Chromatic Number and Immersions of Complete Graphs

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For my Grandfathers,
the engineer and the educator.

Abstract

A classic question in graph theory is: Does a graph with chromatic number d "contain" a complete graph on d vertices in some way? In this dissertation we will explore some attempts to answer this question and will focus on the containment called immersion. In 2003 Abu-Khzam and Langston conjectured that every d-chromatic graph contains an immersion of K_d. This conjecture is true for $d \leq 6$ by the work of Lescure and Meyniel in 1989 and for $d \leq 7$ by the work of DeVos, Kawarabayashi, Mohar, and Okamura in 2010. In each case the conjecture was proved by proving the stronger statement that graphs with minimum degree $d-1$ contain immersions of K_d. DeVos, Dvořák, Fox, McDonald, Mohar, and Scheide show this statement fails for $d = 10$ and $d \geq 12$. In this dissertation we show that the stronger statement is false for $d \geq 8$ and give infinite families of examples with minimum degree $d-1$ and chromatic number $d-3$, $d-2$, or $d-1$ that do not contain an immersion of K_d. Our examples can be up to $(d-2)$-edge-connected. We show, using Hajós' Construction, that there is an infinite class of non-$(d-1)$-colorable graphs that contain an immersion of K_d. We conclude with some open questions, and the conjecture that a graph G with minimum degree $d-1$ and more than $\frac{|V(G)|}{m(d+1)-(d-2)}$ vertices of degree at least md has an immersion of K_d.

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Contents

Introduction

In this dissertation we will explore the classic question in graph theory of: if a graph has chromatic number d does the graph "contain" a complete graph on d vertices in some way? Some attempts to answer this question will be explored in Chapter 2, where we will consider subdivisions and minors as possible answers to the classic question. We will see that subdivisions will not work and that there have been difficulties in proving (or disproving) that minor is the answer. Given these difficulties we choose to explore a different type of containment, namely, graph immersion. Immersions will be introduced in Section 2.3. The concept of immersion was introduced by Nash-Williams [20], when he conjectured that graphs can be well-quasi-ordered by immersion. This conjecture was proved by Robertson and Seymour in [21]. In 2003, Abu-Khzam and Langston [1] conjectured that immersion is the answer to the classic question, that is if a graph has chromatic number d, then it has an immersion of K_d. This conjecture is true for $d \leq 7$, though it was proved, first by Lescure and Meyniel [17] then by DeVos et al. [6], by proving a stronger statement involving minimum degree. We will explore their results in Section 2.4. We will see that the method of considering minimum degree cannot be extended to $d \geq 8$ both through our own work and the work in [7] in Chapter 3.

We show graphs with minimum degree $d - 1$ need not have an immersion of

K_d for $d \geq 8$. We do this by giving infinite families of examples of graphs with minimum degree $d - 1$ and no immersion of K_d. We will begin by describing, in general, properties of graphs with minimum degree $d - 1$ and no immersion of K_d. We will do this by first considering a graph with an immersion of K_d, and exploring where the vertices of that immersion must live within the graph. This will be our Corner Separating Lemma, which appears in Section 3.1. In Section 3.2, we will describe a general construction of docks, bays, and pods that form graphs with minimum degree $d-1$ and no immersion of K_d. In Section 3.3, we construct appropriate pods, bays, and docks. In Section 3.4, we compare our examples to those in [7]. Our examples have chromatic number $d-3$, $d-2$ or $d-1$, showing the importance of some chromatic number bound to have an immersion of K_d. Our examples can be up to $(d-2)$-edge-connected. In Chapter 4, we prove that there is an infinite class of d-colorable graphs that contain immersions of K_d. Finally, we conjecture that if a graph G has minimum degree $d-1$ and more than $\frac{|V(G)|}{m(d+1)-(d-2)}$ vertices of degree at least md for any positive integer m, then it has an immersion of K_d.

We will begin by defining necessary terms and setting the notation that will be used in this dissertation along with some useful lemmas about color-critical graphs in Chapter 1.

Chapter 1

Preliminaries

1.1 Definitions and Notation

We will follow the notation established by West in [25]. A **graph**, G, is a vertex set, $V(G)$, and an edge set, $E(G)$, which is a set of unordered pairs of vertices. A graph is **simple** if it has no loops or multiple edges. In this dissertation all graphs will be finite and simple unless otherwise stated. We will denote the **neighborhood** of a vertex u in a graph G by $N_G(u)$. The **degree** of a vertex u, denoted $d_G(u)$, is the number of edges incident with u in G. When it is clear which graph we are talking about we will write $N(u)$ and $d(u)$. The **minimum degree** among the vertices of a graph G is denoted $\delta(G)$ and the **maximum degree** among the vertices of the graph is denoted $\Delta(G)$. If every vertex has the same degree, say k, we call the graph k**-regular**. A **path** is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A **cycle** is a graph with the same number of edges and vertices whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear as consecutive vertices along the circle. A cycle on n vertices will be denoted C_n. The **length** of a path or cycle is the number of edges in the graph. In our work we will consider complete graphs, a graph on d vertices is a **complete graph** if it is a graph whose vertices are pairwise adjacent, denoted K_d. A **matching** in a graph is a set of edges sharing no endpoints.

We will also be interested in coloring the vertices of our graphs.

1.2 Color Critical Graphs

Definition 1.2.1. *A d-**coloring** of a graph is a map $c : V(G) \to \{1, \ldots, d\}$, where labels $1, \ldots, d$ are called colors. A coloring is **proper** if adjacent vertices receive different colors, that is if $uv \in E(G)$, then $c(u) \neq c(v)$. If a proper d-coloring of a graph G exists, then we say G is d-**colorable**. The **chromatic number** of a graph G, denoted $\chi(G)$, is the minimum d such that G is d-colorable.*

It will be useful to understand some facts about color-critical graphs.

Definition 1.2.2. *If $v \in V(G)$ and $\chi(G - v) < \chi(G)$, v will be called a **critical vertex** of G.*

If v is a critical vertex of G, then $\chi(G - v) \geq \chi(G) - 1$, so $\chi(G - v) = \chi(G) - 1$. Also the degree of v is at least $\chi(G) - 1$.

Definition 1.2.3. *If every vertex of G is critical, then G is called a **critical graph**. In other words, a graph G is d-**color-critical** (or d-critical) if $\chi(G) = d$ and $\chi(H) < d$ for all proper subgraphs H of G. Where a proper subgraph is one in which $V(H) \subset V(G)$ and $E(H) \subset E(G)$.*

Note that, critical graphs are connected.

Definition 1.2.4. *A vertex v is a* ***cut-vertex*** *if the graph contains two distinct vertices, different from v, such that they are connected by a path and v lies on every path connecting them. That is, removal of v disconnects the graph.*

Lemma 1.2.5. *A critical graph has no cut-vertex.*

Proof. Let G be a critical graph, so every vertex in G is critical. If G had a cut-vertex, v, then the chromatic number would be the chromatic number of the component of $G - \{v\}$ requiring the most colors. Then removing a vertex from one of the other components would not decrease the chromatic number, so the graph could not be critical. $\square$

Lemma 1.2.6. *A d-chromatic graph always contains a d-color-critical subgraph.*

Proof. Let G be a d-chromatic graph. If G is d-color-critical then we are done. Let G be the smallest, that is has the fewest number of vertices, d-chromatic graph that is not d-color-critical. Then G has a subgraph H that is d-chromatic. Then H must be a d-color-critical graph, if not then G would not be the smallest d-chromatic graph that is not color-critical. $\Rightarrow\Leftarrow$ $\square$

Lemma 1.2.7. *If G is d-color-critical, then $\delta(G) \geq d - 1$.*

Proof. Suppose not, let G be d-color-critical and suppose there exists $u \in V(G)$ such that $d(u) \leq d - 2$. Since G is color-critical, $G - u$ must be $(d-1)$-colorable. Since $d(u) \leq d - 2$, $G - u$ can be colored with $d - 1$ colors, one of which is not used on $N(u)$, so G is $(d-1)$-colorable. $\Rightarrow\Leftarrow$ $\square$

Lemma 1.2.8. *If G is d-color-critical, then for any vertex u there exists a coloring c in which $c(u) = 1$ and $c(v) \neq 1$ for every $v \in V(G - u)$*

Proof. Let G be d-color-critical. Then $G - u$ is $(d-1)$-colorable. Color $G - u$ with the colors $2, 3, 4, \ldots, d$, so no vertex of $G - u$ is assigned color 1. Use this coloring on G with $c(u) = 1$. This gives a coloring of G in which $c(u) = 1$ and $c(v) \neq 1$ for every $v \in V(G - u)$. □

We will discuss the connectivity of our graphs, so the following definitions will be useful.

Definition 1.2.9. *A* ***vertex cut*** *of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component. The* ***connectivity*** *of G is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph is k-**connected** if its connectivity is at least k.*

Definition 1.2.10. *A* ***disconnecting set*** *of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component. A graph is k-**edge-connected** if every disconnecting set has at least k edges. The* ***edge-connectivity*** *of G, is the minimum size of a disconnecting set. A* ***cut-edge*** *is a single edge whose removal separates the graph.*

While most of this dissertation will focus on vertex coloring it will be useful, in Chapter 4, to understand a bit about edge coloring.

Definition 1.2.11. *A k-**edge-coloring** of G is a map $f : E(G) \to \{1, 2, \ldots, k\}$. A k-edge-coloring is* ***proper*** *if incident edges receive different colors. A graph is k-**edge-colorable** if it has a proper k-edge-coloring. The* ***edge-chromatic number*** *of a loop-less graph G, $\chi'(G)$, is the least k such that G is k-edge-colorable. The edge-chromatic number is sometimes called the* ***chromatic index*** *of G.*

Since edges sharing vertices receive different colors, $\chi'(G) \geq \Delta(G)$. Vizing [23] and Gupta [13] independently proved that, for simple graphs, $\chi'(G) \leq \Delta(G) + 1$.

In Chapter 4, we will use the notion of a homomorphism between two graphs. A **graph homomorphism** from a graph G into a graph H is a map $f : V(G) \to V(H)$ such that if $uv \in E(G)$ then $f(u)f(v) \in E(H)$, that is it is a map that preserves adjacency.

Chapter 2

A Classic Question

A classic question in graph theory is: If a graph, G, has chromatic number d, does it contain a complete graph on d vertices in some way? The chromatic number of the complete graph on d vertices is d and the K_d is the smallest graph with chromatic number d, so this is a natural question to ask. The first thing one might consider in trying to answer this question is: does a graph with chromatic number d have a subgraph of K_d? However, the containment is not as a subgraph. For example, an odd cycle has chromatic number three and no subgraph of K_3. In fact, Mycielski [19] gave a construction for graphs with arbitrarily large chromatic number and no subgraph of K_3.

Mycielski's construction is, beginning with a simple graph G having vertex set $\{v_1, v_2, \ldots, v_n\}$, add vertices $\{u_1, u_2, \ldots, u_n\}$ and one more vertex w. Add edges so that u_i is adjacent to all of $N_G(v_i)$, and let $N(w) = \{u_1, u_2, \ldots, u_n\}$. Figure 2.1 shows the Mycielski construction applied to C_5. Mycielski [19] proved that from a d-chromatic triangle-free graph G, Mycielski's construction produces a triangle-free graph G' with $\chi(G') = d + 1$.

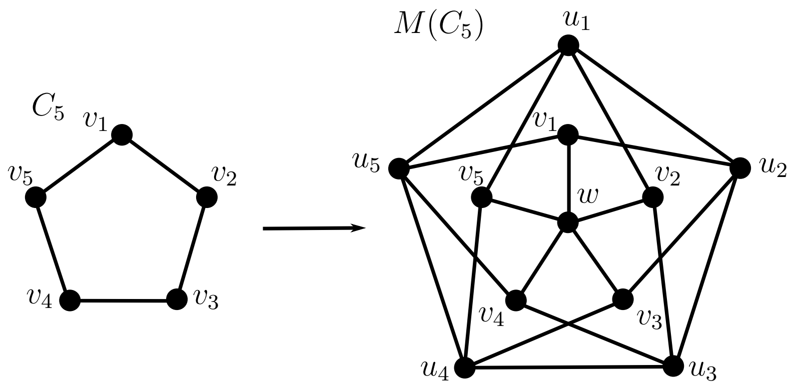

Figure 2.1: Mycielski's construction applied to C_5.

2.1 Minor

Another property to consider in trying to answer this question is graph minor.

Definition 2.1.1. *A graph H is a **minor** of a graph G if a copy of H can be obtained from G by deleting and/or contracting edges of G.*

Figure 2.2 shows the Peterson graph, P, which has a minor of K_5 by contracting edges as, bt, cu, dv, and ew.

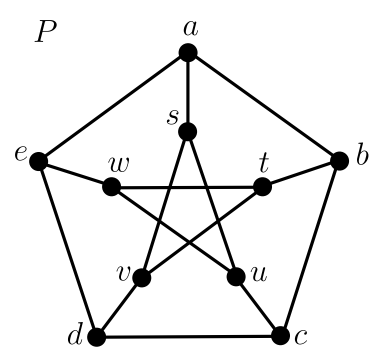

Figure 2.2: P has a minor of K_5.

In 1943, Hadwiger conjectured that minor is the answer to the classic question.

Conjecture 2.1.2 (Hadwiger's Conjecture, [14])**.** *Every loop-less d-chromatic graph has a minor of K_d.*

Hadwiger's Conjecture is true for $d \le 6$. The $d = 1$ and 2 cases are obvious and the cases for $d = 3$ and 4 are true because the statement is true for subdivision which will be discussed in Section 2.2. Wagner [24] showed that the $d = 5$ case of Hadwiger's Conjecture is equivalent to the Four Color Theorem (every planar graph is 4-colorable), so is true by the work of Appel and Haken [3]. Robertson, Seymour, and Thomas [22] proved the case when $d = 6$. The $d = 6$ case came out of the work of Robertson and Seymour in their series of 20 plus Graph Minor papers in which they prove graphs are well-quasi-ordered by the minor relation. The cases $d \ge 7$ are still open. Given the difficulties in proving Hadwiger's Conjecture it seems natural to explore another type of containment.

2.2 Subdivision

By 1950, Hajós conjectured that the necessary containment was subdivision.

Definition 2.2.1. *A graph H is a **subdivision** of a graph G if it is obtained from G by replacing edges with pairwise internally disjoint paths.*

Figure 2.3 shows a subdivision of K_4.

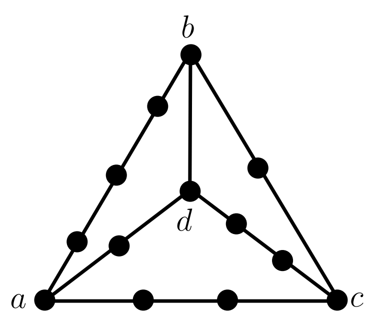

Figure 2.3: A subdivision of K_4.

Notice that, if H is a subdivision of G, then H contains G as a minor, just contract along the paths. However, if H has a minor of G it is not necessarily a subdivision of G, for example the Peterson graph in Figure 2.2 has a minor of K_5, but is not a subdivision of K_5.

Conjecture 2.2.2 (Hajós Conjecture, [15])**.** *A d-chromatic graph always contains a subgraph which is a subdivision K_d.*

This conjecture is trivially true for $d = 1$ and $d = 2$. For $d = 3$ we need the following lemma of Dirac.

Lemma 2.2.3 ([9])**.** *For a d-chromatic graph to contain a subgraph with a subdivision of K_d for $d \geq 3$ it must contain a cycle of length at least d.*

Proof. By Lemmas 1.2.5, 1.2.6, and 1.2.7 it is enough to show, for $d \geq 3$, a graph with no cut-vertex, in which the degree of every vertex is at least $d - 1$, contains a cycle of length at least d. Let G be a graph with minimum degree $d-1$. Let W be one of the longest paths in G and let the vertices of W be $a_1, a_2, \ldots, a_i$. The vertex a_1 is adjacent to all of the vertices on this path, if a_1 were adjacent to a vertex not on this path the path could be made longer contradicting W is one of the longest paths. The degree of a_1 is at least $d-1$, so $i \geq d$. Since a_1 is adjacent to a_i there is a cycle of length at least d. □

Hajós' Conjecture is true for $d = 3$ because a 3-chromatic graph contains an odd cycle which is a subdivision of K_3. Proving Hajós' Conjecture for $d = 4$ requires more work.

Lemma 2.2.4 (The $d = 4$ case of Hajós' Conjecture, [9].)**.** *A 4-chromatic graph always contains a subgraph which is a subdivision of K_4.*

Proof. If $\chi(G) = 4$, then by Lemmas 1.2.5, 1.2.6, and 1.2.7 G has a subgraph H which is 4-critical, $\delta(H) \geq 3$, and H has no cut-vertex. So, it is sufficient to show a graph with no cut-vertex in which every vertex has degree at least 3 has a subdivision of K_4.

Let G be a graph with no cut-vertex in which every vertex has degree at least 3. Let $\mathcal{C}$ be one of the longest cycles in G and let the vertices of $\mathcal{C}$ be, in cyclic order, $a_1, a_2, a_3, \ldots, a_n, a_1$. Note, $n \geq 4$ because every vertex is adjacent to at least 3 others.

We claim, every vertex of $\mathcal{C}$ is connected by a chord to another vertex of $\mathcal{C}$, where a chord is a path between two non-adjacent vertices of a cycle, having only its end vertices in common with the cycle. Without loss of generality consider the vertex a_1. The vertex a_1 is adjacent to a_2 and a_n and to at least one other vertex b, because the minimum degree of G is 3. If b is one of the vertices of $\mathcal{C}$ then a_1 is connected by a chord to another vertex of $\mathcal{C}$. If b is not a vertex of $\mathcal{C}$, then since a_1 is not a cut vertex there must be a path from b to a_2 that does not go through a_1. If the path from b to a_2 has no interior vertices in common with $\mathcal{C}$ then $\mathcal{C}$ would not be one of the longest cycles in G. Therefore at least one of the interior vertices of this path must be a vertex of $\mathcal{C}$. Let a_i be the first vertex where the path and $\mathcal{C}$ intersect. Then $i \neq n$ and $i \neq 2$ because $\mathcal{C}$ is one of the longest cycles in G. The edge $a_1 b$ and the portion of the path from b to a_i is a chord from a_1 to a_i. So every vertex of $\mathcal{C}$ is connected to another vertex of $\mathcal{C}$ by a chord. We will now use this fact to find a subdivision of K_4. There are two cases to consider: (1) There are two chords of $\mathcal{C}$ which join different pairs of vertices and have an interior vertex in common or, (2) no two chords of $\mathcal{C}$ whose end vertices do not coincide have an interior vertex in common.

(1) Let the chord from a_i to a_j and the chord from a_k to a_l intersect at an interior vertex c. There are paths along the cycle from a_i to a_k, a_k to a_j,

and from a_j through a_l to a_i. Then using the chords we have paths from a_i to c, a_j to c, and a_k to c. The paths mentioned form a subgraph of G that is a subdivision of K_4, by replacing the paths with edges.

(2) Every vertex of $\mathcal{C}$ is connected by a chord to a non-neighboring vertex of $\mathcal{C}$ and none of these chords share a vertex. Each chord divides the cycle into two arcs. For example if there is a chord from a_1 to a_4, the cycle is divided into the arcs: a_1, a_2, a_3, a_4 and $a_4, a_5, \ldots, a_n, a_1$. The length of the arc is the number of vertices in the arc. Choose one of the chords that creates one of the shortest arcs. Let this chord have vertices on the cycle a_i and a_j. Since chords do not connect adjacent vertices there is a vertex between a_i and a_j on the cycle in the shorter of the arcs, call this vertex a_k. Every vertex on the cycle is connected by a chord to another vertex of the cycle, let the vertex a_k is connected to by a chord be a_l. Note the chord from a_i to a_j crosses the chord from a_k to a_l and they do not share any vertices. These four vertices form a subdivision of K_4. The paths forming this K_4 are the paths along the cycle from a_i to a_k, a_k to a_j, a_j to a_l, and a_l to a_i and the chords from a_i to a_j and a_k to a_l.

□

To prove if $\chi(G) = 4$, then G contains a subgraph that is a subdivision of K_4, we proved if H has no cut-vertex and minimum degree 3, then H contains a subdivision of K_4. This is equivalent because if $\chi(G) = 4$, then G contains a subgraph H which is 4-critical, that is has no cut-vertex and minimum degree 3. This method does not work for $d \geq 5$. It is not true that a graph with no cut-vertex and minimum degree 4 has a subdivision of K_5. An example of this is the graph of the icosahedron shown in Figure 2.4. The graph of the icosahedron has no cut vertex and every vertex has degree 5, however there is no subdivision of K_5 because the graph of the icosahedron is planar.

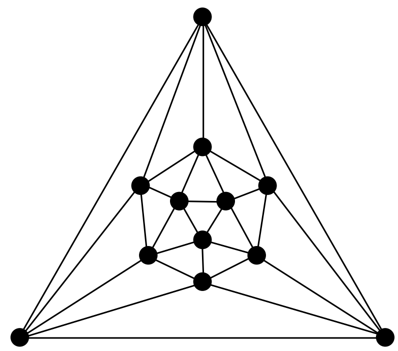

Figure 2.4: A graph of the icosahedron.

Hajós' Conjecture is true for $d \leq 4$ [9] and is open for $d = 5, 6$, however, Catlin [4] showed it is false for $d \geq 7$ by giving a family of counterexamples.

More recently, immersion has been considered as an answer to this classic question. The remainder of this dissertation will focus on immersion.

2.3 Immersion

In 1965, Nash-Williams [20] introduced the concept of immersion as a weakening of subdivision. He conjectured that for every countable sequence G_i $(i = 1, 2, \ldots)$ of graphs, there exist $j > i \geq 1$ such that there is an immersion of G_i in G_j. That is, he conjectured that graphs can be well-quasi-ordered by immersion. Robertson and Seymour proved this conjecture in 2010 [21] in paper 23 of their series of Graph Minors papers. The goal of this series of papers was to prove graphs are well-quasi-ordered by the minor relation.

Informally, a graph H is **immersed** in a graph G if for every edge in H there is a corresponding edge disjoint path in G. Figure 2.5 shows a graph with an immersion of K_4 by using existing edges connecting vertices a, b, c, and d and the path $a - x - d - y - c$ from a to c.

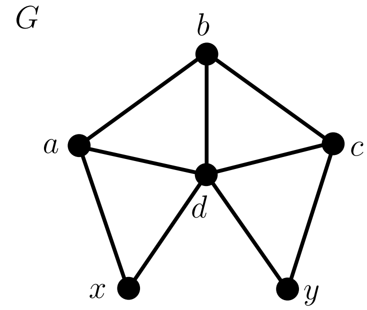

Figure 2.5: G has an immersion of K_4.

By thinking of immersion as edge disjoint paths, we can see that it is a weakening of subdivision, which requires paths to be vertex disjoint. Thus, if a graph has a subdivision of H, then it has an immersion of H. The converse is not necessarily true, for example, the graph in Figure 2.5 has an immersion of K_4, but no subdivision of K_4. Immersion and minor are incomparable. Figure 2.2 shows a graph which has a minor of K_5, but no immersion of K_5 and Figure 2.5 shows a graph with an immersion of K_4, but no minor of K_4.

We formally define immersion by lifting pairs of edges.

Definition 2.3.1. *A pair of adjacent edges uv and vw, $w \neq u$, is* ***lifted*** *by deleting the edges uv and vw and adding the edge uw.*

Definition 2.3.2. *A graph H is* ***immersed*** *in a graph G if and only if a graph isomorphic to H can be obtained from G by lifting pairs of edges and taking a subgraph. If a graph H is immersed in a graph G we may say G has an immersion of H.*

Figure 2.6 shows how K_4 is immersed in a graph G by lifting pairs of edges.

We can think of the pairs of edges that are lifted as paths, which gives an equivalent definition of immersion.

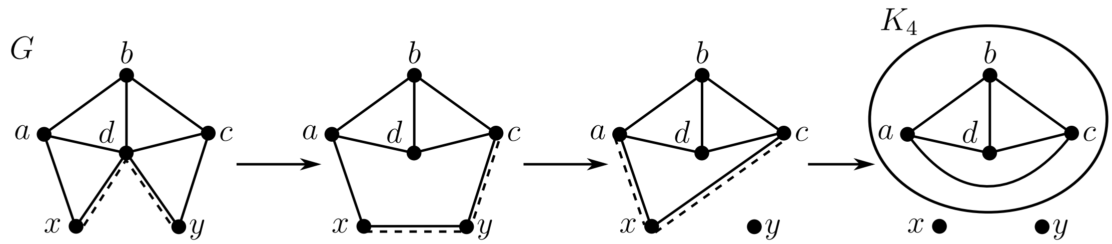

Figure 2.6: A series of lifting of pairs of edges showing K_4 is immersed in G.

Definition 2.3.3. *A graph H is **immersed** in a graph G if and only if there exists an injection $\phi : V(H) \to V(G)$ that can be extended to an injection $\phi_E : E(H) \to \{\text{paths in } G\}$ such that if $u, v \in V(H)$ and $e = uv \in E(H)$ then $\phi_E(e)$ is a path between $\phi(u)$ and $\phi(v)$, and for all $e_1 \neq e_2$, $\phi_E(e_1)$ and $\phi_E(e_2)$ are edge disjoint.*

Definition 2.3.4. *In an immersion we call image vertices under the injection **corner** vertices. We call vertices that are on the paths in G corresponding to edges in H, that are not endpoints of the path, **pegs**.*

Figure 2.7 shows the graph G from Figure 2.5 and the paths, corners, and pegs used to find an immersion of K_4.

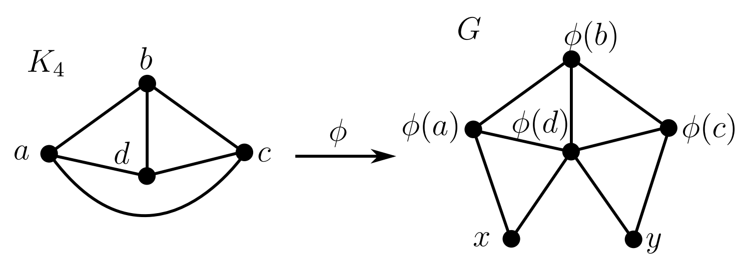

Figure 2.7: Showing K_4 is immersed in G using edge disjoint paths. $\phi(a), \phi(b), \phi(c)$, and $\phi(d)$ are corners, x and y are pegs.

Lemma 2.3.5. *Definitions 2.3.2 and 2.3.3 are equivalent.*

Proof. Let H be a graph that can be obtained from a graph G by lifting pairs of edges and taking a subgraph (Definition 2.3.2). Let $\varphi : V(H) \to V(G)$ be the map that sends vertices of H to the vertices they came from in G via lifting pairs of edges ($V(H) \subseteq V(G)$). This map is injective. We must show that the images of adjacent vertices of H under the map φ are connected in G by edge disjoint paths. We will proceed by induction on the number of liftings of pairs of edges in G needed to obtain H.

If H is obtained from G by no lifts, then H is a subgraph of G and adjacent vertices in H are adjacent vertices in G.

If H is obtained from G by one lift and then taking a subgraph, then in G we have $\varphi(u)w, w\varphi(v) \in E(G)$. When we lift this pair of edges we get the edge $\varphi(u)\varphi(v)$ which corresponds to the edge $uv \in E(H)$. Since we only need to lift one pair of edges to get H the other edges in H correspond to edges in G, so the path $\varphi(u) - w - \varphi(v)$ is edge disjoint from other edges.

Assume if H is immersed in G and can be obtained from G by lifting k pairs of edges and taking a subgraph, then images of adjacent vertices of H are connected in G by edge disjoint paths.

Let H be a graph immersed in G and obtained from G by lifting $k+1$ pairs of edges and taking a subgraph. Let H' be a graph immersed in G obtained from G by performing k of these lifts. So we have a map $\varphi' : V(H') \to V(G)$ and by induction images of adjacent vertices in H' are connected by edge disjoint paths in G. Then we can obtain H from H' by lifting one pair of edges and taking a subgraph. So we have a map $\varphi : V(H) \to V(H')$. Composing the two maps φ' and φ we get a map from $V(H)$ to $V(G)$ that is injective and images of adjacent vertices of H are connected by edge disjoint paths.

Now suppose we have an injection $\varphi : V(H) \to V(G)$ such that images of adjacent vertices are connected by edge disjoint paths in G (Definition 2.3.3). We show H can be obtained from G by lifting pairs of edges and taking a subgraph.

If $uv \in E(H)$ then there is a path in G from $\varphi(u)$ to $\varphi(v)$ that is edge disjoint from the paths corresponding to other edges of H. We must show that the edge $\varphi(u)\varphi(v)$ can be obtained by lifting pairs of edges along the path from $\varphi(u)$ to $\varphi(v)$, then since these paths are edge disjoint we can obtain all edges of H by lifting pairs of edges along their corresponding paths. If the path has length 1 (where we are considering the length of a path to be the number of edges in the path), then we do not need to lift any edges. If the path has length 2, we lift this pair of edges and get the edge $\varphi(u)\varphi(v)$. Assume if a path in G from $\varphi(u)$ to $\varphi(v)$ has length k then we obtain the edge $\varphi(u)\varphi(v)$ by lifting $k-1$ pairs of edges along the path. Suppose the path from $\varphi(u)$ to $\varphi(v)$ has length $k+1$, then we can lift the first two edges in the path which gives a path of length k from $\varphi(u)$ to $\varphi(v)$, by induction we can lift pairs of edges to get the edge $\varphi(u)\varphi(v)$. Hence, Definitions 2.3.2 and 2.3.3 are equivalent. □

In 2003, Abu-Khzam and Langston [1] made the following conjecture as an answer to the classic question of complete graph containment in a d-chromatic graph.

Conjecture 2.3.6. *[1] The complete graph K_d can be immersed in any d-chromatic graph.*

Since Hajós' Conjecture is true for $d \leq 4$ and an immersion is a weakening of subdivision, Conjecture 2.3.6 is true for $d \leq 4$. We will see that the conjecture is true for $d \leq 7$ [6, 17], but we will need a different approach. The new approach will involve the minimum degree of the graph. It follows from the definition of immersion that if H is immersed in G, then the degree of any vertex in H is less than or equal to its degree in G. This is one reason that immersion may be easier to work with than minor in which the degree of a vertex can greatly increase.

Another appealing property of immersion is that in order for a d-chromatic graph, G, to have an immersion of K_d, G must have at least d vertices of degree at least $d-1$. In the next section we will present others work on Conjecture 2.3.6 before presenting our own results in Chapter 3.

2.4 Immersion Results

Since any d-chromatic graph has a d-critical subgraph of minimum degree $d-1$, as shown in Lemmas 1.2.6 and 1.2.7, a proof of the stronger statement that every graph with minimum degree $d-1$ has an immersion of K_d would imply Conjecture 2.3.6. Lescure and Meyniel [17] used this method to prove the conjecture for $d \leq 6$, and DeVos et al. [6] gave a new proof for $d \leq 7$ using the same method. In their paper they prove:

Theorem 2.4.1 ([6]). *Let $f(d)$ be the smallest integer such that every simple graph of minimum degree at least $f(d)$ contains an immersion of K_d. Then $f(d) = d-1$ for $d \in \{5, 6, 7\}$.*

They prove this by proving the following slightly stronger theorem which allows for graphs that are not simple. If two vertices are joined by more than one edge they call the set of edges joining them a **proper parallel class**.

Theorem 2.4.2 ([6]). *Let $d \in \{4, 5, 6\}$, let G be a loop-less graph, and let $u \in V(G)$. Assume G satisfies the following properties*

- $|V(G)| \geq d$
- $d(v) \geq d$ *for every* $v \in V(G) - \{u\}$

- *There are at most $d-2$ proper parallel classes, and every edge in such a parallel class is incident with u.*

Then there is an immersion of K_{d+1} in G.

Theorem 2.4.2 implies Theorem 2.4.1 since graphs with minimum degree $d-1$ fall into the class of graphs that satisfy the required properties in Theorem 2.4.2.

Sketch of proof given in [6]. Suppose G is a counterexample with $|V(G)|+|E(G)|$ minimum. Then prove the following properties of G.

1. $|V(G)| \geq d+1$.
2. Every edge has an end of degree at most d.
3. $d(v) = d$ for every $v \in N(u)$.
4. $d(v) \leq d+1$ for every $v \in V(G) - \{u\}$.
5. $N(u)$ induces a complete graph.
6. $|N(u)| \geq 3$.
7. $|N(u)| \leq d-2$.
8. $d = 6$ and $|N(u)| = 3$.
9. G does not have exactly one proper parallel class.
10. There do not exist $w_1, w_2 \in \bigcap_{v \in N(u)} N(u) - \{u\}$ so that either $w_1 w_2 \in E(G)$ or $d(w_1) = d(w_2) = 6$.
11. $|\bigcap_{v \in N(u)} N(u) - \{u\}| \leq 2$.
12. Every $v \in N(u)$ satisfies $|N(v)| \leq 5$.

Property (12) implies that every neighbor of u is incident with a proper parallel class. Then by Properties (3), (5), and (8) the graph obtained from G by identifying $\overline{N}(u) = N(u) \bigcup \{u\}$ to a single vertex and deleting loops is immersed in G. This new graph is a smaller counterexample. $\Rightarrow\Leftarrow$ $\square$

In the same paper DeVos et al. cite an example of Paul Seymour that shows the stronger statement fails for $d = 10$. Seymour's example is: let G be the graph obtained from K_{12} by deleting the edges of four disjoint triangles. Then G has minimum degree 9, but has no immersion of K_{10}. Note that, $\chi(G) = 4$ and G contains K_4 as a subgraph, so G is not a counterexample to Conjecture 2.3.6. Seymour's example is shown in Figure 2.8.

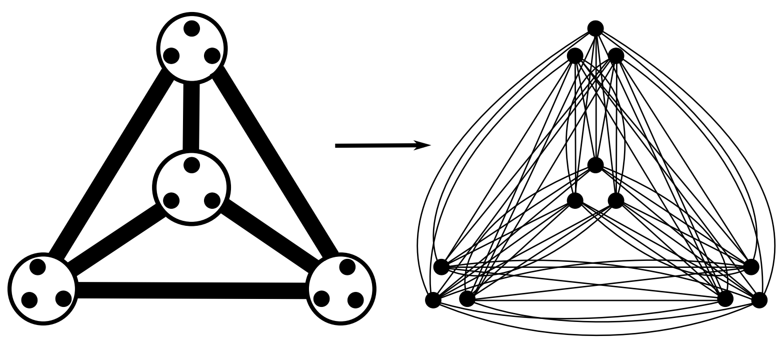

Figure 2.8: In the figure on the left the bold edges mean every possible edge between the circled vertices. The figure on the right shows the graph with all of the edges added.

Following [6], we let $f(d)$ be the smallest integer such that every graph of minimum degree at least $f(d)$ contains an immersion of K_d. In [7], the authors show that $f(d) \leq 200d$ for all d, and that $d \leq f(d)$ for $d = 10$ and $d \geq 12$. Their examples are similar to, but more general than, those given by Seymour and they give a finite number of examples for each d. These examples do not cover the $d = 8, 9$, and 11 cases, and have small chromatic number relative to d. In the next chapter we will resolve these cases and give an infinite number of examples for each d with larger chromatic number.

Chapter 3

Constructing Examples

Recall $f(d)$ is the smallest integer such that every simple graph of minimum degree $f(d)$ has an immersion of K_d. In this chapter we show $f(d) \geq d$ for $d \geq 8$ by giving infinite families of examples of graphs with minimum degree $d-1$ and no immersion of K_d. Our examples have chromatic number $d-3$, $d-2$, or $d-1$ showing the necessity of some chromatic number bound to have an immersion of K_d. Our examples can be up to $(d-2)$-edge-connected. The inspiration for our examples comes from the proof of Theorem 2.4.2 given in [6]. We will begin by describing, in general, properties of graphs with minimum degree $d-1$ and no immersion of K_d. We will do this by first considering a graph with an immersion of K_d, and exploring where the vertices of that immersion must live within the graph. This will be our Corner Separating Lemma, which appears in Section 3.1. In Section 3.2, we will describe a general construction of docks, bays, and pods that form graphs with minimum degree $d-1$ and no immersion of K_d. In Section 3.3 we explicitly construct appropriate docks, bays, and pods and in Section 3.4 we compare these examples to those presented in [7].

3.1 Corner Separating Lemma

We will begin with a lemma that is useful for determining given an immersion where the vertices of that immersion are located in the graph. Recall, that we can think of immersion as a mapping of edges in H to edge disjoint paths in G, where we call the image vertices **corners** and vertices on the paths **pegs**. We will argue that all corner vertices of an immersion of K_d would have to be in one part of the graph. We call a set of edges $C \subseteq E(G)$ such that $G - C$ has more than one component a **cutset of edges**.

Lemma 3.1.1 (Corner Separating Lemma)**.** *Let G be a graph and M a subgraph such that there is a cutset of edges C in G, $|C| \leq d - 2$, and M is a connected component of $G - C$. If G has an immersion of K_d, then all of the corner vertices are in $V(M)$ or all of the corner vertices are in $V(G - M)$.*

Proof. Let G and M be as described in Lemma 3.1.1 and suppose G has an immersion of K_d. Suppose for a contradiction that there are corner vertices in both $V(M)$ and $V(G - M)$. Then these corner vertices must be connected by edge disjoint paths, and any path from $V(M)$ to $V(G - M)$ uses an edge of C. There are d corner vertices partitioned between $V(M)$ and $V(G - M)$. If there are x corner vertices in $V(M)$, where $1 \leq x \leq d - 1$, then there are $x(d - x)$ edge disjoint paths between corners in M and corners in $G - M$. So, we must have $x(d - x) \leq |C| \leq d - 2$. That is,

$$\begin{aligned} x(d-x) &\leq d-2 \\ dx - x^2 &\leq d-2 \\ -x^2 + dx + 2 - d &\leq 0. \end{aligned}$$

The function $-x^2 + dx + 2 - d \leq 0$ is a concave down parabola, so we need only check the endpoints of the interval $1 \leq x \leq d - 1$ to see that this function is positive on the entire interval, giving a contradiction.

At $x = 1$ we have

$$\begin{aligned} -(1)^2 + d(1) + 2 - d &= -1 + d + 2 - d \\ &= 1 > 0. \end{aligned}$$

At $x = d - 1$ we have

$$\begin{aligned} -(d-1)^2 + d(d-1) + 2 - d &= -(d^2 - 2d + 1) + d^2 - d + 2 - d \\ &= -d^2 + 2d - 1 + d^2 - 2d + 2 \\ &= 1 > 0. \end{aligned}$$

Therefore all of the corner vertices must be in $V(M)$ or all in $V(G - M)$. □

The following lemmas will also be useful.

Lemma 3.1.2. *[7] If G has an immersion of K_d on a set of J corners, then G has an immersion of K_d on J in which the edges between adjacent vertices in J are used as the paths between these vertices.*

Proof. [7] Suppose G has an immersion of K_d on a set of J corners. Let $v, w \in J$ and $vw \in E(G)$. Suppose vw is not used as the path, P_{vw}, between v and w in the immersion. Then replace P_{vw} by the edge vw. If vw is on a path, P, between two corners u and u', then replace that portion of P with P_{vw}. This creates a walk from u to u' which contains a path from u to u' that can be used for the immersion. We can do this for each edge in J to obtain an immersion of K_d in which the edges between adjacent vertices in J are used as the paths between the vertices. □

This lemma tells us that if there is an immersion of K_d, then there is an immersion of K_d that uses edges between adjacent corners.

Lemma 3.1.3. *Suppose G has an immersion of K_d. If a corner vertex of the immersion has degree at most d, then it cannot also be used as a peg in the immersion.*

Proof. Let G be a graph that has an immersion of K_d and let v be a corner vertex in this immersion with degree at most d. Since v is a corner in an immersed K_d the degree of v is either $d-1$ or d. There are d corners in the immersion so the immersion uses $d-1$ of the edges incident with v as the paths between v and the other corner vertices. Then there is at most one more edge incident with v, so v cannot be a peg, that is, it cannot be an interior vertex on another path in the immersion. □

Lemma 3.1.4. *Suppose G has an immersion of K_d and $|V(G)| = d+1$, then the immersion of K_d has exactly one peg.*

Proof. Suppose G has an immersion of K_d and $|V(G)| = d + 1$. Since $|V(G)| = d + 1$, $d(v) \leq d$ for all $v \in V(G)$ and by Lemma 3.1.3 corner vertices of the immersion cannot also be used as pegs. To have an immersion of K_d there must be d corners, therefore there is exactly one peg. □

The Corner Separating Lemma tells us that if G has an immersion of K_d, then all the corners must be in a maximally $(d-1)$-edge-connected subgraph G. In the next section, we will construct graphs so that there can be no immersion of K_d with all of the corners in such a subgraph.

3.2 Docks, Bays, and Pods

In general, we would like to be able to construct graphs with minimum degree $d-1$ and no immersion of K_d. Our construction will be formed by docks, bays, and pods defined below. We will see that as long as we can find docks, bays, and pods with the desired characteristics we can create graphs of minimum degree $d-1$ with no immersion of K_d. The hardest part will be to find pods that satisfy the definition. In [6] the authors prove that for $d \leq 7$ a graph with minimum degree d has an immersion of K_d. So we will see it is impossible to find pods for $d \leq 7$.

In each of the following definitions let d be a positive integer.

Definition 3.2.1. *A d-**pod** is a graph in which every vertex has degree at least $d-2$ and no more than $d-2$ vertices have degree exactly $d-2$. In addition, there is no immersion of K_d in a d-pod, even if a maximum matching of the vertices of degree $d-2$ is added.*

Note that, adding a matching between the vertices of degree $d-2$ may create multiple edges. When considering an immersion in a larger graph this matching in the pod will represent paths that can be lifted outside the pod to create more connections between vertices in the pod. An example of an 8-pod is shown in Figure 3.1

Definition 3.2.2. *A d-**bay** is a graph with at most $d-2$ vertices.*

Definition 3.2.3. *A d-**dock** is composed of one or more d-bays arranged in a circle with no more than $d-3$ edges between any two consecutive d-bays, and no edges between nonconsecutive d-bays.*

Note that, a d-dock with two bays can have $2(d-3)$ edges between the bays.

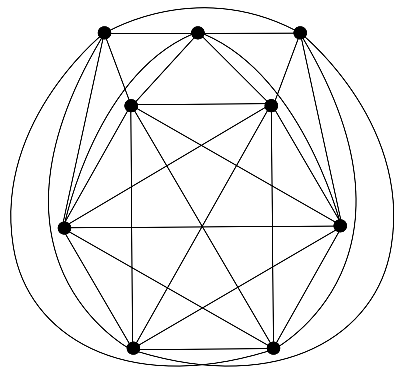

Figure 3.1: Example of an 8-pod.

Definition 3.2.4. *A d-pod is* ***added*** *to a d-bay by adding an edge from each vertex of degree* $d-2$ *in the d-pod to a vertex in the d-bay.*

Note that, a d-pod is connected to exactly one bay.

Definition 3.2.5. *A d-bay in a d-dock is* ***full*** *if d-pods are added in such a way that every vertex in the d-bay has degree at least* $d-1$.

A schematic for combining docks, bays, and pods is shown in Figure 3.2

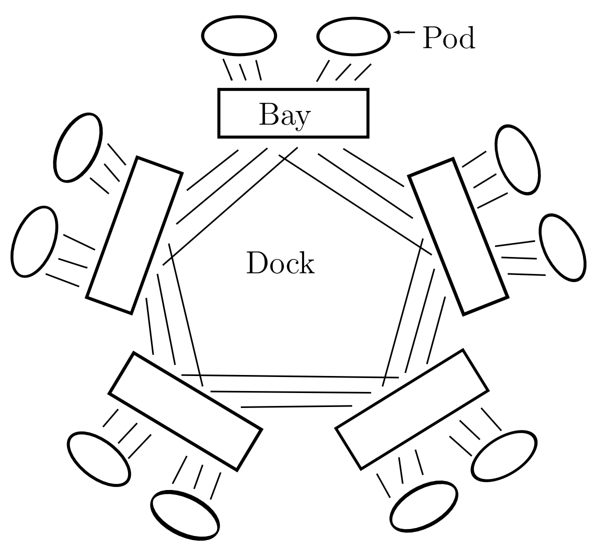

Figure 3.2: Schematic for combining docks, bays, and pods.

When there is no confusion we will drop the d prefix in the terms.

Theorem 3.2.6. *Let G be a d-dock in which every d-bay is full. Then G has minimum degree $d-1$ and no immersion of K_d.*

Proof. Let G be a dock in which every bay is full. Let $v \in V(G)$. Then v is in a pod or a bay. If v is in a pod, then by the definition of a pod it has degree greater than or equal to $d-2$ within the pod. If v has degree $d-2$ within the pod, then by the definition of adding a pod to a bay, there is an edge from v to a bay, so v has degree $d-1$ in G. Thus, if v is in a pod it has degree at least $d-1$. If v is in a bay, then by the definition of a bay being full it has degree at least $d-1$. Therefore, G has minimum degree $d-1$.

Suppose G has an immersion of K_d. Then by multiple applications of the Corner Separating Lemma all of the corner vertices are either in a single pod or in the dock. We can use the Corner Separating Lemma because there are at most $d-2$ edges between any pod and the dock and there are no edges between pods.

The corners cannot all be in a pod because by definition pods have no immersion of K_d. The edges out of a pod may provide paths that can be lifted to give a maximum matching of vertices of degree $d-2$ in the pod, but by definition of a pod even this is not enough to have an immersion of K_d with all of the corners in the pod. Thus all of the corners must be in the dock.

If the dock has fewer than d vertices then it cannot have an immersion of K_d, so assume the dock has at least d vertices. If all of the corners are in the dock, then notice they cannot all be in a single bay because the bays have at most $d-2$ vertices. Thus, the d corners are split between at least two bays. Suppose there is a bay B with k corners where $2 \leq k \leq d-2$. There are at most $2(d-3)$ edges from B to neighboring bays. So, to get edge disjoint paths from the k corners to the remaining corners in the dock we would need

$$\begin{aligned} k(d-k) &\leq 2(d-3) \\ dk - k^2 &\leq 2d - 6 \\ -k^2 + dk - 2d + 6 &\leq 0. \end{aligned}$$

The function $-k^2 + dk - 2d + 6$ has a maximum at $k = \frac{d}{2}$, thus if we check the endpoints of our interval and find the values are positive then $-k^2 + dk - 2d + 6$ is positive on the entire interval we are considering giving a contradiction.
At $k = 2$

$$\begin{aligned} -k^2 + dk - 2d + 6 &= -(2)^2 + 2d - 2d + 6 \\ &= 2 > 0. \end{aligned}$$

At $k = d - 2$

$$\begin{aligned} -k^2 + dk - 2d + 6 &= -(d-2)^2 + d(d-2) - 2d + 6 \\ &= -d^2 + 4d - 4 + d^2 - 4d + 6 \\ &= 2 > 0. \end{aligned}$$

Therefore, there are not enough edge disjoint paths, and hence no immersion of K_d when there are k corners, $2 \leq k \leq d-2$, in a bay.

If there is no bay B with 2 or more corners, then let B_1 be a bay with one corner. Let B_n be the next bay in the clockwise direction containing a corner. Let H be the subgraph of G induced by the union of the B_i, $1 \leq i \leq n$. Then there are at most $2(d-3)$ edges connecting corners in H to corners in $G - H$. There are 2 corners in H so for there to be an immersion of K_d there must be at least $2(d-2)$ edge disjoint paths from H to $G-H$, so we need $2(d-2) \leq 2(d-3)$, a contradiction. Thus there is no immersion of K_d with all of its corners in the dock and so there is no immersion of K_d in G. □

The above theorem tells us that as long as we can find d-bays and d-pods with the desired characteristics we can create a graph of minimum degree $d-1$ with no immersion of K_d. The hardest part is to find d-pods that satisfy the definition. We now give a general construction for a pod with $d+1$ vertices for $d \geq 8$.

Definition 3.2.7. *Let P be a simple graph with $d+1$ vertices and minimum degree $d-2$ with at most $d-2$ vertices of degree exactly $d-2$. Split the vertices into two sets, A and B. Where $A = \{v \in V(P) \mid d(v) = d-2\}$ and $B = V(P) - A$. A **gadget** is*

1. *a missing odd cycle in A, or*

2. *a missing path of length 2 with end vertices in B and middle vertex in A.*

Figure 3.3 shows two examples of graphs with gadgets, dotted lines are missing edges.

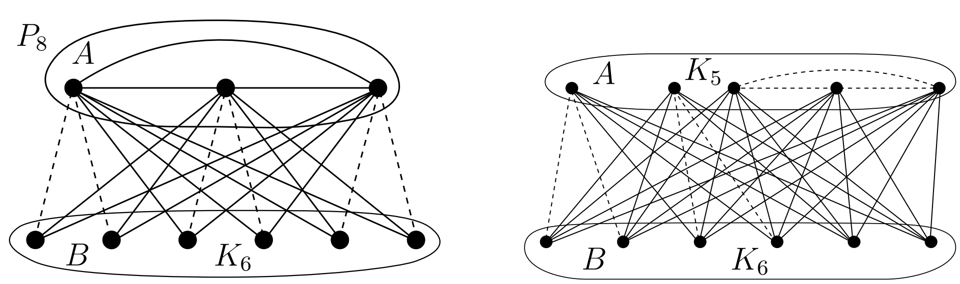

Figure 3.3: Dotted lines show gadgets.

Theorem 3.2.8. *Let P be a simple graph with $d+1$ vertices and minimum degree $d-2$ with at most $d-2$ vertices of degree exactly $d-2$. If P has three or more disjoint gadgets, then P is a d-pod.*

Proof. Let P be as described with at least three gadgets. Since P is simple and $|V(P)| = d+1$, $d(v) \leq d$ for all $v \in V(P)$, so $d-2 \leq d(v) \leq d$ for all $v \in V(P)$. Since $|A| \leq d-2$, that is the number of vertices of degree exactly $d-2$ is at most $d-2$, P satisfies the degree requirements to be a d-pod. Add a maximum matching to the vertices in A. Let this new graph be called P^+. Suppose P^+ has an immersion of K_d for a contradiction. Notice that, since the gadgets are missing odd cycles, or have missing edges between A and B, the addition of a matching to A leaves at least one vertex in each gadget incident with two missing edges. There are at least three gadgets, so there are at least three vertices each incident with two distinct missing edges. Call these vertices x, y, and z. By Lemma 3.1.3 and 3.1.4, P^+ has exactly one peg and it is not a corner, call it w.

Suppose $w \in B$, then there are at least two gadgets of which w is not a part. Without loss of generality say w is not part of the gadget involving x. Then w must be used on edge disjoint paths to replace both missing edges incident with x. However, there is at most one path, in fact an edge, from w to x that is not already used in the immersion. Thus w can be used to replace at most 1 edge incident with x. So $w \notin B$.

Thus, $w \in A$. Then there are at least two gadgets of which w is not a part. Without loss of generality say w is not part of the gadgets containing x and y. For there to be an immersion of K_d we must use edge disjoint paths through the peg w to replace the two missing edges incident with x and the two missing edges incident with y. Without using the matching, there is one unused edge in the graph from w to x and one unused edge from w to y. Thus, to replace all four missing edges there must be a matching edge from w to x and a matching edge from w to y, a contradiction. There is at most one matching edge incident with w. Therefore, there is no immersion of K_d in P^+, that is P is a d-pod. □

We have given general constructions for graphs with minimum degree $d-1$ and no immersion of K_d. In the next section we give specific examples and explore some of the characteristics of these examples.

3.3 Examples

In this section we will give examples of pods and graphs for $d \geq 8$. We will give three types of examples. In Section 3.3.1 we give examples with exactly one bay. These examples will have chromatic number $d-3$ or $d-1$. In Section 3.3.2 we give an infinite number of examples for each d by increasing the number of bays in the graph. These graphs will have chromatic number $d-2$ and will be 3-edge-connected. In Section 3.3.3 we will give examples with edge-connectivity up to $d-2$ and chromatic number $d-2$.

3.3.1 Examples with One Bay

The inspiration for the examples with one bay comes from the proof of Theorem 2.4.2. The authors prove this theorem by supposing there is a minimal (in terms of vertices and edges) counterexample, G. They prove properties about G which lead to no such graph existing. In their proof they have a vertex u that may have degree smaller than d and whose neighbors form a complete graph on three vertices. In our initial graphs the pods will be similar to G without u. We begin by exploring the case where $d = 8$. In giving our example for $d = 8$ we are showing that Theorems 2.4.1 and 2.4.2 cannot be extended to $d = 8$. For the $d = 8$ case we form the following graph, P_8.

Definition 3.3.1. *Define P_8 as follows. Begin with a K_9. Remove three disjoint paths of length 2.*

The graph P_8 is shown in Figure 3.5.

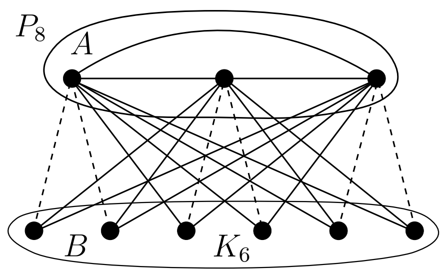

Figure 3.4: An 8-pod, P_8, dotted lines are missing edges.

Lemma 3.3.2. *P_8 is an 8-pod.*

Proof. By construction $|V(P_8)| = 9 = d + 1$. We can split the vertices into two sets A and B. Let $A = \{v \in V(P_8) \mid d(v) = 6\}$ and $B = V(P_8) - A$. Notice that, $|A| = 3 < 6 = d - 2$ and the degree of every vertex in B is $7 = d - 1$. Thus P_8 satisfies the degree requirements of Theorem 3.2.8. The three missing paths of length two are disjoint gadgets because they have end vertices in B and middle vertex in A. Therefore, we can use Theorem 3.2.8 to conclude P_8 is an 8-pod. $\square$

Theorem 3.3.3. *Let G_8 be the graph with K_5 as its only bay, filled with the 8-pods P_8. Then G_8 has minimum degree 7 and no immersion of K_8.*

Proof. The K_5 is a bay because it has $5 = d - 3 < d - 2$ vertices. The statement follows from Theorem 3.2.6. $\square$

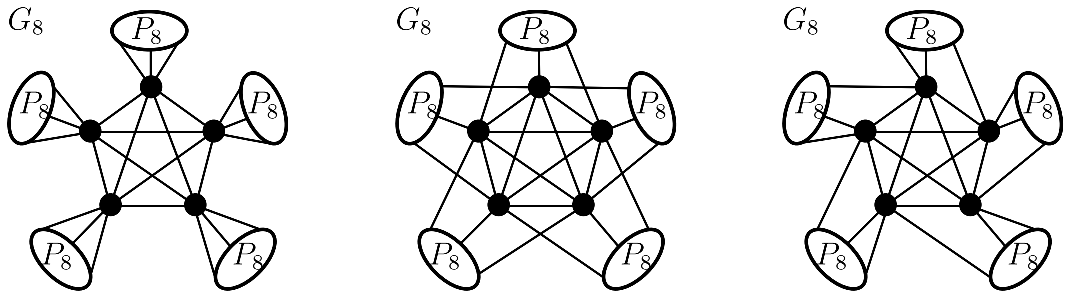

Figure 3.5: Three options for G_8.

Three possible options for G_8 are shown in Figure 3.5. Notice that, the 8-pods in the figure on the left are each added to a single vertex giving an example that is 1-vertex connected. Since the bay in G_8 is the entire K_5 the 8-pods, P_8, may be added so that they connect to multiple vertices, as shown in Figuer 3.5, and/or more pods may be added, giving different examples still satisfying Theorem 3.3.3. The resulting examples could be 2 or 3-vertex-connected. However, all configurations of G_8 will be 3-edge-connected. The chromatic number of G_8 is 6 and G_8 has a subgraph of K_6.

We will build all of our examples in a similar way. We must construct G_d and P_d in general. We construct G_d by starting with a K_{d-3} as the only d-bay in a d-dock and make this bay full by adding copies of the d-pod P_d.

Definition 3.3.4. *For every $d \geq 8$ we construct P_d in the following way. Begin with a K_{d+1}. Remove three disjoint paths of length 2. Remove a maximum matching from the vertices that are not on these paths.*

The maximum matching is removed to make the graph close to $d-1$ regular (if d is odd there will be one vertex in P_d of degree d). It is not necessary to remove this matching to prove that P_d is a pod. The graph P_d, for d odd, is shown in Figure 3.6, the dotted lines represent missing edges.

Lemma 3.3.5. *P_d is a d-pod.*

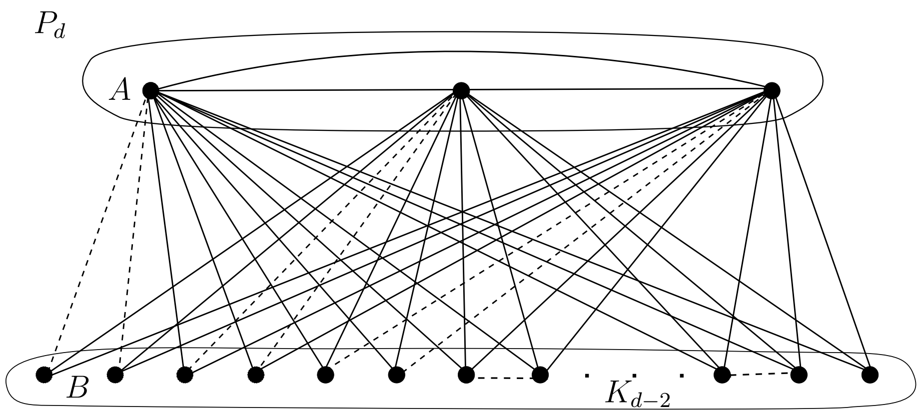

Figure 3.6: A d-pod, P_d, with d odd. Dotted lines are missing edges.

Proof. By construction $|V(P_d)| = d+1$. We can split the vertices into two sets A and B. Let $A = \{v \in V(P_d) \mid d(v) = d-2\}$ and let $B = V(P_d) - A$. The only vertices of degree $d-2$ are those that are the centers of the missing paths, therefore $|A| = 3 < d-2$ for $d \geq 8$. Notice that, the degree of the vertices in B is either $d-1$ or d. Thus, P_d satisfies the degree requirements of Theorem 3.2.8. The three missing paths are disjoint gadgets because they have end vertices in B and middle vertex in A. Therefore, we can use Theorem 3.2.8 to conclude P_d is a d-pod. □

Theorem 3.3.6. *Let G_d be the graph with K_{d-3} as its only bay filled with copies of P_d. The graph G_d has minimum degree $d-1$ and no immersion of K_d.*

Proof. By construction G_d is a d-dock in which every bay is full, thus the statement follows from Theorem 3.2.6. □

Notice that, for $d = 8, 9$, $\chi(G_d) = d-2$ and G_d has a subgraph of K_{d-2}. For $d \geq 10$, $\chi(G_d) = d-3$ and G_d has a subgraph of K_{d-3} and an immersion of K_{d-1}. Therefore G_d is not a counterexample to Conjecture 2.3.6. Given the construction of G_d we can create a 1-vertex-connected graph by attaching pods to exactly one vertex in the bay, or 2 or 3-vertex-connected graphs by attaching pods to multiple vertices in the bay, however in all cases we get a 3-edge-connected graph.

Lemma 3.3.7. *G_d has an immersion of K_{d-1}.*

Proof. We will actually prove that the pods P_d have immersions of K_{d-1}, so G_d has an immersion of K_{d-1}.

Let P_d be as defined in Definition 3.3.4. Label the vertices $\{x, y, z, v_1, v_2, \ldots, v_{d-2}\}$ so that the three disjoint paths of length two that are removed are $v_1 - x - v_2$, $v_3 - y - v_4$, and $v_5 - z - v_6$ and the maximum matching is removed from the set of vertices $\{v_7, \ldots, v_{d-2}\}$. We now find an immersion of K_{d-1}. Let the vertices $\{x, v_1, v_2, \ldots, v_{d-2}\}$ be the corners of our immersed K_{d-1}. To have an immersion of K_{d-1} we must have edge disjoint paths connecting x to v_1, x to v_2, and the matching that was removed from $\{v_7, \ldots, v_{d-2}\}$ (all other required connections are edges in the graph). To get from x to v_1 use the path $x - y - v_1$. To get from x to v_2 use the path $x - z - v_2$, notice that these paths are edge disjoint. The vertex z is adjacent to every vertex in $\{v_7, \ldots, v_{d-2}\}$ and since it is a matching that has been removed using paths through z to replace each of the edges will yield edge disjoint paths. We have described an immersion of K_{d-1}. □

We can also create examples that have chromatic number $d - 1$.

Lemma 3.3.8. *Let G be a graph constructed by adding pods to a K_{d-1} until every vertex in the K_{d-1} has degree at least $d - 1$. G has minimum degree $d - 1$, no immersion of K_d and chromatic number $d - 1$.*

Note that, G as described in Lemma 3.3.8 does not fit into our docks, bays, and pods construction because K_{d-1} has too many vertices to be a bay.

Proof. Let G be as described in Lemma 3.3.8, then by construction and the definitions of pods and adding pods G has minimum degree $d-1$.

Suppose for a contradiction, that G has an immersion of K_d. Then by the Corner Separating Lemma 3.1.1 all of the corners would be in the K_{d-1} or in a single pod. All of the corners cannot be in the K_{d-1} because there are not enough vertices. All of the corners cannot be in a pod by definition. Therefore G has no immersion of K_d.

Since G has a subgraph of K_{d-1}, $\chi(G) \geq d-1$. $\chi(G) = d-1$ because all of our pods have chromatic number less than or equal to $d-2$ and the colors can be permuted to give a proper $(d-1)$-coloring. □

Next we create examples with multiple bays for each $d \geq 8$.

3.3.2 Examples with Multiple Bays

The examples given so far have exactly one bay in a dock. We will now consider several cases where the dock has more than one bay. Let $d \geq 8$ be a fixed integer. We will use the same pods, P_d, as in Definition 3.3.4 and will use bays labeled $B^0, \ldots, B^{n-1}$, each bay B^i will be a copy of K_{d-2}. We form the docks by placing n copies of K_{d-2}, the B^i, in a circle. The idea for connecting consecutive bays, B^i, is to add edges from half of the vertices in B^i to half of the vertices in the next bay, B^{i+1}, and edges from the other half of the vertices in B^i to half of the vertices in the previous bay, B^{i-1}. The following is a precise description of how to connect consecutive bays.

Construction 3.3.9. *Label the vertices of B^i as*

$$a_1^i, a_2^i, \ldots, a_{d-2}^i.$$

Connect consecutive bays by adding edges between B^i and B^{i-1}, where we consider the superscripts $\mod n$*, so that*

$$a^i_j \text{ is adjacent to } a^{i-1}_{d-1-j} \text{ for } 0 \leq i \leq n-1, \text{ and } 1 \leq j \leq \left\lceil \frac{d-2}{2} \right\rceil.$$

A dock with odd d is shown in Figure 3.7.

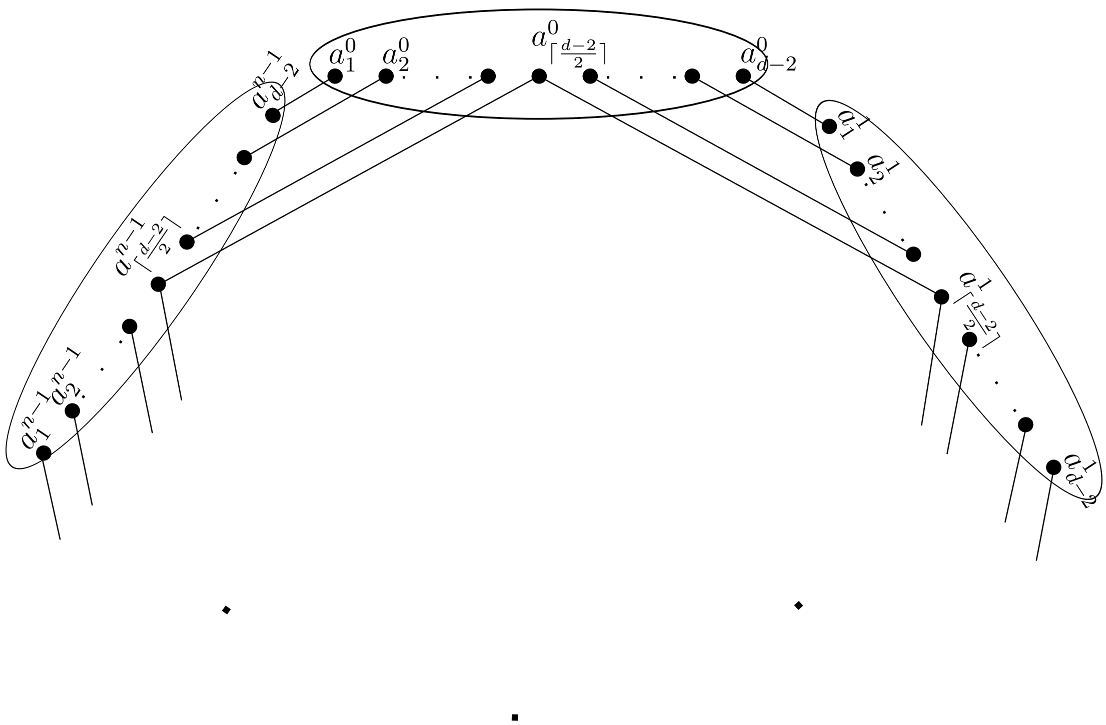

Figure 3.7: Dock for d odd.

Theorem 3.3.10. *Let $d \geq 8$ and G^n_d be a graph with a d-dock formed by connecting the B^i ($1 \leq i \leq n$) as described in Construction 3.3.9. Each B^i is made full by adding copies of the d-pod P_d, as described in Definition 3.3.4. Then G^n_d has minimum degree $d-1$ and no immersion of K_d.*

Proof. We must show that we may apply Theorem 3.2.6. We proved in Lemma 3.3.5 that the P_d satisfy the definition of a d-pod. The B^i are indeed bays because they are copies of K_{d-2}. The number of edges between consecutive bays is $\lceil \frac{d-2}{2} \rceil$ and

$$\left\lceil \frac{d-2}{2} \right\rceil \leq \frac{d}{2} \leq d-3$$

for $d \geq 4$. By construction the bays are full. Now we may apply Theorem 3.2.6 to determine G_d^n is a graph with minimum degree $d-1$ and no immersion of K_d. $\square$

Notice that, since the bays are copies of K_{d-2}, $\chi(G_d^n) \geq d-2$. In fact $\chi(G_d^n) = d-2$ because the pods have chromatic number at most $d-2$ and the colors can be permuted to give a proper $(d-2)$-coloring. A case where $d = 9$ and $n = 4$, that is there are four bays, is shown in Figure 3.8. Note the a_3^i, a_4^i, a_5^i have degree $d = 9$

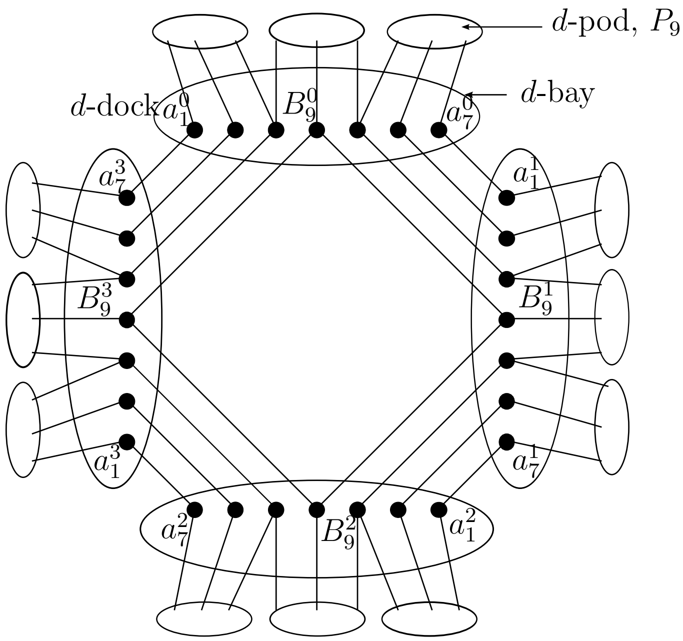

Figure 3.8: An example of G_9^4 with 3 copies of P_9 added to each bay.

for $i \in \{1, 2, 3, 4\}$, but the other vertices in the dock have degree $d - 1 = 8$. This is just one example of G_9^4, we could have more (or fewer) bays and/or more pods, P_9. The example shown is 3-vertex-connected, but if pods are added in a different way we could have examples with vertex-connectivity 1, 2, or 3. However, the graph G_d^n is always 3-edge-connected. Using the d-pods P_d to create a graph G with no immersion of K_d will result in a graph that is at most 3-vertex-connected and exactly 3-edge-connected. Next we create graphs with larger connectivity.

3.3.3 Examples with Greater Edge-Connectivity

First we give examples of graphs with minimum degree $d-1$ and no immersion of K_d that can be up to 5-vertex-connected and are exactly 5-edge-connected. To create these examples we will use the same docks as in Construction 3.3.9, but must modify the pods that are added to the bays.

Definition 3.3.11. *For every $d \geq 8$ construct P_d^5 in the following way. Begin with a K_{d+1}. Remove two disjoint paths of length 2 and a disjoint 3-cycle. Remove a maximum matching from the vertices not involved in a missing path or the missing 3-cycle.*

The superscript of P_d^5 indicates the edge-connectivity of the example of which it is a part. Again, it is not necessary to remove a maximum matching, this is done to make the graph almost $(d-1)$-regular. The graph P_d^5 is shown in Figure 3.9 with the edges that connect it to a bay.

Lemma 3.3.12. *The graph P_d^5 is a d-pod.*

Proof. Split the vertices into two sets A and B, as shown in Figure 3.9. Let $A = \{v \in V(P_d^5) \mid \deg(v) = d-2\}$ and let $B = V(P_d^5) - A$. The only vertices of degree $d-2$ are the centers of the two missing paths and the three vertices in the missing 3-cycle. Therefore $|A| = 5 < d-2$ for $d \geq 8$. The degree of the vertices in B is $d-1$ or d. So, P_d^5 satisfies the degree requirements to be a d-pod. The two missing paths of length 2 and the missing 3-cycle are gadgets and $|V(P_d^5)| = d+1$, so we can use Theorem 3.2.8 to conclude P_d^5 is a d-pod. □

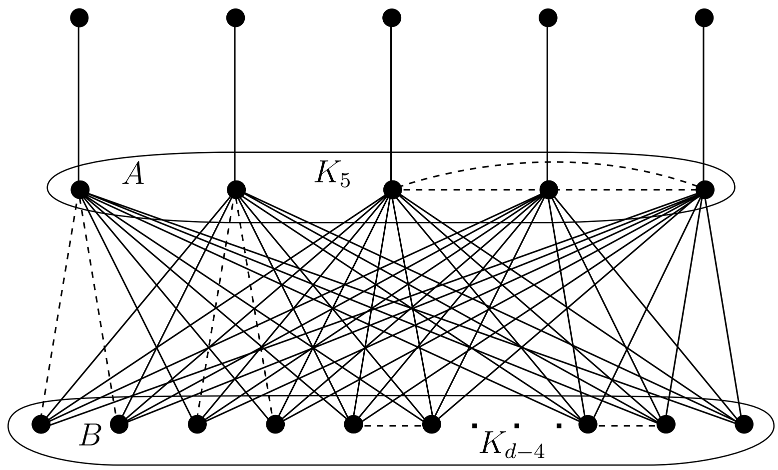

Figure 3.9: 5-vertex-connected pod, P_d^5, with d odd. Missing edges are represented by dotted lines.

Definition 3.3.13. *For $d \geq 8$, construct the graph H_d^5 as follows. Use $n \geq 1$ bays isomorphic to K_{d-2}, connect consecutive bays as in Construction 3.3.9. Make the bays full by adding copies of P_d^5.*

The graphs H_d^5 can be 1, 2, 3, 4, or 5-vertex-connected and are 5-edge-connected. The chromatic number of H_d^5 is $d-2$ and these graphs have subgraphs, and thus an immersion, of K_{d-2}.

Theorem 3.3.14. *The graph H_d^5 has minimum degree $d-1$ and no immersion of K_d.*

Proof. Lemma 3.3.12 tells us that P_d^5 is a pod, then by construction H_d^5 is a d-dock in which every bay is full. Applying Theorem 3.2.6 we see H_d^5 is a graph with minimum degree $d-1$ and no immersion of K_d. □

A similar idea will give us examples with edge-connectivity k where $d \geq 9$ and $7 \leq k \leq d-2$. To do this we will give a new way to construct d-pods with exactly k vertices of degree $d-2$, which we will label P_d^k.

Definition 3.3.15. *Construct P_d^k, for $d \geq 9$, $k = 7$ or $9 \leq k \leq d-2$ as follows. Begin with a K_{d+1}. We continue with two cases depending on the parity of k:*

1. *If k is an odd integer, then remove a path of length two, a 3-cycle, and a C_{k-4}. These should be disjoint. Finally, remove a maximum matching of the vertices not in the path, the 3-cycle, or the C_{k-4}.*

2. *If k is an even integer, then remove three disjoint 3-cycles. If $k = 10$, remove a path of length two disjoint from the 3-cycles. If $k - 9 > 1$ remove a C_{k-9} disjoint from the 3-cycles. Finally remove a maximum matching of the vertices that have not yet been used.*

Examples of five different P_d^k are shown in Figure 3.10. We need not remove the matching to show P_d^k is a d-pod, this is done only to make P_d^k as close to $(d-1)$-regular as possible.

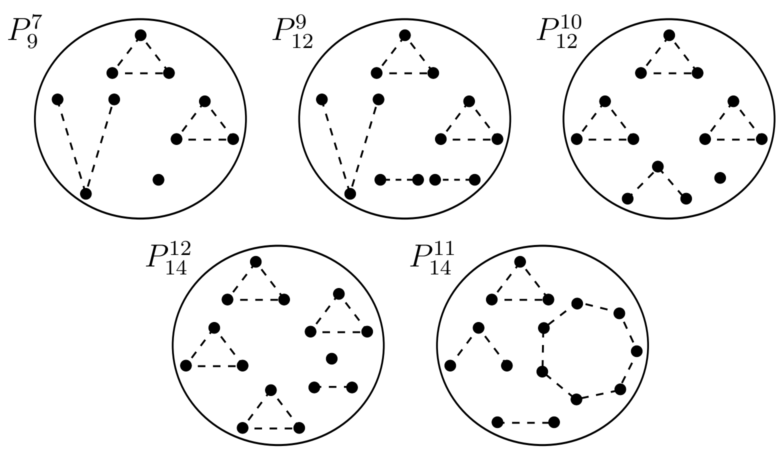

Figure 3.10: Examples of P_d^k, only edges that are not present are shown.

Lemma 3.3.16. *P_d^k is a d-pod.*

Proof. We must show that we may apply Theorem 3.2.8. By construction $|V(P_d^k)| = d+1$. The minimum degree of P_d^k is $d-2$, there are k vertices of degree exactly

$d-2$, and $k \le d-2$ by definition. Let $A = \{v \in V(P_d^k) | \ d(v) = d-2\}$ and $B = V(P_d^k) - A$. By construction P_d^k has three gadgets. Thus by Theorem 3.2.8 P_d^k is a d-pod. □

Now using the same docks as in Construction 3.3.9 we create examples of graphs, which we will label M_d^k, with minimum degree $d-1$, edge-connectivity k ($7 \le k \le d-2$) and no immersion of K_d.

Definition 3.3.17. *For $d \ge 9$ form the graph M_d^k using $n \ge 1$ bays that are isomorphic to K_{d-2}, connect consecutive bays as in Construction 3.3.9. Make the bays full with copies of the d-pod P_d^k.*

Theorem 3.3.18. *The graph M_d^k has minimum degree $d-1$ and no immersion of K_d.*

Proof. Lemma 3.3.16 tells us the P_d^k are d-pods, then by construction M_d^k is a d-dock in which every bay is full. Applying Theorem 3.2.6 we see M_d^k is a graph with minimum degree $d-1$ and no immersion of K_d. □

Note that, M_d^k is k-edge-connected. In fact Definition 3.3.15 tells us that we can create examples that are $(d-2)$-edge-connected for $d = 9$ and $d \ge 11$. We give the following special example for $d = 10$ and G is 8-edge-connected.

Example: Let P be a K_{11} with two disjoint 3-cycles, a disjoint edge, and a disjoint path of length two removed. P is shown in Figure 3.11. The graph P has 7 vertices of degree 8 and 4 vertices of degree 9. Let $A = \{v \in V(P) \mid d(v) = 8\} \cup \{w\}$, where w is one of the vertices of the missing edge. Let $B = V(P) - A$. Use the same docks and bays as those used for the M_d^k defined in 3.3.17. Make the bays full by copies of P where we add one edge from each vertex in A to a bay. Notice

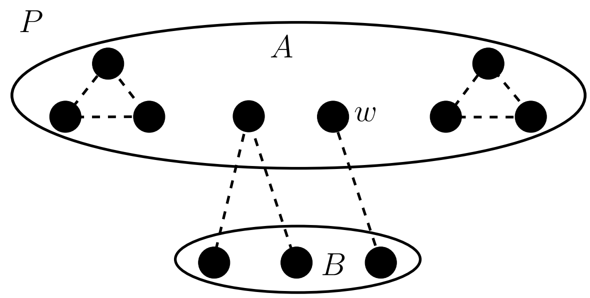

Figure 3.11: Only missing edges are shown as dotted lines, all other edges are present in the graph.

that, P is not quite a pod because we are adding edges from a vertex of degree 9 to the bays. Let this new graph be called G.

Lemma 3.3.19. *G has minimum degree 9, is 8-edge-connected, and has no immersion of K_{10}.*

Proof. G has minimum degree 9 and is 8-edge-connected by construction. Suppose G has an immersion of K_{10}. Using the Corner Separating Lemma we see that all of the corner vertices would be in the dock or in a single copy of P. We know from Theorem 3.2.6 that all of the corners cannot be in the dock. Suppose all of the corners are in a copy of P. Then there is exactly one peg, label the peg x. Suppose $x \in A$. The edges out of P can be used to replace at most one edge in each of the missing 3-cycles. Therefore, there are at least two vertices, say y and z, in A distinct from x that are incident with two missing edges. We know $xy, xz \in E(G)$. To replace all four missing edges we must have another path from x to y and an edge disjoint path from x to z. Each of these paths must use the edge out of P that is incident with x, but there is only one such edge. Therefore $x \notin A$.

Suppose $x \in B$. Then again there is at least one vertex in A, say y, that is incident with two missing edges, each of which must be replaced by an edge disjoint path through x. However, there is only one unused edge incident with y, so at most one of these edges can be replaced, that is $x \notin B$. Thus there is no immersion of K_{10} in G. $\square$

We have now given examples of graphs that have minimum degree $d-1$ that are $(d-2)$-edge-connected, have chromatic number $d-2$, and have no immersion of K_d for $d \geq 9$. In the next section we compare our examples with those given in [7].

3.4 Comparison of Examples

We now compare our examples with those given in [7] where the authors prove the following theorem.

Theorem 3.4.1. *[7] Suppose $H_1, \ldots, H_t$ are simple D-regular graphs, each with chromatic index $D+1$, where $t > \frac{1}{2}D(D+1)$. Let G be the complement of the graph formed by taking the disjoint union of $H_1, \ldots, H_t$. Letting n denote the number of vertices of G, the minimum degree of G is $n-1-D$, but G does not contain an immersion of the complete graph on $n-D$ vertices.*

Recall Seymour's example, discussed in Section 2.4 and shown in Figure 2.8, was a K_{12} minus four disjoint triangles. This is an example of a graph with minimum degree 9 and no immersion of K_{10}. It is an example of the type described in the above theorem, with $t = 4, D = 2$, and $n = 12$.

We have been discussing graphs with minimum degree $d-1$ and no immersion of K_d, in the language of the above theorem $d = n - D$. Theorem 3.4.1 gives examples that show for $d = 10$ and $d \geq 12$ there are graphs with minimum degree $d-1$ that contain no immersion of K_d. We give examples for $d \geq 8$. We will show that the theorem does not include 8, 9, or 11, but first we show there are only a finite number of examples for any given d that satisfy Theorem 3.4.1.

Lemma 3.4.2. *For a fixed d there are a finite number of examples of the type in Theorem 3.4.1.*

Proof. Let G be a graph of the type described in Theorem 3.4.1, then $d = n - D$ and $n = |V(G)| = \sum_{i=1}^{t} |V(H_i)|$. In order for a graph to be D-regular it must have at least $D+1$ vertices, thus we get the following

$$n = \sum_{i=1}^{t} |V(H_i)| \geq (D+1)t \tag{3.1}$$

$$n = d + D \geq (D+1)\left(\frac{1}{2}D(D+1)+1\right) \tag{3.2}$$

$$d + D \geq \frac{1}{2}D^3 + D^2 + \frac{3}{2}D + 1 \tag{3.3}$$

$$d \geq \frac{1}{2}D^3 + D^2 + \frac{1}{2}D + 1 \tag{3.4}$$

$$0 \geq \frac{1}{2}D^3 + D^2 + \frac{1}{2}D + 1 - d. \tag{3.5}$$

The roots of $\frac{1}{2}D^3 + D^2 + \frac{1}{2}D + 1 - d$ are

$$\begin{aligned} D =& \frac{1}{3}\sqrt[3]{3\sqrt{3}\sqrt{27d^2 - 52d + 25} + 27d - 26} + \\ & \frac{1}{\sqrt[3]{3\sqrt{3}\sqrt{27d^2 - 52d + 25} + 27d - 26}} - \frac{2}{3} \\ D =& -\frac{1}{6}(1 - i\sqrt{3})\sqrt[3]{3\sqrt{3}\sqrt{27d^2 - 52d + 25} + 27d - 26} - \\ & \frac{1 + i\sqrt{3}}{\sqrt[3]{3\sqrt{3}\sqrt{27d^2 - 52d + 25} + 27d - 26}} - \frac{2}{3} \end{aligned}$$

$$D = -\frac{1}{6}(1+i\sqrt{3})\sqrt[3]{3\sqrt{3}\sqrt{27d^2-52d+25}+27d-26}-$$
$$\frac{1-i\sqrt{3}}{\sqrt[3]{3\sqrt{3}\sqrt{27d^2-52d+25}+27d-26}}-\frac{2}{3}$$

Since there is only one real root and the function is positive for D greater than the real root, for any fixed d there are only a finite number of positive integer solutions to Inequality (3.5). Therefore, given d there are only a finite number of examples of the type described in Theorem 3.4.1. □

For fixed d, our constructions allow for an infinite number of examples because we can have as many bays and pods as we like.

The proof of Lemma 3.4.2 helps us to see that Theorem 3.4.1 gives examples for $d = 10$ and $d \geq 12$. When $d \leq 9$ the only positive integer solution to Inequality (3.5) is $D = 1$. When $D = 1$, the H_i would be 1-regular graphs with chromatic index 2, but there are no such graphs (1-regular graphs are disjoint unions of edges and have chromatic index 1). Thus, $D = 1$ does not yield an example. If $d = 11$ Inequality (3.5) gives

$$0 \geq \frac{1}{2}D^3 + D^2 + \frac{1}{2}D - 10.$$

The only positive integers for which this is true are $D = 1$ and $D = 2$. We know that $D = 1$ yields no examples. When $D = 2$, the H_i are 2-regular graphs with chromatic index 3 and $t > \frac{1}{2}(2)(2+1) = 3$. Theorem 3.4.1 will yield an example G with minimum degree $n-1-2 = n-3$ and no immersion of K_{n-2}, Where n is the number of vertices in G. Since $d = 11$, we are looking for an example with minimum degree 10 and no immersion of K_{11}, therefore we must have $n = 13$. The smallest 2-regular graph with chromatic index 3 is K_3 which has 3 vertices. For there to be an example $t = 4$. However, there are not four 2-regular graphs with chromatic index 3 that would give a graph with 13 vertices. Thus there is no example of the type described in Theorem 3.4.1 for $d = 11$. For $d = 10$ and $d \geq 12$ examples can be formed using $D = 2$, $t = 4$, and each H_i some odd cycle.

Another difference between our examples and those given in [7] is the chromatic number. In the previous sections we gave examples with chromatic number $d - 3$ that have immersions of K_{d-1}, examples with chromatic number $d - 2$ that have subgraphs, and thus immersions, of K_{d-2}, and examples with chromatic number $d-1$ and subgraphs, and thus immersions, of K_{d-1}. We now consider the chromatic number of the graphs described in Theorem 3.4.1. We know that for each H_i, $\chi(H_i) \leq D + 1$ because each H_i is D-regular. Therefore, $\bar{H}_i$ (the complement of H_i) is $[|V(H_i)| - (D + 1)]$-regular and thus

$$\chi(\bar{H}_i) \leq |V(H_i)| - (D + 1) + 1 = |V(H_i)| - D.$$

Since G is the complement of the disjoint union of the H_i we have,

$$\begin{aligned}
\chi(G) &= \sum_{i=1}^{t} \chi(\bar{H}_i) \\
&\leq \sum_{i=1}^{t} (|V(H_i)| - D) \\
&= n - tD \\
&= n - tD - D + D \\
&= n - D - (t - 1)D \\
\chi(G) &\leq n - D - (t - 1)D.
\end{aligned}$$

Recall, $d = n - D$, so if we can show that $(t - 1)D > 2$ we will have shown that the chromatic number of the examples given in [7] is smaller than the chromatic number of the examples that we presented in the previous sections where for any d we could give an example with chromatic number $d-2$.

$$\begin{aligned}(t-1)D &> \left(\frac{1}{2}D(D+1)-1\right)D \\ &> \left(\frac{1}{2}D^2+\frac{1}{2}D-1\right)D \\ &> \frac{1}{2}D^3+\frac{1}{2}D^2-D \\ &\geq 4 \text{ for } D \geq 2.\end{aligned}$$

So,

$$\chi(G) \leq n - D - 4 = d - 4 < d - 2.$$

Since $D \geq 2$ in all of the graphs in Theorem 3.4.1 the chromatic number of these graphs is smaller than the chromatic number of our graphs. The main conjecture, Conjecture 2.3.6, is that every d-chromatic graph contains an immersion of K_d. Our examples show that chromatic number $d-1$ is not large enough to give an immersion of K_d.

In the next chapter we will prove there is an infinite class of graphs satisfying Conjecture 2.3.6 and discuss ideas for future work.

Chapter 4

Conclusion

4.1 Hajós' Construction

While we have not resolved more cases of Conjecture 2.3.6, we note that there is an infinite class of graphs satisfying Conjecture 2.3.6. This can be seen by considering Hajós's Construction [15] which gives a method for constructing non-k-colorable graphs from non-k-colorable graphs, and in fact every non-k-colorable graphs can be constructed by beginning with a K_{k+1}. Before looking at Hajós' construction we must first prove the following about a specific class of graphs.

Lemma 4.1.1. *[18] Let $\mathcal{K} \neq \emptyset$ be a class of simple graphs satisfying:*

(i) If $G \in \mathcal{K}$ and G has a homomorphism into G', then $G' \in \mathcal{K}$ and

(ii) if G is a graph and $a, b, c \in V(G)$ are such that a, c are adjacent, but b is not adjacent to either a or c, and if $G + ab \in \mathcal{K}$ and $G + bc \in \mathcal{K}$, then $G \in \mathcal{K}$.

Then $\mathcal{K}$ consists of all non-k-colorable graphs for some $k \geq 0$.

Proof. [18] Let $\mathcal{K}$ be a nonempty class of simple graphs satisfying (i) and (ii). We claim, $\mathcal{K}$ contains some complete graphs. For example, if $G \in \mathcal{K}$ and $\chi(G) = n$, then there is a homomorphism from $G \to K_n$, so $K_n \in \mathcal{K}$. In fact, if G is d-colorable there is a homomorphism from $G \to K_d$, so $K_d \in \mathcal{K}$. So we know $\mathcal{K}$ contains some complete graphs.

Let K_{k+1} be the least (in terms of number of vertices) complete graph in $\mathcal{K}$. Let $G \in \mathcal{K}$ and suppose G is k-colorable. Let c be a k-coloring of G, then c gives a homomorphism of G into K_k. Therefore by (i) $K_k \in \mathcal{K}$ contradicting that K_{k+1} is the least complete graph in $\mathcal{K}$. This shows if $G \in \mathcal{K}$, then G is non-k-colorable. Now we need to show every non-k-colorable graph is in $\mathcal{K}$.

Suppose G is a non-k-colorable graph and G is a maximum graph such that $G \notin \mathcal{K}$, that is connecting any pair of non-adjacent vertices in G by an edge results in a graph that is in $\mathcal{K}$. First we claim being non-adjacent is an equivalence relation on $V(G)$.

- If $a \in V(G)$, then a is not adjacent to a because the graph G is simple. Non-adjacency is reflexive.

- If $ab \notin E(G)$, then $ba \notin E(G)$, because our graphs are undirected. Non-adjacency is symmetric.

- Suppose for a contradiction that non-adjacency is not transitive, that is $ab, bc \notin E(G)$ and $ac \in E(G)$. Since G is maximal $G + ab \in \mathcal{K}$ and $G + bc \in \mathcal{K}$, then by (ii) $G \in \mathcal{K} \Rightarrow\Leftarrow$. Therefore if $ab, bc \notin E(G)$, then $ac \notin E(G)$. Non-adjacency is transitive.

We have shown being non-adjacent is an equivalence relation on $V(G)$.

Using the equivalence relation of non-adjacency there is a partition of $V(G)$, $\{V_1, V_2, \ldots, V_m\}$ such that two vertices are adjacent if and only if they belong to distinct classes V_i and V_j, $i \neq j$. Since G is not k-colorable, $m \geq k + 1$. Thus

G contains a K_{k+1}, so there is a homomorphism from K_{k+1} into G, then by (i) $G \in \mathcal{K} \Rightarrow\Leftarrow$. Thus, $\mathcal{K}$ is the class of non-k-colorable graphs. $\square$

Now let us consider Hajós' construction.

Lemma 4.1.2 (Hajós' Construction)**.** *[18] Consider the following operations on simple graphs*

(α) *Addition of edges and/or vertices.*

(β) *Identification of two non-adjacent vertices and cancelation of multiple edges.*

(γ) *For two graphs G_1, G_2 and $x_iy_i \in E(G_i)$, removal of x_iy_i $(i = 1, 2)$, addition of a new edge y_1y_2, and identification of x_1 and x_2.*

These operations produce non-k-colorable graphs from non-k-colorable ones and every non-k-colorable graph arises by the repetition of these operations, from the initial graph K_{k+1}.

Proof. [18] First prove these operations produce non-k-colorable graphs from non-k-colorable graphs. Let G be a non-k-colorable graph with $e \notin E(G)$ and $v \notin V(G)$. Since G is non-k-colorable $\chi(G) > k$.

(α) $\chi(G + e) \geq \chi(G) > k$ and $\chi(G + v) \geq \chi(G) > k$. Addition of edges and/or vertices to a non-k-colorable graph produces a non-k-colorable graph.

(β) Suppose $x, y \in V(G)$ and $xy \notin E(G)$. Let G' be the graph obtained by identifying x and y and deleting multiple edges. Let the vertex obtained by identifying x and y be denoted z. If $a \in V(G), a \neq z$, then we denote a in $V(G')$ as a'. Assume for a contradiction that G' is k-colorable. Let c' be a k-coloring of G'. Let c be a coloring of G defined in the following way: For $a' \neq z$, let $c(a) = c'(a')$, $c(x) = c'(z)$, and $c(y) = c'(z)$. Since $xy \notin E(G)$, c is a proper k-coloring of $G \Rightarrow\Leftarrow$. Therefore in a non-k-colorable graph identification of non-adjacent vertices and cancelation of multiple edges produces a non-k-colorable graph.

(γ) Let G_1, G_2 be non-k-colorable graphs with $x_i y_i \in E(G_i)$. Let G be the graph obtained by deleting $x_i y_i$ $(i = 1, 2)$, adding the edge $y_1 y_2$, and identifying x_1 and x_2. Assume for a contradiction that G is k-colorable. Let c be a k-coloring of G. Use the coloring c on G_1 and G_2. The coloring c assigns the same color to x_1 and x_2. Since $y_1 y_2 \in E(G), c(y_1) \neq c(y_2)$, therefore one of y_1 or y_2 is assigned a different color than x_1 and x_2. Without loss of generality $c(x_1) \neq c(y_1)$, so c is a k-coloring of $G_1 \Rightarrow\Leftarrow$. Therefore performing ($\gamma$) on a non-$k$-colorable graph produces a non-k-colorable graph.

We have shown the operations $(\alpha), (\beta)$, and (γ) produce non-k-colorable graphs from non-k-colorable graphs. Now we must prove every non-k-colorable graph arises from repetition of these operations from the initial graph K_{k+1}. Let $\mathcal{K}$ be the class of all graphs obtained by repeated application of $(\alpha), (\beta)$, and (γ) on the initial graph K_{k+1}. We claim $\mathcal{K}$ satisfies the conditions of Lemma 4.1.1, therefore $\mathcal{K}$ is the class of all non-k-colorable graphs. $K_{k+1} \in \mathcal{K}$, therefore $\mathcal{K} \neq \emptyset$.

(i) Let $G \in \mathcal{K}$ and φ be a homomorphism from G into G'. We want to show $G' \in \mathcal{K}$, that is G' can be obtained by repeated applications of $(\alpha), (\beta)$, and (γ) on the initial graph K_{k+1}. We have a homomorphism $\varphi : V(G) \to V(G')$, so G' is obtained from G by identifying vertices as described in the homomorphism and adding vertices and edges as necessary. This is just repetition of the operations (α) and (β) on G, therefore $G' \in \mathcal{K}$. Thus, if $G \in \mathcal{K}$ and G has a homomorphism into G', then $G' \in \mathcal{K}$.

(ii) Let G be a graph with $a, b, c \in V(G)$, $ac \in E(G)$, and $ab, bc \notin E(G)$ such that $G + ab \in \mathcal{K}$ and $G + bc \in \mathcal{K}$. We want to show that $G \in \mathcal{K}$. Let G' be isomorphic to G and $x' \in V(G')$ corresponds to $x \in V(G)$ for all x. Then $G' + b'c' \in \mathcal{K}$. We will perform (γ) on the graphs $G + ab$ and $G' + b'c'$. Delete the edges ab and bc, identify b and b' and add the edge ac', this is the operation (γ) using $G_1 = G + ab, G_2 = G' + b'c', x_1y_1 = ba$, and $x_2y_2 = b'c'$. In the graph obtained from performing (γ) identify x and x'. Notice x and x' are not adjacent, so we may identify these vertices, this operation is (β). The resulting graph is isomorphic to G, therefore $G \in \mathcal{K}$.

Thus, $\mathcal{K}$ is the class of all non-k-colorable graphs and every non-k-colorable graph is obtained by repeated application of $(\alpha), (\beta)$, and (γ) on the initial graph K_{k+1}.

□

Given this construction a possible approach to proving Conjecture 2.3.6 is to show that immersion is preserved by the operations.

Lemma 4.1.3. *Immersions of K_d are preserved by applications of (α) and (γ).*

Proof. Let G be a graph with an immersion of K_d. Adding edges and/or vertices to this graph does not change the immersion of K_d. Thus, (α) preserves the immersion.

Let G_1 and G_2 be graphs, each of which has an immersion of K_d. Let $x_1y_1 \in E(G_1)$ and $x_2y_2 \in E(G_2)$. Let H be the graph obtained by applying (γ).

If there is a path P_1 from x_1 to y_1 in $G_1 - x_1y_1$, then there is an immersion of K_d in H using the immersion of K_d in G_2 and replacing x_2y_2 by $P_1 + y_1y_2$, if necessary. Similarly, if there is a path P_2 from x_2 to y_2 in $G_2 - x_2y_2$, then there is an immersion of K_d in H using the immersion of K_d in G_1 and replacing x_1y_1 by $P_2 + y_1y_2$, if necessary.

If there is no P_1 in $G_1 - x_1y_1$ and no P_2 in $G_2 - x_2y_2$, then x_1y_1 is a cut-edge in G_1 and x_2y_2 is a cut-edge in G_2. Since these are cut-edges the Corner Separating Lemma 3.1.1 tells us that the corners in the immersion of K_d in G_i are all on one side of the graph, that is the immersion does not use the edge x_iy_i ($i \in \{1,2\}$). Thus, the immersions of K_d in G_i are immersions of K_d in H ($i \in \{1,2\}$). □

This lemma tells us that the class of non-k-colorable graphs obtained from applying operations (α) and (γ), starting with a K_{k+1}, satisfy Conjecture 2.3.6. To prove Conjecture 2.3.6, one would have to prove that immersions of K_d are preserved by the operation (β). Proving this seems quite complicated. Since (β) is the identification of any two non-adjacent vertices, if two non-adjacent corners are identified we would need to show that another vertex in the new graph could become a corner. It seems that, while applying (α) and (γ) result in a graph with a very similar immersion to the original graph (or graphs), applying (β) could result in a very different immersion than that in the original graph.

4.2 Future Work

Given that our Corner Separating Lemma 3.1.1 relies on there being at most $d-2$ edges between different parts of the graph a different approach would be needed to give examples with edge-connectivity greater than $d-2$. This has led us to ask the following questions.

1. Do graphs with large connectivity have to have an immersion of a large complete graph?

   This would not help in proving the conjecture of Abu-Khzam and Langston, since graphs with large chromatic number may have small connectivity, but might shed some light on the structure necessary to have an immersion of a large complete graph.

2. Are there d-pods with more than $d+1$ vertices and what do they look like? All of our examples of pods have exactly $d+1$ vertices, so exactly one peg. So far we have not been able to find examples with more vertices. We have the following lemmas about pods with more vertices.

Lemma 4.2.1. *Let G be a graph with $2d$ vertices, minimum degree $d-2$, with exactly $d-2$ vertices of degree $d-2$ and exactly $d-2$ vertices of degree $2d-1$. Then G is not a d-pod.*

Proof. Notice that, G satisfies the degree requirements to be a d-pod, however, we will find an immersion of K_d by adding a matching to the vertices of degree $d-2$.

Let $U = \{v \in V(G) \mid d(v) = d-2\}$ and $V = \{v \in V(G) \mid d(v) = 2d-1\}$. Then $|U| = |V| = d-2$. Notice that, the vertices of V form a K_{d-2}. This is because in order for a vertex $v \in V$ to have degree $2d-1$ it must be incident with every other vertex because there are only $2d$ vertices. Similarly, for all $u \in U$ and $v \in V$, $uv \in E(G)$ since $d(v) = 2d-1$. Since each $u \in U$ is adjacent to all of the vertices in V, which has size $d-2$, the set U is an independent set of vertices. We form an immersion of K_d using each $v \in V$ as a corner and two of the vertices in U as corners, say u_1 and u_2. The only edge that is not already present between these vertices is the edge u_1u_2, this edge can be replaced my a matching edge. Therefore G has an immersion of K_d when a matching is added to the vertices of degree $d-2$, and thus is not a d-pod. □

Lemma 4.2.2. *Let G be a graph with $2d$ vertices and minimum degree $d-2$, with* *exactly $d-2$ vertices of degree $d-2$ and exactly $d-3$ vertices of degree $2d-1$. Then G is not a d-pod.*

Proof. The graph G satisfies the degree requirements to be a d-pod, however we will be able to find an immersion of K_d when a matching is added to the vertices of degree $d-2$. Notice that, since the graph has minimum degree $d-2$ and has $2d$ vertices $d-2 \leq d(v) \leq 2d-1$ for all $v \in V(G)$.

Let $U = \{u \in V(G) \mid d(u) = d-2\}$, $V = \{v \in V(G) \mid d-1 \leq d(v) \leq 2d-2\}$, and $W = \{w \in V(G) \mid d(w) = 2d-1\}$. Then $|U| = d-2$, $|V| = 5$, and $|W| = d-3$. Say $U = \{u_1, u_2, \ldots, u_{d-2}\}$, $V = \{v_1, v_2, v_3, v_4, v_5\}$, and $W = \{w_1, w_2, \ldots, w_{d-3}\}$. There are two cases to consider (1) two distinct elements $u_i \in U$ are adjacent to the same $v_k \in V$ and (2) each v_k is adjacent to at most one u_i.

(1) Suppose $u_i, u_j, i \neq j$, are both adjacent to some v_k. Then there is an immersion of K_d using $\{u_i, u_j, v_k, w_1, \ldots, w_{d-3}\}$ as the corners of the immersion. All edges between these corners are in G, except maybe u_iu_j which can be filled in with a matching edge. Note that, all other edges are present because each vertex in W is adjacent to every other vertex in G because these vertices have degree $2d-1$.

(2) Suppose v_k is incident with at most one element of U for $k = 1, 2, 3, 4, 5$. This means there is at least one edge between vertices in U, without loss of generality say $u_1u_2 \in E(G)$.

   (a) If there are two edges in U, say $u_1u_2, u_3u_4 \in E(G)$, then there is an immersion of K_d using $\{u_1, u_2, u_3, w_1, w_2, \ldots, w_{d-3}\}$ as corners. All edges are present except maybe u_1u_3 and u_2u_3. We can use a matching edge between u_1 and u_3 and the path $u_2 - u_4 - u_3$, where the edge between u_2 and u_4 that is used in this path is a matching edge.

(b) If there is exactly one edge in U, say $u_1u_2 \in E(G)$, then for each vertex in U to have degree $d-2$ each of $u_3, u_4, \ldots, u_{d-2}$ is adjacent to some v_i. For $d \geq 10$ this would mean two distinct u_i are adjacent to the same v_j putting us back into Case (1). For $d = 8, 9$ two elements of V that are adjacent to elements of U must also be adjacent to each other. Without loss of generality say $u_3v_1, u_4v_2, v_1v_2 \in E(G)$. Then there is an immersion of K_d using $\{u_1, u_2, u_3, w_1, w_2, \ldots w_{d-2}\}$ as corners. All edges are present except maybe u_1u_3 and u_2u_3. Replace u_1u_3 by a matching edge and replace u_2u_3 by the path $u_2 - u_4 - v_2 - v_1 - u_3$ where the edge u_2u_4 on this path may be a matching edge.

□

Perhaps graphs with $2d$ vertices are never pods, but more work still needs to be done on this.

4.3 Conjecture

Finally, we conclude with a conjecture. A **strong immersion** of a graph means that the corner vertices of the immersion are not also used as pegs in the immersion. In [7] the authors prove

Theorem 4.3.1 ([7])**.** *Every simple graph with minimum degree at least* $200d$ *contains a strong immersion of* K_d.

This made us question how many vertices in a graph can have degree md, for m any positive integer, and still have no immersion of K_d?

Suppose we want the vertices in all of the bays to have degree md, m a positive integer. To do this we can add pods P_d^{d-2} to single vertices in a bay. If v is a vertex in a bay, then $d(v) \geq d-2+(d-2)p = (d-2)(1+p)$, where p is the number of pods P_d^{d-2} added by connecting all edges out of the pod to v. If we want v to have degree md, then

$$\begin{aligned}(d-2)(1+p) &= md\\ 1+p &= \frac{md}{d-2}\\ p &= \frac{md}{d-2}-1\\ p &= \frac{md-2m+2m}{d-2}-1\\ p &= \frac{m(d-2)+2m}{d-2}-1\\ p &= (m-1)+\left\lceil\frac{2m}{d-2}\right\rceil.\end{aligned}$$

If the number of bays is b, then the number of vertices in G is

$$\begin{aligned}|V(G)| &= (d-2)b+(d-2)bp(d+1)\\ |V(G)| &= (d-2)b(1+p(d+1)).\end{aligned}$$

The number of vertices of degree md is $(d-2)b$ (all of the vertices in bays).

$$\begin{aligned}(d-2)b &= \frac{|V(G)|}{1+p(d+1)}\\ (d-2)b &= \frac{|V(G)|}{1+\left((m-1)+\left\lceil\frac{2m}{d-2}\right\rceil\right)(d+1)}\end{aligned}$$

So, we can construct graphs with $\frac{|V(G)|}{1+\left((m-1)+\left\lceil\frac{2m}{d-2}\right\rceil\right)(d+1)}$ vertices of degree md, minimum degree $d-1$, and no immersion of K_d. Notice that if d is large compared to m, then $\frac{2m}{d-2}$ is small, so $\left\lceil\frac{2m}{d-2}\right\rceil = 1$. One might think this would mean, if d is large compared to m and a graph G has more than $\frac{|V(G)|}{1+m(d+1)}$ vertices of degree at least md, then G has an immersion of K_d. However, if we consider a graph that has a K_{d-1} in the center, then for all of the vertices in the K_{d-1} to have degree md we need to add a total of P pods, P_d^{d-2}, where

$$\begin{aligned} P(d-2) &\geq (d-1)(md-d+2) \\ P &\geq \frac{(d-1)(md-d+2)}{d-2}. \end{aligned}$$

This means that,

$$\begin{aligned} |V(G)| &= d-1+P(d+1) \\ &\geq d-1+\frac{(d-1)(md-d+2)}{d-2}(d+1) \\ &= (d-1)\left(1+\frac{(d+1)(md-d+2)}{d-2}\right). \end{aligned}$$

Suppose $|V(G)| = (d-1)\big(1+\frac{(d+1)(md-d+2)}{d-2}\big)$. Then,

$$\frac{|V(G)|}{1+m(d+1)} = \frac{(d-1)}{1+m(d+1)}\left(1+\frac{(d+1)(md-d+2)}{d-2}\right).$$

If this quantity is smaller than $d-1$ (the number of vertices of degree md), then we can create an example with more than $\frac{|V(G)|}{1+m(d+1)}$ vertices have degree at least md that has no immersion of K_d.

$$\begin{aligned} \frac{(d-1)}{1+m(d+1)}\left(1+\frac{(d+1)(md-d+2)}{d-2}\right) &< d-1 \\ 1+\frac{(d+1)(md-d+2)}{d-2} &< 1+m(d+1) \\ \frac{(d+1)(md-d+2)}{d-2} &< m(d+1) \end{aligned}$$

$$(d+1)(md-d+2) < m(d+1)(d-2)$$

$$md-d+2 < md-2m$$

$$2m < d-2$$

$$m < \frac{d-2}{2}$$

Thus, for any m we can find examples with more than $\frac{|V(G)|}{1+m(d+1)}$ vertices of degree md and no immersion of K_d.

Perhaps adding $d-1$ to the fraction would be enough. That is, let m be a positive integer, and $d \geq 8$ large compared to m. If a graph G has minimum degree $d-1$ and more than $\frac{|V(G)|}{1+m(d+1)} + (d-1)$ vertices have degree at least md, then does G have an immersion of K_d? Now the examples with K_{d-1} in the center are no longer a problem. Let us consider a graph with b bays. Then we would create a graph with $b(d-2)$ vertices of degree md by adding a total of $P = b(md-d+2)$ pods. Then $|V(G)| = b(d-2)+b(md-d+2)(d+1)$. If $b(d-2)$ is bigger than $\frac{|V(G)|}{1+m(d+1)} + (d-1)$, then we will have created an example with more than $\frac{|V(G)|}{1+m(d+1)} + (d-1)$ vertices of degree md with no immersion of K_d.

$$\frac{b(d-2)+b(md-d+2)(d+1)}{1+m(d+1)} + (d-1) < b(d-2)$$

$$\frac{b(d-2)+b(md-d+2)(d+1)}{1+m(d+1)} < b(d-2)-d+1$$

$$bd-2b+b(md^2+md-d^2-d+2d+1) < (bd-2b-d+1)(1+md+m)$$

$$-bd^2+2bd-b < bd-2b-2mbd-2mb-d-md^2+1+m$$

$$md^2+mbd+2mb-m < bd^2-2bd+b+bd-2b-d+1$$

$$m(d^2+bd+2b-1) < bd^2-bd-b-d+1$$

$$\begin{aligned} m &< \frac{bd^2 - bd - b - d + 1}{d^2 + bd + 2b - 1} \\ &< \frac{bd^2 + 1}{d^2} \\ &\le \frac{bd^2 + d^2}{d^2} = b + 1 \end{aligned}$$

Thus, if m is at most b, then we can construct examples with more than $\frac{|V(G)|}{1+m(d+1)} + (d-1)$ vertices of degree md that have no immersion of K_d. So, it is not enough to add $d-1$ to our fraction.

Let us now suppose we have a graph, G, created by docks, bays, and pods in which enough pods are added to give the vertices in the bays degree β. We will consider what fraction of the vertices, α, have degree β. If v is a vertex in a bay, then it will have degree β. To achieve this we need to had $\mathcal{P}$ pods to each bay, where

$$\begin{aligned} \mathcal{P}(d-2) &= (d-2)(\beta - d + 2) \\ \mathcal{P} &= \beta - d + 2. \end{aligned}$$

If there are b bays, the total number of pods that must be added to G is

$$P = b(\beta - d + 2).$$

So, the number of vertices is

$$\begin{aligned} |V(G)| &= b(d-2) + P(d+1) \\ &= b(d-2) + b(d+1)(\beta - d + 2). \end{aligned}$$

There are $b(d-2)$ vertices of degree β, so we want to find the α that gives

$\alpha|V(G)| = b(d-2)$.

$$\begin{aligned}
\alpha|V(G)| &= b(d-2) \\
\alpha(b(d-2) + b(d+1)(\beta - d + 2)) &= b(d-2) \\
\alpha((d-2) + (d+1)(\beta - d + 2)) &= d-2 \\
\alpha &= \frac{d-2}{(d-2) + (d+1)(\beta - d + 2)} \\
\alpha &= \frac{d-2}{-d^2 + \beta d + 2d + \beta}
\end{aligned}$$

If $\beta = md$, then

$$\begin{aligned}
\alpha &= \frac{d-2}{-d^2 + (md)d + 2d + md} \\
&= \frac{1 - \frac{2}{d}}{-d + md + 2 + m} \\
&> \frac{1}{m(d+1) - (d-2)}.
\end{aligned}$$

The above analysis has led us to make the following conjecture.

Conjecture 4.3.2. *Let m be a positive integer and $d \geq 8$ an integer. If a graph G has minimum degree $d-1$ and more than $\frac{(|V(G)|}{m(d+1)-(d-2)}$ vertices of degree at least md, then G has an immersion of K_d.*

Bibliography

[1] Faisal N. Abu-Khzam and Michael A. Langston, *Graph coloring and the immersion order*, Computing and combinatorics, Lecture Notes in Comput. Sci., vol. 2697, Springer, Berlin, 2003, pp. 394–403. MR 2063516

[2] Thomas Andreae, *On self-immersions of infinite graphs*, J. Graph Theory **58** (2008), no. 4, 275–285. MR 2423447 (2009f:05244)

[3] Kenneth Appel and Wolfgang Haken, *The solution of the four-color-map problem*, Sci. Amer. **237** (1977), no. 4, 108–121, 152. MR 0543796 (58 #27598e)

[4] Paul A. Catlin, *A bound on the chromatic number of a graph*, Discrete Math. **22** (1978), no. 1, 81–83. MR 522914 (80a:05090a)

[5] Maria Chudnovsky, Alexandra Fradkin, and Paul Seymour, *Tournament immersion and cutwidth*, J. Combin. Theory Ser. B **102** (2012), no. 1, 93–101. MR 2871770 (2012m:05152)

[6] Matt DeVos, Ken-ichi Kawarabayashi, Bojan Mohar, and Haruko Okamura, *Immersing small complete graphs*, Ars Math. Contemp. **3** (2010), no. 2, 139–146. MR 2729363 (2011k:05119)

[7] Matt DeVos, Jessica McDonald, Bojan Mohar, and Diego Scheide, *Minimum degree forcing complete graph immersion*, Submitted to Combinatorica.

[8] ———, *Immersing complete digraphs*, European J. Combin. **33** (2012), no. 6, 1294–1302. MR 2921015

[9] G. A. Dirac, *A property of* 4*-chromatic graphs and some remarks on critical graphs*, J. London Math. Soc. **27** (1952), 85–92. MR 0045371 (13,572f)

[10] Michael Ferrara, Ronald J. Gould, Gerard Tansey, and Thor Whalen, *On H-immersions*, J. Graph Theory **57** (2008), no. 3, 245–254. MR 2384023 (2008j:05182)

[11] Petr A. Golovach, Marcin Kamiski, Danil Paulusma, and Dimitrios M. Thilikos, *Lift contractions*, Electronic Notes in Discrete Mathematics **38** (2011), no. 0, 407 – 412, ¡ce:title¿The Sixth European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2011¡/ce:title¿.

[12] Rajeev Govindan and Siddharthan Ramachandramurthi, *A weak immersion relation on graphs and its applications*, Discrete Math. **230** (2001), no. 1-3, 189–206, Paul Catlin memorial collection (Kalamazoo, MI, 1996). MR 1812351 (2001m:05242)

[13] R. P. Gupta, *The chromatic index and the degree of a graph (abstract 66t-429*, Not. Amer. Math. Soc. **13** (1966), 719.

[14] H. Hadwiger, *Über eine Klassifikation der Streckenkomplexe*, Vierteljschr. Naturforsch. Ges. Zürich **88** (1943), 133–142. MR 0012237 (6,281c)

[15] G. Hajós, *Über eine Konstruktion nicht n-farbbarer graphen*, Wiss. Z. Martin Luther Univ. Halle-Wittenberg Math. Naturwiss. Reihe **10** (1961), 116–117.

[16] Nancy G. Kinnersley, *Immersion order obstruction sets*, Proceedings of the Twenty-fourth Southeastern International Conference on Combinatorics, Graph Theory, and Computing (Boca Raton, FL, 1993), vol. 98, 1993, pp. 113–123. MR 1267345 (94m:05173)

[17] F. Lescure and H. Meyniel, *On a problem upon configurations contained in graphs with given chromatic number*, Graph theory in memory of G. A. Dirac (Sandbjerg, 1985), Ann. Discrete Math., vol. 41, North-Holland, Amsterdam, 1989, pp. 325–331. MR 976011 (90c:05089)

[18] László Lovász, *Combinatorial problems and exercises*, second ed., North-Holland Publishing Co., Amsterdam, 1993. MR 1265492 (94m:05001)

[19] J. Mycielski, *Sur le coloriage des graphs*, Colloq. Math. **3** (1955), 161–162. MR 0069494 (16,1044b)

[20] C. St. J. A. Nash-Williams, *On well-quasi-ordering infinite trees*, Proc. Cambridge Philos. Soc. **61** (1965), 697–720. MR 0175814 (31 #90)

[21] Neil Robertson and Paul Seymour, *Graph minors XXIII. Nash-Williams' immersion conjecture*, J. Combin. Theory Ser. B **100** (2010), no. 2, 181–205. MR 2595703 (2011f:05300)

[22] Neil Robertson, Paul Seymour, and Robin Thomas, *Hadwiger's conjecture for K_6-free graphs*, Combinatorica **13** (1993), no. 3, 279–361. MR 1238823 (94i:05037)

[23] V. G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz No. **3** (1964), 25–30. MR 0180505 (31 #4740)

[24] K. Wagner, *Beweis einer Abschwächung der Hadwiger-Vermutung*, Math. Ann. **153** (1964), 139–141. MR 0160202 (28 #3416)

[25] Douglas B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996. MR 1367739 (96i:05001)