \sqrt{LC}$ is the gain and loss parameter, and $\mu = M/L$ is the rescaled mutual inductance. Inspection of Eqs. (1) reveals that they are invariant under a combined parity (i.e., $Q_1 \rightarrow Q_2$) and time-reversal (i.e., $t \rightarrow -t$) transformation.

The theoretical analysis (see the Appendix) relies on a Liouvillian formulation of Eqs. (1) which take the form

$$
\frac{d\Psi}{dt} = \mathcal{L}\Psi, \quad \mathcal{L} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\mu \alpha & \gamma & 0 & 0 \\
\mu \alpha & -\alpha & -\gamma & 0
\end{pmatrix},
$$

where $\Psi = (Q_1, Q_2, \dot{Q}_1, \dot{Q}_2)^T$. Equation (2) can be interpreted as a Schrödinger equation with non-Hermitian effective Hamiltonian $H_{\text{eff}} = i\mathcal{L}$. This Hamiltonian is symmetric with respect to generalized $\mathcal{P}\mathcal{T}\mathcal{R}$ transformations, i.e., $[\mathcal{P}\mathcal{T}\mathcal{R}, H_{\text{eff}}] = 0$,

$$
\mathcal{P} \equiv \begin{pmatrix}
\sigma_t & 0 \\
0 & \sigma_t
\end{pmatrix}, \quad \mathcal{T} \equiv \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

and $\sigma_t$ is the Pauli matrix, $1$ is the $2 \times 2$ identity matrix, and $\mathcal{K}$ denotes the operation of complex conjugation. By a similarity transformation $\mathcal{R}$,

$$
H_{\text{eff}} \mathcal{R} = \mathcal{R}^T H_{\text{eff}}, \quad \mathcal{T} = \mathcal{K} = \mathcal{R} \mathcal{T} \mathcal{R}^{-1},
$$

$H_{\text{eff}}$ can be related to a transposition symmetric, $\mathcal{PT}$-symmetric Hamiltonian $H$. Specifically,

$$
\begin{align*}
H &= H^T = \mathcal{R} H_{\text{eff}} \mathcal{R}^{-1}, \\
\mathcal{PT} H &= 0, \quad \mathcal{P} \mathcal{T} H = \mathcal{R} H_{\text{eff}} \mathcal{R}^{-1},
\end{align*}
$$

where

$$
H = \begin{pmatrix}
0 & b + iy/2 & c + iy/2 & 0 \\
b + iy/2 & 0 & 0 & c - iy/2 \\
c + iy/2 & 0 & 0 & b - iy/2 \\
0 & c - iy/2 & b - iy/2 & 0
\end{pmatrix},
$$

and $b = \sqrt{(\alpha + \alpha^3)/2}$ and $c = -\sqrt{(\alpha - \alpha^3)/2}$. This allows us to make contact with the brachistochrone studies of Refs. [6,8,15].

The eigenfrequencies $\omega_{1,2}$ of system (1) [or equivalently Eq. (2)] are shown as functions of the gain and loss parameter $\gamma$ in Fig. 2 (solid lines). For $\gamma < \gamma_{\text{eff}} = 1/\sqrt{1 - \mu^2} = 1/\sqrt{LC}$ the system is in the exact phase and thus the eigenfrequencies are real [31]. We investigate the signal-energy tachistochrone passage under the constraint of fixed bandwidth $\delta \omega = \omega_1 - \omega_2$. Experimentally, the $\delta \omega$ constraint is implemented in the $LRC$ dimer through adjustment of the mutual inductance.

Figure 2 shows both theoretical and experimental results for the parametric evolution of the eigenfrequencies in the exact phase, for various $\mu$ values. The black dashed lines in Fig. 2 illustrate a path for fixed $\delta \omega/\omega_0 = 0.36$ through the family of eigenfrequencies associated with different mutual inductances $\mu$.

Equation (2) can be solved either analytically or via direct numerical integration in order to obtain the temporal behavior of the capacitor charge $Q_c(t)$ and the displacement current $I_s(t)$ in each of the two circuits of the $\mathcal{PT}$-symmetric dimer. For the investigation of the tachistochrone wave evolution, we consider an initial displacement current in one of the circuits with all other dynamical variables zero. The first passage time $\tau_{fe}$ is then defined as the time interval needed to reach an orthogonal state. In our experiments this corresponds to the condition that the envelope function of the current at the
A typical temporal dynamics of the displacement current $I$ (passive) dimer. (c) The extracted initial condition starting from the gain side while the $-\gamma$ eigenfrequencies vs the gain and loss parameter correspond to the theoretical prediction $\tau_E$. Time is asymmetric with respect to the initially excited circuit. We find that the first passage initially excited circuit is zero. We find that the first passage time is asymmetric with respect to the initially excited circuit. Specifically, we have that

$$\tau_{fe} = \frac{1}{\delta\omega} \arccos \left( \frac{\delta\omega^2 - \gamma^2}{\delta\omega^2 + \gamma^2} \right)$$

(see the Appendix), where the $+$ sign corresponds to an initial condition starting from the gain side while the $-$ sign corresponds to an initial condition starting from the lossy side. For $\gamma \gg \delta\omega$, Eq. (6) takes the limiting values $\tau_{fe} \approx 2\pi/\delta\omega$ and $\tau_{fe} \approx 2/\gamma$, respectively. The latter case indicates the possibility of transforming an initial state to an orthogonal final one, or in more practical terms, transferring energy from one side to the other, in an arbitrarily short time interval. In the opposite limit of $\gamma = 0$, we recover for both initial conditions the Anandan-Aharonov lower bound for the first passage time $\tau_{fe} = \pi/\delta\omega$ [3]. This is the time for which energy is transferred from the initial circuit to its partner according to the constraint of the bandwidth theorem.

Geometrically, one can understand the relation (6) in the following way: the time required for the evolution between two states induced by a Hermitian Hamiltonian is proportional to the length of the shortest geodesic connecting the two states in projective Hilbert space [3]. Non-Hermitian $PT$-symmetric Hamiltonians in the exact $PT$-symmetric domain can be similarity mapped to equivalent Hermitian Hamiltonians. Under such a similarity mapping the corresponding projective Hilbert space undergoes a deformation obtaining a nontrivial metric. This results in an effective contraction or dilation of the corresponding geodesic and with it of the corresponding evolution time [8,9,14].

In Fig. 3 we present some typical measurements for the temporal behavior of the displacement currents. In Fig. 3(a) we show $|I_1(t)|$ for an initial condition corresponding to the case $I_1(0) = 1$ with all other dynamical variables zero. The case where the initial current excitation is at the lossy side, i.e., $I_2(0) = 1$, is shown for contrast in Fig. 3(b). In both cases, agreement between the experiment (circles) and the simulations (lines) is observed. For comparison, we also report with black line the temporal behavior of the displacement current for the case of a passive circuit (i.e., $\gamma = 0$) with the same $\delta\omega$ constraint. We observe that the orthogonal target state is reached faster (or slower) depending on whether the initial excitation is applied to the lossy (or gain) side.

The above results can be verified in more cases by changing the inductive coupling $\mu$ and gain and loss parameter $\gamma$, while keeping constant the frequency difference $\delta\omega = \omega_2 - \omega_1$. A summary of our measured $\tau_{fe}$ versus $\gamma$ is presented in Fig. 3(c).

The experimental data show agreement with the theoretical prediction Eq. (6).

A parallel analysis pertains directly to the study of the energy transport from one side to another. Using the same initial conditions as above we investigate the temporal behavior of the energies

$$E_n(t) = \frac{1}{2} \frac{Q^2}{C} + \frac{1}{2} L_n^2$$

of each $n = 1,2$ circuit. The first passage time can be defined as the time for which the two energies become equal for the first time, i.e., $E_1(t_{fe}^0) = E_2(t_{fe}^0)$. For passive (i.e., $\gamma = 0$)
coupled circuitry, this time is half of the beating time $\tau_{\text{beating}} = \tau_{\text{PT}}(\gamma = 0)/2$ and it is insensitive to the initial preparation. In contrast, for the active PT-symmetric dimer of Fig. 1, we find that the energy transfer from the lossy (gain) side to the gain (lossy) one is faster (slower) than the corresponding passive system with the same $\delta\omega$. In Fig. 3(d), we summarize our measurements for the $\tau_{\text{beating}}^\text{PT}$ versus $\gamma$ under the constraint of fixed frequency bandwidth $\delta\omega$. A similar behavior as the one found for the displacement current is evident.

Our results open a direction toward investigating novel phenomena and functionalities of PT-symmetric arrangements with pseudounitary spatiotemporal evolution. Along these lines, we envision PT-symmetric (nano)antenna configurations and metamaterial or optical microcavator arrays with unidirectional ultrafast communication capabilities. These structures also have potential applications as delay lines, buffers, and switches. Questions like the effects of nonlinearity or the topological complexity of the PT-symmetric structures in the tachistochronic dynamics are open and offer new exciting opportunities, yet to be discovered.

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APPENDIX

From the eigenvalue equation $(H_{\text{PT}} - \omega_0)\mathbf{Z}_k = 0$ with $k = 1, 2, 3, 4$ and the corresponding characteristic polynomial $\omega_0^4 - (2\alpha - \gamma)\omega_0^2 + \alpha = 0$ one obtains $\omega_{1,4} = \pm\sqrt{\Delta\pm\sqrt{\Delta^2 - 4\alpha^2}}$, where

$$\Delta := \omega_0^2 - \alpha^2, \quad \alpha \mu \in \mathbb{R},$$

$$\Omega_\pm := \omega_0^2 \pm \sqrt{\Delta^2 - 4\alpha^2}.$$ (A1)

$$\mathbf{Z}_k = \alpha \mu \left(e^{i\delta_1}, e^{i\delta_2}, -i\omega_0 e^{i\delta_2}, -i\omega_0 e^{i\delta_1}\right)^T.$$ (A1)

For gain and loss parameters $\gamma \in [0, \gamma_{\text{PT}}]$ the eigenvalues are purely real, $\omega_0 \in \mathbb{R}$; it holds $\Phi_0 \in \mathbb{R}$, so that $P_{\text{PT}} T_{\text{PT}} = 1 = T_{\text{PT}} P_{\text{PT}}$ and the $P_{\text{PT}} T_{\text{PT}}$ symmetry is exact. For $\gamma > \gamma_{\text{PT}}$ the eigenvalues $\omega_0$ are not real, but pairwise complex conjugate; one finds $\Phi_0 \not\in \mathbb{R}$, so that $P_{\text{PT}} T_{\text{PT}} \not\equiv 1$ and the $P_{\text{PT}} T_{\text{PT}}$ symmetry is spontaneously broken with $P_{\text{PT}} T_{\text{PT}}$ phase transition at $\gamma = \gamma_{\text{PT}}$. The Hamiltonian $H_{\text{PT}}$ can be brought into the more symmetric form Eq. (5) via a similarity transformation $\mathcal{T}$ given by Eq. (4).

To describe the dynamics in the sector of exact $P_{\text{PT}} T_{\text{PT}}$ symmetry, $\gamma \in [0, \gamma_{\text{PT}}]$, where $\omega_1 = -\omega_4$, $\omega_2 = -\omega_3$, and $\omega_0 \in \mathbb{R}$, we start from an ansatz $\Psi(t) = \sum_{j=1}^{\infty} e^{-i\omega_0 t} A_j Z_j$, $A_j \in \mathbb{C}$, and impose the experimentally required reality constraint $\Psi(t) \in \mathbb{R}$ as $A_1 Z_1 = A_2 Z_2$ and $A_3 Z_3 = -A_4 Z_4$. The solutions of the evolution equation (1), (2) with initial condition on the gain side $\Psi(t = 0) = (0, 0, 1, 0)^T$ are

$$\Psi(t) = \alpha \mu \left(\begin{array}{c}
-\frac{\sin(\omega_1 t + \delta_1)}{\omega_1} + \frac{\sin(\omega_2 t + \delta_2)}{\omega_2} \\
-\frac{\sin(\omega_4 t + \delta_1)}{\omega_4} + \frac{\sin(\omega_3 t + \delta_2)}{\omega_3} \\
-\cos(\omega_1 t + \delta_1) + \cos(\omega_2 t + \delta_2) \\
-\cos(\omega_4 t + \delta_1) + \cos(\omega_3 t + \delta_2)
\end{array}\right),$$

$$\Delta := \omega_0^2 - \alpha^2, \quad \alpha \mu = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\gamma^2}{\delta\omega^2}} \sqrt{1 + \frac{\gamma^2}{\delta\omega^2}},$$

$$\delta\omega := \omega_1 - \omega_4, \quad \tilde{\omega} := \omega_1 + \omega_4,$$

$$\sin(\delta_1) = \frac{\gamma \omega_0}{\alpha \mu}, \quad \cos(\delta_1) = -\frac{\Delta - \gamma^2}{2\alpha \mu},$$

$$\sin(\delta_2) = \frac{\gamma \omega_0}{\alpha \mu}, \quad \cos(\delta_2) = \frac{\Delta + \gamma^2}{2\alpha \mu}.$$ (A2)

The first passage time $\tau_{\text{PT}}$ required for the evolution from the initial state $\Psi(t = 0) = (0, 0, 1, 0)^T$ to an orthogonal final state $\Psi(t = \tau_{\text{PT}}) = (\psi_1, \psi_2, 0, \psi_3)^T$, $\Psi(0)|\Psi(\tau_{\text{PT}}) = 0$ with regard to the slowly evolving enveloping amplitude follows from $\psi_3(t) = \sqrt{1 + \frac{\gamma^2}{\delta\omega^2}} \sqrt{1 + \frac{\gamma^2}{\delta\omega^2}} \sin(\frac{\gamma \omega_0 t + \delta_1}{\delta\omega})$ as

$$\tau_{\text{PT}} = (\delta_2 - \delta_1)/\delta\omega.$$ (A3)

Due to the invariance of Eqs. (1) under simultaneous action of $\mathcal{G}_1 := \mathcal{G}_2$ and $\gamma \to -\gamma$, the evolution from the lossy side $\Psi(t = 0) = (0, 0, 0, 1)^T$ to $\Psi(t = \tau_{\text{PT}}) = (\psi_1, \psi_2, \psi_3, 0)^T$ requires a $\tau_{\text{PT}}$ obtainable from (A2) by sign change $\gamma \to -\gamma$. Explicitly this yields Eqs. (6).

[7] Note though Ref. [6] where these results have been questioned.
[11] For closely related work see also [12–16].
[39] The tachistochrone passage, although closely related to a brachistochrone evolution, may not necessarily reach the brachistochrone limit for a corresponding non-Hermitian system.
Exceptional-point dynamics in photonic honeycomb lattices with $\mathcal{PT}$ symmetry

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We theoretically investigate the flow of electromagnetic waves in complex honeycomb photonic lattices with local $\mathcal{PT}$ symmetries. Such $\mathcal{PT}$ structure is introduced via a judicious arrangement of gain and loss across the honeycomb lattice, characterized by a gain and loss parameter $\gamma$. We found a class of conical diffraction phenomena where the formed cone is brighter and travels along the lattice with a transverse speed proportional to $\sqrt{\gamma}$.

I. INTRODUCTION

Conical refraction phenomena, i.e., the spreading into a hollow cone of an unpolarized light beam entering a biaxial crystal along its optic axis, are fundamental in classical optics and in mathematical physics [1–4]. Originally predicted by Hamilton in 1837 [3] and experimentally observed by Lloyd [4], these phenomena have been intensively studied in recent years by a large community of theorists and experimentalists [1–8]. The physical origin of the phenomenon is associated with the existence of the legendary diabolical points, which emerge along the axis of intersection of the two shells associated with the wave surface. Around a diabolical point the energy dispersion relation is linear while the direction of the group velocity is not uniquely defined. Recently, conical diffraction was observed in two-dimensional photonic honeycomb lattices [5] which share key common features, including the existence of diabolical points, with the band structure of graphene in condensed matter physics literature. In graphene, the electrons around the diabolical points of the band structure behave as massless relativistic fermions, thus resulting in extremely high electron mobility. Both photonic and electronic graphene structures allow us to test experimentally various legendary predictions of relativistic quantum mechanics such as the Klein paradox [9] and the dynamics of optical tachyons [10].

While diabolical points are spectral singularities associated with Hermitian systems, for pseudo-Hermitian Hamiltonians, like those used for the theoretical description of non-Hermitian optics, a topologically different singularity may appear: an exceptional point (EP), where not only the eigenvalues but also the associated eigenstates coalesce. Pseudo-Hermitian optics is a rapidly developing field which aims, via a judicious design that involves the combination of delicately balanced amplification and absorption regions together with the modulation of the index of refraction, to achieve new classes of synthetic metamaterials that can give rise to altogether new physical behavior and novel functionality [11,12]. The idea can be carried out via index-guided geometries with special antilinear symmetries. Adopting a Schrödinger language that is applicable in the paraxial approximation, the effective Hamiltonian that governs the optical beam evolution is non-Hermitian and commutes with the combined parity ($\mathcal{P}$) and time ($\mathcal{T}$) operator [13–15]. In optics, $\mathcal{PT}$ symmetry demands that the complex refractive index obeys the condition $n(\mathbf{r}) = n^*(\mathbf{r})$. It can be shown that for such structures, a real propagation constant (eigenenergies in the Hamiltonian language) exists for some range (the so-called exact phase) of the gain and loss coefficient. For larger values of this coefficient, the system undergoes a spontaneous symmetry breaking, corresponding to a transition from real to complex spectra (the so-called broken phase). The phase transition point shows all the characteristics of an exceptional-point singularity.

$\mathcal{PT}$ symmetries are not only novel mathematical curiosities. In a series of recent experimental papers, $\mathcal{PT}$ dynamics were investigated and key predictions confirmed and demonstrated [16–19]. Symmetry breaking has been experimentally observed in non-Hermitian structures [16–18], while power-law growth—characteristic of phase transitions—of the total energy has been demonstrated close to the exceptional points in Ref. [18]. In a silicon platform claims have been made that nonreciprocal light propagation in a silicon photonic circuit has been recorded [19]. $\mathcal{PT}$-synthetic materials can exhibit several intriguing features. These include, among others, power oscillations and nonreciprocity of light propagation [11,16,20,21], nonreciprocal Bloch oscillations [22], and unidirectional invisibility [23]. More specifically, a recent paper has proposed photonic honeycomb lattices with $\mathcal{PT}$ symmetry [10]. Interestingly, that work has shown that introducing alternating gain and loss to a honeycomb system prohibits $\mathcal{PT}$ symmetry, but adding appropriate strain (direction and strength) restores the symmetry, giving rise to $\mathcal{PT}$-symmetric photonic lattices [10]. Moreover, the mentioned work has found that in such systems, much higher group velocities can be achieved (compared with non-$\mathcal{PT}$-symmetry breaking systems), corresponding to a tachyonic dispersion relation [10]. In the nonlinear domain, such pseudo-Hermitian nonreciprocal effects can be used to realize a new generation of on-chip isolators and circulators [24]. Other results within the framework of $\mathcal{PT}$ optics include the realization of coherent perfect laser absorber [25], spatial optical switches [26], and nonlinear switching structures [27]. Despite the wealth of results on transport properties of $\mathcal{PT}$-symmetric one-dimensional optical structures, the properties of high-dimensional $\mathcal{PT}$ optical lattices (with the exception of few recent studies [11,28]) has remained so far essentially unexplored.

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Recently, it was pointed out [29] that $PT$-symmetric Hamiltonians are a special case of pseudo-Hermitian Hamiltonians, i.e., Hamiltonians that have an antilinear symmetry [30–32]. Such Hamiltonians commute with an antilinear operator $ST$, where $S$ is a generic linear operator. The corresponding Hamiltonian $H$ is termed generalized $PT$ symmetric. In a similar manner as in the case of $PT$ symmetry, one finds that if the eigenstates of $H$ are also eigenstates of the $ST$ operator then all the eigenvalues of $H$ are strictly real and the $ST$ symmetry is said to be exact. Otherwise the symmetry is said to be broken. An example case of a generalized $PT$-symmetric optical structure is the one reported in [33] (see also [20]). The unifying feature of these systems is that they are built of a particular kind of “building block” which below is referred to as a dimer. Each dimer in itself does have $PT$ symmetry and it can be represented as a pair of sites with assigned energies $\epsilon_n, \epsilon_d$. The system is composed of such dimers, coupled in some way. For an arbitrary choice of the site energies $\epsilon_n$ and coupling between the dimers, the system as a whole does not possess $PT$ symmetry (indeed, such global $PT$ symmetry would require precise relation between various $\epsilon_n$ and coupling symmetry between the dimers). On the other hand, since each dimer is $PT$ symmetric, with respect to its own center, there is some kind of “local” $PT$ symmetry (which we shall define as $P_2T$ symmetry). The main message of these papers is that, $PT$ symmetry of each individual dimer.

In this paper we investigate beam propagation in non-Hermitian two-dimensional photonic honeycomb lattices with $PT$ symmetry and probe for the possibility of abnormal diffraction. We find a type of conical diffraction that is associated with the spontaneously $PT$-symmetry-breaking phase transition point. Despite the fact that at the EP the Hilbert space collapses, the emerging cone is brighter and propagates evanescently between the waveguides.

Their arrangement in space is such that they form coupled A-B dimers with inter- and intradimer couplings $t_d$ and $t$, respectively. Such structure, apart from a global $PT$ symmetry, respects also another antilinear symmetry (in Ref. [33] we coined this $P_2T$ symmetry) which is related to the local $PT$ symmetry of each individual dimer.

Without loss of generality we rescale everything in units of $\epsilon_0 = 0$ and $\gamma_0 > 0$.

It is useful to work in the momentum space. To this end, we write the field amplitudes $\Psi_{n,m}$ in their Fourier representation, i.e.,

$$\Psi_{n,m}(x,y) = \frac{1}{(2\pi)^2} \int \frac{d{k_x} d{k_y}}{\gamma} [a_{n,m}(k) \tilde{a}_{n,m}(k) \exp(i{k_x} x + {k_y} y)]$$

$$b_{n,m}(x,y) = \frac{1}{(2\pi)^2} \int \frac{d{k_x} d{k_y}}{\gamma} [b_{n,m}(k) \tilde{b}_{n,m}(k) \exp(i{k_x} x + {k_y} y)]$$

Substitution of Eq. (2) into Eq. (1) leads to the following set of coupled differential equations in the momentum space:

$$i \frac{d}{dz} \tilde{a}_n(z) = H_k \tilde{a}_n(z)$$

where

$$H_k = \begin{pmatrix} -i \gamma & D(k) \\ D(k)^* & i \gamma \end{pmatrix}$$

and

$$D(k) = -(t + 2\epsilon e^{-ikz}, \cos k_z)$$

$$k \equiv (k_x, k_y)$$

FIG. 1. (Color online) Honeycomb photonic lattice structure with intradimer coupling $t$ and interdimer coupling $t_d = 1$. Sublattice (lossy waveguide) $a_{n,m}$ is shown by green (gray in the black-and-white printed version) circles while sublattice (gain waveguide) $b_{n,m}$ is shown by the red (dark in the black-and-white printed version) circles. Each dimer is distinguished by index $n$ and $m$. The field is coupled evanescently between the waveguides.
In other words, because of the translational invariance of the system, the equations of motion in the Fourier representation break up into $2 \times 2$ blocks, one for each value of momentum $k$. The two-component wave functions for different $k$ values are then decoupled, thus allowing a simple theoretical description of the system.

### III. EIGENMODES ANALYSIS

We start our analysis with the study of the stationary solutions corresponding to Eq. (3). Substituting the stationary form

$$\langle \alpha_{n,\alpha}, \beta_{n,\beta} \rangle^T = \exp(-i\varepsilon \tau)(A, B)^T$$

in Eq. (3), we get

$$\varepsilon \begin{pmatrix} A \\ B \end{pmatrix} = H_b \begin{pmatrix} A \\ B \end{pmatrix}.$$  

The spectrum is obtained by requesting a nontrivial solution, i.e., $(A, B) \neq 0$. The corresponding dispersion relation has the form

$$\varepsilon_{\pm} = \pm \sqrt{|D(k)|^2 - \gamma^2}. \quad (8)$$

For $\gamma = 0$ the dispersion relation is

$$\varepsilon = \pm |D(k)|,$$

and we have two bands of width $t + 2$. There are three pairs of diabatic points (DPs):

$$k_{\pm,\pm}^0 = \left( \pm \pi, \pm \arccos \frac{1}{2} \right),$$

$$k_{\pm,\mp}^0 = \left[ 0, \pm \left( \pi - \arccos \frac{1}{2} \right) \right]. \quad (10)$$

Expansion of $D(k)$ up to the first order around the DPs leads to

$$D(k) \approx \xi_{\eta,\eta} \mp \sqrt{4 - t^2} \eta.$$  

Substituting the above expression in the energy dispersion given by Eq. (8), we get the linear relation

$$\varepsilon_k \approx \pm \left[ t^2 \eta_k^2 + (4 - t^2) \eta_k^2 \right]^{1/2}, \quad (12)$$

where $\eta_{\pm,\pm} = k_{\pm,\pm} - (k_{0,0})_{\eta,\eta}$.

The standard passive ($\gamma = 0$) honeycomb lattice (zero strain) corresponds to $t = 1$. In this case, there are three pairs of DPs at [see Fig. 2(a)]

$$k_{\pm,\pm}^0 = \left( \pm \pi, \pm \pi \right),$$

$$k_{\pm,\mp}^0 = \left( 0, \pm \frac{2\pi}{3} \right). \quad (13)$$

For $1 < t < 2$ the two pairs of DPs at $k_{\pm,\pm}^0$ start moving toward each other while the pair at $k_{\pm,\mp}^0$ moves away from one another. At $t = 2$ a degeneracy occurs, i.e., $k_{\pm,\pm}^0 = (\pm \pi, 0)$. At the same time the dispersion relation $\varepsilon$ around $k_{\pm,\pm}^0$ and $k_{\pm,\mp}^0$ is linear only in the $k_x$ direction (and quadratic in the $k_y$).

For $t > 2$ the two energy surfaces move away from each other and a gap between them is created [see Fig. 2(b)]. Therefore for $t \geq 2$ the DPs disappear for $\gamma = 0$, and the conical diffraction is destroyed [6,34].

By introducing gain and loss to the system described by Eq. (1), the resulting effective Hamiltonian which describes the paraxial evolution becomes non-Hermitian. In fact, for $1 \leq t \leq 2$, any value of $\gamma$ results in complex eigenvalues, i.e., the system is in the broken $\mathcal{PT}$-symmetry phase. The resulting dispersion relation resembles the dispersion relation of relativistic particles with imaginary mass, as was recently discussed in Ref. [10].

In the case of $t > 2$ the size of the gap between the two bands can be controlled by manipulating the gain and loss parameter $\gamma$. In this case, there is a $\gamma$ domain, corresponding to the exact phase, for which the energies are real. It turns out from Eq. (8) that the line

$$\gamma_{\mathcal{PT}} = t - 2$$

defines the phase transition from exact to broken $\mathcal{PT}$-symmetry [10]. The mechanism for this symmetry breaking is the crossing between levels, associated with the exceptional points $k_{\pm,\pm}^0 = (0, \pm \pi)$ or $k_{\pm,\mp}^0 = (\pm \pi, 0)$ [see Fig. 2(c)] and belonging to different bands [33]; it follows from Eq. (8), that when $\gamma = \gamma_{\mathcal{PT}}$, the gap disappears and the two (real) levels at the “inner” band edges become degenerate. Evaluation of $D(k_x, k_y)$ to second order in $(\eta_x, \eta_y)$ around the degeneracy points leads to

$$D(\eta_x, \eta_y) \approx -(\gamma_{\mathcal{PT}} + 2i\eta_x + \eta_y)^2, \quad \eta^2 \equiv \eta_x^2 + \eta_y^2, \quad (16)$$

thus resulting in the dispersion relation

$$\varepsilon = \pm \sqrt{2\gamma_{\mathcal{PT}} \eta_x^2 + (2\gamma_{\mathcal{PT}} + 4)\eta_y^2}. \quad (17)$$

For large $\gamma_{\mathcal{PT}}$ values (i.e., $\gamma_{\mathcal{PT}} \gg 2$) one can approximate the above equation to get

$$\varepsilon \approx \pm \sqrt{2\gamma_{\mathcal{PT}}} \eta.$$  

This comment will become important in the analysis of optical beam propagation discussed in Sec. IV.

Next, we turn to the analysis and characterization of the biorthogonal set of eigenvectors of our non-Hermitian system. The target here is to identify the proximity to the exceptional point in the case of finite Hilbert spaces, where finite-size effects might play an important role in the analysis of the dynamics. The latter do not respect the standard (Euclidean) orthonormalization condition. Let $|L_n\rangle$ and $|R_n\rangle$ denote the left and right eigenvectors of the non-Hermitian Hamiltonian.
\[ H = \sum_k \xi_k |k\rangle \langle k | \] for a fixed coupling \( t \). The inset shows representative \( \xi \) for a fixed coupling \( t \).

We have studied the mean (diagonal) Petermann factor, which is defined as [20,36]

\[ K_{nn} = \langle L_n | L_m \rangle \langle R_m | R_n \rangle. \] (22)

We have studied the mean (diagonal) Petermann factor

\[ \gamma = \frac{1}{\mathcal{N}} \sum_{n=1}^{\mathcal{N}} K_{nn}, \] (23)

which takes the value of unity if the eigenfunctions of the system are orthogonal, while it is larger than unity in the opposite case. At the EP, a pair of eigenvectors associated with the corresponding degenerate eigenvalues coalesce, leading to a \textit{collapse} of the Hilbert space. At this point, the Petermann factor diverges as [20,36]

\[ \gamma \sim 1/|\gamma - \gamma_{\text{PT}}|. \] (24)

This indicates strong correlations between the spectrum and the eigenvectors, which can affect drastically the dynamics, as we will see later.

In Fig. 3 we present our calculations for \( \gamma \) for different \((t,\gamma)\) values and various system sizes. We find that as \( \mathcal{N} \) increases, the divergence approaches the line \( \gamma_{\text{PT}} = t - 2 \), which was derived previously [see Eq. (14)] for the case of infinite graphene.

IV. DYNAMICS

Armed with the previous knowledge about the eigenmode properties of the \( \mathcal{PT} \)-symmetric graphene, we are now ready to study beam propagation in \( \mathcal{PT} \)-symmetric honeycomb lattices at the vicinity of the EPs. The question at hand is whether the collapse of the Hilbert space at the EP will affect the conical diffraction (CD) pattern, and if yes, what is the emerging dynamical picture.

A. Numerical analysis

We first study wave propagation in the honeycomb lattice numerically (Fig. 4), by launching a beam with the structure of a Bloch mode associated with the EP, multiplied by a Gaussian envelope. The Bloch modes at the tip can be constructed from pairs of plane waves with \( k \) vectors of opposite pairs of exceptional points. Thus, interfering two plane waves at angles associated with opposite EP yields the phase structure of the modes from these points. Multiplying these waves by an envelope yields a superposition of Bloch modes in a region around these points. Figure 4 shows an example of the propagation of a beam constructed to excite a Gaussian superposition of Bloch modes around an EP. The input beam has a bell-shape structure, which, after some distance, transforms into the ring-like characteristic of conical diffraction [34]. From there on, the ring is propagating in the lattice by keeping its width constant while its radius is increasing linearly with distance. The invariance of the ring thickness and structure manifests a (quasi-) linear dispersion relation above and below the EP (see Fig. 2); hence, the diffraction coefficient for wave packets constructed from Bloch modes in that region is zero (infinite effective mass). This is especially interesting because the ring itself is a manifestation of the dispersion properties of the EP itself, where the diffraction coefficient is infinite (zero-effective mass). As a result, the ring forms a light cone in the lattice. The appearance of CD in the case of \( \mathcal{PT} \) lattices where the eigenvectors are nonorthogonal and coalesce at the EP singularity provides a clear indication that the phenomenon is insensitive to the eigenmodes structure and that it depends only on the properties of the dispersion relation.

The \( \mathcal{PT} \)-symmetric conical diffraction shows some unique characteristics with respect to the CD found in the case of beam propagation around DPs for passive honeycomb lattices...
Eqs. (4), (16), and (17), we find that the evolving amplitude of the conical diffraction by considering the field evolution in the momentum space. We consider for simplicity an initial PT-symmetric diffraction by considering the field evolution in PT and loss parameter at the symmetry-breaking point $\gamma$. This is shown in Fig. 4 where we compare the spreading of a CD ring of light, whose radius expands linearly with time. Numerical simulations approve theoretical predictions of the transverse speed of the cone, the emerging cone is brighter and moves faster than the one found in passive lattices [5] (i.e., the field intensity of the conical wavefront is larger).

Next, we calculate the evolution matrix $U \equiv e^{-i H_\text{B}}$ where $H_\text{B}$ is given by Eq. (4). After a straightforward algebra and using the fact that $H_\text{B}^2 = \mathcal{E}^2 \times 1$, we get

$$ U = \cos(\mathcal{E}^2) \mathbb{1} - i \sin(\mathcal{E}^2) / \mathcal{E} ] H_\text{B}. $$

Equation (28) is the starting point of our analysis. Substituting Eqs. (4), (16), and (17), we find that the evolving amplitude of the field $(a_{n,m}, b_{n,m})$ is

$$ a_{n,m}(z) \approx \sum_{l=1,2} \left[ (-1)^l \left( 1 + g \frac{\phi(n,m,z,g)}{\gamma_{PT}} \right)^{3/2} \right. $$

$$ b_{n,m}(z) \approx \sum_{l=1,2} \left. \left( -1 \right)^{l+1} \left( 1 + g \frac{\phi(n,m,z,g)}{\gamma_{PT}} \right)^{3/2} \right] $$

where

$$ \phi(n,m,z,g) = \left[ g + \left( -1/2 \right)^2 \nu^2 / \left( 2 \gamma_{PT} + 4 \right) \right. $$

$$ \left. + \nu^2 / (2 \gamma_{PT}) \right]. $$

Although our simplified calculations are not able to capture all the features of the propagating cone, the above expression encompasses the main characteristics of the conical diffraction that we have observed in our numerical simulations. At $z = 0$, Eq. (29) resembles a Lorentzian, which slowly transforms into a ring of light, whose radius expands linearly with $z$ with velocity $\gamma_{PT}$, while its thickness remains unchanged. At the same time the field intensity on the ring in the case of PT-symmetric lattices is brighter than the one corresponding to passive honeycomb lattices (i.e., $1/z^2$ vs $1/z^4$ behavior, respectively).

V. SUMMARY AND CONCLUDING REMARKS

We studied numerically and analytically the propagation of waves in PT-symmetric honeycomb photonic lattices, demonstrating the existence of conical diffraction arising solely from the presence of a spontaneous PT-symmetry-breaking phase transition point. In spite of the fact that the eigenvectors are nonorthogonal and there is a collapse of the Hilbert space at the EP, the emerging cone is brighter and moves faster than the corresponding one of the passive structure. Although, the realization of such photonic structures is currently a challenging task, active electronic circuits, like the one proposed in Ref. [18], can be proven useful alternatives that might allow us to investigate experimentally wave propagation in extended PT-symmetric lattices.

These findings raise several intriguing questions. For example, how does nonlinearity affect PT-symmetric conical diffraction? What is the effect of disorder [37]? Is this behavior generic for any system at the spontaneously PT-symmetry-breaking point? These intriguing questions are universal and relate to any field in which waves can propagate in a periodic potential. It is expected that in active metamaterials such phenomena will be present and they may have specific technological importance.

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[34] Here we use the term “conical” diffraction in a rather loose fashion. Specifically, when the lattice is deformed, i.e., \( t \neq 1 \), the CD pattern becomes elliptic [6].


**PT-Symmetric Talbot Effects**

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We show that complex $\mathcal{PT}$-symmetric photonic lattices can lead to a new class of self-imaging Talbot effects. For this to occur, we find that the input field pattern has to respect specific periodicities dictated by the symmetries of the system. While at the spontaneous $\mathcal{PT}$-symmetry breaking point the image revivals occur at Talbot lengths governed by the characteristics of the passive lattice, at the exact phase it depends on the gain and loss parameter, thus allowing one to control the imaging process.

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**Introduction.**—The Talbot effect [1,2], a near-field diffraction phenomenon, in which self-imaging of a periodic structure illuminated by a quasimonochromatic coherent light periodically replicates at certain imaging planes, is an important phenomenon in optics. These imaging planes are located at even integer multiples of the so-called Talbot distance $z_T = 2\lambda^2/a$, where $a$ represents the spatial period of the pattern and $\lambda$ the light wavelength. The simplicity and beauty of Talbot self-imaging have attracted the interest of many researchers. Such effects nowadays find applications in fields ranging from imaging processing and synthesis, photolithography [3], and optical testing and metrology [4] to spectrometry and optical computing [5] as well as in electron optics and microscopy [6]. Similar processes are encountered in other areas of physics involving nonclassical light [7], atom optics [3,8], Bose-Einstein condensates [9], coupled lasers [10], and waveguide arrays [11]. However, all these achievements are limited in view of studying the properties of the input beams and using real gratings for imaging. Overcoming these limitations will not only enrich the conventional self-imaging research, but also offer new methods for imaging technologies. It is therefore extremely desirable to investigate and propose self-imaging architectures that incorporate gain and loss mechanisms.

In the present Letter we study the Talbot revivals in a new setting, namely, a class of active lattices with antilinear symmetries. These structures deliberately exploit notions of (generalized) parity ($\mathcal{P}$) and time ($\mathcal{T}$) symmetry [12,13] in order to achieve new classes of synthetic metamaterials that can give rise to altogether new physical behavior and novel functionality [14–16]. Some of these results have been already confirmed and demonstrated in a series of recent experimental papers [15–17]. In classical optics, $\mathcal{PT}$-symmetries can be naturally incorporated [14] via a judicious design involving the combination of delicately balanced amplification and absorption regions with modulation of the index of refraction. In optics, $\mathcal{PT}$-symmetry demands that the complex refractive index obeys the condition $n(r) = n^*(-r)$. It can be shown that these structures have a real propagation constant (eigenvalues of the paraxial effective Hamiltonian) for some range (the so-called exact phase) of the gain and loss coefficient. For larger values of this coefficient, the system undergoes a spontaneous symmetry breaking, corresponding to a transition from real to complex spectra (the so-called broken phase). The phase transition point shows all the characteristics of an exceptional point (EP) singularity. $\mathcal{PT}$-symmetric matter can exhibit several intriguing features [14–31]. These include, among others, power oscillations and nonreciprocity of light propagation [14,15,19], nonreciprocal Bloch oscillations [20], unidirectional invisibility [28], and a new class of conical diffraction [31]. In the nonlinear domain, such nonreciprocal effects can be used to realize a new generation of optical on-chip isolators and circulators [22]. Other results include the realization of coherent perfect laser-absorber [23,29] and nonlinear switching structures [24].

Here, we define conditions that guarantee the existence of Talbot self-imaging for a class of active $\mathcal{PT}$-symmetric lattices. We find that the nonhermiticity of the Floquet-Bloch modes imposed by the non-Hermitian nature of the dynamics together with the discreteness of the lattice structures imposes strong constraints for the appearance of Talbot recurrences. We show that while at the spontaneous $\mathcal{PT}$-symmetric point the Talbot length $z_T$ is characterized by the structural characteristics of the lattice, in the exact $\mathcal{PT}$-symmetric phase it is controlled by the gain and loss parameter $\gamma$. This allow us to have reconfigurable Talbot lengths for the same initial pattern. Finally, we discuss possible experimental realizations where our predictions can be observed.

**Model.**—We consider a one-dimensional (1D) array of coupled optical waveguides. Each of the waveguides can support only one mode, while light is transferred from waveguide to waveguide through optical tunneling. The
array consists of two types of waveguides: type (A) involving a gain material and type (B) exhibiting an equal amount of loss. Their arrangement in space is such that they form N coupled (A-B) dimers with intra- and interdimer couplings k and c, respectively, such that both couplings are of similar (but not the same) size; i.e., k = c (for example, see Fig. 2, where k = 1.05c). In the tight binding description [32], the diffraction dynamics of the electric field amplitude \( \Psi_n = (a_n, b_n)^T \) at the n-th dimer evolves according to the following Schrödinger-like equation:

\[
\begin{align*}
    i \frac{d a_n(z)}{dz} &= e a_n(z) + k b_n(z) + c b_{n-1}(z) \\
    i \frac{d b_n(z)}{dz} &= e^{2} b_n(z) + k a_n(z) + c a_{n+1}(z),
\end{align*}
\]

where \( e = \epsilon_i + i \gamma \) is related to the complex refractive index \([14]\). Without any loss of generality, we will assume below that \( \epsilon_0 = 0, \gamma > 0 \), and \( c < k \) \([19]\). The effective Hamiltonian that describes the system commutes with an anti-linear operator \([19]\) and is called this \( \mathbb{PT} \)-symmetry) that is related to the local \( \mathbb{PT} \)-symmetry of each individual dimer.

At this point, it is beneficial to adopt a momentum representation \( a_n(z) = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} d q q a_q(z) \exp(i q z) \) (and similarly for \( b_n \)), where the integral is taken over the Brillouin zone \(-\pi \leq q \leq \pi\). Because of the translational invariance of the system \([1]\), the equations of motion in the Fourier representation break up into \( 2 \times 2 \) blocks, one for each value of momentum \( q \),

\[
    \frac{d}{dz} \begin{pmatrix} \tilde{a}_q(z) \\ \tilde{b}_q(z) \end{pmatrix} = H_q \begin{pmatrix} \tilde{a}_q(z) \\ \tilde{b}_q(z) \end{pmatrix}, \quad H_q = \begin{pmatrix} \epsilon_q & v_q \\ v_q^* & -\epsilon_q \end{pmatrix},
\]

with \( v_q = k + c e^{-i q} \). The two-component wave functions for different \( q \) values are decoupled, thus allowing for a simple theoretical description of the system. This allows us to perform the evolution in Fourier space and then evaluate the spatial representation by a backward transformation, i.e.,

\[
    \Psi_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_n(q) e^{iqz} dq,
\]

where \( \Psi_n(z) = (a_n(z), b_n(z))^T \) is the field amplitude for the n-th dimer in the spatial representation and \( \phi_n \) is the corresponding Fourier component.

**Dynamics.**—Substituting the stationary form \( (a_n, b_n)^T = \exp(-iEz)(A, B)^T \) in Eq. (2) and requesting nontrivial solutions of the resulting stationary problem, i.e., \( (A, B) \neq 0 \), we obtain the band structure of this diatomic \( \mathbb{PT} \)-system \([19]\).

\[
    \mathcal{E}_m = \pm \sqrt{(k-c)^2 + 4kc\cos^2(q/2) - \gamma^2}.
\]

For \( \gamma = 0 \), we have two bands of width \( 2c \), centered at \( \mathcal{E} = \pm k \). In this case, the two bands are separated by a gap \( \delta = 2(k-c) \) and the exact \( \mathbb{PT} \) phase persists across a large \( \gamma \) regime. It follows from Eq. (4) that when \( \gamma \geq \gamma_{PT} = \delta/2 \), the gap disappears and the two (real) levels at the ‘‘inner’’ band-edges of the two different bands (corresponding to \( q = \pm \pi \)) become degenerate. The corresponding eigenvectors are also degenerate, resulting in an EP singularity. For \( \gamma > \gamma_{PT} \), the spectrum becomes partially complex \([19]\). Below, we focus our analysis on the domain \( \gamma \leq \gamma_{PT} \).

The eigenvectors associated with the Hamiltonian Eq. (2) are biorthogonal, and therefore do not respect the standard (Euclidian) orthonormalization condition. As a result, the conservation of total field intensity is violated for any \( \gamma \neq 0 \). Denoting by \( |R_q(q)\rangle = \frac{1}{\sqrt{N}} \sum_{m=-N}^{N} c_m^- e^{-iE_m z} |R_m(q)\rangle \) the right eigenvectors corresponding to the eigenvalue \( E_m(q) \), we have that the \( q \)-th momentum components of any initial excitation can be written as \( \psi_q(0) = \sum_{m=-N}^{N} c_m^- |R_m(q)\rangle \). The evolved \( q \)-field component is

\[
    \psi_q(z) = \sum_{m=-N}^{N} c_m^e e^{-iE_m(z)} |R_m(q)\rangle,
\]

where \( c_m^e = (L_q(q) | \psi_q(0)\rangle \) is the expansion coefficient and \( (L_q(q) \) the left eigenvector associated with eigenvalue \( E_m(q) \). The above expansion is valid as long as the Hamiltonian \( H_q \) in Eq. (2) does not have a defective eigenvalue. The latter appears at the spontaneous \( \mathbb{PT} \)-symmetric point \( \gamma_{PT} = k - c \) (EP) for \( q = \pm \pi \). The corresponding evolved \( q \)-field component is then written as

\[
    \psi_{q=\pm \pi}(z) = (c_1 + c_2)(1, -i)^T + c_2(-i/\gamma, 0)^T. \tag{6}
\]

Direct substitution of Eqs. (5) and (6) into Eq. (3) provides the evolution of the field in this system. A note of caution is in order here. For the existence of Talbot revivals, a necessary condition is that the initial preparation must not excite the \( q = \pm \pi \) defective mode. In the opposite case, the field increases linearly with the propagation distance \( z \) [see Eq. (6)], thus destroying the possibility of revivals of any initial pattern.

**Talbot self-imaging.**—We are now ready to analyze the Talbot self-imaging revivals in the case of the \( \mathbb{PT} \)-symmetric structure of Fig. 1. We recall that for the Talbot effect to occur, the input field distribution should be periodic \([11]\), and thus in general \( \Psi_0 = \Psi_{N+N}(0) \), where \( N \) represents the spatial period of the input field. Because of this periodic boundary condition, \( q \) can take values only from the discrete set

\[
    q_m = \frac{2m\pi}{N}, \quad m = 0, 1, 2, \ldots, N - 1. \tag{7}
\]

Substituting the above constrain in Eq. (3), we get the following expression for the evolved field at the \( n \)-th dimer:

\[
    \Psi^{(N)}_n(z) = \sum_{l=\pm m=1}^{N-1} c_{l,m}^e e^{-iE_l(z)} |R_l(q_m)\rangle.
\]

033902-2
It is therefore clear that field revivals are possible at intervals $z$ if $E(q_n)zT = 2\pi\nu$ where $\nu$ is an integer. Therefore, the ratio of any two eigenvalues $\lambda_{m} = E(q_m)$ has to be a rational number, i.e.,

$$\sqrt{(k-c)^2 + 4kccos^2(\frac{\pi\nu}{\lambda_{m}})} - \gamma^2 = \frac{a}{b}, \quad (9)$$

where $a$ and $b$ are relatively prime integers. At the same time, revivals in the field intensity are ensured provided that $(E_{m} - E_{n})/c(c - E_{n}) = a/b$, where the indices belong to the set $\{0, 1, \ldots, N-1\}$ and are taken at least three at a time. It is straightforward to show that this condition is trivially satisfied for the same set of $\lambda_{m}$ input patterns periodicities for the fields.

Next, we consider the field Talbot revivals of input patterns with period $N$ at the spontaneous $\mathcal{PT}$-symmetric point. To this end, we observe that the direct substitution of $\gamma = \gamma_{\mathcal{PT}}$ in Eq. (9) for the ratio $E_{m}/E_{0}$ leads to the simple condition $cos(n\pi/N) = a/b$. The latter is rewritten in terms of the Chebyshev polynomials, defined as $cos(ln(x)) = T_{n}(cos(x)) = \sum_{n=0}^{\infty} c_{n}ln^{n}(cos(x))$, where $n$ represents the integer part of $m$. The Chebyshev coefficients $c_{n}(\nu)$ are integer numbers; of importance to our discussion is the fact that the first one is given by $c_{0}(\nu) = 2^{n-1}$. Given that $c_{n}(\nu)$ are integers, then $cos(ln(x)) = \sum_{n=0}^{\infty} c_{n}ln^{n}(cos(x))$, where $n$ represents the integer part of $m$. Using the Chebyshev identity with $m = N$ (assuming $N$ is an odd number), we obtain the following polynomial in $cos(\frac{\pi}{\lambda_{m}})$:

$$2N^{-1}(\cos^2(\frac{\nu}{\lambda_{m}}))^{N} + \cdots + c_{N}(\nu)\cos(\frac{\nu}{\lambda_{m}}) + 1 = 0 \quad (10)$$

where we have used the fact that $T_{N}(cos(\frac{\nu}{\lambda_{m}})) = cos(N\pi/N) = -1$. By applying the rational root theorem, one can show that the roots of this polynomial in $cos(\pi/N)$ are rational only if $N = 1, 3$. A similar technique leads to the fact that for even values of $N$, the only possibility is $N = 2$ [11]. However, input patterns with $N = 2$ periodicity excite the $q = \pm\pi$ Fourier mode, and therefore based on our previous discussion [see Eq. (6) above] have to be excluded. Therefore, strictly speaking, discrete Talbot revivals at the spontaneous $\mathcal{PT}$-symmetric point are possible only for a finite set of periodicities $N = 1, 3$, where, for example, the $N = 1$ case can represent initial patterns $\{1, 0, 1, \ldots, 1, 0\}$ or $\{0, 1, 0, \ldots, 0\}$ or the more trivial case of a plane wave with $\{1, 1, 1, \ldots, 1\}$. Some representative intensity revivals for $N = 1$ and $3$ periods are depicted in Fig. 2.

Talbot revivals can appear also in the exact phase $\gamma < \gamma_{\mathcal{PT}}$. A simple examination of Eq. (9) indicates that an initial periodic pattern with periodicity $N = 1$ [resulting in eigenvalue index $m = 0$ in Eq. (7)] leads to a rational value $a/b = 1$. In this case, the Talbot length $z_{T}$ depends on the gain and loss parameter, as $z_{T} = 2\pi/L_{0} = 2\pi/\sqrt{\gamma_{\mathcal{PT}}^{2} + 4kc - \gamma^{2}}$, and therefore it varies by changing $\gamma$. Such reconfigurable behavior of the Talbot length is characteristic of the exact phase $\gamma < \gamma_{\mathcal{PT}}$ and can be found also for the $N = 2, 3$-period input patterns. For $N = 2$ [corresponding to eigenvalue indices $m = 0, 1$ in Eq. (7)] one can show that for fixed $k$ and $\gamma_{\mathcal{PT}} = k - c$ such that $\gamma_{\mathcal{PT}}(k + c) = a/b$. Eq. (9) is satisfied provided that $\gamma = \sqrt{\gamma_{\mathcal{PT}}^{2} - 4kc\alpha^{2}}/(\beta^{2} - \alpha^{2})$, we assume that $\alpha < \beta$.
Similarly, for $N = 3$, Talbot revivals are possible provided that $\gamma = \sqrt{\gamma_{PT}^2 + kc[1 - 4(\alpha/\beta)^2]/[1 - (\alpha/\beta)^2]}$ where $0.5 < \alpha/\beta < \sqrt{1 - 3kc/(k + c^2)}$. In both cases, the corresponding Talbot length is $\gamma$-dependent and is given by the largest period $z_T = 2\pi/|\varepsilon| - 2\pi/|\varepsilon_0|$ that results from the eigenvalues involved in the initial pattern. Example cases of Talbot self-imaging revivals for initial periodic patterns with period $N = 1, 2, 3$ and different $\gamma$ values are shown in Figs. 3(a)–3(f), respectively. We see that for the same initial preparation, the revivals are controlled by $\gamma$ and can occur at different Talbot lengths.

In fact, we can show that larger periods $N > 3$ do not result in Talbot self-imaging revivals in the exact $PT$-symmetric domain. Using Eq. (9) for $|\varepsilon_m|/|\varepsilon_0| = \alpha/\beta$ and enforcing the constraint that $\gamma = \gamma_{PT}$, one can obtain the inequality $\cos(\pi N) \leq 0$, which has to be satisfied together with equation Eq. (9) (the equality corresponds to the case $\gamma = \gamma_{PT}$ discussed above). At the same time, $\cos(\pi N)$ has $m = 0, \ldots, N - 1$ roots. Applying the intermediate value theorem, one finds that this inequality cannot be valid for $N > 3$.

**Experimental implementation.**—Finally, we would like to suggest possible experimental implementations of the $PT$-symmetric waveguide arrays, which will allow observation of the reconfigurable Talbot effect. The proposed structures involve MBE-grown quantum wells (QW) that will be patterned to form coupled waveguides. The basic $PT$ structural element of the array shown in Fig. 1 involves two $PT$-symmetric sites (dimer). Such a design is desirable because of its simplicity. The dimensions and index contrast can be such that each waveguide will be single mode. For example, for AlGaAs structures, this can be achieved by a refractive index of $n_0 = 3.35$ operated at 800 nm. Reconfigurable gain can be achieved by running an electric current through a AlGaAs/GaAs QW $p$. $n$ junction. In such structures, one can easily reach gain and loss values as high as 50 cm$^{-1}$. The two site channels in every dimer will be excited at different current levels $I_1$ and $I_2$ so as to establish the antisymmetric gain and loss profile that is necessary to observe $PT$ optical behavior. In practice, this will be done provided current $I_1 \gg I_2$ so that the corresponding regions underneath gain equal amounts of gain and loss. More specifically, $I_2$ will be relatively small so that the associated waveguide site will experience material absorption. Its sole purpose will be fine tuning. Given that $I_1$ and $I_2$ can be interchanged and adjusted, this will allow us to dynamically control the Talbot length $z_T$ of these $PT$-symmetric structures. Of course, special consideration has to be given to the effects of gain and loss on the modal index change in these structures (because of Kramers-Kronig relations).

Finally, we comment on the robustness of Talbot revivals against structural imperfections. For realistic values of positional imperfections (up to 5% of the interdimer coupling), we could confirm numerically that Talbot revivals are only slightly distorted. Specifically, we found that revivals associated with short Talbot lengths $z_T$ are essentially unaffected for moderate propagation distances $z$, while revivals associated with larger lengths $z_T$ are fragile due to the distortion of the delicate balance between the mode amplitudes and phases that eventually dominate the evolution.

**Conclusions.**—In conclusion, we have shown that a class of $PT$-symmetric optical lattices support Talbot self-imaging revivals for input patterns with periodicities dictated by the discreteness of the lattice and the strength of gain and loss parameters. It would be of interest to investigate whether Talbot revivals can also occur in higher dimensions and in the presence of nonlinearity. Our results might be applicable to other areas such as self-imaging of coupled lasers [10] with distributed gain and synchronization of $PT$-symmetric coupled electronic oscillators [16].

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![Figure 3](color online) Talbot intensity carpets for period-$N$ input field patterns at the exact phase $\gamma = \gamma_{PT}$. Everything is measured in units of interdimer coupling $c = 1$, while the interdimer coupling is $k = 4$. (a) $\gamma = 0.1$, while in (b) $\gamma = 2$. In both cases, the input pattern has periodicity $N = 1$, and it is chosen to be $\{1, 0, 1, 0, 1, 0, 1, 0, \ldots\}$; (c) $\gamma = \sqrt{11}/3$ and (d) $\gamma = \sqrt{7}$. Now, the input pattern has periodicity $N = 2$, and it is chosen to be $\{1, 1, 0, 0, 1, 1, \ldots\}$; (e) $\gamma = 5/\sqrt{17}$ and (f) $\gamma = 9/\sqrt{10}$. The input pattern in these cases is $\{1, 1, 1, 0, 0, 1, 1, 1, 0, 0, \ldots\}$ and has periodicity $N = 3$. Different Talbot lengths $z_T$ are observed between (a) and (b), (c) and (d), and (e) and (f).
[32] This approximation holds if the waveguides that the array is made of are single mode, exhibit a strong confinement, and a weak overlap with the modes of the adjacent guides, thus entailing only nearest neighbor interactions (for example, see discussion in [33]).
**PT**-symmetric electronics

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Abstract

We show both theoretically and experimentally that a pair of inductively coupled active LRC circuits (dimer), one with amplification and another with an equivalent amount of attenuation, display all the features which characterize a wide class of non-Hermitian systems which commute with the joint parity-time **PT** operator: typical normal modes, temporal evolution, and scattering processes. Utilizing a Liouvilian formulation, we can define an underlying **PT**-symmetric Hamiltonian, which provides important insight for understanding the behavior of the system. When the **PT**-dimer is coupled to transmission lines, the resulting scattering signal reveals novel features which reflect the **PT**-symmetry of the scattering target. Specifically we show that the device can show two different behaviors simultaneously, an amplifier or an absorber, depending on the direction and phase relation of the interrogating waves. Having an exact theory, and due to its relative experimental simplicity, **PT**-symmetric electronics offers new insights into the properties of **PT**-symmetric systems which are at the forefront of the research in mathematical physics and related fields.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Among the many recent developments in **PT** systems, the application of pseudo-Hermitian ideas into the realm of electronic circuitry not only promises a new generation of electronic structures and devices, but also provides a platform for detailed scrutiny of many new concepts within a framework of easily accessible experimental configurations. A first example of this was the demonstration in [1] that a pair of coupled LRC circuits, one with amplification and the other with equivalent amount of attenuation, provided the simplest experimental realization of a **PT** symmetric system. With a normal mode structure where all dynamical variables are easily measured in the time domain, extensions of the circuit approach will provide a valuable
testing ground for further developments into more sophisticated structures. Moreover, the $PT$-circuitry approach suggested also opens new avenues for innovative electronics architectures for signal manipulation from integrated circuits to antenna arrays, and allows for direct contact with cutting edge technological problems appearing in (nano)-antenna theory, split-ring resonator arrays, and meta-materials.

Examples of $PT$-symmetric systems range from quantum field theories and mathematical physics [2–5] to atomic [6, 7], solid state [8–10] and classical optics [11–19]. A $PT$-symmetric system can be described by a phenomenological ‘Hamiltonian’ $\mathcal{H}$ which may have a real energy spectrum, although in general $\mathcal{H}$ is non-Hermitian. Furthermore, as some parameter $\gamma$ that controls the degree of non-Hermiticity of $\mathcal{H}$ changes, a spontaneous $PT$ symmetry breaking occurs. At this point, $\gamma = \gamma_{PT}$, the eigenfunctions of $\mathcal{H}$ cease to be eigenfunctions of the $PT$-operator, despite the fact that $\mathcal{H}$ and the $PT$-operator commute [2]. This happens because the $PT$-operator is anti-linear, and thus the eigenstates of $\mathcal{H}$ may or may not be eigenstates of $PT$. As a consequence, in the broken $PT$-symmetric phase the spectrum becomes partially or completely complex. The other limiting case where both $\mathcal{H}$ and $PT$ share the same set of eigenvectors corresponds to the so-called exact $PT$-symmetric phase in which the spectrum is real. This result led Bender et al to propose an extension of quantum mechanics based on non-Hermitian but $PT$-symmetric operators [2, 3]. The class of non-Hermitian systems with real spectrum has been extended by other researchers in order to include Hamiltonians with generalized $PT$ (antilinear) symmetries [20].

While the applicability of these ideas in the quantum framework is still being debated, optical systems provide a particularly fertile ground where $PT$-related concepts can be realized [11] and experimentally investigated [13, 14]. In this framework, $PT$ symmetry demands that the complex refractive index obeys the condition $n(\vec{r}) = n^*(-\vec{r})$. $PT$-synthetic materials can exhibit several intriguing features. These include among others, power oscillations and non-reciprocity of light propagation [11, 14, 16], absorption enhanced transmission [13], and unidirectional invisibility [18]. Despite these efforts and the consequent wealth of theoretical results associated with $PT$-structures, until very recently only one experimental realization of a system with balanced gain and loss has been reported [14]. These authors studied the light propagation in two coupled $PT$ symmetric waveguides where the spontaneous $PT$-symmetry breaking ‘phase transition’ was indirectly confirmed. The analysis relied on the paraxial approximation which under appropriate conditions maps the scalar wave equation to the Schrödinger equation, with the axial wavevector playing the role of energy and with a fictitious time related to the propagation distance along the waveguide axis.

This observation led us recently to propose a new set-up based on active LRC circuits where the novel features of $PT$-symmetric structures can reveal themselves and can be studied both theoretically and experimentally in great detail. The system consists of a pair of coupled electronic oscillators, one with gain and the other with loss. This ‘active’ dimer, is implemented with simple electronics, and allow not only for a direct observation of a spontaneous $PT$-symmetric ‘phase transition’ from a real to a complex eigenfrequency spectrum but also for its consequences in the spatio-temporal domain. At the same time the equivalent scattering system, where a localized $PT$ symmetric structure is connected to one or two transmission line (TL) leads allow us to access the validity of recent theoretical predictions [17, 21–27].

This paper presents our recent results pertaining to the $PT$ electronics. We begin with a general discussion of electronics in the context of $PT$ symmetric systems in section 2.

1 We use the term ‘phase transition’ not in the standard thermodynamic sense but rather in the frame of $PT$-literature (see for example [2]).
Then in section 3 we examine the normal mode structure of the simplest such circuit, the $PT$ dimer. We experimentally demonstrate how it displays all the novel phenomena encountered in systems with generalized $PT$-symmetries. Section 4 discusses the unique aspects of $PT$ dynamics exhibited by the dimer, particularly upon passage from the exact to broken phase. In section 5 we investigate the simplest possible scattering situation where the dimer is coupled to a single TL, and derive a non-unimodular conservation relation connecting the left and right reflectances. A direct consequence of this relation is the existence of specific frequencies for which the system behaves either as a perfect absorber or as an amplifier, depending on the side (gain or loss) to which the TL is coupled. In section 6 we demonstrate theoretically and experimentally that a two-port $PT$-symmetric electronic cavity can act as a simultaneous coherent perfect absorber (CPA)-amplifier. Our circuit is the electronic equivalent of a CPA-laser device which was recently proposed in the optics framework, and constitutes the first experimental realization of such devices. Finally, section 7 presents several issues involving practical implementation of $PT$ circuits, along with some related experimental details. Our conclusions are given in section 8.

2. $PT$ electronics

One of the most convenient advantages of an electronic approach is that, at least in the low frequency domain, where the wavelength is significantly greater than the dimensions of the circuit, all spatial symmetry considerations can be reduced to a matter of network topology defined through the application of Kirchoff’s laws. Physical symmetry is irrelevant as long as the network has the desired node topology and the connecting elements are appropriately valued. Analogous to the familiar case of a $PT$-symmetric potential, the parity operation is equivalent to the interchange of labels corresponding to pairs of associated circuit components.

For simplicity, we will restrict our discussion to the usual fundamental physical devices: resistors, capacitors and inductors. Only the resistor, due to its dissipative nature, requires modification upon time-reversal where we include generic ohmic elements with either positive or negative resistance. Negative resistance represents the simplest conceptual inclusion of amplification into electronics since Kirchoff’s laws can be used without modification. Figure 1 illustrates how simple linear amplifiers can be configured to achieve negative resistance. The schematic implementation in (a) results in a single, ground-referenced node, while that in (b) shows a true two-terminal configuration. The former is of greatest utility due to its simplicity and the pervasiveness of ground nodes (defining a common zero potential) in typical circuits. Section 7 discusses further details of the experimental negative resistance converters.
Theoretical analysis of circuits including negative resistance elements, however, requires respecting a subtle condition: any two terminal circuit structure reducing to a pure negative resistance will be undefined unless the structure is placed in parallel with a capacitance. This conclusion results from the divergence of the pole associated with parallel RC combinations (with negative $R$) in the limit of $C \to 0$. This pole only arises if parallel capacitance is initially considered, so is hidden, and often overlooked in the consideration of negative resistance circuits. It is inconsequential with normal, positive resistance, where it’s sign corresponds to exponential decay. For example, the standard series LRC circuit, though it appears to have a mathematically well-behaved solution for negative $R$, is invalid in that realm due to the hidden pole. Our choice of the parallel LRC configuration was dictated by this consideration.

Thus, for a $\mathcal{PT}$-symmetric circuit incorporating these basic elements, it is necessary that (1) all reactive elements either have representation in parity-associated network pairs, or connect parity inverted network nodes, (2) all ohmic elements are paired with opposite sign, and (3) each negative ohmic element has an associated parallel capacitance, or an equivalent, as part of the circuit. Valid $\mathcal{PT}$-circuits of arbitrary complexity can be built up using these simple rules, though their stability needs to be independently determined. In principle, the long-wave approximation could be relaxed with an appropriate inclusion of waveguide connections, however, this would return geometry into the mix of $\mathcal{PT}$ considerations.

3. $\mathcal{PT}$ dimer modes

Figure 2 shows the $\mathcal{PT}$-symmetric dimer, the simplest configuration with a non trivial (more than one mode) pseudo-Hermitian spectrum. Both capacitive and mutual inductive coupling are included for generality, although the experimental results presented throughout this work are exclusively one or the other. The gain side on the left of figure 2 is indicated by $-R$ and was implemented using the configuration of figure 1(a). The loss on the right is achieved with a conventional resistance of the same value, resulting in the gain/loss parameter $\gamma = R^{-\frac{1}{2}} \sqrt{L/C}$ for this system. Further details of the experimental circuit are given in section 7.

Kirchoff’s laws for the dimer with both mutual inductance coupling and capacitive coupling between the oscillators are given for the gain side (equation (1)) and loss side (equation (2)).

\[
V_1 = i\omega(LI_1 + MI_2)
\]
\[
I_1 = \frac{V_1}{R} + i\omega CV_1 + i\omega C_c (V_1 - V_2) = 0
\]
\[ V_2 = i\omega(LI_2 + MI_1) \quad I_2 + \frac{V_2}{R} + i\omega CV_2 + i\omega C_c (V_2 - V_1) = 0. \] (2)

Eliminating the currents from the relations, scaling frequency and time by \( \omega_0 = \sqrt{L/C} \), and taking \( \mu = M/L \) and \( c = C_c/C \) gives the matrix equation:

\[
\begin{pmatrix}
\frac{1}{\omega(1-\mu^2)} - \omega(1+c) - i\gamma \\
\omega c - \frac{\mu}{\omega(1-\mu^2)} - \omega(1+c) + i\gamma
\end{pmatrix}
\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0.
\] (3)

At this point, it is obvious that the system is \( PT \) symmetric: swapping the indices and changing the sign of \( i \) leaves the equations unchanged. This linear, homogeneous system has four normal mode frequencies, as required to fulfill any arbitrary initial condition for voltage and current, given by

\[
\omega_{1,2} = \pm \sqrt{\gamma_c^2 - \gamma^2 + \sqrt{\gamma_{PT}^2 - \gamma^2}} \quad \omega_{3,4} = \pm \sqrt{\gamma_c^2 - \gamma^2 - \sqrt{\gamma_{PT}^2 - \gamma^2}}
\] (4)

with the \( PT \) symmetry breaking point identified as

\[
\gamma_{PT} = \frac{1}{\sqrt{1-\mu}} = \sqrt{\left| \frac{1+2c}{1+\mu} \right|}
\] (5)

and the upper critical point by

\[
\gamma_c = \frac{1}{\sqrt{1-\mu}} + \sqrt{\left| \frac{1+2c}{1+\mu} \right|}.
\] (6)

Note that the given forms explicitly show all of the relationships among the critical points and the real and imaginary parts of the frequencies. The exact phase, \( 0 < \gamma < \gamma_{PT} \), is characterized by four purely real eigenfrequencies coming in two pairs of positive \( (\omega_1, \omega_2 > 0) \) and negative \( (\omega_3, \omega_4 < 0) \) values, while in the broken phase below the upper critical point, \( \gamma_{PT} < \gamma < \gamma_c \) the eigenfrequencies are coming in complex conjugate pairs with non-vanishing real parts, and above \( \gamma_c \), as two purely imaginary complex conjugate pairs. The broken phase of the \( PT \) dimer is unstable, in that it is ultimately dominated by an exponentially growing mode.

The normal modes in the exact phase are characterized by equal magnitudes for the voltage oscillations in the gain and loss sides, which in the \( +\omega \), real part convention allowed by the real eigenfrequencies, are given by

\[
\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_\pm = \frac{1}{\sqrt{2}} \left( 1 - \exp(i\phi_\pm) \right)
\] (7)

with a phase \( \phi_\pm \) of the loss side

\[
\phi_\pm = \pi/2 - \tan^{-1} \left( \frac{1}{\gamma} \left( \frac{(1-\mu^2)}{(1+\mu^2)} \omega_\pm - (1+c)\omega_\pm \right) \right).
\] (8)

As the gain/loss parameter traverses the exact region, \( 0 \leq \gamma \leq \gamma_{PT} \), the phase progresses from the in- and out-of-phase configuration of a Hamiltonian coupled oscillator, to a mode coalescence at \( \gamma_{PT} \) with \( \phi_\pm \sim \pi/2 \) with the real frequency

\[
\omega_+ = \omega_- = [(1-\mu^2)(1+c)]^{-1/4}.
\] (9)

Examination of the inductor currents,

\[
\begin{pmatrix} I_1 \\ I_2 \end{pmatrix}_\pm = \begin{pmatrix} \frac{1}{1-\mu^2} \quad -\frac{\mu}{1-\mu^2} \\ -\frac{1}{1-\mu^2} \quad \frac{1}{1+\mu^2} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}_\pm
\] (10)
reveal phase shifts, relative to the corresponding voltages, that advance on the gain side and retard on the loss side within either mode. This is as required for the net transfer of electrical energy from the gain side to the loss side as the gain/loss parameter increases. This evolutionary behavior is helpful in understanding the spectral and dynamical behavior of the dimer.

An alternate analysis of the dimer is also accomplished by recasting Kirchoff’s laws, equations (1) and (2) into a ‘rate equation’ form by making use of a Liouvillian formalism

\[
\frac{d\Psi}{dt} = \mathcal{L}\Psi; \quad \mathcal{L} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\alpha\beta & \alpha\zeta & (1 + c)\gamma & c\gamma \\
\alpha\zeta & -\alpha\beta & -c\gamma & -(1 + c)\gamma
\end{pmatrix}
\]

(11)

where \(\alpha = 1/(1 - \mu^2)\), \(\beta = 1 + c + c\mu\), \(\zeta = c + \mu + c\mu\) and \(\Psi = (Q_1, Q_2, \dot{Q}_1, \dot{Q}_2)^T\) with \(Q_0 = CV_0\). This formulation opens new exciting directions for applications [28] of generalized \(\mathcal{PT}\)-mechanics [20] as it can be interpreted as a Schrödinger equation with non-Hermitian effective Hamiltonian \(\mathcal{H}_{\text{eff}} = i\mathcal{L}\). This Hamiltonian is symmetric with respect to generalized \(\mathcal{PT}\)-transforms \(\mathcal{P}_0\mathcal{T}_0\). The matrix

\[
\mathcal{P}_0 = \begin{pmatrix}
\sigma_x & 0 \\
0 & \sigma_y
\end{pmatrix}; \quad \mathcal{T}_0 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

(12)

and \(\sigma_i\) is the Pauli matrix, \(I\) is the \(2 \times 2\) identity matrix, and \(\mathcal{K}\) denotes the operation of complex conjugation. By a similarity transformation \(\mathcal{R}\) [28],

\[
\mathcal{R} = \begin{pmatrix}
\frac{2i(b+d)}{1 + \sqrt{1 + 2c}} & \frac{-2i(b+d)}{1 + \sqrt{1 + 2c}} & \frac{-2i(1 + 2c + \sqrt{1 + 2c})}{c} & \frac{-2i(1 + 2c - \sqrt{1 + 2c})}{c} \\
\frac{i(1 + 2c - \sqrt{1 + 2c})(b-d)}{c} & \frac{i(1 + 2c + \sqrt{1 + 2c})(b-d)}{c} & \frac{2}{1 + \sqrt{1 + 2c}} & \frac{-2}{1 + \sqrt{1 + 2c}} \\
\frac{2i(b+d)}{1 + \sqrt{1 + 2c}} & \frac{-2i(b+d)}{1 + \sqrt{1 + 2c}} & \frac{-2i(1 + 2c + \sqrt{1 + 2c})}{c} & \frac{-2i(1 + 2c - \sqrt{1 + 2c})}{c} \\
\frac{i(1 + 2c - \sqrt{1 + 2c})(b-d)}{c} & \frac{i(1 + 2c + \sqrt{1 + 2c})(b-d)}{c} & \frac{2}{1 + \sqrt{1 + 2c}} & \frac{-2}{1 + \sqrt{1 + 2c}}
\end{pmatrix}
\]

(13)

\(\mathcal{H}_{\text{eff}} = \mathcal{R}^{-1}\mathcal{H}\mathcal{R}\) can be related to a transposition symmetric, \(\mathcal{PT}\)-symmetric Hamiltonian \(\mathcal{H} = \mathcal{H}^T = \mathcal{P}\mathcal{H}^\dagger\mathcal{P}, \mathcal{T} = \mathcal{K}\) where \(\mathcal{P} = \mathcal{R}\mathcal{P}_0\mathcal{R}^{-1}\). The matrix \(\mathcal{H}\) is then

\[
\mathcal{H} = \begin{pmatrix}
0 & b + ir & d + ir & 0 \\
b + ir & 0 & 0 & d - ir \\
d + ir & 0 & 0 & d - ir \\
0 & d - ir & d - ir & 0
\end{pmatrix}
\]

(14)

where \(b = \sqrt{\alpha(\beta - \sqrt{\beta^2 - \zeta^2})}/2\), \(d = \sqrt{\alpha(\beta - \sqrt{\beta^2 - \zeta^2})}/2\) and \(r = \frac{1}{2}\sqrt{1 + 2c}\gamma\). The frequencies and normal modes within this framework are identical to equations (4) and (10).

These normal mode properties can be measured in our electronic dimer by simultaneous observation of the node voltages \(V_1\) and \(V_2\) of figure 2. Our set-up allows detailed analysis of gain/loss parameters \(\gamma\) on either side of the \(\mathcal{PT}\)-phase transition point. In the exact phase, time series samples are captured with the dimer slightly unbalanced to marginally oscillate the mode of interest. Beyond the critical point, a transient sample is obtained dominated by the exponentially growing mode. Details are given in section 7.

In figure 3 we report measurements for the dimer frequencies (left) and inter-component phases (right) compared with the theoretical expressions, equations (4) and (8) respectively, for the values \(\mu = 0.2\) and \(c = 0\). The \(\mathcal{PT}\) symmetry imposes the condition that the magnitude of the two voltage components are equal to one-another in the exact phase. This property is also experimentally observed. For \(\gamma = 0\), the phases corresponding to the symmetric and antisymmetric combination are \(\phi_- = 0\) and \(\phi_+ = \pi\), respectively. When \(\gamma\) is subsequently
increased and the system is below the $\mathcal{PT}$ threshold, the eigenstates are not orthogonal and their phases can be anywhere (depending on $\gamma /\gamma_{\mathcal{PT}}$) in the interval $[0, \pi]$.

The value of phase difference at the spontaneous $\mathcal{PT}$-symmetric breaking point $\gamma = \gamma_{\mathcal{PT}}$ can be calculated analytically and it is given by the expression:

$$\phi_{\mathcal{PT}}(\mu) = \arccos \left( \frac{\sqrt{1 - \sqrt{1 - \mu^2}}}{\sqrt{1 + \sqrt{1 - \mu^2}}} \right).$$  \hspace{1cm} (15)

We note that in the limit of $\mu \rightarrow 0$ we get $\phi_{\mathcal{PT}} = \pi /2$, corresponding to a ‘circular’ polarization of the eigenmode. The opposite limit of $\mu \rightarrow 1$ results to $\phi_{\mathcal{PT}} = 0$ corresponding to ‘linear’ polarization.

4. $\mathcal{PT}$ dimer dynamics

The signatures of $\mathcal{PT}$-symmetry and the transition from the exact phase to the broken phase are similarly reflected in the temporal behavior of our system. Equation (11) can be solved either analytically or via direct numerical integration in order to obtain the temporal behavior of the capacitor charge $Q_n(\tau)$ and the displacement current $I_n(\tau)$ in each of the two circuits of the $\mathcal{PT}$-symmetric dimer. As an example of the dimer state evolution, we consider an initial displacement current in one of the circuits with all other dynamical variables zero.

In figure 4 we present some typical measurements for the temporal behavior of circuit voltages along with the corresponding numerical result. We consider a dimer configuration with $\mu = 0.2$ and $e = 0$ (i.e. inductive coupling only). In the left panel of figure 4(a), we show $V_1(\tau)$ and $V_2(\tau)$ for an initial condition having $I_1(0) = 1.2$ mA with all other dynamical variables zero. The right panel shows the same data as a Lissajous plot, with the initial condition trajectory leaving the origin with $V_1$ decreasing, and $V_2$ stationary. Agreement between the experiment (circles) and the simulations (lines) is observed, illustrating that, in spite of the presence of dissipative elements and non-orthogonal states, the beat superposition associated with real frequencies occurs. There is, however, a subtle distinction: since energy
is not conserved, the beat is asymmetric between the gain side and the loss side nodal times, with oscillatory activity spending more time between gain side nodal points as energy grows to a significantly larger size before decaying and growing between the loss side nodal points. However, unlike traditional coupled-oscillator beats, instead of ‘slashing’ between both sides during the course of the beats, a growth and decay energy dance occurs with both sides more or less equally represented except in the vicinity of the nodal points. This behavior is a direct result of the non-orthogonal phase relationships that become more pronounced as $\gamma \rightarrow \gamma_{PT}$. A Hamiltonian dimer would exhibit a perfect half-beat offset between the left and right voltage beat envelopes.

We have also traced these energy dance features by studying the time-dependence of the total capacitance energy:

$$E_{C}^{\text{tot}}(\tau) = \frac{Q_{1}^{2}(\tau) + Q_{2}^{2}(\tau)}{2C}. \quad (16)$$

With the initial condition used in the experiment, we expect power oscillations which are due to the unfolding of the non-orthogonal eigenmodes $[2, 11, 16, 14]$. This universal feature is evident in the temporal behavior of $E_{C}^{\text{tot}}(\tau)$ as can be seen in figure 5. On the other hand, for $\gamma > \gamma_{PT}$ the dynamics is unstable and $E_{C}^{\text{tot}}(\tau)$ grows exponentially with a rate given by the maximum imaginary eigenvalue $\max\{\text{Im}(\omega)\}$ (see figure 5).

The most interesting behavior appears at the spontaneous $\mathcal{PT}$-symmetry breaking point $\gamma = \gamma_{PT}$. At this point the matrix $\mathcal{L}$ has a defective eigenvalue. In this case, the evolution $U = \exp(\mathcal{L}t)$ can be calculated from the Jordan decomposition of $\mathcal{L}$ as $\mathcal{J} = \mathcal{S}\mathcal{L}\mathcal{S}^{-1}$. Having in mind the form of the exponential of a Jordan matrix, it follows immediately that linear growing terms appear in the evolution of the charge vector $(Q_{1}(\tau), Q_{2}(\tau))^{T}$ [29]. This results in a quadratic increase of the capacitance energy i.e. $E_{C}^{\text{tot}}(\tau) \sim \tau^{2}$. Although all systems typically becomes very sensitive to parameters near a critical point, we are able to control the circuit elements sufficiently well to observe the approach to the predicted $\tau^{2}$ behavior of the energy. This time range is limited by the dynamic range of our circuit linearity, as discussed in section 7.
Figure 5. Experimentally measured temporal dynamics of the capacitance energy $E_{C}^{\text{tot}}(\tau)$ of the total system for various $\gamma$-values. As $\gamma \to \gamma_{PT}$ the $\tau^2$ behavior signaling the spontaneous $\mathcal{PT}$-symmetry breaking is observed.

Figure 6. Two experimental configurations associated with a simple $\mathcal{PT}$-symmetric dimer. In the lower and upper circuits, we couple a TL to the gain and loss sides, respectively. Preliminary experimental measurements for the corresponding reflection coefficients are shown (loss-side red, gain-side blue) along with the solid line corresponding to $1/R$ illustrate the reciprocal nature $R_{L}R_{R} = 1$ (see text) of the $\mathcal{PT}$-scattering. Here $\mu = 0.2$, $\gamma = 0.164$ and $Z_0 = 15.5\sqrt{L/C}$.

5. The Janus faces of $\mathcal{PT}$-symmetric scattering

We report our initial scattering studies with the following two reciprocal geometries: in the first case, a TL is attached to the left (amplified) circuit of the dimer load while in the second case, the TL is connected to the right (lossy) circuit of the load (see lower and upper insets of figure 6 respectively). Experimentally, the equivalent of a TL with characteristic impedance $Z_0$ could be attached to either side of the dimer at the RLC circuit voltage node in the form
of a resistance \( R_0 = Z_0 \) in series with a variable frequency voltage source. The right and left traveling wave components associated with the TL would be deduced from the complex voltages on both sides of \( R_0 \). With \( V_{\text{LC}} \) the voltage on the LC circuit, and \( V_0 \) the voltage on the synthesizer side of the coupling resistor \( R_0 \), the right (incoming) wave has a voltage amplitude \( V_L^+ = V_0/2 \) and the left (reflected) wave has a voltage amplitude \( V_L^- = V_{\text{LC}} - V_0/2 \). The voltage source defines the phase of the incoming wave.

At any point along a TL, the current and voltage determine the amplitudes of the right and left traveling wave components \([30]\). The forward \( V_{LR}^+ \) and backward \( V_{LR}^- \) wave amplitudes, and \( V_{1,2} \) and \( I_{1,2} \) the voltage and current at the left or right TL-dimer contacts satisfy the continuity relation

\[
V_{1,2} = V_{LR}^+ + V_{LR}^-; \quad I_{1,2} = \frac{[V_{LR}^+ - V_{LR}^-]}{Z_0}
\]  

which connect the wave components to the currents and voltages at the TL-dimer contact points. Note that with this convention, a positive lead current flows into the left circuit, but out of the right circuit, and that the reflection amplitudes for left or right incident waves are defined as \( r_L \equiv V_L^- / V_L^+ \) and \( r_R \equiv V_R^- / V_R^+ \) respectively.

Application of Kirchoff’s laws at the TL leads allow us to find the corresponding wave amplitudes and reflection. For this analysis, we assume the \( e^{-i\omega t} \) wave convention. For example, the case of the left-attached lead in the lower inset of figure 6 gives

\[
\eta (V_L^+ - V_L^-) = I_L^M - \gamma V_1 - i\omega V_1 \\
V_1 = -i\omega [I_1^M + \mu I_2^H], \quad V_2 = -i\omega [I_2^M + \mu I_1^H] \\
0 = I_L^M + \gamma V_2 - i\omega V_2
\]  

(18)

where \( \eta = \sqrt{LC/Z_0} \) is the dimensionless TL impedance, and \( I_{1,2}^M, I_{1,2}^H \) are the current amplitudes in the left or right inductors. These are equivalent to the simple dimer form equations (1) and (2) with the addition of the contact current and the opposite sign convention for \( i \) more appropriate for the traveling wave analysis. Similar equations apply for the right-attached case shown in the upper inset of figure 6. We are interested in the behavior of the reflectance \( R_{LR} \equiv |r_{LR}|^2 \), as the gain/loss parameter \( \gamma \), and the frequency \( \omega \) changes.

For \( PT \)-symmetric structures, the corresponding scattering signals satisfy generalized unitarity relations which reveal the symmetries of the scattering target. Specifically, in the single-port set up this information is encoded solely in the reflection. To unveil it, we observe that the lower set-up of figure 6 is the \( PT \)-symmetric replica of the upper one. Assuming therefore that a potential wave at the left lead (lower inset) has the form \( V_L(x) = \exp(ikx) + r_L \exp(-ikx) \) (we assume \( V_L^+ = 1 \) and \( V_L^- = r_L \) in equation (17)), we conclude that the form of the wave at the right lead associated with the upper circuit of figure 6 is \( V_R(x) = \exp(-ikx) + r_r \exp(ikx) = V_R^+(x) \). Direct comparison leads to the relation

\[
r_L \cdot r_r = 1 \rightarrow R_L = 1/R_R \quad \text{and} \quad \phi_L = \phi_R
\]  

(19)

where \( \phi_{LR} \) are the left/right reflection phases. Note that equation (19) differs from the more familiar conservation relation \( R = 1 \), which applies to unitary scattering processes as a result of flux conservation. In the latter case left and right reflectances are equal. Instead in the \( PT \)-symmetric case we have in general that \( R_L \neq R_R [31] \).

For the specific case of the \( PT \)-symmetric dimer, we can further analytically calculate the exact expression for the reflection coefficients. From equations (18) we have

\[
r_L(\omega) = -\frac{f(-\eta, -\gamma)}{f(\eta, -\gamma)} \\
r_R(\omega) = -\frac{f(-\eta, \gamma)}{f(\eta, \gamma)}
\]  

(20)

\[
f = 1 - (2 - \gamma m(\gamma + \eta))\omega^2 + m\omega^4 - i\eta\omega(1 - m\omega^2)
\]

with \( m = 1/\sqrt{1 - \mu^2} \).
In the limiting cases of $\omega \to 0, \infty$ the reflection amplitude becomes $r_R \to \mp 1$ and thus unitarity is restored.

In the main panel of figure 6 we plot the reflection coefficients of equation (26) for the two scattering configurations shown in the sub-panels. The measured reflectances $R_L$, and $R_R$ satisfy the generalized conservation relation $R_L \cdot R_R = 1$ as expected from equation (19). The slight deviation from reciprocity in the vicinity of large reflectances can be attributed to nonlinear effects.

A peculiarity of our results is the appearance of a singularity frequency point $\omega(\mu, \gamma)$ for which $R_R \to \infty$, while a reciprocal point for which $R_L = 0$ is also evident. The corresponding $(\omega, \gamma)_{\infty,0}$ are found from equation (20) to be

$$\gamma_{\infty,0} = \frac{1}{2} \left( \sqrt{\eta^2 + \frac{4\mu^2}{(1 - \mu^2)} \mp \eta} \right); \quad \omega = \frac{1}{\sqrt{1 - \mu^2}}. \tag{21}$$

Therefore, our experiment demonstrates that a $\mathcal{PT}$-symmetric load is a simple electronic Janus device that for the same values of the parameters $\omega$, $\mu$, $\gamma$ acts as a perfect signal absorber as well as a signal amplifier, depending on the side (gain or loss) that the TL is coupled to the dimer.

For the more general case of a two-port $\mathcal{PT}$ scattering, it was shown theoretically in [22] and later on confirmed experimentally in [30] that the following conservation relation holds:

$$\sqrt{R_L R_R} = |T - 1|. \tag{22}$$

Equation (19) is a special case of equation (22) once we realize that in the single port case the transmittance $T = 0$.

### 6. Two-port CPA-amplifier

Recent theoretical studies in the optics framework [17] have suggested that a two-port $\mathcal{PT}$-symmetric cavity can act as a simultaneous CPA-laser. In this section we provide the first experimental realization of this proposal using a two-port configuration of our $\mathcal{PT}$-symmetric electronic dimer, and demonstrate it’s action as simultaneous CPA-amplifier. We consider the capacitively coupled case, with $c = C_i/C$ as previously defined, to demonstrate the independence of the generic behavior from the coupling mechanism. Following steps similar to the single-port case, Kirchoff’s laws lead to the following set of equations:

$$\eta(V_R^+ - V_L^-) + (V_L^+ + V_L^-) \left[ i\omega(1 + c) + \frac{1}{i\omega} + \gamma \right] - i\omega(V_R^+ + V_R^-) = 0 \tag{23}$$

$$-\eta(V_R^+ - V_L^-) + (V_R^+ + V_R^-) \left[ i\omega(1 + c) + \frac{1}{i\omega} - \gamma \right] - i\omega(V_L^+ + V_L^-) = 0. \tag{24}$$

The above equations can be written in a more elegant form by making use of the transfer matrix formulation:

$$\begin{pmatrix} V_R^+ \\ V_R^- \end{pmatrix} = \mathcal{M} \begin{pmatrix} V_L^+ \\ V_L^- \end{pmatrix}; \quad \mathcal{M} = \frac{1}{2\omega \eta} \begin{pmatrix} A + iB & iC \\ -iD & A - iB \end{pmatrix} \tag{25}$$

where the transfer matrix elements $\mathcal{M}_i$ are $A = 2\eta \Omega$, $B = \Omega^2 - \eta^2 - \omega^2 c^2 + \gamma^2$, $C = (\gamma - \eta)^2 + \Omega^2 - \omega^2 c^2$, $D = (\gamma + \eta)^2 + \Omega^2 - \omega^2 c^2$, and $\Omega = \omega(1 + c) - 1/\omega$. One can further express the spectral transmission and reflection coefficients for left (L) and right (R) incidence in terms of the transfer matrix elements as [25, 26]

$$t_L = t_R \equiv t = \frac{1}{M_{22}}, \quad r_L = -\frac{M_{21}}{M_{22}}, \quad r_R = \frac{M_{12}}{M_{22}}. \tag{26}$$
where we have used the identity that det$(M) = 1$.

An alternative formulation of the transport problem utilizes the so-called scattering matrix $S$ which connects incoming to outgoing waves and its elements can be written in terms of the transmission/reflection coefficients. Specifically

$$
\begin{pmatrix}
V_R^+ \\
V_L^+
\end{pmatrix} = S
\begin{pmatrix}
V_R^- \\
V_L^-
\end{pmatrix}; \quad S = \frac{1}{M_{22}} \begin{pmatrix} 1 & M_{12} \\ -M_{21} & 1 \end{pmatrix}.
$$

Using the scattering matrix language one can derive conditions in order our $\mathcal{PT}$-symmetric structure to act either as an amplifier or as a perfect absorber. For a laser oscillator without an injected signal, the boundary conditions $V_R^+ = V_R^- = 0$ apply, which imply from equation (27) the condition $M_{22}(\omega) = 0$ [26]. In contrast, for a perfect absorber the boundary conditions, $V_L^- = V_R^+ = 0$, corresponding to zero reflected waves, hold. From equation (27) this implies $M_{11}(\omega) = 0$, while the amplitudes of the incident waves must satisfy the condition $V_L^- = M_{21}(\omega)V_R^+$. In general, the condition for an amplifier/laser system, is not satisfied simultaneously with the condition for a perfect absorber. However for any $\mathcal{PT}$-symmetric structure, one can show from equation (25) that the matrix elements of $M$ satisfy the relation $M_{22}(\omega) = M_{11}(\omega^*)$ [17]. As a result, a real $\omega = \omega_j$ exists, that satisfies the amplifier/laser condition simultaneously with the absorber condition $(M_{22}(\omega_j) = M_{11}(\omega_j) = 0)$. Hence the two-port $\mathcal{PT}$-symmetric dimer can behave simultaneously as a perfect absorber and as an amplifier. This property can be explored using an overall output coefficient $\Theta$ defined as [17]

$$
\Theta = \frac{|V_R^+|^2 + |V_L^-|^2}{|V_L^+|^2 + |V_R^-|^2}.
$$

Note that in the case of a single-port scattering set-up discussed earlier in this section, the $\Theta$-function collapses to the left/right reflectances. We can further simplify the above expression using equation (25), together with the property det$(M) = 1$. We get

$$
\Theta(\omega) = \frac{|V_R^- M_{12}(\omega) + 1|^2 + |V_L^- M_{21}(\omega)|^2}{(1 + |V_L^-|^2) |M_{22}(\omega)|^2}.
$$

At the singularity frequency point $\omega = \omega_j$ and for a generic ratio $V_R^+ / V_L^-$, the $\Theta(\omega)$-function diverges as $\omega \to \omega_j$ and the circuit acts as an amplifier/laser. If on the other hand, we assume that $V_R^- = M_{21}(\omega)V_L^+$ (perfect absorption condition), we get

$$
\Theta(\omega_j) = \frac{|M_{21}(\omega_j) M_{12}(\omega_j) + 1|^2}{(1 + |M_{21}(\omega_j)|^2)|M_{22}(\omega_j)|^2}
= \frac{|M_{22}(\omega_j) M_{11}(\omega_j)|^2}{(1 + |M_{21}(\omega_j)|^2)|M_{22}(\omega_j)|^2} = 0.
$$

In the context of electronics, the two port simultaneous laser/absorber properties are manifest as a delicate balance of the driven, marginally stable circuit. The singular behavior of the theoretical $\Theta$ in figure 7(a), solid curves, illustrate that at the Janus frequency $\omega_j$ the injected signals can result in either amplification or complete attenuation, depending on the relative amplitude and phase of the injected signals. The perfect absorption condition is particularly sensitive to the injection parameters: the deviation of the experimental data, figure 7(a) dots, is characteristic of component imbalance of less than 1%. In fact, the minimally absorbing experimental points near the dip in the attenuation curve of figure 7(a) can only be obtained by an independent determination of the minimal reflectance condition at each frequency. Figure 7(b) shows this extreme sensitivity to the phase of the right input signal near $\omega_j$ and illustrates our current experimental limits to the observation of the Janus condition.
Figure 7. (a) The overall output coefficient $\Theta_1(\omega)$ around the Janus amplification/attenuation frequency $\omega_J$ (vertical dashed line) for a $PT$-symmetric electronic circuit coupled to two ports. The parameters used in this simulation are $\eta = 0.110$, $\gamma = 0.186$ and $\epsilon = 0.161$. The red curve corresponds to the two port coherent input excitation with $V^-_R = M_21(\omega)V^+_r$; the blue curve correspond to a two-port input signal with $V^-_R = V^+_r$. In the former case the system acts as an perfect attenuator while in the latter as an amplifier. The dots are experimental values. (b) Plots of experimental $\Theta_1(\omega_J)$ as the loss side input excitation phase is changed, for several excitation amplitudes. Note the extremely sharp dependence at the Janus condition.

7. Practical considerations

Although the fundamental theoretical aspects of $PT$ electronic circuits is straightforward, it is important to realize that all physical electronic elements deviate from their ideal intended function in two distinctly different ways. First, all have unintentional or stray impedances—resistive and reactive components—that can become significant as frequency changes. Second, all components, particularly amplifiers, are subject to linearity limits.

The experimental dimer, equivalent to that shown in figure 2 with either the inductive or capacitive coupling, consists of a pair of coupled LC circuits, one with amplification in the form of the negative resistance, and the other with equivalent attenuation. The circuit was shown in [1] to be a simple realization of the $PT$-symmetric dimer. Each inductor is wound with 75 turns of #28 copper wire on 15 cm diameter PVC forms in a $6 \times 6$ mm loose bundle for an inductance of $L = 2.32$ mH. The coils are mounted coaxially with a bundle separation adjusted for the desired mutual inductance $M$. The isolated natural frequency of each coil is $\omega_0 = 1/\sqrt{LC} = 2 \times 10^5$ s$^{-1}$.

The actual experimental circuit includes several additions to that of figure 2 acknowledging the physical realities mentioned above. First, a resistive component associated with coil wire dissipation is nulled by an equivalent ohmic gain component applied in parallel to each coil. A discussed in section 2, it is not possible to directly apply a series ohmic gain for this compensation. This is our dominant deviation from ideal behavior, however, we have determined from simulation that compensating for this series loss by a parallel gain has negligible impact on the ideal $PT$ behavior.

Second, additional LF356 op-amps, also used for the negative impedance converters of figure 1(a), are used for voltage followers to buffer the voltages $V_1$ and $V_2$ of figure 2, allowing for a less intrusive capture with the Tektronix DPO2014 oscilloscope used for signal acquisition.

Finally, small capacitance and gain trims are included to aid in circuit balancing.

Our linearity is constrained by the LF356 op-amps in the negative impedance converters of figure 1(a). At the $f \sim 30$ kHz operating frequency, the limits were consistent with
the ±12 V supply voltage used for the circuit. In fact, the op-amp linearity limited the overall operation frequency of the dimer: higher frequency op-amps are available, but their linearity and input impedance suffer.

The linear nature of our system allows an exact balance of the $PT$ symmetry only to the extent that component drift over a time scale necessary to perform a measurement is negligible. In the exact phase, real system modes are not perfectly achievable: in time, any physical linear system ultimately either shrinks to zero or exponentially grows to the physical linearity limit. In the case of our dimer, component precision and drift dictate the accuracy of the $PT$ balance to approximately 0.1%, and all data was obtained respecting the linearity limits and associated transient time scales.

Experimental practice allows for only a marginal determination gain/loss balance. The chosen gain/loss parameter $\gamma = R^{-1}\sqrt{L/C}$ is set by the loss-side resistance $R$ of figure 2, typically in the range 1–10 kΩ for this work. The gain-side $R$ and capacitance balance are set with the help of the gain and capacitance trims. In the exact phase, not too close to $\gamma_{PT}$, the system is trimmed for simultaneous marginal oscillation of both modes with growth or decay times greater than $\sim 1$s, where data is then obtained. The imaginary frequency component is then zero to within $\sim 1$s$^{-1}$.

Very close to the critical point, $\gamma \sim \gamma_{PT}$ attempts to trim the dimer to the marginal configuration result in either $V = 0$ (the gain too small), or a chaotic interplay of the two modes with the op-amp nonlinearity if the gain is larger. This behavior serves as an indication that the critical point has been exceeded. In the vicinity of $\gamma_{PT}$ and beyond, the capacitance trim is kept fixed at its asymptotic value, and the gain trim is numerically set to compensate for any measured deviation of the gain side $R$ chosen from the desired value. The exponential growth or decay rate of transient data obtained then directly gives us the imaginary component. Beyond the $PT$ point, the exponentially growing mode always dominates.

At this point, these experimental techniques ultimately impact the limits to which the theory is applicable, particularly in the vicinity of the symmetry breaking point where small imbalances can drastically impact the dynamics. We anticipate that, due to the stabilizing nature of resistive loads in the form of TLs, the $PT$ dimer will provide many opportunities for incorporation into scattering configurations.

8. Conclusions

The $PT$-symmetric dimer opens a new direction toward investigating novel phenomena and functionalities of $PT$-symmetric systems in the spatio-temporal domain via electronic circuits. This minimal example, which is experimentally simple and mathematically transparent, displays all the universal phenomena encountered in systems with generalized $PT$-symmetries. The direct accessibility to all the dynamical variables of the system enables insight and a more thorough understanding of generic $PT$-symmetric behavior. In addition, we envision new opportunities for inclusion of $PT$ electronics into structures including (nano)-antenna configurations, metamaterials, or microresonator arrays with electronic control over directional signal transmission capabilities and real-time manipulation in the spatio-temporal domain.

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Taming the flow of light via active magneto-optical impurities

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Abstract: We demonstrate that the interplay of a magneto-optical layer sandwiched between two judiciously balanced gain and loss layers which are both birefringent with misaligned in-plane anisotropy, induces unidirectional electromagnetic modes. Embedding one such optically active non-reciprocal unit between a pair of birefringent Bragg reflectors, results in an exceptionally strong asymmetry in light transmission. Remarkably, such asymmetry persists regardless of the incident light polarization. This photonic architecture may be used as the building block for chip-scale non-reciprocal devices such as optical isolators and circulators.

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5. There are several ways to address the problem with absorption. One approach is to replace a uniform magnetic material with a slow-wave magneto-photonic structure [6]. Under certain conditions, such a structure can enhance asymmetric transmittance effects associated with magnetism, while significantly reducing absorption. The problem with the above approach is that it does not apply to infrared and optical frequencies it can only work at MW frequencies. Another approach is to incorporate gain and loss together with non-linearity [19]. In this case, however, optical isolation occurs only for specific power ranges.

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causes deviation of the transmitted light polarization from linear to elliptic (non-reciprocal circularly affect the functionality of the optical devices. For example, an enhanced absorption effects. The most deleterious of all is absorption which under resonance conditions can dra-
structure with feature sizes comparable to the light wavelength. At the same time, such res-
incorporate the magneto-optical material into an optical resonator, which can be a photonic
of novel classes of non-reciprocal integrated photonic devices
1. Introduction
The present rapid development of global communications and computer science demands const-
whereas the progress on PICs is still quite limited. One challenging task that researchers are cur-
mently facing is the realization of novel classes of non-reciprocal integrated photonic devices
At optical frequencies, all non-reciprocal effects (NRE), such as magnetic Faraday rotation,
are very weak. This weakness of NRE results in prohibitively large size of most non-reciprocal
devices. A natural way to enhance a weak NRE, and thus reduce the size of a structure, is to in-
corporate the magneto-optical material into an optical resonator, which can be a photonic
structure with feature sizes comparable to the light wavelength. At the same time, such re-
sonators cause undesirable effects, like enhanced absorption, linear birefringence, and nonlinear
effects. The most deleterious of all is absorption which under resonance conditions can dra-
matically affect the functionality of the optical devices. For example, an enhanced absorption causes deviation of the transmitted light polarization from linear to elliptic (non-reciprocal cir-
cular dichroism), which significantly compromises the performance of optical isolator. In addition, the enhanced absorption can result in a significant power loss. Finally, even moderate absorption can lower the quality factor of the optical resonator by several orders of magnitude and, thereby, significantly compromise its performance as a Faraday rotation enhancer [5].

In this Letter we show that a strongly asymmetric transport at infrared and optical frequencies can be achieved in active magneto-photonic structures in which the spatial distribution of gain and loss displays a special, anti-linear symmetry (see Fig. 1). In classical optics, it involves the combination of delicately balanced gain and loss regions together with the modulation of the index of refraction [7]. A sub-class of such optical antilinear structures are the so-called parity-time ($\mathcal{PT}$) symmetric media for which the complex index of refraction obeys the condition $n(\vec{r}) = n^*(\vec{r})$. Synthetic materials with $\mathcal{PT}$ symmetries are shown to exhibit several intriguing features some of which have been already demonstrated in a series of recent experimental papers [9–13]. These include among others, power oscillations [7–10], absorption enhanced transmission [11], unidirectional transparency and invisibility [12, 14], non-reciprocal Bloch oscillations [15, 16], a new type of conical diffraction [17] and reconfigurable Talbot effects [18]. In the nonlinear domain, such asymmetric transmittance effects can be used to realize a new generation of optical on-chip isolators and circulators [19]. Other results include the realization of coherent perfect laser-absorber [20–22] and nonlinear switching structures [23].

![Fig. 1. (a) Scattering setup of a micro-cavity with $\mathcal{PT}$- symmetric gain/loss distribution and asymmetric transport. The left/right (green/red) slab is a lossy/gain non magnetic layer with in-plane anisotropy. The middle (arsenic) slab is a passive ferromagnetic material with magnetization $M_0$ (indicated with the arrows inside the layer). (b) A generalized $\mathcal{PT}$- symmetric micro-cavity embedded in an anisotropic Bragg reflector.](image-url)

The micro-cavity that we investigate (see Fig. 1(a)) is a generalization of the standard $\mathcal{PT}$-symmetric structures and consists of three components: a central magnetic layer sandwiched between two active (one with gain and the other with loss) anisotropic layers distributed in a way that the whole structure exhibits an antilinear symmetry. The magnetic layer provides a nonreciprocal circular birefringence (magnetic Faraday effect). The role of the magnetic layer is to break the Lorentz reciprocity, which would otherwise impose the symmetry in forward and backward transmission of the layered structure. The magnetic circular birefringence does break the Lorentz reciprocity, but it is not sufficient to provide asymmetry in forward and backward transmission. Another requirement is a broken space inversion symmetry. This is achieved with the use of misaligned birefringent layers as suggested in [24]. Even then, the asymmetry in forward and backward transmission averaged over the input light polarization remains symmetric. In this report, we show that the interplay of the active birefringent elements with the magnetooptical layer can lead to strong asymmetry in forward and backward transmission. Although in such micro-cavities the local unitarity is violated, the balanced gain/loss design that we incor-
porated in our structure imposes generalized unitary relations for the energy flux conservation. The transport asymmetry can be drastically enhanced by embedding our micro-cavity inside an anisotropic Bragg grating (see Fig. 1(b)). We show the formation of broad frequency domains at the pseudo-gaps of the grating which support high-Q micro-cavity modes with enhanced transport asymmetry. The proposed architecture can be used for the creation of highly efficient on-chip non-reciprocal devices such as optical isolators and circulators.

2. Modeling and symmetries

We consider the structure shown in Fig. 1(a). The non-Hermitian permittivity tensor, \( \hat{\varepsilon}(z) \) is assumed to be

\[
\hat{\varepsilon}(z^+) = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{bmatrix}, \quad \hat{\varepsilon}(z^-) = \begin{bmatrix} \varepsilon_{xx}^* & \varepsilon_{xy}^* & 0 \\ \varepsilon_{yx}^* & \varepsilon_{yy}^* & 0 \\ 0 & 0 & \varepsilon_{zz}^* \end{bmatrix},
\]

(1)

where the variables \( z^+ \) take values in the intervals \( -L \leq z^- \leq 0 \) and \( 0 \leq z^+ \leq L \). Above, \( \varepsilon_{xx} = \varepsilon(z) + \delta(z) \cos(2\phi(z)) + i\gamma(z), \varepsilon_{xy} = \delta(z) \sin(2\phi(z)) + i\alpha(z) \) and \( \varepsilon_{yy} = \varepsilon(z) - \delta(z) \cos(2\phi(z)) + i\gamma(z) \). The function \( \delta \) describes the magnitude of in-plane anisotropy, \( \gamma \) is the gain/loss parameter and the angle \( \phi \) defines the orientation of the principle axes in the \( xy \)-plane. The gyrotropic parameter \( \alpha \) is responsible for the Faraday rotation. Outside the scattering region \( z \notin [-L,L] \), we assume that the permittivity takes a constant value \( \varepsilon_0 \) i.e \( \hat{\varepsilon} = \varepsilon_0 \times \mathbf{I} \) where \( \mathbf{I} \) is the \( 2 \times 2 \) identity matrix.

The corresponding permeability tensor, \( \hat{\mu}(z) \) takes the form

\[
\hat{\mu}(-L \leq z \leq L) = \begin{bmatrix} \mu_{xx} & \mu_{xy} & 0 \\ \mu_{yx} & \mu_{yy} & 0 \\ 0 & 0 & \mu_{zz} \end{bmatrix}; \quad \hat{\mu}(z \notin [-L,L]) = \mathbf{I}
\]

(2)

where \( \mu_{xx} = \mu(z) = \mu(-z), \mu_{yy} = i\beta(z) = i\beta(-z) \) and \( \beta \) is another gyrotropic parameter which essentially depends on the static components of the magnetic field \( \hat{H}_0 \), as well as the frequency \( \omega \).

Changing \( \hat{H}_0 \rightarrow -\hat{H}_0 \) and \( \hat{M}_0 \rightarrow -\hat{M}_0 \) implies the following transformation [24]

\[
\hat{\varepsilon} \rightarrow \hat{\varepsilon}, \quad \hat{\mu} \rightarrow -\hat{\mu}, \quad \alpha \rightarrow -\alpha, \quad \beta \rightarrow -\beta
\]

(3)

which allows us to conclude that the micro-cavity of Fig. 1(a) is invariant under the combined \( \mathcal{PT} \) antilinear symmetry:

\[
\mathcal{PT} \hat{\varepsilon}(z)(\mathcal{PT})^{-1} = \hat{\varepsilon}(z)
\]

\[
\mathcal{PT} \hat{\mu}(z)(\mathcal{PT})^{-1} = \hat{\mu}(z)
\]

(4)

The time reversal operator \( \mathcal{T} \) is an anti-linear operator which performs transpose complex conjugation while the linear operator \( \mathcal{P} = \mathcal{PT} \Theta \) consist of the parity operator \( \mathcal{P} \) which represents a spatial inversion \( \hat{r} \rightarrow -\hat{r} \) (note though that in our case spatial inversion is the same as reflection), and the exchange operator \( \Theta \) which changes \( \phi_+ \equiv \phi(z^+) \leftrightarrow \phi_- \equiv \phi(z^-) \). We note that in the case of \( \phi_+ = \phi_- \), i.e. the two layers are not misaligned, the structure of Fig. 1(a) is \( \mathcal{PT} \)-symmetric. For this reason, below we will refer to our structure (with misalignment), as a generalized \( \mathcal{PT} \)-symmetric geometry.

3. Scattering formalism

The electric and magnetic field are designated by the time-harmonic Maxwell equations:

\[
\nabla \times \hat{E}(\hat{r}) = \frac{i\omega}{c} \hat{\mu}(\hat{r}) \hat{H}(\hat{r}), \quad \nabla \times \hat{H}(\hat{r}) = -\frac{i\omega}{c} \hat{\varepsilon}(\hat{r}) \hat{E}(\hat{r})
\]

(5)
with the following solution
\[ \vec{E}(\vec{r}) = e^{i(k_xx + k_yy)} \vec{E}(z), \quad \vec{H}(\vec{r}) = e^{i(k_xx + k_yy)} \vec{H}(z). \] (6)

Assuming normal propagation, i.e. \( k_x = k_y = 0 \), the solutions of Eq. (6) for \( \vec{E}(z) \) in the left (l) and right (r) side of the scattering region, are written in terms of the forward and backward traveling waves:
\[ \vec{E}^{lr}(\alpha, \beta, \phi, z) = A^{lr} e^{ikz} + B^{lr} e^{-ikz}; \] (7)

where
\[ A^{lr} = [A^{lr}(\alpha, \beta, \phi, \phi_+)]^T, \quad B^{lr} = [B^{lr}(\alpha, \beta, \phi, \phi_+)]^T. \] (8)

The corresponding magnetic field \( \vec{H}(z) \) is \( \vec{H}(z) = \frac{1}{2} \vec{E}(z) \), where \( \vec{z} \) is the unit vector in the z-direction.

The \( 4 \times 4 \) transfer matrix \( \mathbf{M} \equiv \mathbf{M}(\alpha, \beta, \phi, \phi_+) \) directly furnishes the relation between the electric field on the left and right sides of the scattering region (below we assume \( c = 1 \) units), i.e.,
\[ \begin{bmatrix} A' \\ B' \end{bmatrix} = \mathbf{M} \begin{bmatrix} A \\ B \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}. \] (9)

Performing the operation \( \mathcal{P} \mathcal{T} \) on the solutions of Eq. (7) results in
\[ \vec{E}^{lr}(\alpha, \beta, \phi, \phi_+) \mathcal{P} \mathcal{T} \left( \vec{E}^{lr}(-\alpha, -\beta, \phi, \phi_-) \right)^* \] (10)

Since our system is invariant under the \( \mathcal{P} \mathcal{T} \)-operation then Eq. (10) is also a solution of the Maxwell equations. Applying once more the transfer matrix \( \mathbf{M}(\alpha, \beta, \phi, \phi_+) \) for the transformed solutions Eq.(10), we get:
\[ \begin{bmatrix} A'(\alpha', -\beta', \phi_+, \phi_-) \\ B'(\alpha', -\beta', \phi_+, \phi_-) \end{bmatrix} = \mathbf{M} \begin{bmatrix} A'(-\alpha, -\beta, \phi_+, \phi_-) \\ B'(-\alpha, -\beta, \phi_+, \phi_-) \end{bmatrix}. \] (11)

It follows from the Eq.(11) and the conjugated form of the Eq.(9) with \( \alpha \rightarrow -\alpha, \beta \rightarrow -\beta, \phi_- \leftrightarrow \phi_+ \) that
\[ \mathbf{M}(\alpha, \beta, \phi, \phi_+) \mathbf{M}^*(\alpha', -\beta', \phi_+, \phi_-) = \mathbf{1}. \] (12)

The transmission and reflection coefficients can be expressed in terms of the transfer matrix elements as
\[ \begin{cases} r' = -M_{12}^1 M_{21}^1, & r' = M_{12}^2 M_{21}^2 \\ t' = M_{11}^1 - M_{12}^2 M_{22}^1 M_{21}^2, & t' = M_{22}^1 \end{cases} \] (13)

which leads us to the conclusion that the transmission/reflection for left or right incidence are not necessarily the same.

From Eqs. (13) one can deduce the form of the scattering matrix \( \mathbf{S} \equiv \mathbf{S}(\alpha, \beta, \phi_+, \phi_-) = \begin{bmatrix} r' & t' \\ t' & r' \end{bmatrix} \), in terms of the \( \mathbf{M} \)-matrix elements. After some straightforward algebra we can show that the scattering matrix should satisfy the following generalized unitary relation:
\[ \mathcal{P} \mathcal{S}^*(-\alpha, -\beta, \phi_+, \phi_-) \mathcal{P} \mathcal{S}(\alpha, \beta, \phi_+, \phi_-) = \mathbf{1} \] (14)

where \( \mathcal{P} = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix} \) is the representation of the parity operator in the channel space. This is a fundamental relation obeyed by \( \mathcal{P} \mathcal{T} \)-symmetric \( \mathbf{S} \) matrices in the presence of magneto-optical
asymmetry quality factor.

4. Asymmetric transport

Next we focus on the transport properties of the simple $\tilde{\mathcal{P}}\mathcal{F}$-symmetric magneto-optical cavity of Fig. 1(a). The misalignment angle $\Delta\phi = \phi_+ - \phi_-$ between the pair of active layers is different from 0 and $\pi/2$ to ensure the asymmetry of forward and backward wave propagation [24]. In the absence of gain and loss, a polarization averaged asymmetric transmission would be prohibited due to the sum rule resulting from the flux conservation and unitarity of the scattering matrix [25]. This prohibition is lifted with the introduction of loss and gain layers in Fig. 1(a).

To confirm the above expectations we present in Fig. 2(a) the result of our numerical simulation for the average (left/right) reflectance $\langle R' \rangle_p$, while in Fig. 2(b) we report the left-right transmittance difference $\langle R^- >_p - \langle R^+ >_p \rangle$ (where $R \equiv |r|^2$ and $T \equiv |t|^2$). In all these simulations the average $\langle \cdot \rangle_p$ is taken over all possible polarizations of the incoming wave. We find that the reflectances and transmittances for left/right incident waves are different from one another. Although an asymmetric left-right reflection is a characteristic property of systems with anti-linear symmetry [14], the asymmetry in the transmittances is a new element which essentially requires the presence of both a magneto-optical material and loss and/or gain materials. To further quantify the asymmetric behavior of our structure we report in Fig. 2(c) the asymmetry quality factor $Q_T$ which is defined as

$$Q_T = \frac{\langle R^- > - \langle R^+ >_p \rangle}{\langle R^+ > + \langle R^- >_p \rangle}. \quad (15)$$

The asymmetry of the $\tilde{\mathcal{P}}\mathcal{F}$-symmetric magneto-optical micro-cavity can be further amplified by embedding it between two identical anisotropic Bragg mirrors, see Fig. 1(b). The anisotropy at the Bragg gratings creates pseudo-gaps at the transmission spectrum as shown in...
Fig. 3(a). We have found that at these frequency windows the non-reciprocity is enhanced. In Fig. 3(b) we report the $Q_T$-factor for a structure shown in Fig. 1(b) with a grating consisting of only 45 layers. The frequency domains of polarization independent asymmetric transport are marked with (green) shadowed areas and coincide with the pseudo-gaps of the grating.

The enhancement of asymmetry can be understood intuitively once we realize that our microring behaves as a high-Q optical resonator filled with magneto-optical material once it is embedded in a Bragg grating. In this case, each individual photon resides in the magnetic material much longer compared to the same piece of magnetic material placed outside the resonator. Since the sign of Faraday rotation is independent of the direction of light propagation, one can assume that the total amount of Faraday rotation is proportional to the photon residence time in the magnetic material.

5. Conclusion

Although the enhancement of non-reciprocal effects, such as Faraday rotation, can be achieved via resonance conditions (see, for example [26–28] and references therein), the same resonance conditions also enhance the absorption, which can ruin the performance of almost any non-reciprocal device. Here we have shown that the use of a judiciously balanced gain and loss unit shown in Fig. 1, or its equivalent, can simultaneously solve two fundamental problems. Firstly, it allows a significant enhancement of the desired non-reciprocal effects without creating the energy loss issue. Secondly, the simultaneous presence of loss/gain components and a magnetic component result in strong polarization-independent transmission asymmetry, which is the defining property of an optical isolator. Our photonic architecture may be used as the building block for chip-scale non-reciprocal devices such as optical isolators and circulators.
Acknowledgments

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Observation of Asymmetric Transport in Structures with Active Nonlinearities

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A mechanism for asymmetric transport which is based on parity-time-symmetric nonlinearities is presented. We show that in contrast to the case of conservative nonlinearities, an increase of the complementary conductance strength leads to a simultaneous increase of asymmetry and transmittance intensity. We experimentally demonstrate the phenomenon using a pair of coupled Van der Pol oscillators as a reference system, each with complementary anharmonic gain and loss conductances, connected to transmission lines. An equivalent optical setup is also proposed.

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Directed transport is at the heart of many fundamental problems in physics. Furthermore it is of importance to engineering where the challenge is to design on-chip integrated devices that control energy and/or mass flows in different spatial directions. Along these lines, the creation of novel classes of integrated photonic, electronic, acoustic or thermal diodes is of great interest and constitutes the basic building blocks for a variety of transport-based devices such as rectifiers, pumps, molecular switches, and transistors.

The idea was originally implemented in the electronics framework, with the construction of electrical diodes that were able to rectify the current flux. This significant revolution motivated researchers to investigate the possibility of implementing this idea of “diode action” to other areas. For example, a proposal for the creation of a thermal diode, capable of transmitting heat asymmetrically between two temperature sources, was suggested in Ref. [1]. Another domain of application was the propagation of acoustic pulses in granular systems [2].

A related issue concerns the possibility of devising an optical diode which transmits light differently along opposite propagation directions. Currently, such unidirectional elements rely almost exclusively on the Faraday effect, where external magnetic fields are used to break space-time symmetry. Generally this requires materials with appreciable Verdet constants and/or large size nonreciprocal devices—typically not compatible with on-chip integration schemes or light-emitting wafers [3]. To address these problems, alternative proposals for the creation of optical diodes have been suggested recently. Examples include optical diodes based on second harmonic generation in asymmetric waveguides [4], nonlinear photonic band-gap materials [5], photonic quasicrystals and molecules [6], or asymmetric nonlinear structures [7]. Most of these schemes, however, suffer from serious drawbacks making them unsuitable for commercial or small-scale applications. Relatively large physical sizes are often needed while absorption or direct reflection dramatically affects the functionality leading to an inadequate balance between figures of merit and optical intensities. In other cases, cumbersome structural designs are necessary to provide structural asymmetry, or the transmitted signal has different characteristics than the incident one.

In this Letter we, experimentally and theoretically, demonstrate a mechanism for asymmetric transport exploiting the coexistence of active elements with distinctive features of nonlinear dynamical systems, such as amplitude-dependent resonances. As a reference model we will use coupled nonlinear electronic Van der Pol (VDP) oscillators [8] with anharmonic parts consisting of a complementary amplifier (gain) and a dissipative conductor (loss) combined to preserve parity-time (PT) symmetry [see Fig. 1(a)]. PT-symmetric structures were inspired by quantum field theories [9]; their technological importance was first recognized in the framework of optics [10], where several intriguing features were found [10–21]. For example, the

FIG. 1 (color online). (a) A nonlinear PT-symmetric electronic dimer. (b) The equivalent optics setup consisting of two nonlinear microcavities, one with gain and another with loss. (c) Gain and loss circuits of the Van der Pol PT-symmetric dimer. (d) Experimental I-V response (circles) for the gain (red) and loss (blue) elements along with the corresponding NGSPICE simulations (solid lines), taken at a frequency of 30 kHz, typical of the active range of the VDP dimer.

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121
theoretical proposal of Refs. [20,21] suggested using non-linearities to induce asymmetric transport. Very recently the idea of creating \( \mathcal{PT} \)-symmetric devices within the electronics framework was proposed and experimentally demonstrated in Ref. [22]. \( \mathcal{PT} \) electronics provides a platform for the detailed scrutiny of many new concepts within a framework of easily accessible experimental configurations [22–24]. Despite all this activity, the vast majority of \( \mathcal{PT} \)-symmetric Hamiltonians introduced in quantum field theory, optics, and electronics have been restricted to conservative anharmonic constituents (if any) with the matched gain and loss exclusively linear (see, however, the theoretical works [25–27]).

Below we exploit the simplicity of the electronic circuitry framework in order to demonstrate with experiment, simulations, and theory, asymmetric transport from \( \mathcal{PT} \)-symmetric structures that belong to a relatively unexplored class of nonlinear systems whose anharmonic parts include the mutually matched gain and loss. We remark, however, that our proposal can be implemented in optics [see Fig. 1(b)] by employing concatenated semiconductor absorber microcavities.

An ideal VDP oscillator has a linear antidamping at low amplitudes which is subsequently overtaken by a cubic dissipation at high amplitudes. In electronics this is an LC oscillator in parallel with a voltage dependent conductance characterized by the \( I-V \) curve \( R(V) = -V/R + bV^3 \) where \( b \) is a nonlinear strength. A negative impedance converter [right circuit of Fig. 1(c)] generates a \(-1/R\) term, and we approximate the cubic turn-around with parallel back-to-back diodes moderated by a resistor. The time-reversed conductance is constructed with the resistor \( R \) and the diode combination interchanged [left circuit of Fig. 1(c)]. The resulting “gain” and “loss” nonlinear conductances refer to their low amplitude character. The respective nonlinear \( I-V \) curves of the gain and loss elements are shown in Fig. 1(d). The slight retrace errors that we observe will be discussed later in connection with the simulations. The \( 0.36R \) in series with the diodes optimizes the experimental-simulation match to the cubic nonlinearity used in the theory. It is important to note that only the parameter \( R \) is used to set the gain or loss parameter \( \gamma = R^{-1/3}L/C \), while the diode turn-on characteristics are fixed. When comparisons are made to theoretical models, the voltage scaling will consequently depend on \( \gamma \).

The schematic of the complete dimer circuit is shown in the circuit of Fig. 1(a). The coupled LC heart of the circuit is identical to that used in a previous work [22] with the gain and loss elements modified by the \( I-V \) nonlinearity. The complementary VDP oscillators are capacitively coupled by \( C_1 \).

Transmission lines (TLs) with impedance \( Z_0 \) are attached to the left (loss) and the right (gain) LC nodes of the dimer to complete the scattering system used to perform our transport measurements. Experimentally, these take the form of resistances \( R_0 = Z_0 \) in series with independent voltage sources, here HP3325A synthesizers, on the right and left sides. The incoming and outgoing traveling wave components associated with a particular TL are deduced from the complex voltages on both ends of \( R_0 \), as sampled by a Tektronix DPO2014 oscilloscope. For example, on the left (lossy) side, with \( V_{LC} \) the voltage amplitude on the left dimer circuit node, and \( V_L \) the voltage amplitude on the synthesizer side of the coupling resistor \( R_0 \), the incident wave on the dimer has a voltage amplitude \( V_L^+ = V_0/2 \) and the outgoing wave has a voltage amplitude \( V_L^- = V_{LC} - V_0/2 \). An equivalent relation for \( V_R^+ \) and \( V_R^- \) holds for the right TL terminal with the ± superscripts interchanged, since they refer to the right or left wave traveling direction regardless of the terminal orientation.

The scattering measurements are performed for a fixed incoming wave amplitude set by \( V_0 \) of the signal generator on either the left or the right side with the other side set to zero. The generator frequency is stepped (up or down), and the three relevant waveforms, \( V_L \) and \( V_{LC} \) on the left and right are simultaneously captured (the \( V_0 \) channel on the transmitted side is zero). Harmonic components of each wave constituent can be independently analyzed for magnitude and relative phase. Instrumentation noise and sample time determine the accuracy of this analysis, which was found to be \(<1\%\).

Circuit behavior was numerically modeled by the ngsPICE simulator [28]. In Fig. 1, circuit analysis was done in the time-domain for the individual gain or loss elements. Using initial dc operating conditions, an oscillating voltage source drives the circuit through a transient regime into steady-state operation, at which point the voltages and currents are recorded. Using SPICE modeling we were able to confirm that the slight retrace errors that are observed in the \( IV \) experimental curves shown in Fig. 1 result from slew effects in the LM356 op-amps serving in the negative impedance converters.

The transmittances \( T(\nu) \) versus the driving frequency \( \nu \) in Fig. 2 are similarly obtained with \( R_0 \) in series with a drive \( V_0 \) standing in for the TL. The steady-state time-domain simulations for the \( \mathcal{PT} \)-symmetric dimer in the scattering configuration are now obtained. Fourier analysis is used to extract the relevant frequency-dependent voltages and currents, which are then used to calculate the scattering parameters. As a check of the accuracy of our numerical approach, we have also extracted the transmittance using a nonlinear harmonic balance circuit analysis [29]. We have confirmed that the results are identical, within numerical accuracy, to the ones obtained from the time-domain analysis. In Fig. 2 we also report the experimental left and right transmittances for the \( \mathcal{PT} \)-symmetric VDP dimer. The overall shape of the measured transmittances reasonably matches the numerical simulations. The deviations are associated with a small parasitic inductive...
The transmittance from left to right $T_{TR}$ associated with the third harmonic arrow at 39.5 kHz shows the position of maximal asymmetry.

The results of the numerical simulations with SPICE for $T_L(T_R)$ transmittances are shown as dashed blue (red) lines. The arrow at 39.5 kHz shows the position of maximal asymmetry.

Upper inset: The ratio between the experimental transmittances associated with the third harmonic $T^{(3)}$ and the fundamental frequency $T^{(1)}$. Here, $\eta = 0.031$ and $\gamma = 0.15$. Lower inset: Experimental $T(v)$ for the same $\eta$ but smaller gain and loss parameter $\gamma = 0.11$. Note that as $\gamma$ decreases, the asymmetry and the transmitted intensity are both reduced.

A striking feature of the results of Fig. 2 is the fact that the transmittance from left to right $T_L(v)$ differs from the transmittance from right to left $T_R(v)$, i.e. $T_L \neq T_R$. The phenomenon is most pronounced in the regions of the resonances distorted by the nonlinearity, indicated by the arrow at 39.5 kHz, and is the main result of this Letter. This asymmetry is forbidden by the reciprocity theorem in the case of linear, time-reversal symmetric systems [30].

In fact, it is not present even in the case of linear $PT$-symmetric structures [23]. At the same time, a conservative nonlinear medium by itself cannot generate such transport asymmetries. Furthermore, we find that increasing the gain or loss parameter $\gamma$ which is responsible for the asymmetric transport, maintains or even enhances the transmitted intensities while it leaves unaffected the resonance position (compare the lower inset of Fig. 2 with the main panel). This has to be contrasted with other proposals of asymmetric transport which are based on conservative nonlinear schemes (see for example Ref. [7]), where increase of asymmetry leads to reduced transmittances.

We have also confirmed via direct measurements that the observed asymmetric transport is not related to the generation of higher harmonics in the output signal. In the upper inset of Fig. 2 we report the ratio $T^{(3)}/T^{(1)}$ between experimental transmittances of the third harmonic to the fundamental. Even harmonics are absent in the transmission spectra due to the nature of VDP anharmonicity, while for higher harmonics $T^{(n>3)}$ the experimental values of $T^{(n)}/T^{(1)}$ are below the noise level of our measurements. These results are also supported by the SPICE simulations (not shown). We emphasize that our definition of a rectifying structure is related to the fact that the transmitted power at fixed incident amplitude and at the same frequency should be sensibly different in the two opposite propagation directions. The role of the nonlinearity is confined primarily to currents internal to the dimer, while their harmonic contribution to the TL signals is suppressed by the resonant nature of the LC elements.

Since the phenomenon is nonlinear, the asymmetry depends on both frequency and amplitude. To quantify its efficiency, we report the rectification factor

$$Q = \frac{T_L - T_R}{T_L + T_R}$$

which is zero for symmetric transport and approaches $\pm 1$ for maximal asymmetry. Some representative experimental rectification factors $Q$ for two different values of $\gamma$ are shown in Fig. 3 together with the SPICE simulations. The measurements and the simulations compare nicely with one another. Note that increasing $\gamma$ broadens the regions in which $|Q|$ is relatively large.

As discussed above the asymmetry between left and right transmittances does not rely on the presence of higher harmonics but rather on the interplay of nonlinear elements with the $PT$ symmetry. To unveil the importance of these features we have developed a theoretical understanding of asymmetric transport by restricting our analysis to the basic harmonic. Application of the first and second Kirchoff’s laws at the TL-dimer contacts allows us to find the current or voltage wave amplitudes $I$, $V$ at the left (L) and right (R) contact. We get

![Experimental and SPICE simulations of the rectification factor $Q$.](image)
\[ \frac{dI_L}{d\tau} = \gamma (1 - V_L^2) \frac{dV_L}{d\tau} + V_L + (1 + c) \frac{d^2 V_L}{d\tau^2} - c \frac{d^2 V_L}{d\tau^2}, \]
\[ \frac{dI_R}{d\tau} = \gamma (1 - V_R^2) \frac{dV_R}{d\tau} - V_R - (1 + c) \frac{d^2 V_R}{d\tau^2} + c \frac{d^2 V_R}{d\tau^2}, \]

where the dimensionless current or voltage amplitudes \( I, V \) at the lead-dimer contacts are defined as \( I_L/R = I_{L/R}/(Z_0 \sqrt{5bR}) \) and \( V = V_{L/R}/\sqrt{5bR} \). The dimensionless time is \( \tau = t/\sqrt{LC} \) and \( \eta = \sqrt{L/C}/Z_0 \) is the dimensionless TL conductance, while we have also introduced the dimensionless capacitance \( \epsilon = C/C_0 \), a measure of the intradimer coupling. Note that Eqs. (2) are invariant under a joint \( P \) (i.e., \( L = R \)) and \( T \) (i.e., \( t \rightarrow -t \)) operation.

At any point along a TL, the current and voltage determine the amplitudes of the right and left traveling wave components. The forward \( V_{L/R}^+ \) and backward \( V_{L/R}^- \) wave amplitudes, and the voltage \( V_{L/R} \) and current \( I_{L/R} \) at the TL-dimer contacts satisfy the continuity relation

\[ V_{L/R} = (V_{L/R}^+ + V_{L/R}^-) e^{-i\omega t} + c.c.; \]
\[ I_{L/R} = (V_{L/R}^+ - V_{L/R}^-) e^{-i\omega t} + c.c. \]

Note that Eqs. (2) contain nonlinear terms on the right-hand side which are responsible for harmonic generation.

\[ (-1)^s i \omega \eta (-V_{L/R}^+ + V_{L/R}^-) = [1 - \omega^2 (1 + c) + i(-1)^s \omega \gamma (1 - |V_{L/R}^+|^2) - (1 + c) \omega^2 + 1] V_{L/R}^+/R_{L/R} + \omega^2 V_{L/R}^-/R_{L/R} \]

where we used the compact notation \( V_{R/L}^{+/-} \) for \( V_{R/L}^{+/-} \) and \( V_L \). The exponent \( s \) above takes the values \( s = 0, 1 \) for \( L \), \( R \) current amplitudes, respectively.

We solve Eq. (4) with the use of a backward transfer map \[31\]. The latter uses the output amplitude \( V_{out} = V_R^+(V_L) \) as an initial condition together with the boundary conditions \( V^R_L(0) = 0 \) for a left (right) incoming wave. Iterating backwards, we calculate the corresponding incident \( V_{in} = V_L^-(V_R) \) and reflected \( V_{refl} = V_L^+(V_R) \) amplitudes for a left (right) incident wave. Representative (\( |V_{in}|^2, |V_{out}|^2 \)) curves are shown in Fig. 4. We have confirmed by numerical integration of the equations of motion [Eqs. (2)] into steady state, that the backward transfer map is accurate to within 1% below an output amplitude of \( \sim 1.15 \). We emphasize that precise agreement between theory and the experimental data is sensitive to the exact form used for the nonlinearity, cubic for the theoretical vs. that shown in Fig. 1 for the experiment, exacerbated by the predominance of the effect in the vicinity of resonance. The associated transmittances are defined as \( T = |V_{out}/V_{in}|^2 \). Straightforward algebra gives:

\[ T_L = \left| \frac{2 \omega \eta \gamma}{\eta \alpha + \gamma (1 - |V_{in}|^2) + \frac{1}{\omega^2} - i \omega (1 + c) + (\omega^2)^2} \right|^2, \]

where \( \alpha = \eta - \gamma (1 - |V_{in}|^2) + (i/\omega) - i \omega (1 + c) \). The transmittance \( T_L \) for a right incident wave is given by the same expression as Eq. (5) with the substitution of \( \gamma \rightarrow -\gamma \) i.e., \( T_R(\gamma) = T_L(-\gamma) = T_L(\gamma) \). Obviously for \( \gamma = 0 \) (linear passive dimer) we have \( T_L = T_R \). We have also checked (inset of Fig. 4) that a nonlinear dimer with uniform distributed gain or loss in both sites, does not result in asymmetric transport. We conclude therefore
that the origin of asymmetric transport is due to the fact that nonlinear resonances are detuned differently for left (i.e., insert from lossy site) and right (i.e., insert from gain site) incident waves, as seen from Eq. (5).

Conclusions.—Using coupled $PT$-symmetric nonlinear VDP oscillators, we have demonstrated experimentally and theoretically an asymmetric wave transport mechanism which is based on the coexistence of nonlinearity and $PT$ symmetry. Contrary to other nonlinear isolation schemes, we find that the transmitted signal remains relatively unpolluted by higher harmonics. At the same time the $PT$ symmetry guarantees a high asymmetric ratio without compromising the intensity of the transmitted signal. Finally it will be interesting to use our simple anharmonic $PT$-symmetric electronic framework in order to experimentally investigate other novel phenomena like $PT$ solitons, nonlinear Fano resonances, and perfect transmission [25,26].

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[29] The nonlinear harmonic balance analysis was performed in Agilent’s proprietary Advanced Design System (ADS) 2011, using the Large Signal S-Parameter package.
Unidirectional Lasing Emerging from Frozen Light in Non-Reciprocal Cavities

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We introduce a class of unidirectional lasing modes associated with the frozen mode regime of non-reciprocal slow-wave structures. Such asymmetric modes can only exist in cavities with broken time-reversal and space inversion symmetries. Their lasing frequency coincides with a spectral stationary inflection point of the underlying passive structure and is virtually independent of its size. These unidirectional lasers can be indispensable components of photonic integrated circuitry.

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In the lasing process, a cavity with gain produces outgoing optical fields with a definite frequency and phase relationship, without being illuminated by coherent incoming fields at that frequency [1]. Instead, the laser is coupled to an energy source (the pump) that inverts the electron population of the gain medium, causing the onset of coherent radiation at a threshold value of the pump. In most cases the first lasing mode can be associated with a passive cavity mode. The latter is determined by the geometry and the electromagnetic constitutive parameters \( \epsilon - \mu \) of the passive cavity. Above the first lasing threshold, lasers have to be treated as nonlinear systems [2], but up to the first threshold they satisfy the linear Maxwell equations with a negative imaginary part of the refractive index, generated by the population inversion due to the pump [1]. This simplification allows for a linear treatment of threshold modes within a scattering matrix formalism [3].

In this Letter, using scattering formalism, we introduce a class of unidirectional lasing modes emerging from frequencies associated to spectrally asymmetric stationary inflection points of the underlying passive photonic structure. A distinctive characteristic of these modes is that, in contrast to traditional Fabry-Perot (FP) resonances, they are virtually independent of the size and geometry of the confined photonic structure [4]. Utilizing these modes we propose to create a Mirrorless Unidirectional Laser (MUL) which emits the outgoing optical field into a single direction. Incorporating a MUL in an optical ring resonator can result in various functionalities. Potential applications include optical ring gyroscopes in which a beat frequency between two oppositely directed unidirectional ring diode lasers is detected to measure the rotation rate, optical logic elements in which the direction of lasing in the ring is the logic state of the device, and optical signal routing elements for photonic integrated circuits where the signals are routed around a ring cavity towards a specific output coupler.

The concept of frozen modes and electromagnetic unidirectionality first emerged within the context of spectral asymmetry of nonreciprocal periodic structures [5–7]. In this regard it was recognized that magnetic photonic crystals satisfying certain symmetry conditions [7, 8] can develop a strong spectral asymmetry \( \omega(\vec{k}) \neq \omega(-\vec{k}) \). An example case of such periodic arrangement is shown in Fig. 1. The basic unit consists of three components: a central magnetic layer “sandwiched” between two misaligned anisotropic layers (red and blue). The misalignment angle \( \phi_1 - \phi_2 \) must be different from 0 and \( \pi/2 \).

The constitutive tensors \( \hat{\epsilon}_{1,2} - \mu \) of the nonmagnetic layers are assumed to be

\[
\hat{\epsilon}_{1,2} = \begin{bmatrix}
\epsilon_A + \delta \cos(2\phi_{1,2}) & \delta \sin(2\phi_{1,2}) & 0 \\
\delta \sin(2\phi_{1,2}) & \epsilon_A - \delta \cos(2\phi_{1,2}) & 0 \\
0 & 0 & \epsilon_{ss}
\end{bmatrix} ; \mu = 1
\]

(1)

where \( \delta \) describes the magnitude of in-plane anisotropy and the angle \( \phi_{1,2} \) defines the orientation of the principle

\[\hat{k}\]
axes in the xy-plane for each of the two nonmagnetic layers. The corresponding constitutive parameters for the magnetic layer are

\[
\begin{bmatrix}
\epsilon_p & i\alpha & 0 \\
-i\alpha & \epsilon_p & 0 \\
0 & 0 & \epsilon_{xx}
\end{bmatrix}
\]

The gyrotropic parameters \(\alpha, \beta\) are responsible for the magnetic Faraday rotation [9]. In the simulations below we use the same set of physical parameters as in [8]. Specifically: \(\epsilon_A = 43.85, \delta = 42.64, \phi_1 = \pi/4, \phi_2 = 0, \epsilon_F = 30.525, \alpha = 0.625, \beta = 1.24317\), and \(\mu_{xx} = 1.29969\). The frequency is measured in units of \(\Omega = cL\), where \(c\) is the speed of light in vacuum and \(L = L_A\) is the thickness of the dielectric layers. The width of the ferromagnetic layer is \(L_F = 0.45L\). Without loss of generality we assume that \(L = 1\).

The dispersion relation \(\omega(\hat{k})\) for the infinite periodic stack of Fig. 1 can be calculated numerically using a standard transfer matrix approach [8, 10]. We find that \(\omega(\hat{k})\) displays asymmetry with respect to the Bloch wave vector \(\hat{k}\). For a given structural geometry, the degree of the spectral asymmetry depends on the magnitude of nonreciprocal circular birefringence of the magnetic layer and linear birefringence of the misaligned dielectric layers. If either of the two birefringences is too small, or too large, the spectral asymmetry becomes small. The choice of the numerical parameters allows us to clearly demonstrate the effect of unidirectional lasing using the simplest example of a periodic layered structure shown in Fig. 1. At optical frequencies, very similar results can be achieved by turning the open cavity of Fig. 1 into a ring-like structure [11].

The property of spectral asymmetry has various physical consequences, one of which is the possibility of unidirectional wave propagation [8]. Let us consider a transverse monochromatic wave propagating along the symmetry direction \(\hat{z}\) of the gyrotropic photonic crystal. The Bloch wave vector \(\hat{k} = \hat{k}_z\), as well as the group velocity \(\vec{v}(\hat{k}) \equiv \partial \omega(\hat{k}) / \partial \hat{k}\) are parallel to \(\hat{z}\). In Fig. 2a we see that one of the spectral branches \(\omega(k)\) develops a stationary inflection point (SIP) for which

\[
\frac{\partial \omega}{\partial k} |_{k=k_0} = 0; \quad \frac{\partial^2 \omega}{\partial k^2} |_{k=k_0} = 0; \quad \frac{\partial^3 \omega}{\partial k^3} |_{k=k_0} \neq 0 \quad (3)
\]

An example of a SIP occurring at \(\omega_0 \approx 5463.5\) is marked in Fig. 2a with a circle. At this frequency there are two propagating Bloch waves: one with \(k_0 \approx 0.613\) and the other with \(k_1 \approx -2.452\). Obviously, only one of the two waves can transfer electromagnetic energy - the one with \(k = k_1\) and corresponding group velocity pointing in the positive \(\hat{z}\) direction i.e. \(\vec{v}(k_1) > 0\). The Bloch eigenmode with \(k = k_0\) has zero group velocity \(\vec{v}(k_0) = 0\) and therefore does not transfer energy. The latter propagating mode is associated with a stationary inflection point Eq. (3) and referred to as the “frozen” mode. Thus a photonic crystal with the dispersion relation similar to that in Fig. 2a displays the property of electromagnetic unidirectionality at \(\omega = \omega_0\). Such a remarkable effect is an extreme manifestation of the spectral asymmetry [8].

Next we turn to the analysis of the open photonic structure and the emergence of unidirectional lasing modes. It can be rigorously shown within semiclassical laser theory that the first lasing mode in any cavity is an eigenvector of the electromagnetic scattering matrix (S-matrix) with an infinite eigenvalue; i.e., lasing occurs when a pole of the S-matrix is pulled “up” to the real axis by including gain as a negative imaginary part of the refractive index [3]. We therefore proceed to evaluate the scattering matrix associated with the open photonic structure of Fig. 1.

The electric \(\vec{E}(\hat{r})\) and magnetic \(\vec{H}(\hat{r})\) fields are designated by the time-harmonic Maxwell equations:

\[
\nabla \times \vec{E}(\hat{r}) = i\frac{\omega}{c} \mu \vec{H}(\hat{r}), \quad \nabla \times \vec{H}(\hat{r}) = -i\frac{\omega}{c} \epsilon \vec{E}(\hat{r}) \quad (4)
\]
with the solution
\[ \vec{E}(\vec{r}) = e^{i(k_x x + k_y y)} \vec{E}(z), \quad \vec{H}(\vec{r}) = e^{i(k_x x + k_y y)} \vec{H}(z) \]
where \( \omega(\vec{k}) \) is the frequency. An “external region” encompassing the resonator, extends for \( |z| > L/2 \). We assume that the permittivity and permeability parameters take constant values \( \epsilon = \epsilon_0 \times 1, \mu = \mu_0 \times 1 \) where 1 is the \( 3 \times 3 \) identity matrix.

Assuming normal propagation, i.e. \( k_x = k_y = 0 \), the solutions of Eq. (5) for \( \vec{E}(z) \) in the left (l) and right (r) side of the scattering region, are written in terms of the forward and backward traveling waves:

\[ \vec{E}^{l,r} = A^{l,r} e^{ik_{0} z} + B^{l,r} e^{-ik_{0} z} \]

The corresponding magnetic field \( \vec{H}(z) \) is \( \vec{H}(z) = \frac{1}{2} \hat{z} \times \vec{E}(z) \), where \( \hat{z} \) is the unit vector in the z-direction.

The relation between the electric field on the left and right sides of the scattering region is described by the 4x4 transfer matrix \( \mathbf{M} \):

\[ \begin{bmatrix} A' \\ B' \end{bmatrix} = \mathbf{M} \begin{bmatrix} A \\ B \end{bmatrix} ; \quad \mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \]

The transmission and reflection coefficients can be then expressed in terms of the transfer matrix elements as:

\[ \begin{align*} & r' = -M_{21} M_{22}^{-1}, \quad r = M_{21} M_{22}^{-1} M_{11}, \quad t' = M_{22}^{-1} M_{11} - M_{12} M_{22}^{-1} M_{21}; \quad t = M_{22}^{-1}. \end{align*} \]

From Eqs. (9), we construct the scattering matrix \( \mathbf{S} \)

\[ \begin{bmatrix} B' \\ A' \end{bmatrix} = \mathbf{S} \begin{bmatrix} A \\ B \end{bmatrix} ; \quad \mathbf{S} \equiv \begin{bmatrix} r' & t' \\ t' & r' \end{bmatrix} \]

which, below the laser threshold, connects the outgoing wave amplitudes to their incoming counterparts. For lossless media the permittivity \( \epsilon = \epsilon' + i\epsilon'' \) is strictly real \( \epsilon'' = 0 \). Addition of gain in the system results in a complex permittivity \( \epsilon = \epsilon' + i\epsilon'' \) with \( \epsilon'' < 0 \). Without loss of generality we further assume that the gain is introduced at the dielectric layers with \( \phi_g = 0 \). We have checked that other periodic arrangements of the gain along the cavity give the same qualitative results.

Following Refs. [3, 12] we analytically continue the scattering matrix \( \mathbf{S} \) to the complex \( -k \) plane. The scattering resonances are then defined by the following (outgoing) boundary conditions, i.e. \( (\mathbf{A}', \mathbf{B}')^T \neq 0 \) while \( (\mathbf{A}', \mathbf{B}')^T = 0 \), and are associated with the poles of the \( \mathbf{S} \) matrix. It follows from Eqs. (9,10) that the poles of the \( \mathbf{S} \)-matrix can be identified with the complex zeros of the secular equation \( \text{det}(\mathbf{M}_{22}) = 0 \).

In a passive structure, where \( \epsilon = \epsilon' \) is real, the complex poles of the \( \mathbf{S} \)-matrix \( k_p = k_R - ik_I \) are located in the lower half part of the complex plane due to causality. The real part \( k_R \equiv \Re(k_p) \) is associated with the resonant frequencies while the imaginary part \( k_I \equiv \Im(k_p) \) describes the fact that the cavity is open [12].

In Fig. 2b we report the motion of the \( \mathbf{S} \)-matrix poles in the complex \( k \)-plane. Introducing gain \( \epsilon'' \) to the system leads to an (almost) vertical movement of the poles towards the real axis. This indicates that the lasing frequency \( \Re(k_p) \) of the system is almost equal to the associated mode of the passive structure. At the critical value \( c_{th}'' \) for which the first one of the poles (marked with a circle in Fig. 2b) crosses the real axis, the system reaches the lasing threshold. We have further confirmed the lasing action at \( c_{th}'' \) by evaluating in Fig. 2c an overall response function, defined as the total intensity of outgoing (reflected or transmitted) waves for either a left or right (single-port) injected wave; that is \( \Theta_L, R(\omega) = T_{L,R} + R_{L,R} \) where \( T_{L,R} \) and \( R_{L,R} \) are the respective left and right transmittances and reflectances averaged over polarization. For a lossless passive medium, one always has \( \Theta(\omega) = 1 \) due to power conservation, whereas \( \Theta(\omega) > 1 \) indicates that an overall amplification has been realized. Near the lasing frequency \( \Theta(\omega) \) takes large values, diverging as the lasing threshold is attained. Furthermore in the insets of Fig. 2c we report the left and right transmittances in the regime of regular FP resonances (left inset) and at a SIP-related frozen mode (right inset). While for FP resonances \( T_L \) and \( T_R \) exhibit a moderate asymmetry, for a SIP-related mode the asymmetry between them increases dramatically \( (T_R \gg T_L \) by more than two orders of magnitude) indicating strongly asymmetric transport.

A comparison between Fig. 2a and Figs. 2b,c indicates that the lasing threshold frequency is very close to \( \omega_0 \) associated with the SIP. While, at this frequency, a right-moving propagating wave (associated to \( k_L \) and having large group velocity \( v(k_L) > 0 \) releases most of its energy outside the photonic structure, the mode associated to \( k \approx k_0 \) has extremely small group velocity and allows for a long residence time of the photons inside the structure. The interaction of these photons with the gain medium results in strong amplification which in turn leads to a lasing action. Once the lasing threshold is reached, the outgoing lasing beam is emitted predominantly from the left side of our structure (opposite side from \( v(k_L) > 0 \) ), therefore producing unidirectional lasing.

We have confirmed the asymmetry in overall power amplification by analyzing the ratio \( \Lambda \equiv \frac{T_R + R_R}{T_L + R_L} \). The latter is shown in the right column of Fig. 3. Specifically, large values of \( \Lambda \) indicate asymmetric power amplification towards the left while smaller than unity \( \Lambda \) indicates power amplification towards the right. However the actual indication of the non-reciprocal nature of our photonic structure is provided by an analysis of the
transmittance asymmetry $T_A = T_R/T_L$. This asymmetry exists only in the case that non-reciprocity and gain are simultaneously present.

The magnitude of $T_A$ will be strongly affected by the existence of the SIP. Specifically we expect that the unidirectional amplification near the SIP can be enhanced further by increasing the size of the periodic structure. In this case, the residence time of photons associated with the left moving (slow velocity) mode becomes even longer thus resulting in stronger transmission asymmetries. A scaling analysis of $T_A$ for an increasing number of periods of the grating (see Fig. 3 right panels) indicates that as the system becomes larger the asymmetry becomes increasingly peaked around the unidirectional lasing mode (associated to the SIP), while other picks (associated to standard FP resonances) remain unchanged or even subside. We observe that the unidirectional lasing frequency emerging from the SIP mode remains fixed and it is insensitive to the size of the periodic structure.

Conclusions - We have introduced a qualitatively new mechanism for unidirectional lasing action which relies on the co-existence of highly non-reciprocal SIP-related frozen modes and gain. The transmittance asymmetry of these modes increases significantly with the size of the structure. As opposed to a conventional lasing cavity utilizing asymmetrically placed reflecting mirrors, a nonreciprocal magneto-photonic structure can produce unidirectional lasing even at the first (linear) lasing threshold. This unique feature is not available in reciprocal active cavities. Besides, a unidirectional magneto-photonic structure does not need mirrors or any other reflectors in order to trap light and reach the lasing threshold.

The nonreciprocal layered structure in Fig. 1 is just a toy model of a periodic array capable of developing a spectral SIP. It was rather used here in order to demonstrate the proof of principle for the MUL. A future challenging research direction is to propose realistic structures that allow for the co-existence of gain and nonreciprocal SIP. While at infrared and optical wavelengths, the gain is not a problem, the creation of a well-defined nonreciprocal SIP is not a simple matter and is the subject of a whole new area of research. In this respect, a promising way towards building a unidirectional laser could be to add magneto-optical and gain components to a periodic optical waveguide displaying a SIP. Examples of such waveguides can be found, for instance, in [13], and references therein. This possibility is currently under investigation.

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Optical Asymmetry Induced by $\mathcal{P}\mathcal{T}$–symmetric Nonlinear Fano Resonances

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We introduce a new type of Fano resonances, realized in a photonic circuit which consists of two nonlinear $\mathcal{P}\mathcal{T}$-symmetric micro-resonators side-coupled to a waveguide, which have line-shape and resonance position that depends on the direction of the incident light. We utilize these features in order to induce asymmetric transport up to 47 dBs in the optical C-window. Our set-up requires low input power and does not compromise the power and frequency characteristics of the output signal.

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The realization of micron scale photonic elements and their integration into a single chip-scale device constitute an important challenge, both from a fundamental and a technological perspective [1]. An important bottleneck towards their realization is achieving on-chip optical isolation, that is the control of light propagation in predetermined spatial directions. Standard approaches for optical isolation rely mainly on magneto-optical (Faraday) effects, where space-time symmetry is broken via external magnetic fields. This approach requires materials with high Verdet constants and/or large size non-reciprocal structures which are incompatible with on-chip integration [1]. Alternative proposals, for the realization of optical diodes, include dynamical modulation of the index of refraction [2], the use of opto-acoustic effects [3], and optical non-linearities [4–8]. Most of these schemes, have serious drawbacks which make them unsuitable for small-scale implementation. In some of these cases, complicated designs that provide structural asymmetry are necessary, or the transmitted signal has different characteristics (e.g. different frequency) than the incident one. In other cases, direct reflection or absorption dramatically affects the functionality leading to an inadequate balance between transmitted optical intensities and figures of merit.

Recently, optical microresonator structures [9] with high-quality factors that trigger non-linear effects, have attracted increasing attention as basic elements for the realization of on-chip optical diodes [7, 8]. The basic geometries used consists of two waveguides coupled with two single mode non-linear cavities. These geometries typically allow for a narrow band transmission channel with a symmetric Lorentzian transmittance lineshape. In Ref. [7] the structure was passive and the diode action was imposed due to the asymmetric coupling of the cavities to the waveguides. The drawback of this protocol is that high degree of asymmetric transport (due to strong asymmetric coupling) is achieved at the expense of low intensity output signals. On the other hand, the proposal of Ref. [8] involved active cavities (one with gain and another with loss) which excite the nonlinear resonances differently, depending on the incident direction.

This latter proposal has been demonstrated recently in a beautiful experiment [10] where gain in the first resonator is supplied by optically pumping Erbium ions embedded in a silica matrix while the second resonator exhibits passive loss. However, the degree of transport asymmetry is moderate and it is achieved only for high values of gain.

In this Letter we investigate the possibility to utilize a new type of nonlinear Fano resonances emerging in a parity-time ($\mathcal{PT}$) symmetric framework in order to create asymmetric transport. The proposed photonic circuit consists of two non-linear $\mathcal{PT}$–symmetric microcavities which are side-coupled to a single waveguide. We show that this system naturally exhibits Fano resonances [11] which, due to the interplay of non-linearity with the active elements, are triggered at different resonance frequencies and have different lineshape,
depending on the direction of the incident light. Fano resonances, sometimes behaving as coupled-resonator-induced transparency [12], were first introduced in the optics framework in Refs. [13, 14]. Their shape is distinctly asymmetric, and differs from the conventional symmetric Lorentzian resonance curves (for a recent review see [15]). This asymmetric resonance profile essentially results from the interference between a direct and a resonance-assisted indirect pathway [12].

A realization of the proposed photonic diode is shown in the inset of Fig. 1. For demonstration purposes, let the core of both the waveguide and of the two microdiscs be of AlGaAs material. The permittivity of both microdisk resonators and of the waveguide is taken to be \( \varepsilon = 11.56 \) while the nonlinear Kerr coefficient for the microdiscs is \( \chi = 1e - 19(m^2/V^2) \). The radius of the microdiscs and their distance of each other are 5\( \mu \)m and 770\( \mu \)m, respectively. Moreover, the width of the waveguide and its coupling distance to the resonators are 460\( \mu \)m and 120\( \mu \)m, respectively. The circuit is operated at the optical communication window at wavelength around \( \lambda \approx 1558600 \) with one disc experiencing gain, while the other one having an equal amount of loss described by the imaginary part of the permittivity \( \varepsilon'' = 0.00063 \). The structure is invariant under \( \mathcal{PT} \) symmetry where the \( \mathcal{P} \) is the parity reflection, with respect to the axis of symmetry located at the middle between the two resonators, and \( \mathcal{T} \) is the time reversal operator which turns loss to gain and vice versa. The concept of \( \mathcal{PT} \) symmetry first emerged within the context of mathematical physics. In this regard, it was recognized that a class of non-Hermitian Hamiltonians that commute with the \( \mathcal{PT} \) operator may have entirely real spectra [16]. Lately, these notions have been successfully migrate and observed in other areas like photonic [17–23] and electronic circuitry [8, 24, 25].

In Fig.1 we show some transport simulations using COMSOL modeling. The input power used in the simulations is \( P = 1.2mW \). We find that the left-to-right transmittance \( T_L(\lambda) \) differs from the right-to-left transmittance \( T_R(\lambda) \), i.e. \( T_L \neq T_R \). The asymmetric transport is most pronounced near the Fano resonances \( \lambda_{F,0} \) of the linear structure, and constitutes our main result. We stress that non-reciprocal transport is strictly forbidden by the Lorentz reciprocity theorem in the case of linear, time-reversal symmetric systems [26]. At the same time, it cannot be achieved neither by a conservative nonlinear medium by itself nor by linear \( \mathcal{PT} \)-symmetric structures (see black filled circles in Fig. 1) [25].

Next we analyze the origin of the asymmetry between left and right transmittances, near the Fano resonances \( \lambda_{F,0}^{\text{PT}} \) of the linear photonic circuit of Fig. 1. To understand better its origin, we first discuss the transport characteristics of a single gain (lossy) non-linear microdisc side-coupled to a waveguide. In the case that the incident light traveling the waveguide couples with a gain resonator it will be amplified substantially because of the interaction with the gain medium, and the high \( Q \) factor of the disc. Consequently, the signal has sufficiently high power to trigger the non-linearity and red-shift the disc’s resonance \( \lambda_{F,0}^{\text{PT}} \), thus allowing it to pass with small (or even not at all) attenuation at the resonance wavelength \( \lambda_{F,0}^{\text{PT}} \) of the gain cavity (dashed line in the inset of Fig 2). On the other hand, when light couples to a lossy microdisc, the optical energy stored in this disc is not high enough to appreciably red-shift (via non-linearity) the resonance because of the power reduction due to the losses. As a result the transmittance has a resonance dip at \( \lambda \approx \lambda_{F,0}^{\text{PT}} \). Obviously in both cases discussed here, we have symmetric transport i.e. \( T_L(\lambda) = T_R(\lambda) \).

When both linear microdiscs, i.e. the one with gain (left) and the other one with loss (right), are side-coupled to the waveguide the transmittance shows a peak in the middle of the resonant dip. This phenomenon is an optical analogue [27] of electromagnetically induced transparency (EIT) and it is known as coupled-resonator-induced transparency [12]. It is associated with the interaction between two Fano resonances with spectral widths which are comparable to or larger than the frequency separation between them. These Fano resonances have been formed due to coherent interferences between the two coupled resonators. Still we observe that left and right transmittance are equal i.e. \( T_L(\lambda) = T_R(\lambda) \).

When nonlinearities are considered, the transmission near the Fano resonances is asymmetric and depends strongly on the direction of the incident light. Below we concentrate on the wavelength domain on the left of the transparent window where, for our set up, the asymmetry is stronger. In this case the incident light entering the waveguide from the left is first coupled to the gain resonator which amplify the light intensity; thus inducing optical nonlinearity of the material. As a result, the reso-
The asymmetric transport is maintained for a broad range of input power levels. Dashed lines correspond to $T_L(\lambda)$ (gain side) while solid lines to $T_R(\lambda)$ (lossy side). The transport asymmetry is as high as 46.5 dB without compromising the outgoing optical intensity which is as high as $-5\,\text{dBs}$. The asymmetric transport to the left port through the resonance and experience a significantly red-shift the resonance. Thus light is transmitted the accumulated energy there is not enough to appreciate red-shift the resonance. Therefore the outgoing wavelength which is as high as $-5\,\text{dBs}$.

Eqs. (1) describe the interaction of two subsystems. The first one is a linear chain of couple sites with coupling constant $C$ and on-site complex field amplitudes $\phi_n$. This system supports propagating plane waves with dispersion $\omega(k) = 2C\cos k$. The second subsystem consists of two defect states $| \phi_G(\text{gain}) \rangle$ and $| \phi_L(\text{loss}) \rangle$ with on-site energy $E \mp i\gamma$ respectively. The two subsystems interact with one another at the sites $n = 0, N$ via the coupling coefficients $V_G/L$.

We assume elastic scattering processes for which the stationary solutions take the form $\phi_n = A_n e^{i\omega t}; \phi_G = A_e e^{i\omega t}; \phi_L = A_L e^{i\omega t}$. Substitution in Eqs. (1) leads to

$$\begin{align*}
\omega A_n &= C(A_{n-1} + A_{n+1}) + V_G |A_G|^2 \delta_{n,0} + V_L |A_L|^2 \delta_{n,N} \\
\omega A_G &= E A_G - i\gamma A_G + \chi |A_G|^2 A_G + V_G A_0 \\
\omega A_L &= E A_L + i\gamma A_L + \chi |A_L|^2 A_L + V_L A_N
\end{align*} \tag{2}$$

We consider a left incident wave. In this case we have

$$A_n = \begin{cases} 
I e^{i\omega t} & n \leq 0 \\
A_{n-1} + A_{n+1} & 0 \leq n \leq N \\
N e^{i\omega t} & N \leq n
\end{cases} \tag{3}$$

where $I, r, t$ represent the incident, reflected and transmitted wave amplitudes far from the defect sites. Substituting the above scattering conditions in Eqs. (2) and using the continuity conditions at the defect sites $n = 0, N$, we get after some straightforward algebra

$$r_L = i \frac{V_G A_G + V_L A_L e^{-i\omega N}}{2C \sin q}; t_L = (I + i) \frac{V_G A_G + V_L A_L e^{-i\omega N}}{2C \sin q} \tag{4}$$

The unknown amplitudes $A_G, A_L$ can be found in terms of the input amplitude $I$ by utilizing Eqs. (2, b, c). Specifically we get the following set of nonlinear equations

$$\begin{align*}
(\omega + E - i\gamma) A_G + \chi |A_G|^2 A_G + V_G(I + r_L) &= 0 \\
(\omega + E + i\gamma) A_L + \chi |A_L|^2 A_L + V_L e^{i\omega N} t_L &= 0
\end{align*} \tag{5}$$

which can be solved numerically, after substituting $r_L$ and $t_L$ from Eqs. (4).

The transmittance and reflectance for a left incident wave is defined as $T_L = |t_L|^2$ and $R_L = |r_L|^2$ respectively. In a similar manner one can also define the transmittance $T_R = |t_R|^2$ and reflectance $R_R = |r_R|^2$ for a right incident wave. The associated $T_R, R_R$ are given by the same expressions as Eq. (4, 5) with the substitution of $\gamma \rightarrow -\gamma; V_G \rightarrow V_L$ and $V_L \rightarrow V_G$. 

![FIG. 3](Color online) Transmittance for different input powers. The asymmetric transport is maintained for a broad range of input power levels. Dashed lines correspond to $T_L(\lambda)$ (gain side) while solid lines to $T_R(\lambda)$ (lossy side). The transport asymmetry is as high as 46.5 dB without compromising the outgoing optical intensity which is as high as $-5\,\text{dBs}$. 

The asymmetric transport at the wavelength $\lambda^{PT'}$ of the non-linear $PT$-structure is strongly red-shifted with respect to the resonance wavelength $\lambda^{PT}$. Thus at incident wavelength $\lambda = \lambda^{PT'}$ the photonic circuit of Fig. 1 is almost transparent. Moreover, the outgoing signal is strong due to the amplification at the gain disc and despite the fact that some attenuation will take place at the lossy resonator. For backward propagation, light will first couple to the lossy resonator where it will experience attenuation. When the light reaches the gain microdisc, the accumulated energy there is not enough to appreciably red-shift the resonance. Thus light is transmitted to the left port through the resonance and experience a transmission dip at $\lambda^{PT'}$. Therefore a non-reciprocal light transport at the wavelength $\lambda^{PT'}$ is observed. The asymmetric transport is further amplified due to the fact that the Fano resonance lineshape for a left and right incident waves can be different.

In Fig. 3 we further analyze the dependence of the transmittance on the level of the input power. We find that the circuit is stable to variations of the input power; a feature that is desirable from the engineering perspective. For these simulations we assume an imaginary permittivity index $\epsilon' = 0.00073$ i.e. slightly larger than the one used in Fig. 1. This allow us to enhance further the transport asymmetry to values as high as 46.5 dBs. At the same time the outgoing signal $T_L(\lambda = \lambda^{PT'})$ is further amplified ($\approx -5\,\text{dBs}$) with respect to the one found in Fig. 1 ($\approx -8\,\text{dBs}$). This has to be contrasted with passive protocols where an increase figure of merit for isolation might lead to weaker outgoing signal [6, 7].

The asymmetric transport generated by the interplay of the nonlinear Fano resonances with $PT$-symmetric elements calls for a simple theoretical understanding. The following heuristic model, similar in spirit to the so-called Fano-Anderson model [15, 28] that is used to describe the creation of (non-linear) Fano resonances, provides some quantitative understanding of the COMSOL simulations shown in Figs. 1, 3. Our model is described by the following sets of differential equations:

$$\begin{align*}
\dot{\phi}_n &= -(C(\phi_{n-1} + \phi_{n+1}) + V_G(\phi_G \delta_{n,0} + V_L \phi_L \delta_{n,N}) \\
\dot{\phi}_G &= -(i(E - i\gamma)\phi_G + \chi |\phi_G|^2 \phi_G + V_G \phi_0) \\
\dot{\phi}_L &= -(i(E + i\gamma)\phi_L + \chi |\phi_L|^2 \phi_L + V_L \phi_N)
\end{align*} \tag{1}$$

Spintronic elements calls for a simple theoretical understanding. The following heuristic model, similar in spirit to the so-called Fano-Anderson model [15, 28] that is used to describe the creation of (non-linear) Fano resonances, provides some quantitative understanding of the COMSOL simulations shown in Figs. 1, 3. Our model is described by the following sets of differential equations:

$$\begin{align*}
\dot{\phi}_n &= -(C(\phi_{n-1} + \phi_{n+1}) + V_G(\phi_G \delta_{n,0} + V_L \phi_L \delta_{n,N}) \\
\dot{\phi}_G &= -(i(E - i\gamma)\phi_G + \chi |\phi_G|^2 \phi_G + V_G \phi_0) \\
\dot{\phi}_L &= -(i(E + i\gamma)\phi_L + \chi |\phi_L|^2 \phi_L + V_L \phi_N)
\end{align*} \tag{1}$$

The unknown amplitudes $A_G, A_L$ can be found in terms of the input amplitude $I$ by utilizing Eqs. (2, b, c). Specifically we get the following set of nonlinear equations

$$\begin{align*}
(\omega + E - i\gamma) A_G + \chi |A_G|^2 A_G + V_G(I + r_L) &= 0 \\
(\omega + E + i\gamma) A_L + \chi |A_L|^2 A_L + V_L e^{i\omega N} t_L &= 0
\end{align*} \tag{5}$$

which can be solved numerically, after substituting $r_L$ and $t_L$ from Eqs. (4).

The transmittance and reflectance for a left incident wave is defined as $T_L = |t_L|^2$ and $R_L = |r_L|^2$ respectively. In a similar manner one can also define the transmittance $T_R = |t_R|^2$ and reflectance $R_R = |r_R|^2$ for a right incident wave. The associated $T_R, R_R$ are given by the same expressions as Eq. (4, 5) with the substitution of $\gamma \rightarrow -\gamma; V_G \rightarrow V_L$ and $V_L \rightarrow V_G$. 

![FIG. 3](Color online) Transmittance for different input powers. The asymmetric transport is maintained for a broad range of input power levels. Dashed lines correspond to $T_L(\lambda)$ (gain side) while solid lines to $T_R(\lambda)$ (lossy side). The transport asymmetry is as high as 46.5 dB without compromising the outgoing optical intensity which is as high as $-5\,\text{dBs}$.
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transmission amplitude in Eq. (4) consists of two terms:
asymmetric transport.

away from the Fano resonance regime and thus does not affect
\( \lambda \)

\( T \)

\( \approx \)

\( R \)

\( 0 \)

\( G \)

\( \approx \)

\( 0 \).

The red-shadowed area on the right of the graph around
\( \lambda \approx 4.5 \), corresponds to a bi-stability behavior which however is
away from the Fano resonance regime and thus does not affect asymmetric transport.

An analysis of the structure of Eq. (4) can explain the
origin of Fano resonances. Specifically, we note that the transmission amplitude in Eq. (4) consists of two terms:
the first one is associated with a scattering process
associated with a propagating wave that directly passes
through the chain without coupling to any of the defect states. The second term describes an indirect path for
which the wave will first visit the two defects, thus excit-
ing the Fano states, return back, and continue with the
propagation. These two paths are the ingredients of the
Fano resonances observed in Figs. 1.3.

In Fig. 4 we report a representative set of transmission
curves for a left/right incident wave for the model of Eq.
(1). The model captures the qualitative features and ori-
gin of the asymmetric transport observed in the case of the
photonic circuit of Fig. 1. Specifically, we find that
both the shape and the position of the Fano resonances
depend on the direction of the incident wave. Moreover
for a left (gain-side) incoming wave, a red-shift in the
transmittance resonances is found (see the neighborhood
of the second Fano resonance in Fig. 4) which leads to
an asymmetric transport.

**Conclusions** - In conclusion, we have introduce a new
type of \( \mathcal{PT} \)-symmetric Fano resonances with a line-shape
and a resonance position that depends on the direction
of the incident wave. The photonic circuit that allows
for such resonances consists of two \( \mathcal{PT} \)-symmetric
microdisks side-coupled to a waveguide. The proposed
configuration guarantee not only high asymmetry but also a
significant level of transmittance. Our proposal utilizes
existing materials already used in optical integrated cir-
cuity processing and does not require magnetic fields, or
other external elements like polarizers. The efficiency of
the asymmetric transport for a broad input power range
and low values of input power within which our device
performs may be sufficient for on-chip photonic applica-
tions. A problem with the simple design of Fig. 1 is
that the asymmetric transport only occurs in the vicin-
ity of the Fano resonances (narrow band). This problem
can be addressed by using more sophisticated photonic
structures, for instance, those involving more than one
\( \mathcal{PT} \)-symmetric dimer-like resonators side coupled to the
waveguide bus.

**References**

[26] Reciprocity violations can occur in linear magnetoactive media, see for example: H. Ramezani et al., Opt. Express 20, 26200 (2012).
Appendix B

Complete List of Publications

B.1 Nonlinear $\mathcal{PT}$-symmetric optical diode
Hamidreza Ramezani, T. Kottos, R. El-Ganainy, and D. N. Christodoulides In Frontiers in Optics, OSA Technical Digest (CD) (Optical Society of America, 2010), paper FWG4

B.2 Unidirectional Nonlinear $\mathcal{PT}$-symmetric Optical Structures

B.3 Broad band unidirectional invisibility using $\mathcal{PT}$ symmetry
Hamidreza Ramezani, Zin Lin, Toni Eichelkraut, Tsampikos Kottos, Hui Cao, and Demetrios Christodoulides In Quantum Electronics and Laser Science Conference, OSA Technical Digest (CD) (Optical Society of America, 2011), paper QMA3

B.4 Optical diodes in nonlinear structures with parity-time symmetries

B.5 Unidirectional invisibility induced by $\mathcal{PT}$-symmetric periodic structures
Z. Lin, Hamidreza Ramezani, T. Eichelkraut, T. Kottos, H. Cao and D. N. Christodoulides,
Appendix B. Complete List of Publications


B.6 Ab initio description of nonlinear dynamics of coupled microdisk resonators with application to self-trapping dynamics

B.7 Unidirectional invisibility of photonic periodic structures induced by $\mathcal{PT}$-Symmetric arrangements
T. Kottos, Hamidreza Ramezani, Z. Lin, T. Eichelkraut, H. Cao, D. Christodoulides
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